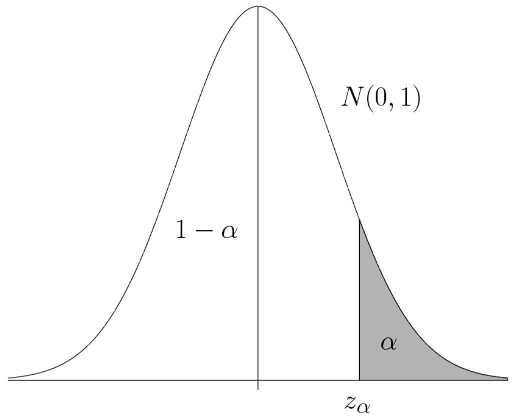
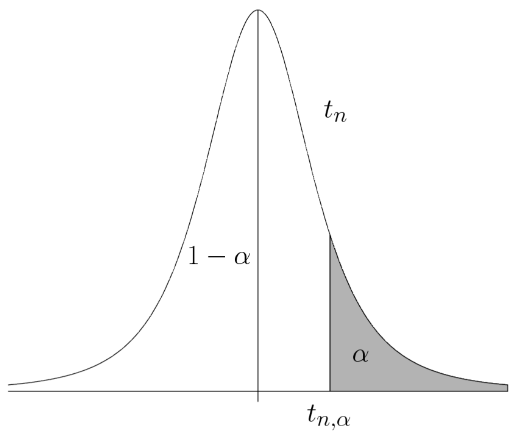
Confidence intervals and Hypothesis testing





$(x_1, \dots, x_n)$ random sample of X.

Sample mean: $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, Sample variance: $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

Confidence intervals

1. $\boxed{X \sim N(\mu, \sigma)}$

$1-\alpha$ confidence interval for μ: $\mathrm{CI}_{1-\alpha}(\mu) = \left(\bar{x} \mp t_{n-1;\alpha/2}\frac{s}{\sqrt{n}}\right)$

2. $\boxed{X \sim \text{Bernoulli}(p)}$ (large samples)

$1-\alpha$ confidence interval for p: $\mathrm{CI}_{1-\alpha}(p) = \left(\bar{x} \mp z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$, where $\hat{p} = \bar{x}$

3. $\boxed{\text{Non-Gaussian } X}$ with $E(X) = \mu$ and $V(X) = \sigma^2$ (large samples)

$1-\alpha$ confidence interval for μ: $\mathrm{CI}_{1-\alpha}(\mu) = \left(\bar{x} \mp z_{\alpha/2}\frac{s}{\sqrt{n}}\right)$

4. Two independent normal populations, $X \sim N(\mu_1, \sigma_1)$ and $Y \sim N(\mu_2, \sigma_2)$ $(\sigma_1 = \sigma_2)$

$(x_1, \dots, x_{n_1})$ random sample from X; $\bar{x}$ and s_1^2 sample mean and variance for X.

$(y_1, \dots, y_{n_2})$ random sample from Y; $\bar{y}$ and s_2^2 sample mean and variance for Y.

$$s_p^2 = \frac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2}{n_1 + n_2 - 2}$$

$1 - \alpha$ confidence interval for $\mu_1 - \mu_2$:

$$\mathrm{CI}_{1-\alpha}(\mu_1 - \mu_2) = \left(\bar{x} - \bar{y} \mp t_{n_1+n_2-2;\alpha/2} \sqrt{s_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)} \right)$$

5. Two independent non-Gaussian populations X and Y with $E(X) = \mu_1$ and $E(Y) = \mu_2$ (large samples $n_1 \geq 20$ and $n_2 \geq 20$)

$(x_1, \dots, x_{n_1})$ random sample from X; $\bar{x}$ and s_1^2 sample mean and variance for X.

$(y_1, \dots, y_{n_2})$ random sample from Y; $\bar{y}$ and s_2^2 sample mean and variance for Y.

