Singularities of the space of arcs

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Outline of the talk

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3. Embedding codimension

4. Embedding codimension of the space of arcs

Review: Differentials on the space of arcs

Jets and arcs

Let K be a field of arbitrary characteristic and X a variety over K.

An n-jet on X is a morphism $\operatorname{Spec} K[t]/(t^{n+1}) \to X$. The n-th jet space X_n of X is the K-scheme parametrizing jets on X, i.e.

$$\operatorname{Hom}_K(\operatorname{Spec} K, X_n) \simeq \operatorname{Hom}_K(\operatorname{Spec} K[t]/(t^{n+1}), X).$$

An arc on X is a morphism $\operatorname{Spec} K[[t]] \to X$. The arc space X_∞ of X is defined as $X_\infty = \varprojlim_n X_n$ and satisfies

$$\operatorname{Hom}_K(\operatorname{Spec} K, X_\infty) \simeq \operatorname{Hom}_K(\operatorname{Spec} K[[t]], X).$$

Higher derivations and arcs

Let R, S be K-algebras. A higher derivation $D : R \to S$ of order $n \in \mathbb{N} \cup \{\infty\}$ over K is given by $D_i : R \to S$, $i \leqslant n$, satisfying

1. $D_0 : R \to S$ is a K-algebra map, and
2. the *higher Leibniz rules*, that is,

$$D_i(r_1 r_2) = \sum_{j+l=i} D_j(r_1) D_l(r_2).$$

The universal object for higher derivations is the Hasse–Schmidt algebra $R_n := \mathrm{HS}^n_K(R)$. If $X = \operatorname{Spec} R$, then $X_n = \operatorname{Spec} R_n$ for $n \in \mathbb{N} \cup \{\infty\}$.

Arc spaces and singularities

Since the work of Nash, jet and arc spaces are known to be deeply connected to the structure of singularities of algebraic varieties.

E.g. Nash problem: components of $\pi^{-1}(\operatorname{Sing} X)$ in correspondence with exceptional components of resolution of singularities.

Relatively little is known about the singularities of X_∞ itself and how they relate to the singularities of X.

Differentials on the space of arcs

Theorem (de Fernex, Docampo)

Let $X = \mathrm{Spec}(R)$ *be an affine variety over* K *and* $n \in \mathbb{N} \cup \{\infty\}$. *Then*

$$\Omega_{X_n/K} \simeq \Omega_{X/K} \otimes_R Q_n,$$

for some R*-module* Q_n *which in addition is a free* R_n*-module.*

Remark: Q_n has an interpretation in terms of higher derivations of modules.

Singularities of the space of arcs

Formally smooth arcs

Recall that X_∞ is infinite-dimensional if $\dim X > 0$, therefore the definition of regular points does not make sense anymore.

Theorem (Bourqui, Sebag)

Let $\alpha \in X_\infty \setminus (\operatorname{Sing} X)_\infty$. Then:

1. *Let $X_0 \subset X$ be the unique formal branch at $\alpha(0)$ containing $\alpha(\eta)$. Then*
$$\widehat{\mathcal{O}_{X_\infty,\alpha}} \simeq \widehat{\mathcal{O}_{(X_0)_\infty,\alpha}}.$$
2. *$\mathcal{O}_{X_\infty,\alpha}$ formally smooth over K iff the formal branch containing $\alpha(\eta)$ is smooth.*

The Drinfeld–Grinberg–Kazhdan theorem

Theorem (Grinberg, Kazhdan 2000; Drinfeld 2002)

Let X be a variety over a field K and $\alpha \in X_\infty \setminus (\operatorname{Sing} X)_\infty$. Then there exists a (nonreduced) scheme Y of finite type over K and a point y such that

$$\widehat{\mathcal{O}_{X_\infty,\alpha}} \simeq \widehat{\mathcal{O}_{Y,y}} \mathbin{\widehat{\otimes}}_K K[[z_i \mid i \in \mathbb{N}]].$$

$\rightsquigarrow$ Singularity at α decomposes into a finite-dimensional part $\widehat{\mathcal{O}_{Y,y}}$ and into an infinite-dimensional smooth part.

