Finite Element Approximation of 2D Parabolic Optimal Design Problems

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Summary. In this paper we consider a problem of parabolic optimal design in 2D for the heat equation with Dirichlet boundary conditions. We introduce a finite element discrete version of this problem in which the domains under consideration are polygons defined on the numerical mesh. The discrete optimal design problem admits at least one solution. We prove that, as the mesh size tends to zero, any limit in H^c of discrete optimal shapes is an optimal domain for the continuous optimal design problem. We work in the functional and geometric setting introduced by V. Šveràk in which the domains under consideration are assumed to have an a priori limited number of holes. We present in detail a numerical algorithm and show the efficiency of the method through various numerical experiments.

1 Introduction

We consider a problem of optimal control in which the control variable is the domain on which a partial differential equation is posed. The function we want to minimize depends on Ω through the solution of the PDE. In the present paper we analyze the heat equation in 2D with Dirichlet boundary conditions extending previous works by D. Chenais and the second author on the elliptic problem in [6] and [7].

We focus on the problem of numerical approximation of optimal shapes. We build a finite element approximation of the optimal design problem and prove that, as the mesh size tends to zero, in the H^c-topology, every limit of discrete optimal shapes is an optimal shape for the continuous equation. We work in the functional setting introduced by Šveràk [25] in which the domains under consideration have an a priori limited finite number of holes, later adapted to the finite element setting in [6] and [7].

Let us describe more precisely the problem under the consideration.

- $\mathcal{C}$ is a non-empty bounded Lipschitz open set of $\mathbf{R}^2$.
- $\mathcal{O}$ is the set of all open subsets of $\mathcal{C}$.
- For all $\Omega \in \mathcal{O}$ and $T > 0$, we consider the heat equation in Ω

$$\begin{cases} u_t - \triangle u = f & \Omega \times [0,T], \\ u = 0 & \partial\Omega \times [0,T], \\ u(0) = \psi_0 & \Omega, \end{cases} \tag{1}$$

where $f \in L^2(0,T;L^2(\mathbf{R}^2))$ and $\psi_0 \in L^2(\mathbf{R}^2)$. The variational formulation of (1) is as follows (see [4]):

$$\begin{cases} \text{To find } u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \text{ such that} \\ \frac{d}{dt}(u,\varphi) + a(u,\varphi) = (f,\varphi), \qquad \forall \varphi \in H_0^1(\Omega), \\ u(0) = \psi_0, \end{cases} \tag{2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx,$$

and $(\cdot,\cdot)$ stands for the scalar product in $L^2(\Omega)$.
- We also consider the functional $J : \mathcal{O} \to \mathbf{R}$ to be minimized. Typically in applications J is defined as an integral involving the solution u of (1). Therefore, the continuity of J (with respect to the H^c convergence of domains) requires the continuity of the solutions of (1) with respect to the domain. For that to be the case one often needs to restrict the functional to a suitable subclass of domains.
To be more precise we consider functionals of the form

$$J(\Omega) = \int_0^T \int_\Omega L(t,x,u,\nabla u) dx dt, \tag{3}$$

where $L(t,x,z,s)$ is assumed to be non-negative, continuous in (t,x,z,s), strictly convex in s and and such that there exists $c > 0$ such that

$$|L(t,x,z,s)| \leq c(|z|^2 + |s|^2).$$

In (3) u denotes the solution of (1) in Ω.
These assumptions may be greatly simplified in specific applications. We do not intend to describe the most general framework but only give a few relevant examples in which our developments apply.
Let us give some examples of functionals $J(\Omega)$ which often arise in applications and fulfill the previous requirements:
  – The first one concerns the compliance of the system (1). It is defined by

$$J(\Omega) = \int_0^T \int_\Omega f u dx dt.$$

  The assumptions are fulfilled when $f = f(x,t)$ is continuous although our methods apply when $f \in L^2(0,T;L^2(\mathbf{R}^2))$.

