Sea $\mathbf{X} = (X_1, X_2, X_3)'$ un vector aleatorio normal tal que la correlación entre X_1 y X_2 es $\rho_{12} = 0.5$,

$$\begin{aligned} X_2|X_3 &\sim N(2X_3, 24), \\ X_3|X_1 &\sim N(2X_1 + 3, 14), \\ X_1 &\sim N(1, 4). \end{aligned}$$

Determinar la esperanza y la matriz de covarianzas de $\mathbf{X}$.

Solución:

Sean $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)' = \mathbb{E}(\mathbf{X})$ y $\boldsymbol{\Sigma} = (\sigma_{ij})_{i,j=1}^3 = \mathbb{V}(\mathbf{X})$. Sabemos que $\mu_1 = 1$ y $\sigma_{11} = 4$.

$$2X_1 + 3 = \mathbb{E}(X_3|X_1) = \mu_3 + \frac{\sigma_{13}}{\sigma_{11}}(X_1 - \mu_1) = \frac{\sigma_{13}}{4}X_1 + \mu_3 - \frac{\sigma_{13}}{4} \Rightarrow \begin{cases} \sigma_{13} = 8 \\ \mu_3 = 5 \end{cases}$$

$$14 = \mathbb{V}(X_3|X_1) = \sigma_{33} - \frac{\sigma_{13}^2}{\sigma_{11}} = \sigma_{33} - 16 \Rightarrow \sigma_{33} = 30$$

$$2X_3 = \mathbb{E}(X_2|X_3) = \mu_2 + \frac{\sigma_{23}}{\sigma_{33}}(X_3 - \mu_3) = \frac{\sigma_{23}}{30}X_3 + \mu_2 - \frac{\sigma_{23}}{6} \Rightarrow \begin{cases} \sigma_{23} = 60 \\ \mu_2 = 10 \end{cases}$$

$$24 = \mathbb{V}(X_2|X_3) = \sigma_{22} - \frac{\sigma_{23}^2}{\sigma_{33}} = \sigma_{22} - 120 \Rightarrow \sigma_{22} = 144$$

$$\sigma_{12} = \rho_{12}\sqrt{\sigma_{11}\sigma_{22}} = 12$$

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 10 \\ 5 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 4 & 12 & 8 \\ 12 & 144 & 60 \\ 8 & 60 & 30 \end{pmatrix}$$