Confidence intervals and Hypothesis testing

\[(x_1, \ldots, x_n)\text{ random sample of } X.\]

Sample mean: \[\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i,\]
Sample variance: \[s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2\]

Confidence intervals

1. \(X \sim N(\mu, \sigma)\)

1 \(\alpha\) confidence interval for \(\mu\): \(\text{CI}_{1-\alpha}(\mu) = \left(\bar{x} \pm t_{n-1;\alpha/2} \frac{s}{\sqrt{n}}\right)\)

2. \(X \sim \text{Bernoulli}(p)\) (large samples)

1 \(\alpha\) confidence interval for \(p\): \(\text{CI}_{1-\alpha}(p) = \left(\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)\), where \(\hat{p} = \bar{x}\)

3. Non-Gaussian \(X\) with \(E(X) = \mu\) and \(V(X) = \sigma^2\) (large samples)

1 \(\alpha\) confidence interval for \(\mu\): \(\text{CI}_{1-\alpha}(\mu) = \left(\bar{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}\right)\)
4. Two independent normal populations, $X \sim N(\mu_1, \sigma_1)$ and $Y \sim N(\mu_2, \sigma_2)$ ($\sigma_1 = \sigma_2$)

$(x_1, \ldots, x_{n_1})$ random sample from $X$; $\bar{x}$ and $s_1^2$ sample mean and variance for $X$.
$(y_1, \ldots, y_{n_2})$ random sample from $Y$; $\bar{y}$ and $s_2^2$ sample mean and variance for $Y$.

$$s_p^2 = \frac{(n_1 - 1) s_1^2 + (n_2 - 1) s_2^2}{n_1 + n_2 - 2}$$

$1 - \alpha$ confidence interval for $\mu_1 - \mu_2$:

$$CI_{1-\alpha}(\mu_1 - \mu_2) = \left( \bar{x} - \bar{y} \mp t_{n_1+n_2-2,\alpha/2} \sqrt{s_p^2 \left( \frac{1}{n_1} + \frac{1}{n_2} \right)} \right)$$

5. Two independent non-Gaussian populations $X$ and $Y$ with $E(X) = \mu_1$ and $E(Y) = \mu_2$

(large samples $n_1 \geq 20$ and $n_2 \geq 20$)

$(x_1, \ldots, x_{n_1})$ random sample from $X$; $\bar{x}$ and $s_1^2$ sample mean and variance for $X$.
$(y_1, \ldots, y_{n_2})$ random sample from $Y$; $\bar{y}$ and $s_2^2$ sample mean and variance for $Y$.

$1 - \alpha$ confidence interval for $\mu_1 - \mu_2$:

$$CI_{1-\alpha}(\mu_1 - \mu_2) = \left( \bar{x} - \bar{y} \mp z_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

6. Comparison of two population proportions $p_1$ and $p_2$

(large samples $n_1 \geq 20$ and $n_2 \geq 20$)

$X \sim \text{Bernoulli}(p_1)$, $Y \sim \text{Bernoulli}(p_2)$, independent.

$(x_1, \ldots, x_{n_1})$ random sample from $X$; $\hat{p}_1 = \bar{x}$
$(y_1, \ldots, y_{n_2})$ random sample from $Y$; $\hat{p}_2 = \bar{y}$

$1 - \alpha$ confidence interval for $p_1 - p_2$:

$$CI_{1-\alpha}(p_1 - p_2) = \left( \hat{p}_1 - \hat{p}_2 \mp z_{\alpha/2} \sqrt{\frac{\hat{p}_1 (1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2 (1-\hat{p}_2)}{n_2}} \right)$$
Hypothesis testing

- $\alpha$ = significance level of the test  
- $n$ = sample size.  
- $H_0$ = null hypothesis.  
- $R$ = critical (reject) region for $H_0$.

1. $X \sim N(\mu, \sigma)$ (unknown $\mu$ and $\sigma$)

   $H_0 : \mu = \mu_0$  
   $H_1 : \mu \neq \mu_0$

   $H_0 : \mu \leq \mu_0$  
   $H_1 : \mu > \mu_0$

   $H_0 : \mu \geq \mu_0$  
   $H_1 : \mu < \mu_0$

   $R = \{ |t| > t_{n-1,\alpha/2} \}$

   $R = \{ t > t_{n-1,\alpha} \}$

   $R = \{ t < -t_{n-1,\alpha} \}$

   where $t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$ is the test statistic.

2. $X \sim \text{Bernoulli}(p)$ (large sampling: $n \geq 20$)

   $H_0 : p = p_0$  
   $H_1 : p \neq p_0$

   $H_0 : p \leq p_0$  
   $H_1 : p > p_0$

   $H_0 : p \geq p_0$  
   $H_1 : p < p_0$

   $R = \{ |z| > z_{\alpha/2} \}$

   $R = \{ z > z_{\alpha} \}$

   $R = \{ z < -z_{\alpha} \}$

   where $z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}}$ is the test statistic and $\hat{p} = \bar{x}$.