– A second important example concerns shape identification problems. Let us consider a subdomain $E \in \mathcal{O}, E \neq \emptyset$. We suppose that a function u_E has been measured on E, which is a known or accesible part of the set Ω which is unknown and has to be identified.
  In this case, the functional to be minimized is, for example, of the form

$$J(\Omega) = \frac{1}{2}\int_0^T \int_\Omega |\nabla(u - \widetilde{u}_E)|^2 dxdt.$$

Here and in the sequel we denote by $\widetilde{u}$ the extension by zero of u so that $\widetilde{u} = 0$ in $\mathcal{C} \setminus \Omega$. The assumptions above are satisfied by this functional too.
The continuous optimal design problem we consider is as follows:

$$\text{To find } \Omega^* \in \mathcal{O} \text{ such that } J(\Omega^*) = \min_{\Omega \in \mathcal{O}} J(\Omega). \tag{4}$$

In practice, often, this problem is formulated in a suitable subset of $\mathcal{O}$ in order to guarantee the compactness and continuity properties that are needed for the minimum to be achieved. The results by Šveràk [25] guarantee that this occurs when working in the subclass of domains with complementary sets with at most a finite prescribed number of connected components. We shall denote by $\mathcal{O}^N$ that class where $\sharp(\Omega^c) \leq N$ for all $\Omega \in \mathcal{O}^N$, N being a finite number and $\sharp(K)$ the number of connected components of K.
In other words, we shall be mainly concerned with the following minimization problem:

$$\text{To find } \Omega^* \in \mathcal{O}^N \text{ such that } \mathcal{I} := J(\Omega^*) = \min_{\Omega \in \mathcal{O}^N} J(\Omega). \tag{5}$$

The question we address in this paper is the numerical approximation of the optimal design problem (5). In particular we address the issue of whether the discrete optimal shapes for a suitable discretization of the above problem converge in H^c (see Section 2 for the precise definition), to an optimal shape for the continuous one as the mesh-size tends to zero. This problem was successfully formulated and solved by D. Chenais and the second author in [6] and [7] for the elliptic case and this article is aimed to give an extension to the parabolic one.

In order to do this, we now introduce a discretization of this problem as follows.

- For any $h > 0$, we consider a triangulation $\mathcal{T}_h = \left\{(\tau_i^h)_{i \in I_h}\right\}$ of $\mathcal{C}$ made of finite elements τ_i^h so that

$$\mathcal{C} = \overbrace{\bigcup_{i \in I_h} \tau_i^h}^{\circ},$$

  where $\overbrace{A}^{\circ}$ denotes the interior of $A \subset \mathbf{R}^2$. To this end, we suppose that the triangulations are uniformly regular, that is

$$\exists \sigma > 0 \text{ s.t. } \forall h > 0, \quad \tau_i^h \in \mathcal{T}_h, \quad 0 < \frac{h}{\rho_i} \leq \sigma,$$

where the grid size h is defined as the maximum diameter of the elements τ_i^h and ρ_i is the radius of the largest ball contained in τ_i^h.

- $\mathcal{O}_h$ is the set of open subsets of $\mathcal{C}$ constituted by unions of triangles of the triangulation $\mathcal{T}_h$ and $\mathcal{O}_h^N = \mathcal{O}_h \cap \mathcal{O}^N$, the subset of those polygonal domains for which the number of connected components of the complement is a priori bounded by N.
- We use the implicit Euler method with time step $\triangle t = T/M$, for some $M \in \mathbf{N}$, to discretize the heat equation (1) in time and a P1 finite element approximation for the elliptic component. For doing that we consider the P1 finite element space $X_h \subset H_0^1(\Omega_{h,\triangle t})$, and we denote by $u_{h,\triangle t}^k$ the discrete solution in the time step k, $u_{h,\triangle t}^k \sim u(x, t_k)$ where $t_k = k\triangle t$. We also denote by $U_{h,\triangle t} := (u_{h,\triangle t}^k)_{k=1}^M$ the vector-valued solution containing the solution for all time-steps. The discrete solution we consider is characterized by the following system:

$$\begin{cases} \text{To find } u_{h,\triangle t}^k \in X_h \text{ such that} \\ \left(\dfrac{u_{h,\triangle t}^k - u_{h,\triangle t}^{k-1}}{\triangle t}, \varphi_h \right) + a(u_{h,\triangle t}^k, \varphi_h) = (f^k, \varphi_h), \, \forall \varphi_h \in X_h, k = 1, \ldots, M \\ u_h^0 = \psi_{0,h}, \end{cases} \tag{6}$$

where

$$f^k = \frac{1}{\triangle t} \int_{t_k - \triangle t}^{t_k} f(t) dt,$$

and $\psi_{0,h}$ is the orthogonal projection of ψ_0 over X_h.
- We approximate $J(\Omega)$ by a well-chosen functional $J_h^{\triangle t}(\Omega_{h,\triangle t}) : \mathcal{O}_h^N \to \mathbf{R}$. In practice this is done by keeping the same structure of the functional as in (3) in what concerns its x-dependence and replacing the time-integral by a discrete sum.

