Basic Statistics and Probability (2018-19)
Science & Engineering Program Boston University-Faculty of Science UAM

MID-TERM. Solutions

Throughout the whole exam and, in general, in this subject we use the basic rule that the probability that a continuous r.v. X is exactly equal to a number is 0, so $P\{X = a\} = 0$, $P\{X \leq b\} = P\{X < b\}$ or $P\{a \leq X \leq b\} = P\{a < X < b\}$ for any a and b.

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1. We denote the different kinds of antibiotic as follows: Te = tetracycline, Pe = penicillin, Mi = minocycline, Ba = Bactrim, St = streptomycin, Zi = Zithromax.

a) The sample points for the experiment of withdrawing a single capsule from the box are

$$\{\text{Te}\}, \{\text{Pe}\}, \{\text{Mi}\}, \{\text{Ba}\}, \{\text{St}\}, \{\text{Zi}\}.$$

The probabilities of each sample point are:

$$P\{\text{Te}\} = \frac{15}{300} = 0.05,\ P\{\text{Pe}\} = \frac{30}{300} = 0.1,\ P\{\text{Mi}\} = \frac{45}{300} = 0.15,$$
$$P\{\text{Ba}\} = \frac{60}{300} = 0.2,\ P\{\text{St}\} = \frac{70}{300} \simeq 0.23,\ P\{\text{Zi}\} = \frac{80}{300} \simeq 0.27.$$

b) The probability that the capsule selected is either penicillin or streptomycin is

$$P(\text{Pe} \cup \text{St}) = P(\text{Pe}) + P(\text{St}) = 0.1 + 0.23 = 0.33$$

c) The probability that the capsule selected is neither Zithromax nor tetracycline is

$$1 - P(\text{the capsule is either Zi or Te}) = 1 - (0.27 + 0.05) = 0.68.$$

d) The probability that the capsule selected is not penicillin is $P(\text{Pe}^c) = 1 - P(\text{Pe}) = 1 - 0.1 = 0.9$.

e) The probability that two Zithromax capsules are drawn is

$$P(\text{1st capsule is Zi, 2nd capsule is Zi}) = \frac{80}{300}\frac{79}{299} = 0.07.$$

The probability that one of capsules is Zithromax and the other one is not is

$$P(\text{1st capsule is Zi, 2nd capsule is not Zi}) + P(\text{1st capsule is not Zi, 2nd capsule is Zi})$$
$$= \frac{80}{300}\frac{299-79}{299} + \frac{300-80}{300}\frac{80}{299} = 0.20 + 0.20 = 0.4.$$

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2. a) The probability that the die will roll a two if we choose the unfair die is

$$P(\text{rolling a 2}|\text{not rolling a 6})\, P(\text{not rolling a 6}) = \frac{1}{5}\frac{50}{100} = 0.1.$$

b) If we pick up one die at random, the probability of rolling a six is

$$P(6) = P(\text{roll a 6}|\text{fair die})\, P(\text{fair die}) + P(\text{roll a 6}|\text{unfair die})\, P(\text{unfair die}) = \frac{1}{6}0.99 + \frac{1}{2}0.01 = 0.17.$$

c) By the Bayes Rule, if we roll one of the two dice at random and obtain a six, the probability that the rolled die is the unfair one is

$$P(\text{unfair die}|\text{rolling a 6}) = \frac{P(\text{rolling a 6}|\text{unfair die})\, P(\text{unfair die})}{P(\text{rolling a 6})} = \frac{\frac{1}{2}0.01}{0.17} = 0.0294.$$