Definition: The formal scheme $\widehat{Y}_y := \operatorname{Spf} \widehat{\mathcal{O}_{Y,y}}$ is called a formal model of α.

Remark: Proof is constructive (c.f. Bourqui, Sebag), but difficult to understand the resulting formal model.

Describing the formal neighborhood by deformations

Idea: Use functorial description of $\widehat{(X_\infty)}_\alpha := \operatorname{Spf} \widehat{\mathcal{O}_{X_\infty,\alpha}}$.

Let $(A, \mathfrak{m})$ be a test ring, that is, a K-algebra such that $A/\mathfrak{m} = K$ and $\mathfrak{m}^n = 0$ for some n.

Fact: $\widehat{(X_\infty)}_\alpha$ is determined by A-deformations of α, that is, morphisms $\widetilde{\alpha} : \operatorname{Spec} A[[t]] \to X$ such that $\widetilde{\alpha} \equiv \alpha$ modulo $\mathfrak{m}$. In diagrammatic notation:

$$\begin{array}{ccc} \operatorname{Spec} A[[t]] & \xrightarrow{\widetilde{\alpha}} & X \\ \uparrow & \nearrow_{\alpha} & \\ \operatorname{Spec} K[[t]] & & \end{array}$$

Drinfeld's example

Let $X = V(yz - x^2)$ and $\alpha(t) = (0, 0, t)$. Let $(A, \mathfrak{m})$ be a test ring and $\widetilde{\alpha}$ an A-deformation of α. That is,

$$\widetilde{\alpha}(t) = (x(t), y(t), z(t) + t),\ x(t), y(t), z(t) \in \mathfrak{m}[[t]].$$

Weierstrass preparation: $z(t) + t = (t - a)u(t)$; $a \in \mathfrak{m}$, $u(t) \in 1 + \mathfrak{m}[[t]]$.
Substituting:

$$y(t)(t - a) - x^2(t)u^{-1}(t) = 0.$$

Given a and $u(t), x(t)$, then there exists $y(t)$ solving this equation if and only if a is a root of $x(t)^2 = 0$. Writing $v(t) := x(t - a)$ this translates to $v(0)^2 = 0$.

Drinfeld's example continued

Thus an A-deformation $(x(t), y(t), z(t) + t)$ is the same as $u(t) \in 1 + \mathfrak{m}[[t]]$, $v(t) \in \mathfrak{m}[[t]]$ satisfying $v(0)^2 = v_0^2 = 0$. Hence

$$\widehat{\mathcal{O}_{X_\infty,\alpha}} \simeq K[[v_0]]/(v_0^2) \,\widehat{\otimes}_K\, K[[u_i, v_i \mid i \geqslant 1]].$$

$\rightsquigarrow$ A formal model for $\alpha(t) = (0, 0, t)$ on $X = V(yz - x^2)$ is given by

$$\widehat{Y}_y = \operatorname{Spf} K[[v_0]]/(v_0^2).$$

Remark: Same argument works for $yz - f(x_1, \ldots, x_n)$ and gives as a formal model $K[[v_1, \ldots, v_n]]/f(v_1, \ldots, v_n)$.

Further questions

Theorem (Grinberg, Kazhdan 2000; Drinfeld 2002)

Let $\alpha \in X_\infty \setminus (\operatorname{Sing} X)_\infty$. *Then:*

$$\widehat{\mathcal{O}_{X_\infty,\alpha}} \simeq \widehat{\mathcal{O}_{Y,y}} \mathbin{\widehat{\otimes}_K} K[[z_i \mid i \in \mathbb{N}]].$$

Questions:

1. What happens for $\alpha \in (\operatorname{Sing} X)_\infty$?
2. Can the statement be extended beyond the formal completions to a more global one?
3. How does the formal model of α relate to the singularity at $\alpha(0) \in X$?

Formal neighborhood of degenerate arcs

Question 1: What happens for $\alpha \in (\operatorname{Sing} X)_\infty$?

Partial results by Bourqui, Sebag and C., Hauser for *constant* arcs.