Thus, the discrete problem we consider is

$$\text{To find } \Omega_{h,\triangle t}^* \in \mathcal{O}_h^N \text{ such that } \mathcal{I}_h^{\triangle t} := J_h^{\triangle t}(\Omega_{h,\triangle t}^*) = \min_{\Omega_{h,\triangle t} \in \mathcal{O}_h^N} J_h^{\triangle t}(\Omega_{h,\triangle t}). \tag{7}$$

As indicated above, this is a natural extension to the parabolic setting of the elliptic optimal design problem addressed in [6] and [7].
The main result of this paper asserts that, for any fixed N, the discrete optimal design problems (7) converge towards (5) as $h \to 0$ and $\triangle t \to 0$ in the sense that the minima converge and that the limits of $\Omega_{h,\triangle t}^*$ are optimal domains for the continuous optimization problem (5).
The techniques we employ and the results we obtain in this article can be adapted and extended to other discretization schemes. In particular this can

be done for the semi-discrete approximation and other time-discretization methods of (1).

This paper is divided in five sections after this introduction. In Section 2 we recall some definitions and properties concerning Hausdorff topology, γ-convergence, Mosco-convergence and some useful results from previous papers. In Section 3 we prove the convergence of the numerical scheme. In Section 4 we prove the convergence of discrete optimal shapes. In Section 5 we develop a classical optimization algorithm to obtain the optimal design in the continuous and the discrete time cases respectively. In particular, we present a fully discrete numerical algorithm allowing to obtain an approximation of the optimal domain. Moreover, we present in detail some numerical experiments that allow checking the efficiency of the method. Finally, Section 6 is devoted to summarize the main results of the paper.

2 Preliminaries

2.1 Hausdorff convergence

In this section we recall some notations and basic results.
The Hausdorff distance between two compact sets K_1 and K_2 of $\mathbf{R}^2$ is defined by

$$d(K_1, K_2) = \max\left(\sup_{x\in K_1}\inf_{y\in K_2}||x-y||, \sup_{x\in K_2}\inf_{y\in K_1}||x-y||\right).$$

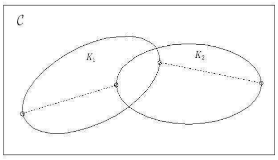


Fig. 1. Hausdorff distance between two compact sets

Definition 1. *The complementary Hausdorff distance between two open subsets Ω_1 and Ω_2 of $\mathcal{C}$ is defined by*

$$d_{H^c}(\Omega_1, \Omega_2) = d_H(\overline{\mathcal{C}}\backslash\Omega_1, \overline{\mathcal{C}}\backslash\Omega_2).$$

We denote by H^c the corresponding convergence of sets, i.e., $\Omega_n \xrightarrow{H^c} \Omega$ if only if $d_{H^c}(\Omega_n, \Omega) \to 0$.

In addition to the set $\mathcal{O}^N$ defined above, for any open non-empty subset ω of $\mathcal{C}$ we define the class $\mathcal{O}^N_\omega$ of domains of $\mathcal{O}^N$ containing ω, i.e.

$$\mathcal{O}^N_\omega = \{\Omega \in \mathcal{O}^N : \omega \subset \Omega\}.$$

The following result on the H^c-compactness of the sets $\mathcal{O}^N$ and $\mathcal{O}^N_\omega$ will be useful for addressing the optimal design problems above.

Lemma 1. *([25], [12]) For any finite N, and ω open subset of $\mathcal{C}$, the sets $\mathcal{O}^N$ and $\mathcal{O}^N_\omega$ are H^c-compact.*

2.2 Dependence of the Dirichlet problem with respect to the domain

For each function $\varphi \in H^1_0(\Omega)$, we define $\widetilde{\varphi}$ its extension by zero to $\mathcal{C}$ so that $\widetilde{\varphi} \in H^1_0(\mathcal{C})$ (see [4]).
We recall the definition of γ-convergence and Mosco-convergence.

