ANALYTIC GROUPS OVER GENERAL PRO-p DOMAINS

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ABSTRACT. While the theory of Lie groups is highly developed in characteristic 0, comparatively little is known about analytic groups over local fields of positive characteristic p. More generally, one can in fact consider groups which are analytic over pro-p domains R, without restricting the Krull dimension to 1. Natural examples are rings of the form $R = \mathbb{F}_p[\![t_1, \ldots, t_m]\!]$.

Experience shows that analytic pro-p groups, also in this general sense, possess interesting properties, and one faces the challenge of developing a systematic structure theory for such objects. With this long term goal in mind, we collect several fundamental results which provide information about the structure of R-analytic pro-p groups. In particular, we establish a useful criterion for isolating $\mathbb{F}_p[\![t]\!]$-analytic groups within the class of finitely generated pro-p groups.

1. Introduction and Summary of Results

1.1. **Introduction.** The study of p-adic Lie groups has led to a beautiful theory with many important applications; cf. [4]. At the same time we know little about the structure of groups which are analytic over pro-p domains R of positive characteristic or higher Krull dimension, e.g. rings of the form $R = \mathbb{F}_p[\![t_1, \ldots, t_m]\!]$ or $R = \mathbb{Z}_p[\![t_1, \ldots, t_m]\!]$. The definition of an analytic group over a complete discrete valuation ring appears in [3] and [18]; the concept over more general domains is defined in [4, Chapter 13].

At the heart of the success story of p-adic Lie groups, as presented in [4], lie various equivalent descriptions of the class of $\mathbb{Z}_p$-analytic pro-p groups in purely algebraic terms. Indeed, the problem of determining which pro-p groups are p-adic analytic can be considered as "Hilbert's fifth problem for p-adic Lie groups" and was first solved by Lazard in the 1960s. For more general pro-p domains R, in particular for $R = \mathbb{F}_p[\![t]\!]$, nobody has as yet discovered any similar characterization of the class of R-analytic pro-p groups, even on the conjectural level. Worse, hardly any useful group-theoretic conditions are known which have to be satisfied in order that a given pro-p group possibly admits an R-analytic structure. Such criteria would be of considerable interest, for instance, in connection with Boston's extension of the Fontain-Mazur conjecture, stating that Galois groups of certain maximal pro-p extensions of number fields have no infinite analytic quotients, also in positive characteristic; cf. [2, Section 3].

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There is considerable evidence that pro-p groups which are analytic over more general pro-p domains possess interesting algebraic properties. In their pioneering work [12], Lubotzky and Shalev very successfully studied R-perfect groups, a special class of R-analytic pro-p groups. Amongst other things they showed that R-perfect groups satisfy a Golod-Shafarevich inequality and they proved that the subgroup growth of R-perfect groups is not too far from polynomial. In [8], the first author has shown that finitely generated R-analytic pro-p groups are linear if $\operatorname{char} R = 0$. It remains an outstanding problem to prove such a linearity result in positive characteristic, where the corresponding Lie theory (based on the exponential and logarithm map) is not available; cf. [7] for partial results in this direction. In [1], complemented by subsequent, unpublished work of the first author, one finds improved bounds for the subgroup growth of certain R-analytic pro-p groups. All this encourages us to suspect that much more about R-analytic groups is yet to be discovered.

1.2. **Summary of Results.** Throughout this subsection let R be a *pro-p domain*, i.e. let R be an infinite commutative, noetherian complete local domain, whose residue field is finite of characteristic p. Of special interest is the case where R is of characteristic p and Krull dimension 1, e.g. $R = \mathbb{F}_p[\![t]\!]$.

In this paper we address several open problems regarding compact R-analytic pro-p groups. Foremost among these is the question: to what extent is the underlying ring R of a non-discrete R-analytic group G determined by the algebraic structure of G?

The following is known: if G is a non-discrete topological group which admits the structure of an R-analytic group, then G also supports a p-adic analytic structure if and only if R is a finitely generated integral extension of the p-adic integers $\mathbb{Z}_p$; see [4, Theorem 13.23]. We complement this result by a counterpart in positive characteristic.

Theorem 1.1. *Let R be a pro-p domain, and let G be a non-discrete, finitely generated, compact topological group which admits the structure of an R-analytic group. Then G supports an $\mathbb{F}_p[\![t]\!]$-analytic structure if and only if R is a finitely generated integral extension of $\mathbb{F}_p[\![t]\!]$.*

We remark that in this theorem the assumption that G is finitely generated is somehow essential: as topological groups, the $\mathbb{F}_p[\![t]\!]$-analytic group $(\mathbb{F}_p[\![t]\!], +)$ and the $\mathbb{F}_p[\![t_1, t_2]\!]$-analytic group $(\mathbb{F}_p[\![t_1, t_2]\!], +)$ are isomorphic to one another.

Note that the Cohen Structure Theorem [4, Theorem 6.42] implies that R is a finitely generated integral extension of $\mathbb{Z}_p$ (respectively $\mathbb{F}_p[\![t]\!]$) if and only if R has characteristic 0 (respectively p) and Krull dimension 1. Thus, if R_1, R_2 are pro-p domains such that R_1 has Krull dimension 1 and if G is a non-discrete, finitely generated, compact topological group which supports both an R_1- and an R_2-analytic structure, then R_1 and R_2 share the same characteristic and Krull dimension.

It is tempting to speculate that, quite generally, the underlying ring R of a non-discrete, finitely generated R-analytic group G is determined, up to a finitely generated integral extension, by the algebraic structure of G. In view of the Cohen Structure Theorem [loc. cit.], a decisive step would consist in the verification of

Conjecture 1.2. *Let R_1, R_2 be pro-p domains, and let G be a non-discrete, finitely generated, compact topological group. If G supports both an R_1- and an R_2-analytic structure, then R_1 and R_2 share the same characteristic and Krull dimension.*

The proof of Theorem 1.1 suggests that, generally, the Krull dimension of an underlying ring R may be hidden in the lattice of normal subgroups of the R-analytic group G. As a "caveat" we remark that the attempt to define, in a much wider setting, a dimension theory for t-linear pro-p groups, as proposed in [7, Section 4], fails due the fact that $\mathbb{F}_p[\![t_1, \ldots, t_m]\!]$ can be embedded in $\mathbb{F}_p[\![t_1, t_2]\!]$ for any $m \in \mathbb{N}$; cf. [23, VII, § 10, Remark 4].

Our next result shows that groups which are "small", in the sense that they satisfy a group identity, are at best p-adic analytic.

Theorem 1.3. *Let R be a pro-p domain, and let G be a non-discrete, finitely generated, compact R-analytic group which satisfies a group identity. Then R is a finitely generated integral extension of the p-adic integers $\mathbb{Z}_p$.*

The proof of this theorem is based on the particular case where G is soluble, and our approach involves amongst other things the notion of Hausdorff dimension; see Theorem 3.4. In characteristic p, the seemingly weaker assertion for soluble groups can be regarded as a special instance of the following conjecture which, if true, would provide a powerful tool for isolating R-analytic groups.

Conjecture 1.4 (cf. [10, Chapter VIII, Problem 1.5])**.** *Let R be a pro-p domain of characteristic p, and let G be a finitely generated, compact R-analytic group. Then G does not map onto any non-discrete p-adic analytic group.*

First, we discuss the assumptions listed in the conjecture. The condition of R having characteristic p is essential: $\mathrm{SL}_2^1(\mathbb{Z}_p[\![t]\!])$ is a finitely generated, compact $\mathbb{Z}_p[\![t]\!]$-analytic group and maps onto $\mathrm{SL}_2^1(\mathbb{Z}_p)$ via specialization. In Conjecture 1.4 (as in Theorem 1.3) the assumption that G is finitely generated is needed: the multiplicative group $(1 + t\mathbb{F}_p[\![t]\!], \cdot)$ of 1-units is free abelian and thus maps onto $\mathbb{Z}_p$. Finally, we are naturally interested in compact groups, and at least in Theorem 1.3 we cannot omit the assumption of G being compact: for example,

$$G = \left\{ \begin{pmatrix} 1 & f \\ 0 & t^m \end{pmatrix} \mid m \in \mathbb{Z}, f \in \mathbb{F}_p(\!(t)\!) \right\}$$

is an $\mathbb{F}_p[\![t]\!]$-analytic group which is non-discrete, finitely generated and soluble.

Next, we provide some evidence for the correctness of Conjecture 1.4: it holds true in the important case where R is a pro-p domain of Krull dimension 1.

Theorem 1.5. *Let R be a pro-p domain of characteristic p and Krull dimension* 1*, and let G be a non-discrete, finitely generated, compact R-analytic group. Then G does not map onto any non-discrete p-adic analytic group.*

In order to prove Theorems 1.1 and 1.5, we establish a general structure theorem, utilizing the fundamental work of Ershov [5], Lubotzky [11] and Pink [16]. We require some terminology. It is known that every R-analytic group contains an open subgroup which is *R-standard*; this notion is briefly described in the paragraph before Theorem 1.8, more details are given in Section 2. It is also a fact that, over a local field such as $\mathbb{F}_p(\!(t)\!)$, algebraic groups are analytic; cf. [13, Proposition I.2.5.2]. By a group of *semisimple type in characteristic p* we mean a topological group which is isomorphic to a compact open subgroup of $\mathcal{G}(F)$, where $\mathcal{G}$ is a (connected) simply-connected semisimple algebraic group, defined over a local field F of characteristic p. By our remark every group of semisimple type in characteristic p naturally admits the structure of an $\mathbb{F}_p[\![t]\!]$-analytic group.

Theorem 1.6. *Let R be a pro-p domain of characteristic p and Krull dimension 1, and let G be a non-discrete, finitely generated, compact R-analytic group. Then G contains an open subgroup H, which is $\mathbb{F}_p[\![t]\!]$-standard and contains a normal analytic subgroup N such that*

(1) *H/N is an $\mathbb{F}_p[\![t]\!]$-analytic group of semisimple type in characteristic p,*
(2) *N is nilpotent of finite exponent, and $N/[N,H]$ is finite.*

This theorem somewhat bridges the gap between analytic and algebraic groups over $\mathbb{F}_p(\!(t)\!)$, and it is useful in a wide range of settings. To illustrate this, we give another application, extending [9, Theorem 1]. Recall that the verbal subgroup $w(G)$ of a group G, corresponding to a word w, has *finite width* if there exists $r \in \mathbb{N}$ such that every element of $w(G)$ can be written as a product of at most r elements of G which are w- or w^{-1}-values. This notion plays a prominent role in the fundamental work of Nikolov and Segal [15].

