# A counterexample to the fake degree conjecture

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Let $J$ be a finite dimensional nilpotent algebra over a finite field $\mathbb{F}$. Then the set $G = 1 + J$ forms a finite group. The groups constructed in this way will be called algebra groups. In [1] it is proved that the character degrees of $G$ are powers of $q$, where $q$ is the order of $\mathbb{F}$.

The group $G$ acts by conjugation on $J$. This induces an action of $G$ on the dual space $J^*$. It has been noted that there exists a relation between the characters of $G$ and the orbits of $J^*$. For example, if $J^p = 0$, there exists an explicit expression that gives a bijective correspondence between the characters of $G$ and the orbits of $J^*$ ([2]). In particular, when $J^p = 0$, we obtain that the character degrees of $G$, counting multiplicities, are the square roots of the sizes of the orbits of $J^*$. It was conjectured that the same holds also in the general case:

**Conjecture 1.** *(Fake degree conjecture) In every algebra group $G = 1 + J$ the character degrees coincide, counting multiplicities, with the square roots of the cardinals of the orbits of $J^*$.*

Note that the immediate corollary of this conjecture is that the orders of $[J, J]_L$ and $[1+J, 1+J]$ have to be equal (in this work we write $[a, b] = a^{-1}b^{-1}ab$ for group commutators and $[a, b]_L = ab - ba$ for Lie brackets). The purpose of this note is to show that there exists a finitely dimensional nilpotent $\mathbb{F}_2$-algebra $J$, such that the order of $[1 + J, 1 + J]$ is greater than the order of $[J, J]_L$, so in the given form Conjecture 1 is not true.

We fix our attention on graded algebras. If $J = \oplus_{i\in\mathbb{N}} J_i$ is a graded $\mathbb{F}$-algebra and $0 \neq a \in J_n$ is a homogeneous element, then we put $\deg a = n$.

**Lemma 1.** *Let $J = \oplus_{i\in\mathbb{N}} J_i$ be a finitely dimensional nilpotent graded algebra over a finite field $\mathbb{F}$ and suppose that $J_k = \{0\}$ for every $k > n$. If $a \in J_n \cap [J, J]_L$, then $1 + a \in [1 + J, 1 + J]$.*

*Proof.* Since $J$ is graded and the element $a$ is homogeneous, there are homogeneous elements $a_i, b_i \in J$ such that

$$a = \sum_i [a_i, b_i]_L$$

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and $\deg a_i + \deg b_i = \deg a = n$. Then

$$1 + a = \prod_i [1 + a_i, 1 + b_i] \in [1 + J, 1 + J].$$

□

**Corollary 1.** *If $J = \oplus_{i=1}^n J_i$ is a finitely dimensional nilpotent graded algebra over a finite field $\mathbb{F}$, then the order of $[1+J, 1+J]$ is at least the order of $[J, J]_L$.*

*Proof.* We will prove the corollary by induction on order of $J$. Let $q$ be the order of $\mathbb{F}$ and $0 \neq a \in J_n$. We define $\bar{J} = J/\mathbb{F}a$. Then we have two possibilities: $a \in [J, J]_L$ and $a \notin [J, J]_L$.

If $a \in [J, J]_L$, then, by the previous lemma, $1 + a \in [1 + J, 1 + J]$. Applying the inductive hypothesis, we have

$$|[1 + J, 1 + J]| = |[1 + \bar{J}, 1 + \bar{J}]|q \geq |[\bar{J}, \bar{J}]_L|q = |[J, J]_L|.$$

If $a \notin [J, J]_L$, then again by induction,

$$|[1 + J, 1 + J]| \geq |[1 + \bar{J}, 1 + \bar{J}]| \geq |[\bar{J}, \bar{J}]_L| = |[J, J]_L|.$$

□

Let $F$ be a free nilpotent associative $\mathbb{F}_2$-algebra of nilpotency index 5 ($F^5 = 0$) on 4 generators $x_1, x_2, x_3, x_4$. We have the next relation in $F$:

**Lemma 2.** *For every $x, y \in F$ the following equality holds:*

$$[1+x, 1+y] = 1 + [x, y]_L + [xy, x]_L + [y, yx]_L + [y^2x, y]_L + [x, x^2y]_L + xy[x, y]_L.$$