Definition 2. *([12]) Given a sequence $(\Omega_n)_n \subset \mathcal{O}$ and a domain $\Omega \in \mathcal{O}$, Ω_n γ-converges to Ω, and we denote it as $\Omega_n \stackrel{\gamma}{\to} \Omega$, if*

$$\forall f \in H^{-1}(\mathcal{C}), \quad \widetilde{u}_{\Omega_n} \to \widetilde{u}_\Omega \text{ strongly in } H^1_0(\mathcal{C}),$$

where $u_{\Omega_n} \in H^1_0(\Omega_n)$ is defined as the solution of the Dirichlet elliptic problem in Ω_n:

$$a(u_{\Omega_n}, \varphi) =< f, \varphi >_{H^{-1}(\Omega_n)\times H^1_0(\Omega_n)}, \quad \forall \varphi \in H^1_0(\Omega_n).$$

Definition 3. *([18]) Ω_n Mosco-converges to Ω and we denote it as $\Omega_n \stackrel{\text{Mosco}}{\to} \Omega$, if*

1. *For all $\varphi \in H^1_0(\Omega)$, there exists $\varphi_n \in H^1_0(\Omega_n)$ such that $\widetilde{\varphi}_n \to \widetilde{\varphi}$ strongly in $H^1_0(\mathcal{C})$.*
2. *For all subsequence of domains $(\Omega_{n_k})_k$, and for all $\varphi_{n_k} \in H^1_0(\Omega_{n_k})$, one has*

$$\{\widetilde{\varphi}_{n_k} \rightharpoonup w \text{ weakly in } H^1_0(\mathcal{C})\} \Rightarrow \{\exists \varphi \in H^1_0(\Omega) \text{ such that } w = \widetilde{\varphi}\}.$$

It is by now well known that these two notions coincide (see [12]), i.e. $\Omega_n \stackrel{\gamma}{\to} \Omega$ if and only if $\Omega_n \stackrel{\text{Mosco}}{\to} \Omega$.
Now, let us recall some relations between H^c-convergence and γ-convergence.

Lemma 2. *([5]) If a sequence H^c-converges, then the first point of the definition of the Mosco convergence is satisfied. In other words, if Ω_n converges to Ω in H^c, then, for all $\varphi \in H^1_0(\Omega)$, there exists $\varphi_n \in H^1_0(\Omega_n)$ such that $\tilde{\varphi}_n \to \tilde{\varphi}$ strongly in $H^1_0(\mathcal{C})$.*

In general, H^c-convergence does not imply γ-convergence, nevertheless, several situations are known where this implication holds true. In [5], a list of subsets $\mathcal{U}$ of $\mathcal{O}$ on which H^c-convergence implies γ-convergence is given. The following one is due to V. Šveràk [25]:

Theorem 1. *In two space dimensions, for any finite N, H^c-convergence and γ-convergence are equivalent properties on $\mathcal{O}^N$.*

In order to deal with the time-dependent continuous and discrete heat equations we have to work with functions depending on the time variable. The following technical result is a natural consequence of γ-convergence for sequences of functions depending both on x and t.

Lemma 3. *Assume that $\Omega_j \xrightarrow{\gamma} \Omega$ and consider a sequence of functions u_j in $L^\infty(0,T;L^2(\Omega_j)) \cap L^2(0,T;H_0^1(\Omega_j))$ satisfying*

$$\widetilde{u}_j \rightharpoonup w \quad \text{weakly } * \text{ in } L^\infty(0,T;L^2(\mathcal{C})) \cap L^2(0,T;H_0^1(\mathcal{C})). \tag{8}$$

Then $w = \widetilde{y}$, with $y \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$

Proof (of Lemma 3). As $\Omega_j \xrightarrow{\gamma} \Omega$ we know that $\Omega_j \overset{\text{Mosco}}{\rightarrow} \Omega$.
Let $\theta \in L^2(0,T)$ be given. We obtain

$$\widetilde{u}_j^\theta(x,t) = \int_0^T \theta(t)\widetilde{u}_j(x,t)dt \rightharpoonup w^\theta(x,t) = \int_0^T \theta(t)w(x,t)dt \text{ in } H_0^1(\mathcal{C}).$$

