Resolution of singular plane curves via geometric invariants

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The topic belongs to the field of resolution of singularities, itself a subfield of algebraic geometry. We consider a plane curve $X \subseteq \mathbb{A}^2_{\mathbb{C}}$ given by the polynomial equation $f(x,y)=0$. It is a classical and very famous result in resolution of singularities that there exists a smooth curve X' and an almost isomorphism $\pi : X' \to X$ such that X can be interpreted as a "shadow" of X' under projection. The resolution via geometric invariants uses certain numerals reflecting the geometry of the singular curve X to find the map π.

We propose to associate to each regular point a of X a height function $a \mapsto h(a)$ such that the space curve obtained as the (Zariski-) closure of the graph of h is a smooth space curve. The idea now is to refine the concept of Nash modification by looking at the (modified) curvature of X at a. It is given by the formula

$$\kappa_1(f) = \frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{(f_x - f_y)^3},$$

or equivalently by

$$\kappa_1(t) = \frac{x''y' - y''x'}{(x' + y')^3}$$

if $(x(t), y(t))$ is a parametrization of X at a. The parametric expression is one example of a geometric invariant of X. These are rational expressions in $x(t), y(t)$ and their derivatives that are invariant under reparametrization. However, as computation shows, the modified curvature is as a height function not always sufficient to yield a resolution of the curve. Hence, we look at higher modified curvatures which are defined iteratively via

$$\kappa_i = \frac{\frac{\partial}{\partial t}(\kappa_{i-1})}{x' + y'}, i > 1.$$

These provide already enough geometric numerals to define the resolution for (unibranched) plane curves in the above way. Evenmore, one can show that the field of all geometric invariants is generated by the higher modified curvatures.