*Proof.* This an easy exercise. □

Define

$$t(x, y) = [xy, x]_L + [y, yx]_L + [y^2x, y]_L + [x, x^2y]_L.$$

Then $t(x, y) \in F^3 \cap [F, F]_L$ for any $x, y \in F$ and, by Lemma 2,

$$[1 + x, 1 + y] = 1 + [x, y]_L + t(x, y) + xy[x, y]_L.$$

We need the following lemma:

**Lemma 3.** *If $b \in F^3 \cap [F, F]_L$, then $1 + b \in [1 + F, 1 + F]$.*

*Proof.* Let $b = b_1 + b_2$, where $\deg(b_1) = 3$ and $\deg(b_2) = 4$. Then $1 + b = (1 + b_1)(1 + b_2)$. There are homogeneous elements $c_i, d_i \in F$ such that $b_1 = \sum_i [c_i, d_i]_L$ and $\deg c_i + \deg d_i = 3$. Thus,

$$\prod_i [1 + c_i, 1 + d_i] = (1 + b_1)(1 + \sum_i t(c_i, d_i)).$$

Since $b_2$, and $\sum_i t(c_i, d_i) \in F^4 \cap [F, F]_L$, using Lemma 1, we obtain that

$$1 + b_2 \text{ and } \prod_i [1 + c_i, 1 + d_i] \in [F, F].$$

Hence $1 + b_1 \in [F, F]$ and so $1 + b \in [1 + F, 1 + F]$. □

Let $I$ be the ideal of $F$ generated by $[x_1, x_2]_L + [x_3, x_4]_L$ and $J = F/I$. Since $J$ is generated by a homogeneous element, $J$ is a graded $\mathbb{F}_2$-algebra. If $z \in F$, then we write the element $z + I$ from $J$ as $\bar{z}$.

**Lemma 4.** *If $a = x_1x_2[x_1, x_2]_L + x_3x_4[x_3, x_4]_L$, then $1 + \bar{a} \in [1 + J, 1 + J]$.*

*Proof.* We have:

$$\begin{aligned}
[1 + x_1, 1 + x_2][1 + x_3, 1 + x_4] &= (1 + [x_1, x_2]_L + t(x_1, x_2) + x_1x_2[x_1, x_2]_L) \\
&\quad (1 + [x_3, x_4]_L + t(x_3, x_4) + x_3x_4[x_3, x_4]_L) \\
&= 1 + a + t(x_1, x_2) + t(x_3, x_4) + [x_1, x_2]_L[x_3, x_4]_L \\
&= (1 + a)(1 + b)(1 + [x_1, x_2]_L^2)(1 + c),
\end{aligned}$$

where

$$b = t(x_1, x_2) + t(x_3, x_4) \in F^3 \cap [F, F]_L$$

and

$$c = [x_1, x_2]_L([x_1, x_2]_L + [x_3, x_4]_L) \in I.$$

Thus, in order to prove that $1 + \bar{a} \in [1 + J, 1 + J]$, we have to show that $1 + b$ and $1 + [x_1, x_2]_L^2 \in [1 + F, 1 + F]$.

By Lemma 3, $1 + b \in [1 + F, 1 + F]$, and since

$$[x_1, x_2]_L^2 = [x_1 + x_2, x_2(x_1 + x_2)^2] + [x_1, x_2x_1^2]_L + [x_1x_2^2, x_2]_L \in [F, F]_L \cap F^4,$$

we obtain, by Lemma 1 that $1 + [x_1, x_2]_L^2 \in [1 + F, 1 + F]$. Hence $1 + \bar{a} \in [1 + J, 1 + J]$. □

Next, we are aiming to show that $\bar{a}$ does not belong to $[J, J]_L$. First, we define a linear map $\sigma$ by:

$$\begin{aligned}
\sigma: \quad F^4 &\rightarrow F^4 \\
x_ix_jx_kx_l &\mapsto x_jx_kx_lx_i.
\end{aligned}$$

We consider $F^4$ as an $\mathbb{F}_2[\sigma]$-module.

**Lemma 5.** *The following equalities hold:*

1. $(1 - \sigma)F^4 = F^4 \cap [F, F]_L$.
2. $(1 - \sigma)^3(F^4 \cap [F, F]_L) = 0$.