Theorem 1 (C., Hauser)

Let $x \in X$ and $\alpha_x \in X_\infty$ the constant arc centered in x. Assume $\operatorname{char}(K) = 0$. *Then there exists a decomposition of the form*

$$\widehat{\mathcal{O}_{X_\infty,\alpha}} \simeq \widehat{\mathcal{O}_{Y,y}} \,\widehat{\otimes}_K\, K[[z]]$$

if and only if there exists such a decomposition for $\widehat{\mathcal{O}_{X,x}}$.

Corollary 1

If $x \in \operatorname{Sing} X$, then there does not exist a Drinfeld–Grinberg–Kazhdan decomposition for α_x.

Remark: Statement can be derived from the formula for the sheaf of differentials.

Globalization of Drinfeld–Grinberg–Kazhdan theorem

Question 2: Can the statement be extended beyond the formal completions to a more global one?

Attempts made by Bouthier, Ngo, Sakellaridis,... to extend the Drinfeld–Grinberg–Kazhdan theorem.

Problem: Proof of Drinfeld crucially makes use of Weierstrass preparation, which holds only over complete local rings. In the language of arc spaces: the morphism

$$Q_d \times (\mathbb{G}_m)_\infty \to (\mathbb{A}^1)_\infty, \; (q(t), u(t)) \mapsto q(t)u(t),$$

where Q_d is the space of monic polynomials of degree d and $(\mathbb{G}_m)_\infty$ is the space of invertible series, is only an isomorphism at the completion at points $(t^d, u(t))$.

The minimal formal model of an arc

Question 3: How does the formal model of α relate to the singularity at $\alpha(0) \in X$?

Definition: A minimal formal model of $\alpha \in X$ is a formal model $\widehat{Y}_y$ which is *indecomposable*, i.e. there does not exist an isomorphism

$$\widehat{\mathcal{O}_{Y,y}} \simeq \widehat{\mathcal{O}_{Z,z}} \widehat{\otimes}_K K[[u]].$$

Fact: For any $\alpha \in X_\infty \setminus (\operatorname{Sing} X)_\infty$ there exists a minimal formal model and it is unique up to isomorphism.

Bourqui, Sebag described explicitly the minimal formal model of certain plane curves. For example, for $X = V(y^2 - x^{n+1})$ and $\alpha(t) = (t^2, t^{n+1})$ the formal model is given by

$$\widehat{Y}_y = \operatorname{Spf} K[[u]]/(u^{n/2-1}).$$

Embedding codimension

Embedding codimension of algebraic varieties

Problem: Given a variety X over K, measure the "size" of a singularity $x \in X$.

Idea: Recall that, for a local ring $(A, \mathfrak{m})$, $\operatorname{embdim} A := \dim_K \mathfrak{m}/\mathfrak{m}^2$. Consider the difference

$$\operatorname{embcodim} \mathcal{O}_{X,x} := \operatorname{embdim} \mathcal{O}_{X,x} - \dim \mathcal{O}_{X,x},$$

which is called embedding codimension or regularity defect (c.f. Lech).

Fact: $\operatorname{embcodim} \mathcal{O}_{X,x} = 0$ iff $x \in X$ is regular.

Fact: $\operatorname{embcodim} \mathcal{O}_{X,x} = 1$ iff X is locally at x an intersection of hypersurfaces.

Embedding codimension of algebraic varieties cont.

Recall: for a variety X with $\operatorname{embdim} \mathcal{O}_{X,x} = d$ there exists a surjective map $K[[x_1, \ldots, x_d]] \to \widehat{\mathcal{O}_{X,x}}$, a *formal embedding*. In fact, we have more:

Theorem

Assume K infinite and $x \in X$ singular. If $\operatorname{embdim} \mathcal{O}_{X,x} = d$, then there exists $U \subset X$ open neighborhood of x such that $U \hookrightarrow \mathbb{A}^d$.

Remark: $\operatorname{embcodim} \mathcal{O}_{X,x}$ measures the codimension of X with respect to a *minimal* embedding.