Theorem 1.7. *Let R be a pro-p domain of characteristic p and Krull dimension 1, and let G be a finitely generated, compact R-analytic group. Then for every non-trivial word w of an abstract free group, the verbal subgroup $w(G)$ has finite width and is, in fact, open in G.*

In a slightly different direction, we clarify the notion of a (finite dimensional) analytic representation of an R-analytic group G. This is relevant in connection with the fundamental question under what conditions one and the same topological group can carry two distinct R-analytic group structures. In this context it is known that the analytic structure of a p-adic analytic group is uniquely determined by its topological group structure; see [4, Chapter 9]. A faithful analytic representation ρ of G in the naive sense is a faithful analytic homomorphism from G into the R-analytic group $\mathrm{GL}_m(R)$. The problem is that, over a general pro-p domain R, the image of such a map ρ does not necessarily form an analytic subgroup of $\mathrm{GL}_m(R)$. A basic example illustrating this point is the homomorphism from the $\mathbb{F}_p[\![t]\!]$-analytic group $\mathrm{GL}_2(\mathbb{F}_p[\![t]\!])$ into itself which is induced by the ring homomorphism $\mathbb{F}_p[\![t]\!] \to \mathbb{F}_p[\![t]\!]$, $f(t) \mapsto f(t^p)$.

It is known that every R-analytic group G contains an open subgroup which is *R-standard.* Such a group is isomorphic, as an analytic group, to the group $\mathrm{Gr}_{\mathbf{F}}(R)$ associated to the pro-p domain R under a group scheme $\mathrm{Gr}_{\mathbf{F}}$, based on a formal group law $\mathbf{F}$; see Section 2 for details. To accommodate for the features inherent in the above example, we introduce in Section 7 the notion of a *quasi-Frobenius map* between R-standard groups, which is a special kind of analytic homomorphism given by taking pth powers in suitable coordinate systems. We further require the notion of a *virtually strongly faithful* homomorphism between compact R-analytic groups. Loosely speaking, an analytic homomorphism is virtually strongly faithful if its restriction to a suitable open R-standard subgroup remains faithful under finitely generated integral extensions of the underlying ring R. We prove

Theorem 1.8. *Let R be a pro-p domain of characteristic p and Krull dimension 1, and let G be a compact R-analytic group which has a virtually strongly faithful analytic representation into $\mathrm{GL}_m(R)$ for some $m \in \mathbb{N}$. Then there exist*

(1) *an open R-standard subgroup of G, isomorphic to $\mathrm{Gr}_{\mathbf{F}}(R)$, say,*
(2) *a pro-p domain S which is a finitely generated integral extension of R and satisfies $S \cong R$,*
(3) *an S-standard group H of the same dimension as G, which can be embedded as an analytic subgroup into $\mathrm{GL}_n(S)$ for suitable $n \in \mathbb{N}$,*
(4) *an S-analytic homomorphism φ from H to the S-standard group $\mathrm{Gr}_{\mathbf{F}}(S)$ which is the composition of finitely many quasi-Frobenius maps.*

We conclude this summary by answering a long-standing question of Shalev. Taking advantage of Zelmanov's solution of the Restricted Burnside Problem, Shalev provides in [19] a characterization of p-adic analytic pro-p groups in terms of wreath products. In a subsequent survey [20] he poses the following problem, emphasizing the positive consequences of an affirmative answer: if G is a finitely generated pro-p group and does not involve the pro-p wreath product $C_p \,\hat{\wr}\, \mathbb{Z}_p$ as a closed section, does it follow that G is p-adic analytic? – It turns out that a counterexample can be found within the class of $\mathbb{F}_p[\![t]\!]$-analytic groups.

Theorem 1.9. *Let $\mathbb{D}$ be a division algebra of index 2 over the local field $\mathbb{F}_p(\!(t)\!)$. Then the norm-1 group $\mathrm{SL}_1(\mathbb{D})$ contains an open pro-p subgroup G which is finitely generated and $\mathbb{F}_p[\![t]\!]$-analytic, hence not p-adic analytic, but at the same time does not involve $C_p \,\hat{\wr}\, \mathbb{Z}_p$ as a closed section.*

2. Preliminaries

Throughout the paper let p be a prime and let R denote a *pro-p domain*; i.e. let $R = (R, \mathfrak{m})$ be an infinite commutative, noetherian complete local domain, whose residue field $R/\mathfrak{m}$ is finite of characteristic p. Important examples of pro-p domains are rings of formal power series over the p-adic integers $\mathbb{Z}_p$ or a finite field $\mathbb{F}_q$, i.e. $\mathbb{Z}_p[\![t_1, \ldots, t_m]\!]$ with $m \in \mathbb{N}_0$ and $\mathbb{F}_q[\![t_1, \ldots, t_m]\!]$ with $m \in \mathbb{N}$.

The concept of an R-analytic group is defined in [4, Chapter 13], where it is shown that every such group contains an open subgroup which is *R-standard*; see [loc. cit., Theorem 13.20] and [8, Remark 10]. Let G be an R-standard group. This means that G admits a global atlas, consisting of a *standard chart* (G, ψ, d): the d-tuple of coordinate functions $\psi = (\psi_1, \ldots, \psi_d)$ yields a homeomorphism from G onto $(\mathfrak{m}^l)^{(d)}$, where $l \in \mathbb{N}$ is called the *level* of the particular chart and $d \in \mathbb{N}_0$ is the *dimension* of G. Moreover, $\psi(1) = \mathbf{0}$ and the group operation is given by a *formal group law*, i.e. a d-tuple $\mathbf{F} = (F_1, \ldots, F_d)$ of power series over R in $2d$ variables, as follows: for all $x, y \in G$ we have $\psi(xy) = \mathbf{F}(\psi(x), \psi(y))$. More suggestively, we can "identify" the underlying set of G with $(\mathfrak{m}^l)^{(d)}$ via ψ so that for all $\mathbf{x}, \mathbf{y} \in G = (\mathfrak{m}^l)^{(d)}$ we have

$$\mathbf{x} \cdot \mathbf{y} = (F_1(\mathbf{x}, \mathbf{y}), \ldots, F_d(\mathbf{x}, \mathbf{y})).$$

For basic properties of R-analytic and R-standard groups we refer to [4, Chapter 13]. We only state one auxiliary result which will be used frequently.

Proposition 2.1 ([4, Examples 13.9]). *Let R be a pro-p domain of characteristic p and Krull dimension 1, and let G be an R-analytic group. Then G also admits an $\mathbb{F}_p[\![t]\!]$-analytic structure.*

Fundamental work of Pink [16] shows that compact groups which are linear over the local field $\mathbb{F}_p(\!(t)\!)$ have surprisingly restricted structure. For convenience we state some consequences, taking also into consideration recent work of Ershov [5] and Lubotzky [11]. Following [6], we use the term *semisimple algebraic group* to denote a non-trivial, connected algebraic group whose soluble radical is trivial. By a group of *semisimple type in characteristic p* we mean a topological group which is isomorphic to a compact open subgroup Γ of $\mathcal{G}(F)$, where $\mathcal{G}$ is a simply-connected semisimple algebraic group, defined over a local field F of characteristic p. Note that in this setting, Γ is Zariski-dense in $\mathcal{G}$; cf. [13, Proposition I.2.5.3]. Moreover, every group of semisimple type in characteristic p naturally admits the structure of an $\mathbb{F}_p[\![t]\!]$-analytic group; cf. [loc. cit., Proposition I.2.5.2]. Consequently, we speak of $\mathbb{F}_p[\![t]\!]$-analytic groups of semisimple type.

Theorem 2.2 (Pink, cf. [16, Corollary 0.5]). *Let $n \in \mathbb{N}$, and let Γ be a finitely generated compact subgroup of $\mathrm{GL}_n(\mathbb{F}_p(\!(t)\!))$. Then there exist closed normal subgroups $\Gamma_2 \subseteq \Gamma_1 \subseteq \Gamma$ such that*

(1) Γ/Γ_1 *is finite,*
(2) Γ_1/Γ_2 *is trivial or an $\mathbb{F}_p[\![t]\!]$-analytic group of semisimple type,*
(3) Γ_2 *is a soluble group of derived length at most n.*

Proof. Since Γ is finitely generated, [16, Corollary 0.5] allows us to choose closed normal subgroups $\Gamma_2 \subseteq \Gamma_1 \subseteq \Gamma$ such that

- Γ/Γ_1 is finite,
- Γ_1/Γ_2 is either trivial, or there exist a local field E of characteristic p, an adjoint semisimple algebraic group $\overline{\mathcal{H}}$ over E, with universal covering $\pi : \mathcal{H} \to \overline{\mathcal{H}}$, and an compact open subgroup $\Delta \leq \mathcal{H}(E)$ such that $\Gamma_1/\Gamma_2 \cong \pi(\Delta)$ as topological groups,
- Γ_2 is a soluble group of derived length at most n.

