Reading Group on Groups, Trees and Projective Modules

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Brief introduction.

A good deal of famous results in combinatorial/geometric group theory can be thought of as trying to answer the questions *When is a group free? When does it arise as some sort of "free construction" (free products with amalgamation, HNN extensions, etc)?*

Bass-Serre theory gives a beautiful answer, by viewing all such "free constructions" (the technical term is a *fundamental group of a graph of groups*) as groups acting on trees with certain conditions on stabilisers. Inspired by the theory of covering spaces in topology, the group is then viewed as the fundamental group of the quotient graph with some extra "group decorations". The question is then reduced to finding an appropriate tree for the group to act on.

There are many ways of achieving this, which we will learn about over the course of the reading group. One involves looking at subsets of the group that are almost invariant under right multiplication by elements of the group (the symmetric difference between the subset and its image is finite – an example is the set of all reduced words in a free group that begin with one of the generators). A theorem (the most general version being due to Dunwoody) says that if a finitely generated group has infinite almost right-invariant subsets satisfying certain conditions, then a tree can be made out them, on which the group acts. Another involves considering the number of ends of the group; that is, the number of ways that one can "go to infinity" in the Cayley graph of the group. A tree, the Cayley graph of a free group, has infinitely many ends. It turns out that out of infinitely many ends one can create a tree (the idea is the same as in the theorem mentioned above: from ends, one obtains almost-invariant subsets and then proceeds as before). This yields a famous theorem of Stallings.

Almost right-invariant sets can also be viewed algebraically, in several equivalent ways: as non-trivial elements of $H^1(G, \mathbb{Z}/2\mathbb{Z})$, the first cohomology group of G with coefficients in the ring $\mathbb{Z}/2\mathbb{Z}$; as *outer derivations* from G to the module $\mathbb{Z}/2\mathbb{Z}G$; as homomorphisms from the augmentation ideal of $\mathbb{Z}/2\mathbb{Z}G$ to $\mathbb{Z}/2\mathbb{Z}$. Don't worry if you don't know what this means, we will learn about it. We will also see that we can replace $\mathbb{Z}/2\mathbb{Z}$ with any other ring with unit.

This means that we can tell whether a group is the fundamental group of a graph of groups by looking at algebraic invariants (which secretly tell us whether the group acts on a tree). For instance, if the augmentation ideal of a group splits over that induced by a subgroup, then the subgroup is a free factor. Another famous theorem (this one due to Stallings and Swan) that follows from the above and a bit of algebra says that free groups are exactly those of cohomological dimension 1 over $\mathbb{Z}$.

The goal of the reading group is to understand how all these seemingly unrelated concepts are actually two (several) sides of the same coin. [1]

We will **follow *Groups, Trees and Projective Modules*, by Warren Dicks** [1]. All references in the schedule are to this book, unless stated otherwise. This book does not explicitly include the theory of ends, so we will follow another source for that connection (either [2, IV.6.] or [3]). The classic reference for Bass-Serre theory is the first part of [4], still an excellent source. For a more topological point of view, and more about ends, [5] is also a good reference.

Suggested schedule

The aim is to **understand**. Adjustments in the schedule may need to be made to ensure that everyone is following at the right pace. Doing more examples in talks and consulting other literature is encouraged.

[1]What is a many-sided coin??

Talk 1 Groups acting on graphs. From a group acting on a graph to a graph of groups. Definition of the fundamental group of a graph of groups. Examples. (I.1 – I.4)

Talk 2 Standard graph of a graph of groups; examples (I.4). Structure theorems of Bass-Serre theory: G is the fundamental group of a graph of groups iff it acts on a tree (I.6). Proof that the standard graph of a graph of groups is a tree, using derivations (I.5.).

Talk 3 (I.8.) The connection between finite edge stabilisers, vertex stabilisers and almost-invariant subsets of a group acting on a tree.

Going the other way, we want to show that if there are almost-invariant subsets, the group acts on a tree, and is therefore the fundamental group of a graph of finite groups. The first step is (I.9.): obtaining trees from partially ordered sets. We will later see how to obtain these partially ordered sets from almost-invariant subsets.

Talk 4 (II.1, II.3.1, II.3.3, II.3.5) Some fundamental groups of graphs of groups. II.1 is about the trivial case, where all vertex groups are trivial, gives free groups (Nielsen-Schreier theorem, free subgroups of fundamental groups of graphs of groups). II.3 is about the faithful case, where all incidence homomorphisms are injective. We only need 3.1, 3.3 and 3.5 for later, but other results in this section are worth mentioning for general knowledge (and reaping the rewards of Bass-Serre theory).

Talk 5 Connections between cuts, almost-invariant subsets, ends, first cohomology. III.2 defines cuts and links them to almost-invariant subsets of a group. It then contains results that will be needed later (notably, 2.8). Dicks does not talk about ends in this book, but it's worth learning about them, and this is probably the best point to do so. Follow Section 2 of Cohen [3]. I recommend going through pp.17–19 (definition of the number of ends in terms of $H^1(G, \mathbb{Z}/2\mathbb{Z})$, link to almost invariant sets, Examples 1, 2), then prove propositions 2.14 and 2.15.

Talk 6 Decomposition theorems. III.3 (you may need III. 1.2).

Talk 7 III.4. The relationship with derivations.

Talk 8 Consequences. III.5.2. Whether or not the augmentation ideal of a group splits over that of a subgroup tells us something about free factors of the group ([2, IV.5.]).

Talk 9 More consequences. Stallings' theorem on ends of groups ([2, IV.6.10] – use results that we have seen in the group instead of the "Almost Stability Theorem").

Talk 10 More consequences. Groups of cohomological dimension one are free (IV.3)

References

[1] W. Dicks. Groups, Trees and Projective Modules. *Springer*, 1980

[2] W. Dicks and M.J. Dunwoody. Groups acting on graphs. *CUP*, 1989.

[3] D.E. Cohen. Groups of cohomological dimension one. *Springer*, 1972.

[4] J.-P. Serre. Trees. *Springer*, 1980.

[5] P. Scott and T. Wall, Topological methods in group theory. In Homological Group Theory, *London Math. Soc. Lecture Notes*, vol. 36, 1979, pp. 137–204.