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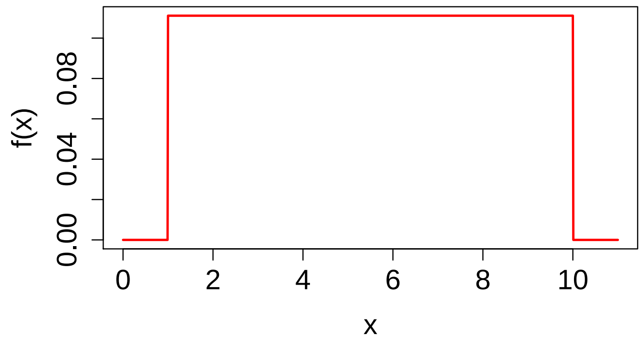
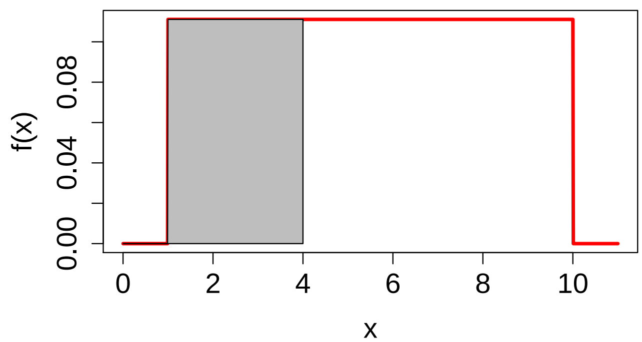
3. Suppose X, the lifetime (in years) of an electrical device, is a random variable following a uniform probability distribution on the interval $[c, d] = [1, 10]$.

a) The probability density f of X is constant in the interval [1,10] and takes the value $1/(10-1) = 1/9 \simeq 0.1111$ along this interval. Outside the interval [1,10] the density is 0.



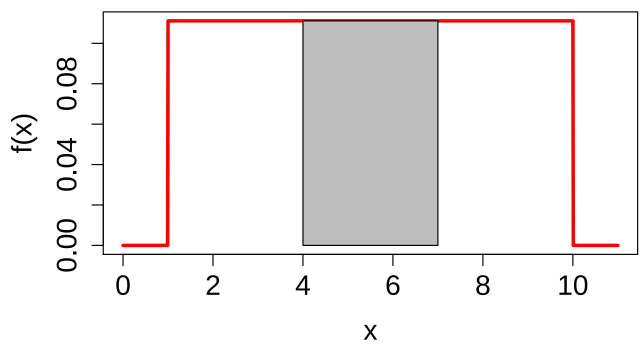
b) The probability that the device lasts less than 4 years is the area underneath the density f and to the left of 4. Since f takes the value 0 to the left of 1, we compute the area of the rectangle with height 1/9 and width between 1 and 4:

$$P\{X < 4\} = \frac{3}{9} = \frac{1}{3}.$$



c) The probability that the lifetime of the device is between 4 and 7 years is the area under the density f and between 4 and 7, so it is the area of the rectangle with height 1/9 and width between 4 and 7:

$$P\{4 < X < 7\} = \frac{3}{9} = \frac{1}{3}.$$



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4. We are considering the Bernoulli trial of asking an American if he/she considers that the International Space Station was a good investment (success) or not (failure). Thus, the parameter $p = 0.8$ is the probability that an American chosen at random considers that it was a good investment.

a) Clearly, since X counts the number of individuals in the group of 10 who think it was a good investment, it is a discrete r.v. The probability distribution of X is Binomial($n = 10$,$p = 0.8$).

b) The probability mass function of X is

$$p(x) = \binom{10}{x} 0.8^x \, 0.2^{10-x}, \qquad \text{for } x = 0, \dots, 10,$$

where

$$\binom{10}{x} = \frac{10!}{x!(10-x)!}.$$

c)

$$p(7) = P\{X = 7\} = \binom{10}{7} 0.8^7 \, 0.2^3 = 120 \cdot 0.2097 \cdot 0.008 = 0.2013.$$

d) The probability that all the ten Americans surveyed think the space station has been a good investment is

$$p(10) = P\{X = 10\} = 0.8^{10} = 0.1074.$$

e) The probability that X equals 7 or 10 is $P\{X = 7\} + P\{X = 10\} = 0.2013 + 0.1074 = 0.3087$.

