Worksheet 6. Vector spaces and linear maps.

1) Consider the vector space $\mathbb{R}^3$.

i) Is it possible to write the vectors $(1,2,3),(1,1,1)$ as linear combinations of the vectors in $S=\{(1,0,1),(0,2,2)\}$?

ii) Compute x and y, if possible, so that $(1,2,x,y)$ is a linear combination of the vectors $(1,2,0,2)$ and $(1,1,2,3)$.

2) Decide whether the following families of vectors in $\mathbb{R}^3$ are linearly independent or not.

i) $\{(1,2,1),(1,0,1),(2,3,2)\}$.

ii) $\{(1,0,0),(2,1,1),(0,1,1),(1,-1,-1)\}$.

3) Is the vector $(1,0,1,0)$ a linear combination of the system $\{(1,1,0,1),(1,0,1,1),(1,0,0,-1)\}\subseteq\mathbb{R}^4$

4) Are the following sets vector subspaces of $\mathbb{R}^4$?

i) $A=\{(3x_2,x_2,x_4+x_2,x_4)\mid x_2,x_4\in\mathbb{R}\}$.

ii) $B=\{(x_1,x_2,x_3,x_4)\mid x_1\cdot x_2=0\}$.

iii) $C=\{(x_1,x_2,x_1,x_2)\mid x_1=1\}$.

iv) $D=\{(x_1,x_1,x_1,x_1)\mid x_1\in\mathbb{R}\}$.

v) $E=\{(x_1,x_2,x_3,x_4)\mid x_1=t,x_2=t,x_3=2t,x_4=5t,t\in\mathbb{R}\}$.

5) Let $\mathbb{S}$ be the subspace of $\mathbb{R}^3$ generated by $\{(1,2,1),(1,0,1),(2,3,2)\}$. Write $\mathbb{S}$ as the set of solutions of some homogeneous system of linear equations.

6) Compute the coordinates of the vectors $(1,2,3)$ and $(1,1,0)$ in the basis $\mathbb{B}=\{(1,-1,2),(0,1,1),(1,0,4)\}$ of $\mathbb{R}^3$.

7) Compute the dimension of the following vector subspaces of $\mathbb{R}^4$ by finding, in each case, a basis.

a) $\mathbb{W}_1:\begin{cases} x_1-x_2+x_3-x_4=0\\ x_1-x_2=0\\ x_3-x_4=0\\ x_1-x_4=0\end{cases}$

b) $\mathbb{W}_2:\begin{cases} x_1-x_2+x_3-x_4=0\\ x_1-x_2-2x_3-x_4=0\end{cases}$

8) Find a basis of the following vector subspaces:

a) $\mathbb{S}\subseteq\mathbb{R}^4$ generated by $\{(1,-1,0,0),(2,0,-1,-1),(1,0,0,-1),(1,-1,1,-1)\}$.

b) $\mathbb{S}\subseteq\mathbb{R}^3$ generated by $\{(1,-1,0),(1,0,-1),(2,-1,-1),(-1,-1,2)\}$.

9) Let U be the subspace of $\mathbb{R}^4$ generated by $(1,1,-1,2)$ and $(-1,2,3,3)$. Let V be the subspace of $\mathbb{R}^4$ generated by $(1,7,3,12)$ and $(-4,5,1,1)$. Let $W=U+V$.

a) Find a basis of W.

b) Write W as the solution of a system of linear equations. Is this system unique? What is the minimum number of equations that your need to describe W?

c) Show that $(2,-3,5,7)\notin W$.

d) Show that $w=(27,0,27,57)\in W$. Compute $u\in U$ and $v\in V$ so that $w=u+v$.

10) Let $\mathbb{W}_1$ and $\mathbb{W}_2$ be the subspaces of $\mathbb{R}^4$ from exercise 7.

(a) Compute a basis of $\mathbb{W}_1 \cap \mathbb{W}_2$ and a bais of $\mathbb{W}_1 + \mathbb{W}_2$. Verify that Grassmann's Identity holds.