Since $\widetilde{u}_j^\theta \in H_0^1(\Omega_j)$, by the γ-convergence of the sets Ω_j, we get $w^\theta \in H_0^1(\Omega)$. By the Lebesgue Differentiation Theorem we have

$$w(x,t_0) = \lim_{j\to 0} \int_0^T \frac{1}{2j}\chi_{[t_0-j,t_0+j]}(t)w(x,t)dt \quad \text{a.e. } t_0 \in [0,T].$$

Therefore, $w(t) \in H_0^1(\Omega)$ a.e. $t \in [0,T]$.
Now, we have to prove that the function $w : [0,T] \mapsto H_0^1(\Omega)$ is measurable. Since $H_0^1(\Omega)$ is separable, it is sufficient to prove (see [8]) that w is weakly measurable, i.e., that for any $\varphi \in C_c^\infty(\Omega)$, the function $t \mapsto \int_\Omega w(x,t)\varphi(x)dx$ is measurable. According to (8), $\int_\Omega w(x,t)\varphi(x)dx$ is the weak $*$ limit in $L^\infty(0,T)$ of $\int_\Omega u_j(x,t)\varphi(x)dx$, and, in particular, it is measurable with respect to t.
This completes the proof of the Lemma. □

3 Preliminaries on the convergence of the numerical scheme

We first define the set of discrete admissible domains. This set is independent of $\triangle t$.

Definition 4. *For each $h > 0$, we consider the set $\mathcal{O}_h$ of subdomains of $\mathcal{C}$ constituted by elements of the triangulations $\mathcal{T}_h$. Then we set*

$$\mathcal{O}_h^N = \{\Omega_h \in \mathcal{O}_h : \sharp(\Omega_h^c) \leq N\}.$$

For all $\Omega_h \in \mathcal{O}_h^N$, we consider the P1 finite element space $X_h \subset H_0^1(\Omega_h)$. We use the implicit Euler method to discretize in time and we get the discrete system (6). At each time step k, it consist on solving a linear system of the form

$$(\mathcal{M} + \triangle t\mathcal{A})\xi^k = \eta^{k-1},$$

where $\eta^{k-1} = \mathcal{M}\xi^{k-1} + F^k$ is known, with $F^k = (f^k, \varphi_j)$, $\mathcal{M} = (\varphi_i, \varphi_j)$ is the mass matrix, $\mathcal{A} = a(\varphi_i, \varphi_j)$ is the stiffness matrix for $i, j = 1, \dots, S$, and $\xi^k = (\xi_j^k)_{j=1}^S$ is the vector of the coefficients of the solution on the finite-elements basis, i.e.

$$u_{h,\triangle t}^k = \sum_{j=1}^{S} \xi_j^k \varphi_j,$$

$(\varphi_j)_{j=1}^S$ being the basis functions for X_h. Obviously $\mathcal{M} + \triangle t\mathcal{A}$ is symmetric and positive definite so that the system above is solvable.
We recall that, for a fixed bounded domain Ω with Lipschitz boundary, the fully discrete solutions $u_{h,\triangle t}^k$ converge to the solution u of the continuous heat equation (1) as $h \to 0$ and $\triangle t \to 0$. The proof of this result is based on the classical consistency plus stability analysis. In particular, the implicit method (6) is unconditionally stable with respect to the $L^2(0, T; H_0^1(\Omega))$-norm.
The following estimate on the rate of convergence is also well known. Given $\psi_0 \in H^2(\Omega)$, for each $k = 1, ..., M$ it follows that (see Section 11.3, pp. 394, Corollary 11.3.1, [21]):

$$\begin{aligned} ||u_{h,\triangle t}^k - u(t_k)||_{L^2(\Omega)} &\leq ||\psi_{0,h} - \psi_0||_{L^2(\Omega)} \\ &+ Ch^2\Big(|\psi_0|_{H^2(\Omega)} + \int_0^{t_k} \Big|\partial_t u(s)\Big|_{H^2(\Omega)}\Big) + \triangle t\Big(\int_0^{t_k} \Big|\Big|\partial_t^2 u(s)\Big|\Big|_{L^2(\Omega)}\Big), \quad (9) \end{aligned}$$

where the seminorm in $H^2(\Omega)$ is denoted by $|\cdot|_{H^2(\Omega)}$ and the norm in $L^2(\Omega)$ is denoted by $||\cdot||_{L^2(\Omega)}$.