If Γ_1/Γ_2 is trivial, there is nothing further to prove. So suppose that Γ_1/Γ_2 is not trivial. As the kernel of π is finite and as Δ is residually-finite, we may replace Γ_1 and Δ by subgroups of finite index, if necessary, so that $\Delta \cap \ker(\pi) = 1$. Then $\Gamma_1/\Gamma_2 \cong \pi(\Delta) \cong \Delta$ is an $\mathbb{F}_p[\![t]\!]$-analytic group of semisimple type. $\square$

Theorem 2.3 (Ershov [5], Lubotzky [11])**.** *Every $\mathbb{F}_p[\![t]\!]$-analytic group of semisimple type is finitely presented as a profinite group.*

Proof. Let $\mathcal{G}$ be a simply-connected semisimple algebraic group, defined over a local field F of characteristic p, and let Γ be a compact open subgroup of $\mathcal{G}(F)$. We have to show that Γ is finitely presented as a profinite group. The group $\mathcal{G}$ is the direct product of simple groups, and each simple factor can be obtained as the Weil restriction of an absolutely simple group along a finite separable field extension; cf. [13, I.0.24 and Section I §1.7]. Thus we may write $\mathcal{G}(F)$ as a direct product

$$\mathcal{G}(F) \cong \mathcal{G}_1(F_1) \times \ldots \times \mathcal{G}_r(F_r),$$

where, for each $i \in \{1, \ldots, r\}$, $\mathcal{G}_i$ denotes a suitable simply-connected absolutely simple algebraic group, defined over a finite extension field F_i of F. Then Γ intersects each factor in a compact open subgroup, and the claim follows from [5, 11]. □

Theorem 2.4 (Pink, cf. [16, Theorem 7.2])**.** *Let Γ be a finitely generated Zariski-dense compact subgroup of $\mathcal{G}(F)$, where $\mathcal{G}$ is an absolutely simple algebraic group, defined over a local field F of characteristic p. Then Γ is virtually a hereditarily just-infinite $\mathbb{F}_p[\![t]\!]$-analytic group of semisimple type.*

Proof. Let $\overline{\mathcal{G}}$ denote the adjoint group corresponding to $\mathcal{G}$, and let $\overline{\Gamma}$ denote the image of Γ in $\overline{\mathcal{G}}(F)$ under the covering map $\pi : \mathcal{G} \to \overline{\mathcal{G}}$. The kernel of π is finite and Γ is residually-finite. Hence, replacing Γ by a subgroup of finite index, if necessary, we may assume that π restricts to an isomorphism from Γ onto $\overline{\Gamma}$.

Applying Pink's reduction procedure [16], we find a closed subfield E of F, an adjoint absolutely simple algebraic group $\overline{\mathcal{H}}$ over E and a compact Zariski-dense subgroup $\overline{\Delta}$ of $\overline{\mathcal{H}}(E)$ such that

- the data $(F, \overline{\mathcal{G}}, \overline{\Gamma})$ and $(E, \overline{\mathcal{H}}, \overline{\Delta})$ are linked by an isogeny $\varphi : \overline{\mathcal{H}} \times_E F \to \overline{\mathcal{G}}$ such that $\varphi(\overline{\Delta}) = \overline{\Gamma}$,
- the datum $(E, \overline{\mathcal{H}}, \overline{\Delta})$ is minimal in Pink's terminology.

Passing to subgroups of finite index, if necessary, we may assume that φ induces an isomorphism from $\overline{\Delta}$ onto $\overline{\Gamma}$; cf. [16, Corollary 3.8]. Let $\rho : \mathcal{H} \to \overline{\mathcal{H}}$ denote the universal covering. It is known that $\overline{\mathcal{H}}(E)/\rho(\mathcal{H}(E))$ is abelian of finite exponent; cf. [13, Section I §1.4]. Consider the normal subgroup $\overline{\Delta}_0 := \overline{\Delta} \cap \rho(\mathcal{H}(E))$ of $\overline{\Delta}$. As Γ is finitely generated, so is $\overline{\Delta}$, and hence $\overline{\Delta}_0$ has finite index in $\overline{\Delta}$. Moreover, $(E, \overline{\mathcal{H}}, \overline{\Delta}_0)$ is minimal by [16, Corollary 3.8], and [16, Theorem 7.2] shows that $\Delta_0 := \rho^{-1}(\overline{\Delta}_0)$ is open in $\mathcal{H}(E)$. Passing to open subgroups once more, we may assume that ρ restricts to an isomorphism from Δ_0 onto $\overline{\Delta}_0$; cf. [16, Corollary 3.8]. Since Δ_0 is isomorphic to a subgroup of finite index in Γ, the group Γ is virtually an $\mathbb{F}_p[\![t]\!]$-analytic group of semisimple type.

Now we prove that Γ is virtually just infinite. For this it suffices to show that Δ_0 is hereditarily just-infinite. Suppose that $\Omega \leq \Delta_0$ is a non-trivial subgroup whose normalizer is open in Δ_0. We have to show that Ω is open in Δ_0. In view of [16, Corollary 3.8] we may assume that Ω is normal in Δ_0. Recall that Δ_0 is Zariski-dense in $\mathcal{H}$ and intersects trivially the center. As $\mathcal{H}$ is simple, this implies that Ω is Zariski-dense in $\mathcal{H}$. Therefore [16, Theorem 7.2] shows that Ω is open in Δ_0. □

3. Groups which Satisfy an Identity

In this section we prove Theorem 1.3, using the concepts of Hausdorff dimension and pro-p identities. The notion of Hausdorff dimension for closed subgroups of a profinite group G is explained in [21, Section 4]. Note that the Hausdorff dimension function implicitly depends on the choice of a particular filtration for G. In our case $G = (\mathfrak{m}^l)^{(d)}$ is R-standard, and we shall use the natural filtration G_n, $n \in \mathbb{N}_{\geq l}$, defined as follows. For every $n \in \mathbb{N}_{\geq l}$ let $G_n := (\mathfrak{m}^n)^{(d)} \subseteq G$. Then G_n, $n \in \mathbb{N}_{\geq l}$, gives a filtration of open normal subgroups of G which form a base for the neighborhoods of the identity. With respect to this filtration the Hausdorff dimension of a closed subgroup H of G is given as

$$\operatorname{hdim}(H) := \operatorname{hdim}_G(H) = \liminf_{n\to\infty} \frac{\log |HG_n : G_n|}{\log |G : G_n|}.$$

Proposition 3.1. *Let G be an R-standard group, and let N be a closed normal subgroup of G such that G/N is p-adic analytic. Suppose that R has characteristic p or Krull dimension at least 2. Then* $\operatorname{hdim}(N) = 1$.

Proof. We "identify" G with $(\mathfrak{m}^l)^{(d)}$. The formal group law associated to G also defines on $\widetilde{G} := \mathfrak{m}^{(d)}$ the structure of an R-standard group. Clearly, $G = \widetilde{G}_l$ is an open subgroup of $\widetilde{G}$, and $\widetilde{N} := \bigcap\{N^g \mid g \in \widetilde{G}\}$ is open in N. Therefore, replacing G by $\widetilde{G}$ and N by $\widetilde{N}$, we may assume without loss of generality that $l = 1$. Since the proposition obviously holds for $G = 1$, we now assume that $d > 0$.

Recall that the Hilbert-Samuel function of R is defined as

$$h : \mathbb{N}_0 \to \mathbb{N},\ n \mapsto \sum_{i=0}^{n} \dim_{R/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}).$$

Following the proof of [4, Theorem 13.30], there exists a polynomial $\hat{h}(X) \in \mathbb{Q}[X]$ such that for large $n \in \mathbb{N}$ we have $h(n) = \hat{h}(n)$, and the degree of $\hat{h}$ is equal to the Krull dimension D of R. Therefore, writing

$$|G : G_n| = p^{e(n)}, \quad n \in \mathbb{N},$$

we find $c_1 \in \mathbb{Q}_{>0}$ such that

$$|e(n) - c_1 n^D| = O(n^{D-1}).$$

Since G/N is p-adic analytic, the rank r of G/N is finite. Now for every $n \in \mathbb{N}$ the group G_n/G_{n+1} is elementary abelian, and hence by induction

$$|G : G_n N| \leq p^{r(n-1)}.$$

If R has Krull dimension $D \geq 2$, we thus obtain

$$\begin{aligned} 1 - \mathrm{hdim}(N) &= \limsup_{n\to\infty} \frac{\log |G : G_n N|}{\log |G : G_n|} \\ &\leq \limsup_{n\to\infty} \frac{r(n-1)}{e(n)} \\ &= 0. \end{aligned}$$

Now suppose that R has characteristic p. Then for every $n \in \mathbb{N}$ we have $G^{p^n} \leq G_{p^n}$, and for all large $n \in \mathbb{N}$ we have $|G^{p^n} N : G^{p^{n+1}} N| = p^r$. So we find $c_2 \in \mathbb{R}_{>0}$ such that for all $n \in \mathbb{N}$,

$$|G : G_{p^n} N| \leq |G : G^{p^n} N| \leq c_2 p^{r(n-1)}.$$

As $D \geq 1$, this implies

$$\begin{aligned} 1 - \mathrm{hdim}(N) &\leq \limsup_{n\to\infty} \frac{\log |G : G_{p^{n+1}} N|}{\log |G : G_{p^n}|} \\ &\leq \limsup_{n\to\infty} \frac{rn}{e(p^n)} \\ &= 0. \end{aligned}$$

□

In a special case, namely when $R = \mathbb{F}_p[\![t]\!]$, it is tempting to hope for a positive answer to

Question 3.2. *Let G be a finitely generated $\mathbb{F}_p[\![t]\!]$-standard group, and let H be a closed subgroup of G with* $\mathrm{hdim}(H) = 1$*. Does it follow that H is open in G?*

We remark that the kernel of the natural projection $\mathrm{SL}_2^1(\mathbb{F}_p[\![t_1, t_2]\!]) \to \mathrm{SL}_2^1(\mathbb{F}_p[\![t]\!])$ via specialization has Hausdorff dimension 1 and infinite index in $\mathrm{SL}_2^1(\mathbb{F}_p[\![t_1, t_2]\!])$. A positive answer of Question 3.2, in conjunction with Proposition 3.1, would yield an alternative proof of Theorem 1.5. On the other hand, from Theorem 1.6 it can be shown that the answer to Question 3.2 is yes, if H is normal in G.

Next we recall the concept of a pro-p identity. Let w be an element of a free pro-p group F on finitely many generators. Then a pro-p group G *satisfies the pro-p identity* w if for every homomorphism $\varphi : F \to G$ we have $\varphi(w) = 1$. The key step toward proving Theorem 1.3 is

Theorem 3.3. *Let G be an R-standard group, and let H be a closed subgroup of G. Suppose that H satisfies a pro-p identity w, whereas G does not satisfy w. Then* $\mathrm{hdim}(H) < 1$.

Proof. Arguing as in the proof of Proposition 3.1, we may "identify" G with $\mathfrak{m}^{(d)}$ where $d > 0$. Let $r \in \mathbb{N}$ such that w is an element of a free pro-p group on r generators. We consider the group $G^{(r)}$ and its subgroup $H^{(r)}$. Clearly, $G^{(r)} = \mathfrak{m}^{(dr)}$ is an R-standard group of dimension dr, and $H^{(r)}$ is a closed subgroup of $G^{(r)}$.