$1 - \alpha$ confidence interval for $\mu_1 - \mu_2$:

$$\mathrm{CI}_{1-\alpha}(\mu_1 - \mu_2) = \left(\bar{x} - \bar{y} \mp z_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

6. Comparison of two population proportions p_1 and p_2 (large samples $n_1 \geq 20$ and $n_2 \geq 20$)

$X \sim$ Bernoulli(p_1), $Y \sim$ Bernoulli(p_2), independent.

$(x_1, \dots, x_{n_1})$ random sample from X; $\hat{p}_1 = \bar{x}$

$(y_1, \dots, y_{n_2})$ random sample from Y; $\hat{p}_2 = \bar{y}$

$1 - \alpha$ confidence interval for $p_1 - p_2$: $\mathrm{CI}_{1-\alpha}(p_1 - p_2) = \left(\hat{p}_1 - \hat{p}_2 \mp z_{\alpha/2} \sqrt{\frac{\hat{p}_1\,(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2\,(1-\hat{p}_2)}{n_2}} \right)$

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Determining the sample size in two-sample inference (assume $n_1 = n_2 = n$)

- To estimate $\mu_1 - \mu_2$ to within a sampling error SE and with confidence level $1 - \alpha$, take

$$n = \frac{(z_{\alpha/2})^2 (s_1^2 + s_2^2)}{(\mathrm{SE})^2},$$

where s_1^2 and s_2^2 are sample variances from prior pilot samples of X and Y.

- To estimate $p_1 - p_2$ to within a sampling error SE and with confidence level $1 - \alpha$, take

$$n = \frac{(z_{\alpha/2})^2 (\hat{p}_1(1 - \hat{p}_1) + \hat{p}_2(1 - \hat{p}_2))}{(\mathrm{SE})^2},$$

where $\hat{p}_1$ and $\hat{p}_2$ are estimates of p_1 and p_2 respectively from prior pilot samples of X and Y.

Hypothesis testing

- α = significance level of the test
- n = sample size.
- H_0 = null hypothesis.
- R = critical (reject) region for H_0.

1.- $\boxed{X \sim N(\mu, \sigma)}$ (unknown μ and σ)

$$\begin{array}{ll} H_0 : \mu = \mu_0 & R = \left\{|t| > t_{n-1;\alpha/2}\right\} \\ H_1 : \mu \neq \mu_0 & \end{array}$$

$$\begin{array}{ll} H_0 : \mu \leq \mu_0 & R = \{t > t_{n-1;\alpha}\} \\ H_1 : \mu > \mu_0 & \end{array}$$

$$\begin{array}{ll} H_0 : \mu \geq \mu_0 & R = \{t < -t_{n-1;\alpha}\} \\ H_1 : \mu < \mu_0 & \end{array}$$

where $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$ is the test statistic.

2.- $\boxed{X \sim \text{Bernoulli}(p)}$ (large sampling: $n \geq 20$)

$$\begin{array}{ll} H_0 : p = p_0 & R = \left\{|z| > z_{\alpha/2}\right\} \\ H_1 : p \neq p_0 & \end{array}$$

$$\begin{array}{ll} H_0 : p \leq p_0 & R = \{z > z_{\alpha}\} \\ H_1 : p > p_0 & \end{array}$$

$$\begin{array}{ll} H_0 : p \geq p_0 & R = \{z < -z_{\alpha}\} \\ H_1 : p < p_0 & \end{array}$$

where $z = \dfrac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}}$ is the test statistic and $\hat{p} = \bar{x}$.

3.- $\boxed{\text{Non-Gaussian } X}$ with $E(X) = \mu$ and $V(X) = \sigma^2$ (large sampling: $n \geq 20$)

$$\begin{array}{ll} H_0 : \ \mu = \mu_0 & R = \left\{|z| > z_{\alpha/2}\right\} \\ H_1 : \ \mu \neq \mu_0 & \end{array}$$

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where $z = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$ is the test statistic.

