Worksheet 7. Symmetric matrices. Eigenvalues and eigenvectors. Diagonalization.

1) Find the real eigenvectors of the following matrices:

$$a)\begin{pmatrix} 4 & 6 \\ -3 & -5 \end{pmatrix} \quad b)\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} \quad c)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$d)\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \quad e)\begin{pmatrix} 0 & 0 & -1 \\ 1 & -2 & -1 \\ -2 & 3 & 1 \end{pmatrix} \quad f)\begin{pmatrix} 2 & 2 & -1 \\ 0 & -2 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$g)\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \quad h)\begin{pmatrix} 0 & -1 & 2 \\ 0 & -1 & 0 \\ -1 & 1 & -3 \end{pmatrix} \quad i)\begin{pmatrix} 2 & 2 & -1 \\ 0 & 2 & 0 \\ 3 & 1 & 1 \end{pmatrix}$$

When possible, find a basis of eigenvectors for $\mathbb{R}^n$.

2) Find the eigenvalues and eigenvectors of the linear maps given by the following matrices:

$$a)\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix} \quad b)\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix} \quad c)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

3) If possible, diagonalize the following matrices:

$$a)\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad b)\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad c)\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad d)\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}.$$

4) If possible, diagonalize the following matirces; in that case, find also a matrix P such that $A = PDP^{-1}$:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}; \qquad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

5) Given $A = \begin{pmatrix} 5/2 & -1 \\ 3 & -1 \end{pmatrix}$, find A^{1438} and $\text{lím}_{n\longrightarrow\infty} A^n$.

6) Given $A = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$, find A^{2016}.

7) Consider $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ a & -1 & 0 & 0 \\ b & d & 1 & 0 \\ c & e & f & 1 \end{pmatrix}$.

a) Find the eigenvalues of A.

b) For which values of a, b, c, d, e, f is A diagonalizable?

c) When A is diagonalizable, find the eigensubspaces associated to the eigenvalues.

d) Find the eigendecomposition of A.

8) Prove or disprove the following statements:

a) All invertible matrices are diagonalizable.

b) If A is diagonalizable, then A^n is diagonalizable for $n \in \mathbb{N}$.

c) If A and B are diagonalizable, then so are $A + B$ and AB.

9) Find the values of a and b in $\mathbb{R}$ for which the matrix $\begin{pmatrix} a & b & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is diagonalizable in $\mathbb{R}$.

10) Find the values of c for which the matrix $\begin{pmatrix} 1 & -2 & -2-c \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ is diagonalizable.

11) Let $f \in \text{End}(\mathbb{R}^3)$ be such that:

a) $f(1,1,1) = (0,0,0)$ and

b) 2 y 3 are eigenvalues of f.

Is f diagonalizable?

12) Find the values of $a, b \in \mathbb{R}$ for which the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix}$ is diagonalizable in $\mathbb{R}$.