Note that w naturally induces a map $\omega : G^{(r)} \to G$. Since the group operation in $G = \mathfrak{m}^{(d)}$ is given by a formal group law, ω is given by a d-tuple $\mathbf{W} = (W_1, \ldots, W_d)$ of power series over R in dr variables. By assumption, ω is not the trivial map, so at least one of the formal power series $W_1, \ldots, W_d \in R[\![X_1, \ldots, X_{dr}]\!]$ is non-trivial. On the other hand, w is a pro-p identity on H, hence $W_1(H^{(r)}) = \ldots = W_d(H^{(r)}) = 0$. Denote by $\mathfrak{I}$ the ideal of $R[\![X_1, \ldots, X_{dr}]\!]$ consisting of all power series W such that $W(H^{(r)}) = 0$. We have just seen that $\mathfrak{I} \neq \{0\}$.

Let D denote the Krull dimension of R, and suppose that $R/\mathfrak{m} \cong \mathbb{F}_q$. By the Cohen Structure Theorem [4, Theorem 6.42], the domain R is a finitely generated integral extension of a regular pro-p domain S, i.e. either $S \cong \mathbb{F}_q[\![t_1, \ldots, t_D]\!]$ or $S \cong S_0[\![t_1, \ldots, t_{D-1}]\!]$ where S_0 is a finite extension of $\mathbb{Z}_p$ with residue field $\mathbb{F}_q$. Write $\mathfrak{n}$ for the maximal ideal of S, and let $\mathrm{gr}(S) := \oplus_{i=0}^{\infty} \mathfrak{n}^i/\mathfrak{n}^{i+1}$ denote the graded ring associated to S. Note that $\mathrm{gr}(S)$ is a domain and, like R and S, of Krull dimension D. Let $h_R : \mathbb{N}_0 \to \mathbb{N}$, $n \mapsto \log_q |R/\mathfrak{m}^{n+1}|$ and $h_S : \mathbb{N}_0 \to \mathbb{N}$, $n \mapsto \log_q |S/\mathfrak{n}^{n+1}|$ denote the Hilbert-Samuel functions of R and S, respectively.

Since $R[\![X_1, \ldots, X_{dr}]\!]$ is integral over $S[\![X_1, \ldots, X_{dr}]\!]$ and $\mathfrak{I} \neq \{0\}$, we find a non-zero power series $W \in \mathfrak{I} \cap S[\![X_1, \ldots, X_{dr}]\!]$. Write $W = \sum_{i=1}^{\infty} W^{[i]}$ as a sum of homogeneous polynomials, where $W^{[i]}$ has total degree i, and let $j \in \mathbb{N}$ be the smallest suffix such that $W^{[j]}$ is non-trivial. We find $\mathbf{x} \in \mathfrak{n}^{(dr)}$ such that $W^{[j]}(\mathbf{x}) \neq 0$, hence $W^{[j]}(\mathbf{x}) \in \mathfrak{n}^a \setminus \mathfrak{n}^{a+1}$ for some $a \in \mathbb{N}_0$.

Subclaim 1 (cf. the Artin-Rees Lemma [14, Theorem 8.5]). For every $m \in \mathbb{N}_0$ we have $\mathfrak{n}^m = R\mathfrak{n}^m \cap S$.

Subproof. Let $m \in \mathbb{N}_0$, and suppose, for a contradiction, that $x \in (R\mathfrak{n}^m \cap S) \setminus \mathfrak{n}^m$. Then $m \geq 1$ and we may assume that $x \in \mathfrak{n}^{m-1}$. As R is a finitely generated S-module, $M := \oplus_{i=0}^{\infty} R\mathfrak{n}^i/R\mathfrak{n}^{i+1}$ is a finitely generated $\mathrm{gr}(S)$-module. Denote by $\bar{x}$ the element $x + \mathfrak{n}^m \in \mathfrak{n}^{m-1}/\mathfrak{n}^m \subseteq \mathrm{gr}(S)$. Then $\bar{x}M = 0$. Thus, the Krull dimension of M is less than the Krull dimension of $\mathrm{gr}(S)$, and so

$$h_R(n) = \log_q |R/\mathfrak{m}^{n+1}| \leq \log_q |R/R\mathfrak{n}^{n+1}| = O(n^{D-1}).$$

This is a contradiction, because the Krull dimension of R is D and, accordingly $h_R(n)$ behaves like a polynomial function of degree D for large $n \in \mathbb{N}_0$.

Subclaim 2. Let $m \in \mathbb{N}_{>a}$, $s \in \mathfrak{n}^m \setminus \mathfrak{n}^{m+1}$ and $\mathbf{y} \in (R\mathfrak{n}^{m+a+1})^{(dr)}$. Then

$$W(s\mathbf{x} + \mathbf{y}) = W(sx_1 + y_1, \ldots, sx_{dr} + y_{dr}) \neq 0.$$

Subproof. An easy calculation shows that

$$W(s\mathbf{x} + \mathbf{y}) \equiv s^j W^{[j]}(\mathbf{x}) \pmod{R\mathfrak{n}^{jm+a+1}}.$$

Since the graded ring $\mathrm{gr}(S)$ is an integral domain, we have $s^j W^{[j]}(\mathbf{x}) \in \mathfrak{n}^{jm+a} \setminus \mathfrak{n}^{jm+a+1}$. Applying Subclaim 1, we obtain Subclaim 2.

Let $b \in \mathbb{N}$ such that $\mathfrak{m}^b \subseteq R\mathfrak{n}$. Then $\mathfrak{m}^{bm} \subseteq R\mathfrak{n}^m$ for all $m \in \mathbb{N}_0$.

Subclaim 3. Let $m \in \mathbb{N}_{>a}$ and $k \in \mathbb{N}_{\geq m+a+1}$. Let $s, \tilde{s} \in \mathfrak{n}^m$ such that $s - \tilde{s} \notin \mathfrak{n}^{m+1}$. Then

$$(s\mathbf{x}) \cdot H^{(r)} G_{bk}^{(r)} \neq (\tilde{s}\mathbf{x}) \cdot H^{(r)} G_{bk}^{(r)}.$$

Subproof. Note that

$$(\tilde{s}\mathbf{x})^{-1}(s\mathbf{x}) \cdot G^{(r)}_{b(m+a+1)} \subseteq (s - \tilde{s})\mathbf{x} + (R\mathfrak{n}^{m+a+1})^{(dr)}.$$

Since $W(H^{(r)}) = 0$, Subclaim 2 yields Subclaim 3.

From Subclaim 3 it follows that for every $k \in \mathbb{N}_{>2a+1}$,

$$\log_q |\{s\mathbf{x} \cdot H^{(r)}G^{(r)}_{bk} \mid s \in S\}| \geq h_S(k - a - 1) - h_S(a).$$

Since R and S have Krull dimension D, the functions $h_R(n)$ and $h_S(n)$ behave like polynomial functions of degree D for large $n \in \mathbb{N}_0$. Therefore we obtain

$$\begin{aligned}
1 - \mathrm{hdim}_G(H) &= 1 - \mathrm{hdim}_{G^{(r)}}(H^{(r)}) \\
&= \limsup_{n\to\infty} \frac{\log |G^{(r)} : H^{(r)}G^{(r)}_n|}{\log |G^{(r)} : G^{(r)}_n|} \\
&\geq \limsup_{k\to\infty} \frac{\log |G^{(r)} : H^{(r)}G^{(r)}_{bk}|}{\log |G^{(r)} : G^{(r)}_{bk}|} \\
&\geq \limsup_{k\to\infty} \frac{h_S(k - a - 1) - h_S(a)}{dr(h_R(bk) - h_R(0))} \\
&> 0,
\end{aligned}$$

and thus $\mathrm{hdim}_G(H) < 1$. □

Now we are ready to prove the central case of Theorem 1.3.

Theorem 3.4. *Let R be a pro-p domain, and let G be a non-discrete, finitely generated, compact R-analytic group which is soluble. Then R is a finitely generated integral extension of the p-adic integers $\mathbb{Z}_p$.*

Proof. Every R-analytic group has an open subgroup which is R-standard. So we may assume without loss of generality that G is R-standard. The derived subgroup G' of G is soluble of derived length strictly less than G. Hence we find a commutator word w such that G' satisfies the (pro-p) identity w, whereas G does not satisfy w. Applying Theorem 3.3, we obtain that $\mathrm{hdim}(G') < 1$. Since G/G' is p-adic analytic, Proposition 3.1 now shows that R has characteristic 0 and Krull dimension 1, thus R is a finitely generated integral extension of $\mathbb{Z}_p$ by the Cohen Structure Theorem [4, Theorem 6.42]. □

Proof of Theorem 1.3. As in the proof of Theorem 3.4, we may assume without loss of generality that G is R-standard. Then, by [7, Proposition 5.1], the group $G/Z(G)$ is R-linear; cf. Section 4. There exists a finitely generated (as an abstract group), dense subgroup H of $G/Z(G)$. Clearly, H satisfies a group identity. By the Tits alternative [22], this implies that H is virtually soluble. So G is soluble, and we can apply Theorem 3.4. □

4. Analytic Subgroups of $\mathbb{F}_q[\![t]\!]$-Analytic Groups

Many of the conventional analytic concepts, e.g. the notion of a tangent space or of an immersion, are also available for the study of analytic manifolds and analytic groups over a local field of positive characteristic p; see [3] and [18]. In the current section we prove in this setting the analogue of a familiar fact, namely that every homogeneous analytic subset of a manifold is a submanifold; see Corollary 4.2. Then we go on to define the soluble radical of a standard group, and we show that the center and the soluble radical are analytic subgroups; see Corollaries 4.3 and 4.4.

Let F be a local field of characteristic p, i.e. let $F = \mathbb{F}_q(\!(t)\!)$ for some power q of p, with valuation ring $\mathbb{F}_q[\![t]\!]$. Let M be a manifold over F, and let $S \subseteq M$. Then S is called an *analytic subset* of M, if for every $\mathbf{x} \in S$ there exist an open neighborhood U of $\mathbf{x}$ in M and analytic functions $f_1, \ldots, f_r$, where $r = r_{\mathbf{x}}$, defined on U such that

$$S \cap U = \{\mathbf{y} \in U \mid f_i(\mathbf{y}) = 0 \text{ for } 1 \leq i \leq r\}.$$

Suppose that S is an analytic subset of M, and let $\mathbf{x} \in S$. Then $\mathbf{x}$ is a *regular point* of S if there exist an open neighborhood U of $\mathbf{x}$ in M and analytic functions $f_1, \ldots, f_r$ defined on U such that

- $S \cap U = \{\mathbf{y} \in U \mid f_i(\mathbf{y}) = 0 \text{ for } 1 \leq i \leq r\}$,
- the differentials $d_{\mathbf{x}} f_1, \ldots, d_{\mathbf{x}} f_r$ at $\mathbf{x}$ are linearly independent (inside the cotangent space of M at $\mathbf{x}$).