3. Non-Gaussian $X$ with $E(X) = \mu$ and $V(X) = \sigma^2$ (large sampling: $n \geq 20$)

   $H_0 : \mu = \mu_0$  
   $H_1 : \mu \neq \mu_0$

   $H_0 : \mu \leq \mu_0$  
   $H_1 : \mu > \mu_0$

   $H_0 : \mu \geq \mu_0$  
   $H_1 : \mu < \mu_0$

   $R = \{ |z| > z_{\alpha/2} \}$

   $R = \{ z > z_{\alpha} \}$

   $R = \{ z < -z_{\alpha} \}$

   where $z = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$ is the test statistic.
4. Two independent normal populations, \( X \sim N(\mu_1, \sigma_1) \) and \( Y \sim N(\mu_2, \sigma_2) \) (\( \sigma_1 = \sigma_2 \))

\( (x_1, \ldots, x_{n_1}) \) random sample from \( X \); \( \bar{x} \) and \( s_1^2 \) sample mean and variance for \( X \).

\( (y_1, \ldots, y_{n_2}) \) random sample from \( Y \); \( \bar{y} \) and \( s_2^2 \) sample mean and variance for \( Y \).

Pooled sample variance: \( s_p^2 = \frac{(n_1 - 1) s_1^2 + (n_2 - 1) s_2^2}{n_1 + n_2 - 2} \)

\[
H_0 : \mu_1 - \mu_2 = D_0 \\
H_1 : \mu_1 - \mu_2 \neq D_0 \\
H_0 : \mu_1 - \mu_2 \leq D_0 \\
H_1 : \mu_1 - \mu_2 > D_0 \\
H_0 : \mu_1 - \mu_2 \geq D_0 \\
H_1 : \mu_1 - \mu_2 < D_0
\]

\[
R = \{ |t| > t_{n_1+n_2-2;\alpha/2} \}
\]

\[
R = \{ t > t_{n_1+n_2-2;\alpha/2} \}
\]

\[
R = \{ t < -t_{n_1+n_2-2;\alpha} \}
\]

where \( t = \frac{(\bar{x} - \bar{y}) - D_0}{\sqrt{s_p^2 \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}} \) is the test statistic.

5. Two independent non-Gaussian populations \( X \) and \( Y \) with \( E(X) = \mu_1 \) and \( E(Y) = \mu_2 \)

\( (x_1, \ldots, x_{n_1}) \) random sample from \( X \); \( \bar{x} \) and \( s_1^2 \) sample mean and variance for \( X \).

\( (y_1, \ldots, y_{n_2}) \) random sample from \( Y \); \( \bar{y} \) and \( s_2^2 \) sample mean and variance for \( Y \).

\[
H_0 : \mu_1 - \mu_2 = D_0 \\
H_1 : \mu_1 - \mu_2 \neq D_0 \\
H_0 : \mu_1 - \mu_2 \leq D_0 \\
H_1 : \mu_1 - \mu_2 > D_0 \\
H_0 : \mu_1 - \mu_2 \geq D_0 \\
H_1 : \mu_1 - \mu_2 < D_0
\]

\[
R = \{ |z| > z_{\alpha/2} \}
\]

\[
R = \{ z > z_{\alpha} \}
\]

\[
R = \{ z < -z_{\alpha} \}
\]

where \( z = \frac{(\bar{x} - \bar{y}) - D_0}{\sqrt{s_1^2/n_1 + s_2^2/n_2}} \) is the test statistic.

6. Comparison of two population proportions \( p_1 \) and \( p_2 \)

\( X \sim \text{Bernoulli}(p_1) \), \( Y \sim \text{Bernoulli}(p_2) \), independent.

\( (x_1, \ldots, x_{n_1}) \) random sample from \( X \); \( \hat{p}_1 = \bar{x} \)

\( (y_1, \ldots, y_{n_2}) \) random sample from \( Y \); \( \hat{p}_2 = \bar{y} \)

\[
\hat{p} = \frac{n_1 \hat{p}_1 + n_2 \hat{p}_2}{n_1 + n_2} = \frac{\sum_{i=1}^{n_1} x_i + \sum_{i=1}^{n_2} y_i}{n_1 + n_2}
\]

\[
H_0 : p_1 = p_2 \\
H_1 : p_1 \neq p_2 \\
H_0 : p_1 \leq p_2 \\
H_1 : p_1 > p_2 \\
H_0 : p_1 \geq p_2 \\
H_1 : p_1 < p_2
\]

\[
R = \{ |z| > z_{\alpha/2} \}
\]

\[
R = \{ z > z_{\alpha} \}
\]

\[
R = \{ z < -z_{\alpha} \}
\]

where \( z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p}) \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}} \) is the test statistic.