4.- Two independent normal populations, $X \sim N(\mu_1, \sigma_1)$ and $Y \sim N(\mu_2, \sigma_2)$ $(\sigma_1 = \sigma_2)$

$(x_1, \ldots, x_{n_1})$ random sample from X; $\bar{x}$ and s_1^2 sample mean and variance for X.
$(y_1, \ldots, y_{n_2})$ random sample from Y; $\bar{y}$ and s_2^2 sample mean and variance for Y.

Pooled sample variance: $s_p^2 = \dfrac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2}{n_1 + n_2 - 2}$

$$\begin{array}{ll} H_0 : \mu_1 - \mu_2 = D_0 & \\ H_1 : \mu_1 - \mu_2 \neq D_0 & R = \left\{|t| > t_{n_1+n_2-2;\alpha/2}\right\} \end{array}$$

$$\begin{array}{ll} H_0 : \mu_1 - \mu_2 \leq D_0 & \\ H_1 : \mu_1 - \mu_2 > D_0 & R = \{t > t_{n_1+n_2-2;\alpha}\} \end{array}$$

$$\begin{array}{ll} H_0 : \mu_1 - \mu_2 \geq D_0 & \\ H_1 : \mu_1 - \mu_2 < D_0 & R = \{t < -t_{n_1+n_2-2;\alpha}\} \end{array}$$

where $t = \dfrac{(\bar{x} - \bar{y}) - D_0}{\sqrt{s_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$ is the test statistic

5.- Two independent non-Gaussian populations X and Y with $E(X) = \mu_1$ and $E(Y) = \mu_2$ (large samples $n_1 \geq 20$ and $n_2 \geq 20$)

$(x_1, \ldots, x_{n_1})$ random sample from X; $\bar{x}$ and s_1^2 sample mean and variance for X.
$(y_1, \ldots, y_{n_2})$ random sample from Y; $\bar{y}$ and s_2^2 sample mean and variance for Y.

$$\begin{array}{ll} H_0 : \mu_1 - \mu_2 = D_0 & \\ H_1 : \mu_1 - \mu_2 \neq D_0 & R = \left\{|z| > z_{\alpha/2}\right\} \end{array}$$

$$\begin{array}{ll} H_0 : \mu_1 - \mu_2 \leq D_0 & \\ H_1 : \mu_1 - \mu_2 > D_0 & R = \{z > z_{\alpha}\} \end{array}$$

$$\begin{array}{ll} H_0 : \mu_1 - \mu_2 \geq D_0 & \\ H_1 : \mu_1 - \mu_2 < D_0 & R = \{z < -z_{\alpha}\} \end{array}$$

where $z = \dfrac{(\bar{x} - \bar{y}) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$ is the test statistic

6.- Comparison of two population proportions p_1 and p_2 (large samples $n_1 \geq 20$ and $n_2 \geq 20$)

$X \sim$ Bernoulli(p_1), $Y \sim$ Bernoulli(p_2), independent.

$$\left.\begin{array}{l} (x_1, \ldots, x_{n_1}) \text{ random sample from } X;\ \hat{p}_1 = \bar{x} \\ (y_1, \ldots, y_{n_2}) \text{ random sample from } Y;\ \hat{p}_2 = \bar{y} \end{array}\right\} \rightsquigarrow \quad \hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2} = \frac{\sum_{i=1}^{n_1} x_i + \sum_{i=1}^{n_2} y_i}{n_1 + n_2}$$

$$\begin{array}{ll} H_0 : p_1 = p_2 & \\ H_1 : p_1 \neq p_2 & R = \left\{|z| > z_{\alpha/2}\right\} \end{array}$$

$$\begin{array}{ll} H_0 : p_1 \leq p_2 & \\ H_1 : p_1 > p_2 & R = \{z > z_{\alpha}\} \end{array}$$

$$\begin{array}{ll} H_0 : p_1 \geq p_2 & \\ H_1 : p_1 < p_2 & R = \{z < -z_{\alpha}\} \end{array}$$

where $z = \dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1 - \hat{p}) \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$ is the test statistic