Theorem (Lech 1964)

Let $(A, \mathfrak{m})$ be an excellent Noetherian local ring. For any prime $\mathfrak{p}$ of A we have $\operatorname{embcodim} A_{\mathfrak{p}} \leqslant \operatorname{embcodim} A$. In particular, the function

$$x \in X \mapsto \operatorname{embcodim} \mathcal{O}_{X,x} \in \mathbb{N}_0$$

is upper semicontinuous.

Formal embedding codimension of non-Noetherian rings

Problem: If $(A, \mathfrak{m})$ is not Noetherian, then in general $\operatorname{embdim} A = \infty$ and $\dim A = \infty$. So the difference $\operatorname{embdim} A - \dim A$ is not defined.

Idea: Assume A contains a field K such that $A/\mathfrak{m} = K$. Choose minimal system of generators a_i, $i \in I$, for $\mathfrak{m}$. This gives a surjection

$$\tau : K[[x_i \mid i \in I]] \to \widehat{A},\ x_i \mapsto a_i.$$

Definition: The formal embedding codimension of A is defined as

$$\operatorname{f.embcodim} A := \operatorname{ht}(\ker \tau).$$

Remark: This is independent of choice of generators a_i.

Formal embedding codimension continued

Definition: $\operatorname{f.embcodim} A := \operatorname{ht}(\ker \tau)$, where $\tau : K[[x_i \mid i \in I]] \to \widehat{A}$.

Fact: If A Noetherian, then

$$\operatorname{f.embcodim} A = \operatorname{ht}(\ker(K[[x_1, \ldots, x_d]] \to \widehat{A})),$$

where $d = \operatorname{embdim} A$. Since $\dim \widehat{A} = \dim A$, we have $\operatorname{f.embcodim} A = \operatorname{embcodim} A - \dim A$.

Fact: We have $\operatorname{f.embcodim} A = 0$ iff A is formally smooth over K. In this case, $\widehat{A} \simeq K[[x_i \mid i \in I]]$.

Towards a different viewpoint of embedding codimension

There are two issues with the definition of formal embedding codimension:

1. Difficult in case A is of mixed characteristic.
2. The ring $K[[x_i \mid i \in I]]$ has a lot of pathologies if $|I| = \infty$.

Observations: For A Noetherian we have $\dim A = \dim \widehat{A} = \dim \operatorname{gr} A$, where $\operatorname{gr} A$ is the associated graded of A.

If $\widehat{A} = K[[x_i \mid i \in I]]$, then $\operatorname{gr} A = K[x_i \mid i \in I]$, which is a much easier ring to study than the former.

Embedding codimension of non-Noetherian rings

Definition: Let $(A, \mathfrak{m})$ be a local ring with $K = A/\mathfrak{m}$. Consider the natural surjection

$$\gamma : \mathrm{Sym}_K(\mathfrak{m}/\mathfrak{m}^2) \to \bigoplus_n \mathfrak{m}^n/\mathfrak{m}^{n+1} = \mathrm{gr}\, A.$$

Then the embedding codimension of A is defined as

$$\mathrm{embcodim}\, A := \mathrm{ht}(\ker \gamma).$$

Fact: If A Noetherian, then $\mathrm{embcodim}\, A = \mathrm{embdim}\, A - \dim A$.

Theorem (EGA IV)

A local K-algebra A is formally smooth over K iff $\mathrm{embcodim}\, A = 0$.

A consequence of the Drinfeld–Grinberg–Kazhdan theorem

Theorem 2 (C., de Fernex, Docampo)

Let $\alpha \in X_\infty \setminus (\operatorname{Sing} X)_\infty$ *and let*

$$\widehat{\mathcal{O}_{X_\infty,\alpha}} \simeq \widehat{\mathcal{O}_{Y,y}} \mathbin{\widehat{\otimes}_K} K[[t_i \mid i \in \mathbb{N}]].$$

Then $\operatorname{embcodim} \mathcal{O}_{X_\infty,\alpha} = \operatorname{embcodim} \mathcal{O}_{Y,y}$ *and similarly for* $\operatorname{f.embcodim}$. *In particular,* $\operatorname{embcodim} \mathcal{O}_{X_\infty,\alpha} < \infty$.