*Proof.* 1. From the definition of $\sigma$ it follows immediately that $(1 - \sigma)F^4 \subseteq F^4 \cap [F, F]_L$. Now, let $b \in F^4 \cap [F, F]_L$. Then $b$ is a linear combination of the elements $[x_i, x_jx_kx_l]_L = (1 - \sigma)(x_ix_jx_kx_l)$ and $[x_ix_j, x_kx_l]_L = (1 - \sigma^2)(x_ix_jx_kx_l)$. Hence $F^4 \cap [F, F]_L \subseteq (1 - \sigma)F^4$.

2. Since $(1 - \sigma)^4F^4 = 0$, we obtain $(1 - \sigma)^3(F^4 \cap [F, F]_L) = 0$. □

Let $F_{(i_1,i_2,i_3,i_4)}$ be the subspace of $F$ generated by the monomials in which each $x_k$ appears $i_k$ times. Then we have the next decomposition:

$$F^4 = \bigoplus_{i_1+i_2+i_3+i_4=4} F_{(i_1,i_2,i_3,i_4)}.$$

Let $\pi_{(i_1,i_2,i_3,i_4)}$ be the projection of $F^4$ onto $F_{(i_1,i_2,i_3,i_4)}$. Set $K = \{x_1, x_2, x_3, x_4\}$.

**Lemma 6.** *The element $a$ does not belong to $[F, F]_L + I$.*

*Proof.* We will prove the lemma by way of contradiction. Suppose that $a \in [F, F]_L + I$, i.e., there exists $c \in F^4 \cap [F, F]_L$ such that

$$a + \sum_{(k_1,k_2)\in K^2} \alpha_{k_1,k_2} k_1 k_2([x_1, x_2]_L + [x_3, x_4]_L) + c = 0.$$

Multiplying the equality by $(1 - \sigma)^3$, we obtain

$$d = (1-\sigma)^3(a + \sum_{(k_1,k_2)\in K^2} \alpha_{k_1,k_2} k_1 k_2([x_1, x_2]_L + [x_3, x_4]_L)) = 0. \qquad (1)$$

Since

$$\begin{aligned} \pi_{(2,2,0,0)}(d) &= (1-\sigma)^3((1+\alpha_{x_1,x_2})x_1x_2[x_1,x_2]_L + \alpha_{x_2,x_1}x_2x_1[x_1,x_2]_L) \\ &= (1+\alpha_{x_1,x_2}+\alpha_{x_2,x_1})(x_1^2x_2^2 + x_1x_2^2x_1 + x_2^2x_1^2 + x_2x_1^2x_2), \end{aligned}$$

we conclude that $\alpha_{x_1,x_2} + \alpha_{x_2,x_1} = 1$. The same argument gives $\alpha_{x_3,x_4} + \alpha_{x_4,x_3} = 1$. Suppose first that $\alpha_{x_1,x_2} = \alpha_{x_3,x_4} = 1$ and $\alpha_{x_2,x_1} = \alpha_{x_4,x_3} = 0$. Then we have

$$\begin{aligned} \pi_{(1,1,1,1)}(d) &= (1-\sigma)^3(x_3x_4[x_1,x_2]_L + x_1x_2[x_3,x_4]_L) \\ &= (1-\sigma)^3(x_3x_4(x_1x_2 - x_2x_1) + x_1x_2(x_3x_4 - x_4x_3)) \\ &= (1-\sigma)^3(x_3x_4x_2x_1 + x_1x_2x_4x_3) \neq 0. \end{aligned}$$

But this contradicts (1). The remaining cases can be proved in the same way. □

**Theorem 1.** *The order of $[1 + J, 1 + J]$ is greater than the order of $[J, J]_L$.*

*Proof.* Let $a$ be as above. Define $\bar{J} = J/\mathbb{F}_2\bar{a}$. We have that

$$|[\bar{J}, \bar{J}]_L| = |[J, J]_L|,$$

because, by the previous lemma, $\bar{a} \notin [J, J]_L$. By Corollary 1,

$$|[\bar{J}, \bar{J}]_L| \leq |[1 + \bar{J}, 1 + \bar{J}]|.$$

On the other hand, since $1 + \bar{a} \in [1 + J, 1 + J]$, the order of $[1 + \bar{J}, 1 + \bar{J}]$ is less than the order of $[1 + J, 1 + J]$. Hence the order of $[1 + J, 1 + J]$ is greater than the order of $[J, J]_L$. □

## References

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[2] J. Sangroniz, *Clases de conjugación y caracteres en grupos finitos*, Ph.D. Thesis, Universidad del País Vasco, Bilbao, 2001.