Recall that S is a submanifold of M if and only if S is an analytic subset of M such that every point of S is regular; see [18, Part II, Sections III §10 §11 and IV §2]. Below we prove

Proposition 4.1. *Let S be a non-empty analytic subset of M. Then S has at least one regular point.*

We call $S \subseteq M$ *homogeneous* if under the action of the automorphism group of M the set S is contained in a single orbit. Under this action regular points are mapped to regular points, thus we obtain

Corollary 4.2. *Every homogeneous analytic subset of M is a submanifold. In particular, if G is an analytic group and H a subgroup as well as an analytic subset of G, then H is an analytic subgroup of G.*

Corollary 4.3. *Let G be an $\mathbb{F}_q[\![t]\!]$-standard group. Then the center $Z(G)$ is an analytic subgroup of G.*

Proof. From [7, Proof of Proposition 5.1] we know that $Z(G)$ is an analytic subset of G, hence it forms an analytic subgroup of G by Corollary 4.2. □

Next we explain that every $\mathbb{F}_q[\![t]\!]$-standard group G contains a maximal normal soluble subgroup, and we prove that this group is an analytic subgroup of G. This permits us to speak of the *soluble radical* of G, which is subsequently denoted by $\mathrm{Rad_s}(G)$. Indeed, if G is an $\mathbb{F}_q[\![t]\!]$-standard group, then $G/Z(G)$ acts faithfully via conjugation on G. Considering the dual action of G on the ring of functions from G to $\mathbb{F}_q[\![t]\!]$, we can regard $G/Z(G)$ as a subgroup of $\mathrm{GL}_n(\mathbb{F}_q[\![t]\!])$ for suitable $n \in \mathbb{N}$; see [7, Proposition 5.1]. Let $\overline{F}$ denote the algebraic closure of $F = \mathbb{F}_q(\!(t)\!)$, and let $\mathcal{G}$ denote the Zariski-closure of $G/Z(G)$ inside $\mathrm{GL}_n(\overline{F})$. Let $\mathcal{N}$ denote the unique largest normal soluble subgroup of $\mathcal{G}$; its connected component is the so-called soluble radical of $\mathcal{G}$. Then, clearly, $\mathrm{Rad_s}(G) := \mathcal{N} \cap (G/Z(G))$ is the maximal normal soluble subgroup of $G/Z(G)$.

Corollary 4.4. *Let G be an $\mathbb{F}_q[\![t]\!]$-standard group. Then the soluble radical* $\mathrm{Rad_s}(G)$ *is an analytic subgroup of G.*

Proof. We use the notation of the preceding paragraph. By [7, Proof of Proposition 5.1], we can construct a matrix representation of $G/Z(G)$ over $\mathbb{F}_q[\![t]\!]$ such that all matrix entries are given by analytic functions. Then $\mathcal{N}$ is given by a finite set of equations with coefficients in some finite extension of $\mathbb{F}_q(\!(t)\!)$. Since $G/Z(G)$ is a subgroup of $\mathrm{GL}_n(\mathbb{F}_q[\![t]\!])$, the soluble radical $\mathrm{Rad_s}(G)$ is given by a finite number of equations with coefficients from $\mathbb{F}_q[\![t]\!]$. Thus $\mathrm{Rad_s}(G)$ is an analytic subset of G, hence it forms an analytic subgroup of G by Corollary 4.2. □

Proof of Proposition 4.1. For every $\mathbf{x} \in S$ let $\dim_{\mathbf{x}}(S) := d_{\mathbf{x}} - r_{\mathbf{x}}$, where

- $d_{\mathbf{x}} := \dim T_{\mathbf{x}}(M)$ denotes the dimension of the tangent space of M at $\mathbf{x}$ and
- $r_{\mathbf{x}}$ denotes the maximal number r such that there exist an open neighborhood U of $\mathbf{x}$ in M and analytic functions $f_1, \ldots, f_r$ defined on U satisfying: (i) $f_i(\mathbf{y}) = 0$ for all $i \in \{1, \ldots, r\}$ and $\mathbf{y} \in U \cap S$, (ii) the differentials $d_{\mathbf{x}} f_1, \ldots, d_{\mathbf{x}} f_r$ are linearly independent.

Choose $\mathbf{x} \in S$ with $\dim_{\mathbf{x}}(S) = \min\{\dim_{\mathbf{y}}(S) \mid \mathbf{y} \in S \cap U\}$ for some open neighborhood U of $\mathbf{x}$ in M. We claim that $\mathbf{x}$ is a regular point of S.

Put $d := \dim T_{\mathbf{x}}(M)$, $n := \dim_{\mathbf{x}}(S)$, and $r := d - n$. We find a coordinate system $(X_1, \ldots, X_d)$ for a (possibly smaller) neighborhood $V \subseteq U$ of $\mathbf{x}$ such that $f_1 := X_{n+1}, \ldots, f_r := X_d$ satisfy $f_i(\mathbf{y}) = 0$ for all $i \in \{1, \ldots, r\}$ and $\mathbf{y} \in U \cap S$. Note that the differentials $d_{\mathbf{y}} f_1, \ldots, d_{\mathbf{y}} f_r$ are linearly independent not only for $\mathbf{y} = \mathbf{x}$ but for all $\mathbf{y} \in V$. Our claim is that in some neighborhood $W \subseteq V$ of $\mathbf{x}$,

$$S \cap W = \{\mathbf{y} \in W \mid f_i(\mathbf{y}) = 0 \text{ for } 1 \leq i \leq r\}.$$

Let $\mathcal{O}_{\mathbf{x}}$ denote the local ring at $\mathbf{x}$ with maximal ideal $\mathcal{M}_{\mathbf{x}} = (\bar{X}_1, \ldots, \bar{X}_d)$. Recall that $\mathcal{O}_{\mathbf{x}}$ is isomorphic to the ring of convergent power series in n variables $\bar{X}_1, \ldots, \bar{X}_d$, that is the subring of $F[\![\bar{X}_1, \ldots, \bar{X}_d]\!]$ consisting of all formal power series which converge on suitably small neighborhoods of 0. (In the following we do not distinguish typographically between X_i and $\bar{X}_i$.) Let $\mathcal{I}_{\mathbf{x}}(S)$ denote the ideal of all (germs of) functions in $\mathcal{O}_{\mathbf{x}}$ which vanish everywhere on the intersection between S and some suitably small neighborhood of $\mathbf{x}$.

Subclaim. Let $f \in \mathcal{I}_{\mathbf{x}}(S) \cap F[\![X_1, \ldots, X_n]\!]$ and $i \in \{1, \ldots, d\}$. Then the partial derivative $\partial_i f$ of f with respect to X_i lies in $\mathcal{I}_{\mathbf{x}}(S)$.
Subproof. For a contradiction, suppose that $\partial_i f$ does not vanish at $\mathbf{y} \in V \cap S$. Then clearly $i \leq n$ and the choice of V implies that $d_{\mathbf{y}} f, d_{\mathbf{y}} f_1, \ldots, d_{\mathbf{y}} f_r$ are linearly independent. This gives $\dim_{\mathbf{x}}(S) > \dim_{\mathbf{y}}(S)$, contradicting the choice of $\mathbf{x}$.

We are now ready to verify that $\mathbf{x}$ is indeed regular. It suffices to show that $\mathcal{I}_{\mathbf{x}}(S)$ is generated by $f_1 = X_{n+1}, \ldots, f_r = X_d$. For a contradiction, suppose this is not so. Then we choose $f \in \mathcal{I}_{\mathbf{x}}(S) \cap F[\![X_1, \ldots, X_n]\!] \setminus \{0\}$ of minimal degree $\deg(f) = \sup\{k \in \mathbb{N}_0 \mid f \in \mathcal{M}_{\mathbf{x}}{}^k\}$. Clearly, we have $\deg(f) \geq 1$.

Let us write $A := \{0, 1, \ldots, p-1\}^n$ and

$$f = \sum_{\alpha \in A} f_\alpha(X_1, \ldots, X_n) X_1^{\alpha_1} \cdots X_n^{\alpha_n},$$

where each f_α lies in $\mathcal{O}_{\mathbf{x}} \cap F[\![X_1^p, \ldots, X_n^p]\!]$. It is an easy consequence of the subclaim that $f_\alpha \in \mathcal{I}_{\mathbf{x}}(S)$ for all $\alpha \in A$. Choose $\alpha \in A$ such that $\deg(f_\alpha) \leq \deg(f)$. Next we extend the ground field $F = \mathbb{F}_q(\!(t)\!)$ to $E = \mathbb{F}_q(\!(s)\!)$ where $s^p = t$. Note that $F \leq E$ is a totally ramified extension of degree p, a basis for E as F-vector space is given by $(1, s, \ldots, s^{p-1})$.

We can write $f_\alpha = g^p$ for a suitable element $g \in E[\![X_1, \ldots, X_n]\!]$. Because absolute values are multiplicative, g is locally convergent at $\mathbf{x}$ (over E). Moreover we find $g_0, \ldots, g_{p-1} \in \mathcal{O}_{\mathbf{x}} \subseteq F[\![X_1, \ldots, X_d]\!]$ such that

$$g = \sum_{i=0}^{p-1} g_i(X_1, \ldots, X_n) s^i.$$

It follows that $g_0, \ldots, g_{p-1} \in \mathcal{I}_{\mathbf{x}}(S)$. Choose $i \in \{0, 1, \ldots, p-1\}$ such that $\deg(g_i) = \deg(g)$. Then the inequality $\deg(g_i) = p^{-1} \deg(f_\alpha) \leq p^{-1} \deg(f)$ contradicts the choice of f. □

5. The Structure of $\mathbb{F}_p[\![t]\!]$-Analytic Groups

In this section we study groups which are analytic over a pro-p domain of characteristic p and Krull dimension 1. We prove Theorems 1.6, 1.5 and 1.1, in this order, taking advantage of the theorems listed in Section 2. We begin with a variant of a classical result of Schur [17, Proposition 10.1.4].