Remark 1. We choose the implicit method for the time-discretization because it is unconditionally stable, so that the choice of $\triangle t$ is dictated from accuracy requirements only. Recall that, by the contrary, explicit methods are conditionally stable, and, therefore, they require the time-step $\triangle t$ to be sufficiently small with respect to the spatial mesh size h.

In the analysis of the convergence of the optimal design problems we will need to pass to the limit in the solution of the discrete problems towards those of the continuous heat equation when the domain varies. The following Proposition provides the needed convergence result:

Proposition 1. *Let $\Omega \in \mathcal{O}^N$ be given. Let $\Omega_{h,\triangle t} \in \mathcal{O}_h^N$ be a sequence such that $\Omega_{h,\triangle t} \stackrel{H^c}{\to} \Omega$.*
Then $\widetilde{u}^k_{h,\triangle t} \to \widetilde{u}$ strongly in $L^2(0,T;H_0^1(\mathcal{C}))$ when $h \to 0$ and $\triangle t \to 0$.

Remark 2. This convergence property holds for the piecewise constant or linear extension of $\widetilde{u}^k_{h,\triangle t}$ to all $t \in [0,T]$. For the sake of simplicity we denote it simply as $\widetilde{u}^k_{h,\triangle t}$.

Proof (of Proposition 1). Let us denote by $\widetilde{X}_h$ the vector space of all functions of X_h extended by zero to $\mathcal{C}$.
Let $\varphi(x,t) = \sigma(t)\omega(x) \in C_c^\infty(\Omega \times [0,T])$ be given. We define the time discrete test function:

$$\varphi^k = \sigma^k \omega(x) = \sigma(t_k)\omega(x), \quad k = 1,...,M$$

and $\widetilde{\varphi}^k = \sigma^k \widetilde{\omega}(x)$ its extension by zero to $\mathcal{C}$.
The equation (6) can be rewritten as follows,

$$\int_{\mathcal{C}} \frac{\widetilde{u}^k_{h,\triangle t} - \widetilde{u}^{k-1}_{h,\triangle t}}{\triangle t}\widetilde{\varphi}^k dx + \int_{\mathcal{C}} \nabla \widetilde{u}^k_{h,\triangle t} \cdot \nabla \widetilde{\varphi}^k dx = \int_{\mathcal{C}} f^k \widetilde{\varphi}^k dx, \quad k = 1,...,M.$$

Taking the test function $\widetilde{\varphi}^k = \widetilde{u}^k_{h,\triangle t}$ and rewriting

$$\widetilde{u}^k_{h,\triangle t} = \frac{\triangle t}{2}\frac{\widetilde{u}^k_{h,\triangle t} - \widetilde{u}^{k-1}_{h,\triangle t}}{\triangle t} + \frac{\widetilde{u}^k_{h,\triangle t} + \widetilde{u}^{k-1}_{h,\triangle t}}{2},$$

we get

$$\frac{||\widetilde{u}^k_{h,\triangle t}||^2_{L^2(\mathcal{C})} - ||\widetilde{u}^{k-1}_{h,\triangle t}||^2_{L^2(\mathcal{C})}}{2\triangle t} + ||\widetilde{u}^k_{h,\triangle t}||^2_{H_0^1(\mathcal{C})} \le \frac{1}{2}||f^k||^2_{L^2(\mathcal{C})} + \frac{1}{2}||\widetilde{u}^k_{h,\triangle t}||^2_{L^2(\mathcal{C})}.$$