Remark: Proof for embcodim is a trivial consequence of the theorem of Drinfeld, Grinberg and Kazhdan theorem.

For f. embcodim the proof is much harder and requires extensions of classical results of commutative algebra such as flatness of completion.

Comparing the two notions of embedding codimension

Theorem 3 (C., de Fernex, Docampo)

For any local equicharacteristic ring $(A, \mathfrak{m})$ *we have*

$$\operatorname{embcodim} A \leqslant \operatorname{f.embcodim} A.$$

Remark: Proof uses degeneration to the extended Rees algebra.

Remark: We do not know of an example where this inequality is strict. As we will see, for any arc $\alpha \in X_\infty$ we have equality for $A = \mathcal{O}_{X_\infty,\alpha}$.

Embedding codimension of the space of arcs

The main result

Theorem 4 (C., de Fernex, Docampo)

Let K be a perfect field and X a variety over K. Let $\alpha \in X_\infty$. Then the following are equivalent:

1. $\alpha \in X_\infty \setminus (\operatorname{Sing} X)_\infty$.
2. $\operatorname{embcodim} \mathcal{O}_{X_\infty,\alpha} < \infty$.
3. $\operatorname{embcodim} \mathcal{O}_{X_\infty,\alpha} \leqslant \operatorname{ord}_\alpha \operatorname{Jac}_X$.

Here Jac_X denotes the Jacobian ideal of X.

Remark: The proof does not make use of the theorem of Drinfeld, Grinberg, Kazhdan; nor of Weierstrass preparation.

Drinfeld's example revisited

Let $X = V(yz - x^2)$ and $\alpha(t) = (0,0,t)$. We have seen that a formal model for α is given by

$$\widehat{Y}_y = \mathrm{Spf}(K[v]/(v^2)).$$

Clearly $\operatorname{embcodim} \mathcal{O}_{Y,y} = 1$. On the other hand,

$$\mathrm{Jac}_X = (2x, y, z)$$

and thus $\operatorname{ord}_\alpha \mathrm{Jac}_X = 1$. We see that in this case

$$\operatorname{embcodim} \mathcal{O}_{Y,y} = \operatorname{ord}_\alpha \mathrm{Jac}_X .$$

Hence the bound provided by the previous theorem is sharp.

Strategy of the proof

Aim: Illustrate the idea of the proof of $\operatorname{embcodim} \mathcal{O}_{X_\infty,\alpha} \leqslant \operatorname{ord}_\alpha \operatorname{Jac}_X$ in several steps:

1. Start with a property on jet spaces and study its asymptotics.
2. Construct a morphism of the underlying variety $f : X \to Y$ whose induced map $f_\infty : X_\infty \to Y_\infty$ "constructs" this property.
3. Show that one may pass to the limit via

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y_\infty \\ \downarrow{\scriptstyle \pi_n^X} & & \downarrow{\scriptstyle \pi_n^Y} \\ X_n & \xrightarrow[f_n]{} & Y_n . \end{array}$$

Ingredient I: Asymptotics of jet spaces

Theorem (de Fernex, Docampo)

Let $\alpha \in X_\infty \setminus (\operatorname{Sing} X)_\infty$ *and* $\alpha_n = \pi_n(\alpha) \in X_n$. *Then, for* $n \gg 0$,

$$\operatorname{embdim} \mathcal{O}_{X_n,\alpha_n} \leqslant (n+1)\dim X + \operatorname{ord}_\alpha \operatorname{Jac}_X,$$

Remark: This is a consequence of the formula for the sheaf of differentials of X_∞.

Observation: If $n \gg 0$ and $\alpha \in X_\infty \setminus (\operatorname{Sing} X)_\infty$, then

$$\dim \mathcal{O}_{X_n,\alpha_n} \geqslant (n+1)\dim X.$$

Ingredient II: A geometric construction

Situtation: Let X be a variety with $\dim X = n$ and $X \subset \mathbb{A}^d$. Consider a general linear projection $\mathbb{A}^d \to \mathbb{A}^n$ and the induced map $f : X \to \mathbb{A}^n =: Y$. Write $f_n : X_n \to Y_n$ for $n \in \mathbb{N} \cup \{\infty\}$.