Lemma 5.1. *Let G be a pro-p group with nilpotent normal subgroup N. Let $Z := Z(G)$ denote the center of G and suppose that NZ/Z has finite exponent. Then $[N, G]$ has finite exponent.*

Proof. The proof proceeds by induction on the nilpotency class of N and uses the following consequence of the Hall-Petrescu formula; cf. [4, Lemma 11.9]. For all $x, y \in G$ and $j \in \mathbb{N}$,

$$[x^{p^j}, y] \equiv [x, y]^{p^j} \mod \gamma_2(H)^{p^j} \prod_{k=1}^{j} \gamma_{p^k}(H)^{p^{j-k}},$$

where $H = \langle x, [x, y] \rangle$. □

Proof of Theorem 1.6. Applying Proposition 2.1, we may assume that G is $\mathbb{F}_p[\![t]\!]$-analytic and standard. By Corollaries 4.3 and 4.4, the center $Z := Z(G)$ and the soluble radical $N := \mathrm{Rad_s}(G)$ are analytic subgroups of G. Moreover, we can regard G/Z as a Zariski-dense subgroup of an algebraic group $\mathcal{G}$ defined over $\mathbb{F}_p(\!(t)\!)$ such that N/Z is the intersection of G/Z with $\mathcal{N}(\mathbb{F}_p(\!(t)\!))$, where $\mathcal{N}$ denotes that unique largest soluble normal subgroup of $\mathcal{G}$. Let $\mathcal{N}^\circ$ be the connected component of $\mathcal{N}$, i.e. the soluble radical of $\mathcal{G}$. Passing to an open subgroup, if necessary, we may assume that $\mathcal{G}$ is connected and that $N/Z \subseteq \mathcal{N}^\circ(\mathbb{F}_p(\!(t)\!))$.

First we examine the quotient G/N of G/Z. By Theorem 3.4, the group G/N is non-trivial. Applying Theorem 2.2 to the compact group $\Gamma := G/N$, which possesses no non-trivial soluble normal subgroups, we see that G/N is virtually $\mathbb{F}_p[\![t]\!]$-analytic of semisimple type. Passing to an open subgroup, if necessary, we may assume that G/N itself is $\mathbb{F}_p[\![t]\!]$-analytic of semisimple type.

Now we start to investigate the normal subgroup N of G. By Theorem 2.3, the group G/N is finitely presented as a profinite group. This implies that $N/[N,G]Z$ is a finitely generated abelian, hence p-adic analytic group. It is a well-known fact that the algebraic group $[\mathcal{N}^\circ, \mathcal{G}]$ is unipotent; cf. [6, Lemma 19.5]. As we are working in characteristic p and $N/Z \subseteq \mathcal{N}^\circ(\mathbb{F}_p(\!(t)\!))$, this implies that $[N,G]Z/Z$ is nilpotent and of finite exponent. By [18, Part II, Section IV §5], the group N/Z, being a quotient of an $\mathbb{F}_p[\![t]\!]$-analytic group by an analytic subgroup, is $\mathbb{F}_p[\![t]\!]$-analytic. We apply Proposition 3.1 and Theorem 3.3 to deduce that every group identity which holds in $[N,G]Z/Z$ also holds in the larger group N/Z. Therefore N/Z is nilpotent and of finite exponent. In particular, this shows that N is nilpotent. Moreover, applying Lemma 5.1, we see that $[N,G]$ has finite exponent. Since G/N is finitely presented, $N/[N,G]$ is p-adic analytic. Thus Proposition 3.1 and Theorem 3.3 show that every group identity which holds in $[N,G]$ also holds in N. We conclude that N has finite exponent and that $N/[N,G]$ is finite. □

Proof of Theorem 1.5. By Proposition 2.1, we may regard G as an $\mathbb{F}_p[\![t]\!]$-analytic group. Let $G \to H$ be a homomorphism onto a p-adic analytic group H. We have to show that H is finite.

Descending to subgroups of finite index, if necessary, we may assume that the torsion elements of H form a finite subgroup; cf. [4, Theorem 4.20]. Hence, descending again to open subgroups, we may assume, by Theorem 1.6, that G is of semisimple type. The corresponding semisimple algebraic group is a finite product of simple components. Thus we may assume further that G is an open subgroup of $\mathcal{G}(F)$, where $\mathcal{G}$ is a simply-connected absolutely simple algebraic group, defined over a local field F of characteristic p. Because G itself cannot be p-adic analytic (cf. [4, Theorem 13.23]), Theorem 2.4 shows that its p-adic analytic image H is indeed finite. □

Proof of Theorem 1.1. If R is a finitely generated integral extension of $\mathbb{F}_p[\![t]\!]$, then Proposition 2.1 implies that G supports an $\mathbb{F}_p[\![t]\!]$-analytic structure.

To prove the converse, suppose that G admits both, an R- and an $\mathbb{F}_p[\![t]\!]$-analytic structure. Passing to an open subgroup, if necessary, we may further assume that G is R-standard. By the Cohen Structure Theorem [4, Theorem 6.42], the domain R is a finitely generated integral extension of $S \cong \mathbb{F}_p[\![t_1,\ldots,t_m]\!]$ with $m \in \mathbb{N}$ or $S \cong \mathbb{Z}_p[\![t_1,\ldots,t_m]\!]$ with $m \in \mathbb{N}_0$. First we rule out the possibility that R has characteristic 0: if $S \cong \mathbb{Z}_p[\![t_1,\ldots,t_m]\!]$, then specialization yields a ring homomorphism from S onto $\mathbb{Z}_p$, which by the Going Up Theorem (cf. [23, V §2]) extends to a ring homomorphism from R onto a finitely generated integral extension of $\mathbb{Z}_p$. This ring homomorphism induces a homomorphism from G onto a non-discrete p-adic analytic group, in contradiction to Theorem 1.5. Hence R has necessarily characteristic p, and we assume for a contradiction that $S = \mathbb{F}_p[\![t_1, \ldots, t_m]\!]$ with $m \geq 2$.

Again we consider specializations. For every $f \in t\mathbb{F}_p[\![t]\!]$, the ring epimorphism $\eta_f : \mathbb{F}_p[\![t_1, \ldots, t_m]\!] \to \mathbb{F}_p[\![t]\!]$, given by $\eta_f(t_1) = t$ and $\eta_f(t_2) = \ldots = \eta_f(t_m) = f$ extends by the Going Up Theorem (loc. cit.) to a ring homomorphism from R onto a finitely generated integral extension of $\mathbb{F}_p[\![t]\!]$. By Proposition 2.1, the latter induces a homomorphism from G onto a non-discrete $\mathbb{F}_p[\![t]\!]$-analytic group with kernel K_f, say. Moreover, by Theorem 1.6, each of the quotients G/K_f has just-infinite images which are not virtually soluble. Note that for $f_1 \neq f_2$, the ideal $\ker(\eta_{f_1}) + \ker(\eta_{f_2})$ is of finite index in S, and consequently the normal subgroup $K_{f_1}K_{f_2}$ is of finite index in G. This shows that G has infinitely many pairwise non-commensurable normal subgroups such that the corresponding quotient groups are just-infinite and not virtually soluble.

On the other hand, remembering that G also supports an $\mathbb{F}_p[\![t]\!]$-analytic structure, Theorems 1.6 and 2.4 imply that G has at most finitely many pairwise non-commensurable normal subgroups such that the corresponding quotient group is just-infinite and not virtually soluble. This is the required contradiction. □

6. Word Width in $\mathbb{F}_p[\![t]\!]$-Analytic Groups

In this section we prove Theorem 1.7. Let G be a pro-p group. A *pro-p word* w is an element of a free pro-p group F on finitely many generators. By a *w-value* in G we mean the image $\varphi(w)$ of w under a homomorphism $\varphi : F \to G$. The set of all w-values in G is denoted by $G^{\{w\}}$; this set is automatically closed. In contrast, the *verbal subgroup* $w(G)$, which is the (abstract) subgroup of G generated by $G^{\{w\}}$, need not be closed. We denote the *topological closure* of an abstract subgroup $H \leq G$ by $\overline{H}$. Recall that $w(G)$ has *finite width* , if there exists $r \in \mathbb{N}$ such that every element of $w(G)$ can be written as a product of at most r elements of $G^{\{w\}} \cup G^{\{w^{-1}\}}$.

Thus every verbal subgroup $w(G)$ of finite width in a pro-p group G is the continuous image of a compact set, hence itself compact and therefore closed in G, i.e. $w(G) = \overline{w(G)}$. This explains the interest in words w such that $w(G)$ has finite width for a large class of pro-p groups G.

Lemma 6.1. *Let G be a finitely generated pro-p group, and let w be a pro-p word such that $\overline{[w(G), w(G)]}$, the closed commutator group of the verbal subgroup $w(G)$, is open in G. Then the verbal subgroup $w(G)$ has finite width. In particular, this implies that $w(G)$ is open in G.*

Proof. Suppose that w is an element of a free pro-p group on d generators. Clearly, $\overline{w(G)}$ is open in G, hence finitely generated: $\overline{w(G)} = \langle x_1, \ldots, x_d \rangle$ where $x_i \in G^{\{w\}}$ for $i \in \{1, \ldots, d\}$. Note that for every $x \in G^{\{w\}}$ and $y \in G$ the commutator $[x, y] = x^{-1}x^y$

is the product of a w^{-1}-value and a w-value in G. It follows that

$$\begin{aligned}\overline{[w(G), w(G)]} &= \{[x_1, y_1] \cdots [x_d, y_d] \mid y_1, \ldots, y_d \in \overline{w(G)}\} \\ &\subseteq \{z_1 \cdots z_{2d} \mid z_1, \ldots, z_{2d} \in G^{\{w\}} \cup G^{\{w^{-1}\}}\};\end{aligned}$$

cf. [4, Proof of Proposition 1.19]. As $\overline{w(G)}/\overline{[w(G), w(G)]}$ is finite, the claim follows. □

Proof of Theorem 1.7. Clearly, we may assume that G is non-discrete. Let w be a non-trivial word. In view of Lemma 6.1, it suffices to show that $H := \overline{[w(G), w(G)]}$ is open in G. We apply Theorem 1.6: replacing G by an open subgroup, if necessary, we find a normal subgroup N of G such that

- G/N is an $\mathbb{F}_p[\![t]\!]$-analytic group of semisimple type,
- N is nilpotent of finite exponent, and $N/[N, G]$ is finite.