(b) Find a system of linear equations that describes $\mathbb{W}_1 \cap \mathbb{W}_2$ and another whose solution is $\mathbb{W}_1 + \mathbb{W}_2$. Make sure that in each case, the number of equations is the minimum needed for such descriptions.

11) Let $a \in \mathbb{R}$. Let U and V be vector subspaces of $\mathbb{R}^4$:

$$U: \left.\begin{array}{rcl} x_1 + 2x_2 - x_3 + 2x_4 & = & 0 \\ 2x_1 + x_2 + x_3 - x_4 & = & 0 \end{array}\right\} \qquad V: \left.\begin{array}{rcl} 3x_1 + x_2 + 2x_4 & = & 0 \\ -x_1 + x_3 + ax_4 & = & 0 \end{array}\right\}$$

a) Find the values of a for which $U \cap V \neq \{0\}$. For those values of a, compute a basis of $U \cap V$ and a system of equations describing $U \cap V$ with the minimum number of equations.

b) For $a = -4$, find $u \in U$ and $v \in V$ such that $(3, 1, -2, 2) = u + v$.

12) Decide which of the following functions are linear maps. For those that are linear maps, write the corresponding matrix.:

a) $f_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f_1(v) = \lambda_0 v$ with $\lambda_0 \in \mathbb{R}$ constant.
b) $f_2 \colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f_2(v) = v_0 - v$ with $v_0 \in \mathbb{R}^3$ constant.
c) $f_3 \colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f_3(x_1, x_2, x_3) = (x_1, 1, x_3)$.
d) $f_4 \colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f_4(x_1, x_2, x_3) = (x_1^2 - x_2^2, 0, 0)$.
e) $f_5 \colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f_5(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1 + x_2, x_3)$.
f) $f_6 \colon \mathbb{R}^3 \to \mathbb{R}^2$ given by r $f_6(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3)$.

13) Ler $f : V \to W$ be a linear map. Write the matrix of f if:

a) $V = \mathbb{R}^2$, $W = \mathbb{R}$, $f(1, 0) = 2$, $f(0, 1) = 1$.

b) $V = \mathbb{R}^2$, $W = \mathbb{R}^2$, $f(1, 0) = (2, 0)$, $f(0, 1) = (1, 1)$.

14) For each case, decide whether there is a linear map satisfying the required conditions:

a) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $\quad T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$.

b) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\quad T(\alpha_i) = \beta_i$ $(i = 1, 2, 3)$ with

$\alpha_1 = (1, -1)$, $\alpha_2 = (2, -1)$, $\alpha_3 = (-3, 2)$, $\beta_1 = (1, 0)$, $\beta_2 = (0, 1)$ and $\beta_3 = (1, 2)$.

15) Consider the linear maps $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1 + x_2, x_3)$ and $g \colon \mathbb{R}^3 \to \mathbb{R}^2$ given by $g(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3)$.

a) Let V be the subspace $\langle (1, -1, 0), (1, 1, 1) \rangle$. Compute $f(V)$ and $g(V)$. Compute $f^{-1}(0, 0, 0)$ and $f^{-1}(2, 2, 1)$.
b) Compute $f^{-1}(W)$, where $W = \{(x_1, x_2, x_3) \colon x_1 = 3\lambda,\ x_2 = 2\lambda,\ x_3 = \lambda,\ \lambda \in \mathbb{R}\}$.

16) The following matrices determine linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ for different values of $n, m > 0$. For each one of them, compute a basis of the kernel, equations of the image and check that the identity for the dimensions: (n = dim (kernel) + dim(Image)) holds.

$$i)\ \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}; \quad ii)\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}; \quad iii)\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}; \quad iv)\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}; \quad v)\begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 0 & -1 \end{pmatrix};$$

$$vi)\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad vii)\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad viii)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}; \quad ix)\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}; \quad x)\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$