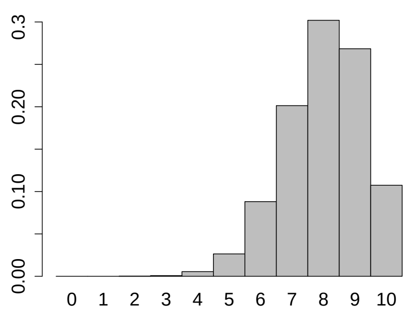
f)

$$\begin{aligned} P\{X \le 8\} \quad &= 1 - P\{X > 8\} = 1 - (P\{X = 9\} + P\{X = 10\}) = 1 - \left(10\, 0.8^9\, 0.2 + 0.1074\right) \\ &= 1 - (0.2684 + 0.1074) = 0.6242. \end{aligned}$$

g) The population mean and variance for X are respectively

$$\mu = E(X) = n\,p = 10 \cdot 0.8 = 8 \qquad \text{and} \qquad \sigma^2 = V(X) = n\,p\,(1-p) = 10 \cdot 0.8 \cdot 0.2 = 1.6.$$

h) The histogram corresponding to the probability mass function of X is the one on the left.



One way to see it is that $p(10) = P\{X = 10\} = 0.1074$ and this is the only histogram in which the mass function attains that value.

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5. We are considering two r.v.'s: X = length (in cm) of a healthy newly born female baby in Spain after a full-term pregnancy $\sim$ N($\mu = 50$,$\sigma = 1.5$), and $Z \sim$ N(0,1).

a) Here we use the table of the N(0,1) straightaway:

$$P\{0 < Z < 0.67\} = 0.2486.$$

b) Here we use the result in **a)** and the fact that the probability that $Z \sim$ N(0,1) is larger than 0 is 0.5:

$$P\{Z > 0.67\} = P\{Z > 0\} - P\{0 < Z < 0.67\} = 0.5 - 0.2486 = 0.2514.$$

c) Here we use the result in **a)**, the symmetry of the N(0,1) density with respect to 0 and then the table of the N(0,1) again:

$$\begin{aligned} P\{-1.33 < Z < 0.67\} \quad &= P\{-1.33 < Z < 0\} + P\{0 < Z < 0.67\} \\ &= P\{0 < Z < 1.33\} + P\{0 < Z < 0.67\} = 0.4082 + 0.2486 = 0.6568. \end{aligned}$$

d) If $P\{Z > c\} = 0.15$, then $P\{Z \leq c\} = P\{0 < Z\} - P\{Z > c\} = 0.5 - 0.15 = 0.35$. We look up the **probability** 0.35 in the N(0,1) table and find that

$$P\{0 < Z < 1.03\} = 0.3485 \qquad \text{and} \qquad P\{0 < Z < 1.04\} = 0.3508.$$

So c is approximately 1.04.

e) By symmetry of the N(0,1) density with respect to 0 we have

$$0.7 = P\{-c < Z < c\} = P\{-c < Z < 0\} + P\{0 < Z < c\} = 2\,P\{0 < Z < c\}.$$

So $P\{0 < Z < c\} = \dfrac{0.7}{2} = 0.35$ and, by the result in **d)**, c is approximately 1.04 again.

f) The probability that the length of a newly born girl is less than 52 cm is

$$\begin{aligned} P\{X < 52\} \quad &= P\left\{\frac{X-50}{1.5} < \frac{52-50}{1.5}\right\} = P\{Z < 1.33\} \\ &= P\{0 < Z\} + P\{0 < Z < 1.33\} = 0.5 + 0.4082 = 0.9082. \end{aligned}$$

g) The probability that the length of a newly born girl is between 49 and 52 cm is

$$\begin{aligned} P\{49 < X < 52\} \quad &= P\left\{\frac{49-50}{1.5} < \frac{X-50}{1.5} < \frac{52-50}{1.5}\right\} \\ &= P\{-0.67 < Z < 1.33\} = P\{-0.67 < Z < 0\} + P\{0 < Z < 1.33\} \\ &= P\{0 < Z < 0.67\} + P\{0 < Z < 1.33\} = 0.2486 + 0.4082 = 0.6568. \end{aligned}$$

h) First we standardize X:

$$0.15 = P\{X > c\} = P\left\{\frac{X-50}{1.5} > \frac{c-50}{1.5}\right\} = P\left\{Z > \frac{c-50}{1.5}\right\}.$$

Now we apply **d)**, where we obtained that $P\{Z > 1.04\} = 0.15$, so

$$\frac{c-50}{1.5} = 1.04 \Rightarrow c = 50 + 1.5 \cdot 1.04 = 51.56 \text{ cm}$$