It suffices to show that (i) HN/N is open in G/N and (ii) $H \cap N$ is open in N.

(i) By Theorem 2.4, the group G/N is virtually the product of finitely many normal subgroups which are just-infinite and $\mathbb{F}_p[\![t]\!]$-analytic. By Theorem 1.3, the group HN/N intersects each of these factors in a non-trivial normal subgroup. Hence HN/N is open in G/N.

(ii) Replacing G by the open subgroup HN, we may assume that $G = HN$. Then $N/[N, NH] = N/[N, G]$ is finite. This shows that the abelianization of the nilpotent group $N/(H \cap N)$ is finite, hence $N/(H \cap N)$ itself is finite and $H \cap N$ is open in N. □

7. Faithful Analytic Representations

In this section we prove Theorem 1.8. To begin with, R denotes an arbitrary pro-p domain with maximal ideal $\mathfrak{m}$. Later we will specialize to the case where R has characteristic p and Krull dimension 1. By an R-standard group we mean an R-standard group which admits a standard chart of level 1.

There is a functorial procedure which associates to any formal group law $\mathbf{F}$ over R and any pro-p domain S, extending R, an S-standard group $\mathrm{Gr}_{\mathbf{F}}(S)$. We explain this method in simple terms. If $\mathbf{F}_1$, $\mathbf{F}_2$ denote formal group laws over R of dimension d_1, d_2 respectively, then a *formal morphism* $\mathbf{f} : \mathbf{F}_1 \to \mathbf{F}_2$ is a d_2-tuple $\mathbf{f} = (f_1, \ldots, f_{d_2})$, whose components are formal power series $f_j \in R[\![\mathbf{X}]\!] = R[\![X_1, \ldots, X_{d_1}]\!]$ with zero constant term, such that

$$\mathbf{F}_2(\mathbf{f}(\mathbf{X}), \mathbf{f}(\mathbf{Y})) = \mathbf{f}(\mathbf{F}_1(\mathbf{X}, \mathbf{Y})).$$

Next suppose that S is a pro-p domain, extending R, with maximal ideal $\mathfrak{n}$. To any formal group law $\mathbf{F}$ over R of dimension d, we naturally associate the S-standard group $\mathrm{Gr}_{\mathbf{F}}(S)$ of dimension d, whose underlying set is "identified" via a standard chart with $\mathfrak{n}^{(d)}$ such that the group multiplication is determined by $\mathbf{F}$. Moreover, any formal morphism $\mathbf{f} : \mathbf{F}_1 \to \mathbf{F}_2$ between formal group laws naturally yields an analytic homomorphism $\mathrm{Gr}_{\mathbf{f}}(S) : \mathrm{Gr}_{\mathbf{F}_1}(S) \to \mathrm{Gr}_{\mathbf{F}_2}(S)$ between S-standard groups.

Let G_1, G_2 be R-standard groups of dimensions d_1, d_2 respectively. Choosing standard charts, we may regard G_1, G_2 as the R-standard groups $\mathrm{Gr}_{\mathbf{F}_1}(R)$, $\mathrm{Gr}_{\mathbf{F}_2}(R)$ associated to formal group laws $\mathbf{F}_1$, $\mathbf{F}_2$ over R respectively. An analytic homomorphism $\varphi : G_1 \to G_2$

is said to be *strongly faithful*, if φ is induced by a formal morphism $\mathbf{f} : \mathbf{F}_1 \to \mathbf{F}_2$ of formal group laws and if, moreover, $\mathrm{Gr}_{\mathbf{f}}(S) : \mathrm{Gr}_{\mathbf{F}_1}(S) \to \mathrm{Gr}_{\mathbf{F}_2}(S)$ is faithful for every finitely generated integral ring extension S of R. We will be concerned with strongly faithful linear representations of R-standard groups. For this we recall that, for every $n \in \mathbb{N}$, we may consider $\mathrm{GL}_n^1(R)$ as an R-standard group; cf. [4, Examples 13.18 and 13.19].

Next suppose that G_1, G_2 have the same dimension $d = d_1 = d_2$. Then a *quasi-Frobenius map* from G_1 to G_2 is an analytic homomorphism $\varphi : G_1 \to G_2$ such that there exist $n \in \{1, \ldots, d\}$ and standard charts (G_1, ψ^1, d), (G_2, ψ^2, d) with the property that for every $j \in \{1, \ldots, d\}$ and all $x \in G_1$,

$$\psi_j^2(\varphi(x)) = \begin{cases} \psi_j^1(x) & \text{if } 1 \le j \le n, \\ (\psi_j^1(x))^p & \text{if } n < j \le d. \end{cases}$$

Identify each of the groups G_1, G_2 via the corresponding standard chart with $\mathfrak{m}^{(d)}$ such that the group multiplications are prescribed by formal group laws $\mathbf{F}_1$, $\mathbf{F}_2$ respectively. Writing $\overline{\mathbf{z}} = (z_1, \ldots, z_n, z_{n+1}^p, \ldots, z_d^p)$, the latter condition can be expressed more suggestively as follows: for all $\mathbf{x}, \mathbf{y} \in \mathfrak{m}^{(d)}$,

$$\mathbf{F}_2(\overline{\mathbf{x}}, \overline{\mathbf{y}}) = \overline{\mathbf{F}_1(\mathbf{x}, \mathbf{y})}.$$

Theorem 1.8 follows immediately from

Proposition 7.1. *Let R be a pro-p domain of characteristic p and Krull dimension 1, and let $G = \mathrm{Gr}_{\mathbf{F}}(R)$ be a an R-standard group of dimension d, associated to a formal group law $\mathbf{F}$ over R. Suppose that G admits a strongly faithful representation into $\mathrm{GL}_m^1(R)$ for some $m \in \mathbb{N}$.*

Then there exist

1. *a pro-p domain S which is a finitely generated integral extension of R and satisfies $S \cong R$,*
2. *an S-standard group $\mathrm{Gr}_{\mathbf{F}'}(S)$ of dimension d, associated to a formal group law $\mathbf{F}'$ over S, which can be embedded as an analytic subgroup into $\mathrm{GL}_n^1(S)$ for suitable $n \in \mathbb{N}$,*
3. *an S-analytic homomorphism $\varphi : \mathrm{Gr}_{\mathbf{F}'}(S) \to \mathrm{Gr}_{\mathbf{F}}(S)$ which is the composition of finitely many quasi-Frobenius maps.*

Proof. As usual we identify G with $\mathfrak{m}^{(d)}$ via a standard chart so that the group multiplication is given by $\mathbf{F}$. Accordingly, elements of G are written as $\mathbf{x}, \mathbf{y} \in \mathfrak{m}^{(d)}$ and, specifically, we denote the unit element of G by $\mathbf{e}$.

Fix a strongly faithful representation $\vartheta : G \to \mathrm{GL}_m^1(R)$, $m \in \mathbb{N}$. Our first aim is to construct from ϑ a more convenient strongly faithful representation of G on a free R-module $\mathcal{V}$ of finite rank. Choosing standard coordinates on $\mathrm{GL}_m^1(R)$, reflecting the matrix description $\mathrm{GL}_m^1(R) = 1 + \mathrm{Mat}_m(\mathfrak{m})$, we "identify" $\mathrm{GL}_m^1(R)$ with $\mathfrak{m}^{(m^2)}$ so that ϑ as given by coordinate functions $\vartheta_{i,j} \in R[\![X_1, \ldots, X_d]\!]$ with zero constant term. Let $\mathcal{V}$ denote the R-module generated by the coordinate functions $\vartheta_{i,j} \in R[\![X_1, \ldots, X_d]\!]$; evidently, this is a free R-module of finite rank contained in $R[\![X_1, \ldots, X_d]\!]$. Regarding

$R[\![X_1,\ldots,X_d]\!]$ as a ring of functions on $G = \mathfrak{m}^{(d)}$, we consider the two regular actions, ρ and λ, of G on $R[\![X_1,\ldots,X_d]\!]$, induced by right and left multiplication:

$$(\rho_{\mathbf{x}} f)(\mathbf{y}) = f(\mathbf{y}\mathbf{x}) \text{ and } (\lambda_{\mathbf{x}} f)(\mathbf{y}) = f(\mathbf{x}^{-1}\mathbf{y}) \quad \text{for } f \in R[\![X_1,\ldots,X_d]\!],\ \mathbf{x},\mathbf{y} \in G.$$

Clearly, $\mathcal{V}$ is closed under both these actions. Since ϑ is strongly faithful, $\mathcal{V}$, regarded as a G-module via ρ say, also pertains to a strongly faithful representation $G \to \mathrm{GL}_n^1(R)$ for some $n \in \mathbb{N}$. In particular, $\{\mathbf{x} \in G \mid \forall f \in \mathcal{V} : f(\mathbf{x}) = 0\} = \{\mathbf{e}\}$.

Next we associate two numerical invariants, $n(\mathcal{V})$ and $\nu(\mathcal{V})$, to the representation of G on $\mathcal{V}$. For every $\mathbf{x} \in G$, let $n_{\mathbf{x}}(\mathcal{V}) := d - r_{\mathbf{x}}$, where $r_{\mathbf{x}}$ denotes the maximal number $r \in \{0,\ldots,d\}$ such that there exist analytic functions $f_1,\ldots,f_r \in \mathcal{V}$ with the property that the differentials $d_{\mathbf{x}}f_1,\ldots,d_{\mathbf{x}}f_r$ are linearly independent (inside the cotangent space of G at $\mathbf{x}$). Since G is a homogeneous manifold, the map $\mathbf{x} \mapsto n_{\mathbf{x}}(\mathcal{V})$ is in fact constant, and we denote its common value by $n(\mathcal{V}) := n_{\mathbf{e}}(\mathcal{V}) \in \{0,\ldots,d\}$.

Suppose for the moment that $n(\mathcal{V}) = 0$. Then, according to [18, Part II, Sections III §10 §11 and IV §2], the representation $G \to \mathrm{GL}_n^1(R)$ pertaining to $\mathcal{V}$ is a faithful immersion, hence we may embed G as an analytic subgroup into $\mathrm{GL}_n^1(R)$ and there is nothing further to prove.