Therefore, we conclude that

$$||\widetilde{u}^k_{h,\triangle t}||_{L^2(0,T;H_0^1(\mathcal{C}))} \le c_1[||\widetilde{f}||_{L^2(0,T;L^2(\mathcal{C}))} + ||\widetilde{\psi}_{0,h}||_{L^2(\mathcal{C})}]$$

for any h and $\triangle t$.
Thus, up to the extraction of subsequences, $\widetilde{u}^k_{h,\triangle t}$ weakly converges in $L^2(0,T;H_0^1(\mathcal{C}))$ to w. We have to show that its limit coincides with $\widetilde{u}$, u being the solution of (1), to later prove strong convergence.
By Lemma 3, we know that there exists $y \in L^2(0,T;H_0^1(\Omega))$ such that $w = \widetilde{y}$.
Now, let us prove that $y = u$ and that the convergence holds in the strong topology.
First we prove that $y = u$. Observe that the solution u of (1) is characterized by the fact that $u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ and

$$\begin{cases} -\int_0^T \int_\Omega u\sigma_t\omega dxdt + \int_\Omega \psi_0\sigma(0)\omega(x)dx - \int_0^T \int_\Omega \nabla u \cdot \sigma\nabla\omega dxdt \\ \qquad\qquad\qquad\qquad\qquad\qquad = \int_0^T \int_\Omega f\sigma\omega dxdt, \\ \forall \sigma(t)\omega(x) \in H^1(\Omega \times (0,T)) \text{ s.t. } \sigma(t)\omega(x)|_{\partial\Omega\times[0,T]} = 0 \text{ and } \sigma(T) \equiv 0. \end{cases}$$

We have to prove that y is the solution of the previous equation.

We have

$$\sum_{k=1}^{M}\int_{\mathcal{C}}\frac{\widetilde{u}^k_{h,\triangle t}-\widetilde{u}^{k-1}_{h,\triangle t}}{\triangle t}\sigma^k\widetilde{\omega}_h dx+\sum_{k=1}^{M}\int_{\mathcal{C}}\nabla\widetilde{u}^k_{h,\triangle t}\cdot\sigma^k\nabla\widetilde{\omega}_h dx=\sum_{k=1}^{M}\int_{\mathcal{C}}f^k\sigma^k\widetilde{\omega}_h dx.$$

By Lemma 3.1 pp. 17, [7], there exists $\widetilde{\omega}_h\in X_h$ such that $\widetilde{\omega}_h\to\widetilde{\omega}$ strongly in $H^1_0(\mathcal{C})$ as $h\to 0$.
Adding by parts we get

$$-\int_0^T\int_{\mathcal{C}}\widetilde{u}^k_{h,\triangle t}\frac{\sigma^{k+1}-\sigma^k}{\triangle t}\widetilde{\omega}_h dxdt-\int_{\mathcal{C}}\widetilde{\psi}_{0,h}\sigma^0\widetilde{\omega}_h dx$$
$$+\int_0^T\int_{\mathcal{C}}\nabla\widetilde{u}^k_{h,\triangle t}\cdot\nabla\widetilde{\omega}_h\sigma^k dxdt=\int_0^T\int_{\mathcal{C}}f^k\sigma^k\widetilde{\omega}_h dxdt. \quad (10)$$

On the other hand

$$\Big|\frac{\sigma^{k+1}-\sigma^k}{\triangle t}-\sigma_t(t)\Big|\le\Big|\frac{\sigma^{k+1}-\sigma^k}{\triangle t}-\sigma_t(t_k)\Big|+\Big|\sigma_t(t)-\sigma_t(t_k)\Big|$$
$$\le C(\triangle t)||\sigma_{tt}||_{L^\infty(0,T)}.$$

Furthermore, we know that $\widetilde{u}^k_{h,\triangle t}\rightharpoonup\widetilde{y}$ weakly in $L^2(0,T;H^1_0(\mathcal{C}))$. Thus, we can pass to the limit in equation (10) and get

$$-\int_0^T\int_{\mathcal{C}}\widetilde{y}\sigma_t\widetilde{\omega}dxdt-\int_{\mathcal{C}}\widetilde{\psi}_0\sigma(0)\widetilde{\omega}dx+\int_0^T\int_{\mathcal{C}}\nabla\widetilde{y}\cdot\sigma\nabla\widetilde{\omega}dxdt=\int_0^T\int_{\mathcal{C}}f\sigma\widetilde{\omega}dxdt.$$

