Worksheet 3. The arithmetic of the integers.

1) Let a and b be non-zero integers. Then we know that there are prime numbers $p_1, \ldots, p_s$ such that:

$$a = \pm p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s} \quad \text{and} \quad b = \pm p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_s^{\beta_s}$$

with $\alpha_i \geq 0$, $\beta_i \geq 0$.

i) Express $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$ in terms of the previous factorizations.

ii) Compute the greatest common divisor of 1547 and 3059 (gcd(1547, 3059)) using (i). Compute gcd(1547, 3059) using Euclid's algorithm.

iii) Compute the least common multiple of 363 y 55 (lcm(363, 55)) using (i). Compute lcm(363, 55) using Euclid's algorithm.

2) Let $d = \gcd(a, b)$. Show that: $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

3) Let $a, b \in \mathbb{Z}$ be non-simultaneously zero. Define $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as $f\big((X, Y)\big) = aX + bY$. Show that $\operatorname{Im}(f) = d\mathbb{Z} = \{dm \mid m \in \mathbb{Z}\}$, where $d = \gcd(a, b)$.

4) Find the pairs $a, b \in \mathbb{Z}$ such that $\gcd(a, b) = 10$ and $\operatorname{lcm}(a, b) = 100$.

5) Show that the equation $6x + 20y = 7$ has no solutions in $\mathbb{Z}$. How about $25x + 45y = 3$?

6) Find all the integer solutions of the following equations:

a) $111x + 36y = 15$, b) $10x + 26y = 1224$ c) $6x + 10y = 20$.

7) Find all the integer solutions of $11x - 13y = 1$ such that $max\{|x|, |y|\} < 13$.

8) Find two natural numbers that add to 81 and are multiples of 12 and 15 respectively.

9) It takes 225, 365 and 687 days for Venus, Earth and Mars, respectively, to go around the Sun. Suppose the three planets were aligned today. How long would it take them to be aligned again?

10) Show that any integer n can be written as a multiple of 1001 plus a multiple of 30.

11) A laboratory spends 50340 euros buying computers of two different brands: IBM and HP. Suppose that one IBM computer costs 1500 euros and that one HP computer costs 1080 euros. How many computers of each brand has the laboratory bought?

12) Some friends went to the movies and spend 150 euros buying the tickets. The ticket for movie A cost 11 euros, while the ticket for movie B was 13 euros. How many friends watched movie A and how many watched movie B?

13) Suppose there are two concerts in Madrid: A and B. The ticket for concert A is $31,80$ euros while for concert B is 15 euros. A group of friends spend $600,60$ euros buying the tickets. How many tickets did they buy?

14) We say that a point with integer coordinates $(x, y) \in \mathbb{Z}^2$ can be seen from $(0, 0)$ if the segment joining (x, y) and $(0, 0)$ does not contain any other point with integer coordinates. What is $\gcd(x, y)$ if (x, y) can be seen from $(0, 0)$? How many points with integer coordinates don't let us see $(42, 45)$ from $(0, 0)$?