Theorem 5 (C., de Fernex, Docampo)

Let $\alpha \in X_\infty \setminus (\operatorname{Sing} X)_\infty$ and assume that K is perfect. Write $\beta = f_\infty(\alpha)$. Then the induced map on Zariski cotangent spaces

$$(T_\alpha f_\infty)^* : \mathfrak{m}_\beta / \mathfrak{m}_\beta^2 \to \mathfrak{m}_\alpha / \mathfrak{m}_\alpha^2$$

is an isomorphism.

Remark: Proof of this theorem uses the formula for the sheaf of differentials. Compare to Bourqui, Sebag and Ein, Mustata.

Ingredient II continued

Since $Y = \mathbb{A}^n$, we have

$$\mathrm{gr}(\mathcal{O}_{Y_\infty,\beta}) \simeq \mathrm{Sym}_K\, \mathfrak{m}_\beta/\mathfrak{m}_\beta^2$$

Then the map

$$(T_\alpha f_\infty)^* : \mathfrak{m}_\beta/\mathfrak{m}_\beta^2 \to \mathfrak{m}_\alpha/\mathfrak{m}_\alpha^2$$

being an isomorphism implies that:

Corollary

$\mathrm{embcodim}(\mathcal{O}_{X_\infty,\alpha}) = \mathrm{ht}(\ker \mathrm{gr}(f_\infty))$, *where*

$$\mathrm{gr}(f_\infty) : \mathrm{gr}(\mathcal{O}_{Y_\infty,\beta}) \to \mathrm{gr}(\mathcal{O}_{X_\infty,\alpha}).$$

Ingredient III: Passing to the limit

Consider the diagram

$$\begin{array}{ccc} \mathrm{gr}(\mathcal{O}_{Y_\infty,\beta}) & \xrightarrow{gr(f_\infty)} & \mathrm{gr}(\mathcal{O}_{X_\infty,\alpha}) \\ {\scriptstyle \mathrm{gr}(\pi_n^X)}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{gr}(\pi_n^Y)} \\ \mathrm{gr}(\mathcal{O}_{Y_n,\beta_n}) & \xrightarrow[\mathrm{gr}(f_n)]{} & \mathrm{gr}(\mathcal{O}_{X_n,\alpha_n}). \end{array}$$

Fact: $\mathrm{ht}(\ker \mathrm{gr}(f_\infty)) = \limsup_n \mathrm{ht}(\ker \mathrm{gr}(f_n))$.

Fact: $\mathrm{ht}(\ker \mathrm{gr}(f_n)) \leqslant \mathrm{embcodim}\, \mathcal{O}_{X_n,\alpha_n}$.

This concludes the proof of the theorem. □

Remark: Here the proof relies on the fact that $K[x_i \mid i \in I]$ is the colimit of all its finite-variate polynomial rings.

Outlook

The explicit bound was already shown to imply results on *Mather–Jacobi discrepancies*, which are invariants used in higher-dimensional birational geometry.

In addition, the approach presented here may be applied to study similar properties of singularities of jet and arc spaces. In particular, one hope is to obtain a full description of the singular structure of the arc space, see:

- A. Bouthier: “Cohomologie étale des espaces d’arcs”.
- D. Bourqui, J. Sebag: “Finite formal model of toric singularities”.

References

- C. Chiu, T. de Fernex, and R. Docampo. *Embedding codimension of the space of arcs.* 2020. arXiv: `2001.08377 [math.AG]`.
- V. Drinfeld. *On the Grinberg–Kazhdan formal arc theorem.* 2002. arXiv: `math/0203263 [math.AG]`.
- M. Grinberg and D. Kazhdan. "Versal deformations of formal arcs". In: *Geom. Funct. Anal.* 10.3 (2000), pp. 543–555. ISSN: 1016-443X. DOI: `10.1007/PL00001628`. URL: `https://doi.org/10.1007/PL00001628`.