Now assume that $n(\mathcal{V}) > 0$. For short we write $n := n(\mathcal{V})$ and $r := r_{\mathbf{e}} = d-n$. Choose $f_1,\ldots,f_r \in \mathcal{V}$ such that $d_{\mathbf{e}}f_1,\ldots,d_{\mathbf{e}}f_r$ are linearly independent. Changing our standard coordinate system, if necessary, we may assume that $f_1 = X_{n+1},\ldots,f_r = X_d$. (Such a change of coordinates is possible, because we are working over a complete ultrametric field. It may involve rescaling in the sense of [4, Section 13.6].)

Let $\mathcal{M}$ denote the ideal generated by $X_1,\ldots,X_d$ in $R[\![X_1,\ldots,X_d]\!]$, and for $g \in R[\![X_1,\ldots,X_d]\!]$ define $\deg g := \sup\{k \in \mathbb{N}_0 \mid g \in \mathcal{M}^k\}$. Let

$$\nu(\mathcal{V}) := \min\{\deg g(X_1,\ldots,X_n,0,\ldots,0) \mid g \in \mathcal{V}\}.$$

As $n > 0$ and as $\mathbf{e}$ is the only common zero of $\mathcal{V}$, we see that $\nu(\mathcal{V})$ is finite so that $(n(\mathcal{V}),\nu(\mathcal{V})) \in \mathbb{N}_0 \times \mathbb{N}$. On the latter set we consider the lexicographic order defined by $(n_1,\nu_1) < (n_2,\nu_2)$, if (a) $n_1 < n_2$ or (b) $n_1 = n_2$ and $\nu_1 < \nu_2$. This is a linear order with smallest element $(0,1)$.

Note that the field of fractions of R is of the form $F = \mathbb{F}_q(\!(t)\!)$ for some p-power q. We extend F to $E = \mathbb{F}_q(\!(s)\!)$ where $s^p = t$. Taking advantage of the obvious isomorphism between F and E, we find in $\mathbb{F}_q[\![s]\!]$ a subring S which contains R, is isomorphic to R and has the property that every element of R is a pth power in S. Next we construct

- an S-standard group $H = \mathrm{Gr}_{\mathbf{F}'}(S)$ of dimension d, associated to a formal group law $\mathbf{F}'$ over S,
- an S-analytic homomorphism $\varphi : \mathrm{Gr}_{\mathbf{F}'}(S) \to \mathrm{Gr}_{\mathbf{F}}(S)$ which is a quasi-Frobenius map,
- a free R-module $\mathcal{W}$ of finite rank together with an H-action, pertaining to a strongly faithful representation of H, such that $(n(\mathcal{W}),\nu(\mathcal{W})) < (n(\mathcal{V}),\nu(\mathcal{V}))$.

Repeating this process finitely many times, we arrive at a proof of the proposition.

Subclaim. For every $f \in \mathcal{V}$ there exists $\hat{f} \in R[\![X_1,\ldots,X_d]\!]$ such that

$$f = \hat{f}(X_1^p,\ldots,X_n^p,X_{n+1},\ldots,X_d).$$

Subproof. Let $f \in \mathcal{V}$ and $i \in \{1, \ldots, n\}$. It suffices to show that the partial derivative $\partial_i f$ of f with respect to X_i is equal to 0. For a contradiction, suppose that $\partial_i f \neq 0$. Then we find $\mathbf{y} \in G$ such that $\partial_i f(\mathbf{y}) \neq 0$. Since $f_1 = X_{n+1}, \ldots, f_r = X_d$, this implies that the differentials $d_{\mathbf{y}} f, d_{\mathbf{y}} f_1, \ldots, d_{\mathbf{y}} f_r$ are linearly independent, and gives $r_{\mathbf{y}} > r$, a contradiction.

We want to use the subclaim to rewrite the last r coordinate functions $F_{n+1}, \ldots, F_d \in R[\![X_1, \ldots, X_d, Y_1, \ldots, Y_d]\!]$ of the formal group law $\mathbf{F}$. As $\mathcal{V}$ is closed under the action of G via ρ, λ and as $f_1 = X_{n+1}, \ldots, f_r = X_d \in \mathcal{V}$, we obtain for $i \in \{n+1, \ldots, d\}$,

$$F_i(\mathbf{X}, \mathbf{y}) = \rho_{\mathbf{y}}(X_i) \in \mathcal{V} \text{ and } F_i(\mathbf{y}, \mathbf{X}) = \lambda_{\mathbf{y}^{-1}}(X_i) \in \mathcal{V} \quad \text{for all } \mathbf{y} \in G.$$

By the subclaim, this implies that, for $i \in \{n+1, \ldots, d\}$, we may write

$$F_i(\mathbf{X}, \mathbf{Y}) = \hat{F}_i(X_1^p, \ldots, X_n^p, X_{n+1}, \ldots, X_d, Y_1^p, \ldots, Y_n^p, Y_{n+1}, \ldots, Y_d)$$

with $\hat{F}_{n+1}, \ldots, \hat{F}_d \in R[\![X_1, \ldots, X_d, Y_1, \ldots, Y_d]\!]$. Writing

$$\overline{\mathbf{X}} = (X_1, \ldots, X_n, X_{n+1}^p, \ldots, X_d^p),$$

we find $\mathbf{F}' \in S[\![X_1, \ldots, X_d, Y_1, \ldots, Y_d]\!]$ such that

$$\mathbf{F}(\overline{\mathbf{X}}, \overline{\mathbf{Y}}) = \overline{\mathbf{F}'(\mathbf{X}, \mathbf{Y})}.$$

One easily checks that $\mathbf{F}'$ is a formal group law, giving rise to an S-standard group $H = \mathrm{Gr}_{\mathbf{F}'}(S)$. Plainly, we have a quasi-Frobenius map $\varphi : \mathrm{Gr}_{\mathbf{F}'}(S) \to \mathrm{Gr}_{\mathbf{F}}(S), \mathbf{x} \mapsto \overline{\mathbf{x}}$.

It remains to manufacture a free R-module $\mathcal{W}$ of finite rank together with an H-action, pertaining to a strongly faithful representation of H, such that $(n(\mathcal{W}), \nu(\mathcal{W})) < (n(\mathcal{V}), \nu(\mathcal{V}))$.

Note that the Frobenius map $\varphi : \mathrm{Gr}_{\mathbf{F}'}(S) \to \mathrm{Gr}_{\mathbf{F}}(S)$ induces a map φ^* between the corresponding rings of functions. Let $\mathfrak{n}$ denote the maximal ideal of the pro-p domain S. By virtue of our identifications, $\mathrm{Gr}_{\mathbf{F}'}(S) = \mathfrak{n}^{(d)}$ and $\mathrm{Gr}_{\mathbf{F}}(S) = \mathfrak{n}^{(d)}$, we have

$$(\varphi^* f)(\mathbf{X}) = f(\overline{\mathbf{X}}) \quad \text{for all } f \in S[\![X_1, \ldots, X_d]\!].$$

Recall that in any integral domain of characteristic p, the extraction of pth roots, if at all possible, is unique. We define $\mathcal{W}$ to be the S-module generated by $\varphi^*\mathcal{V} \cup (\varphi^*\mathcal{V} \cap S[\![X_1^p, \ldots, X_d^p]\!])^{1/p}$. Then it is clear that $\mathcal{W}$ is a free S-module of finite rank and that H acts strongly faithfully on $\mathcal{W}$ via the actions induced by left and right multiplication.

Finally, we show that $(n(\mathcal{W}), \nu(\mathcal{W})) < (n(\mathcal{V}), \nu(\mathcal{V}))$. First, note that $n(\mathcal{W}) \leq n(\mathcal{V})$, because $X_{n+1}, \ldots, X_d \in \mathcal{W}$. Now, let $g \in \mathcal{V}$ such that $\nu(\mathcal{V}) = \deg g(X_1, \ldots, X_n, 0, \ldots, 0)$. Then $g = \hat{g}(X_1^p, \ldots, X_n^p, X_{n+1}, \ldots, X_d)$ for suitable $\hat{g} \in R[\![X_1, \ldots, X_d]\!]$ by the subclaim, hence $h = (\varphi^* g)^{1/p} = (\hat{g}(X_1^p, \ldots, X_d^p))^{1/p} \in \mathcal{W}$ and $\deg h(X_1, \ldots, X_n, 0, \ldots, 0) = \nu(\mathcal{V})/p$. □

8. Characterization of p-adic Analytic Groups – a Counterexample

In this section we prove Theorem 1.9.

Proof of Theorem 1.9. Let $\mathbb{D}$ be a division algebra of index 2 over the local field $F = \mathbb{F}_p(\!(t)\!)$. Let $\mathcal{G}$ denote the 3-dimensional simply-connected absolutely simple F-algebraic group pertaining to the norm-1 group $\mathrm{SL}_1(\mathbb{D})$, and let $\overline{\mathcal{G}}$ denote the corresponding adjoint group. By Theorem 2.2, every compact open subgroup of $\mathcal{G}(\mathbb{F}_p(\!(t)\!))$ is finitely generated and $\mathbb{F}_p[\![t]\!]$-analytic. Let Γ be a pro-p subgroup of $\mathcal{G}(\mathbb{F}_p(\!(t)\!))$. It suffices to show that Γ does not map onto an open subgroup of $C_p \hat{\wr} \mathbb{Z}_p$. Clearly, we may assume that Γ is finitely generated and not virtually abelian.

Subclaim. Γ is Zariski-dense in $\mathcal{G}$.
Subproof. Let $\mathcal{H}$ denote the Zariski-closure of Γ in $\mathcal{G}$ and note that this group is F-defined. It suffices to show that $\mathcal{H}$ has dimension 3. Since every connected 1-dimensional algebraic group is abelian, we have to rule out that $\mathcal{H}$ has dimension 2. Indeed, if $\mathcal{H}$ was 2-dimensional, it would constitute an F-defined Borel subgroup. But $\mathcal{G}$ is of type A_1 and cannot be quasi-split; cf. [6, §35.2].

Now Theorem 2.4 implies that Γ is virtually just-infinite. As $C_p \hat{\wr} \mathbb{Z}_p$ is not virtually just-infinite, Γ cannot map onto an open subgroup of $C_p \hat{\wr} \mathbb{Z}_p$. □

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