Using that $\widetilde{y}$ is vanishes on $\mathcal{C}\backslash\Omega$ and $\widetilde{y}=y$ on Ω we have that y satisfies the same equation on Ω. So $y=u$.
Now, we prove the strong convergence in $L^2(0,T;H^1_0(\mathcal{C}))$. For $\widetilde{u}$ the energy estimate yields

$$\int_0^T\int_{\mathcal{C}}|\nabla\widetilde{u}|^2dxdt+\frac{1}{2}||\widetilde{u}(T)||^2_{L^2(\mathcal{C})}=\frac{1}{2}||\widetilde{u}(0)||^2_{L^2(\mathcal{C})}+\int_0^T\int_{\mathcal{C}}f\widetilde{u}dxdt. \quad (11)$$

For $\widetilde{u}^k_{h,\triangle t}$ taking as test function $\widetilde{\varphi}^k_h=\widetilde{u}^k_{h,\triangle t}$ we get

$$\triangle t\sum_{k=1}^{M}\int_{\mathcal{C}}|\nabla\widetilde{u}^k_{h,\triangle t}|^2dx+\frac{1}{2}\int_{\mathcal{C}}|\widetilde{u}^M_{h,\triangle t}|^2dx=\frac{1}{2}\int_{\mathcal{C}}|\widetilde{u}^0_{h,\triangle t}|^2dx$$
$$+\triangle t\sum_{k=1}^{M}\int_{\mathcal{C}}f^k\widetilde{u}^k_{h,\triangle t}dx. \quad (12)$$

Under the assumptions on the initial data and the weak convergence in $L^2(0,T;H^1_0(\mathcal{C}))$ we can easily pass to the limit in the right hand side term of

(12). On the other hand, by weak convergence of the solutions and the weak lower semi-continuity of norms, we have

$$\begin{aligned}||\nabla\widetilde{u}||^2_{L^2(0,T;L^2(\mathcal{C}))} + \frac{1}{2}||\widetilde{u}(T)||^2_{L^2(\mathcal{C})} \\ \leq \liminf_h \left[||\nabla\widetilde{u}^k_{h,\triangle t}||^2_{L^2(0,T;L^2(\mathcal{C}))} + \frac{1}{2}||\widetilde{u}^M_{h,\triangle t}||^2_{L^2(\mathcal{C})}\right] \\ = \frac{1}{2}||\widetilde{u}(0)||^2_{L^2(\mathcal{C})} + \int_0^T \int_{\mathcal{C}} f\widetilde{u}dxdt.\end{aligned}$$

On the other, by the energy identity (11) for the heat equation (1) we deduce that

$$||\nabla\widetilde{u}^k_{h,\triangle t}||^2_{L^2(0,T;L^2(\mathcal{C}))} + \frac{1}{2}||\widetilde{u}^M_{h,\triangle t}||^2_{L^2(\mathcal{C})} \to ||\nabla\widetilde{u}||^2_{L^2(0,T;L^2(\mathcal{C}))} + \frac{1}{2}||\widetilde{u}(T)||^2_{L^2(\mathcal{C})}.$$

This, together with weak convergence, implies the strong convergences:

$$\begin{aligned}\widetilde{u}^k_{h,\triangle t} \to \widetilde{u} \quad &\text{in} \quad L^2(0,T;H^1_0(\mathcal{C})). \\ \widetilde{u}^M_{h,\triangle t} \to \widetilde{u}(T) \quad &\text{in} \quad L^2(\mathcal{C}).\end{aligned}$$

Note that, in the proof above, we have used the weak convergence of $\widetilde{u}^M_{h,\triangle t}$ towards $\widetilde{u}(T)$ in $L^2(\mathcal{C})$. This is due to the uniform bounds on (12), the weak convergence in $L^2(0,T;H^1_0(\mathcal{C}))$ and a classical compactness argument which uses the Aubin-Lions Lemma and the equation satisfied by $\widetilde{u}^M_{h,\triangle t}$ which allows getting uniform bounds on the time-derivative of its piecewise linear and continuous extension in time in $L^2(0,T;H^{-1}(\mathcal{C}))$. □

4 Convergence of discrete optimal shapes

The question we address here is the numerical approximation of the optimal design problem (5). In particular, we address the issue of whether the discrete optimal shapes for a suitable discretization of the above problem converge in H^c to a continuous optimal shape. As we shall see, the answer to this question is positive if the discrete optimization problem is conveniently built, as above, in the context of finite element approximations.

The triangulation $\mathcal{T}_h$ being fixed, for any $h > 0$, the number of triangular domains in $\mathcal{O}^N_h$ under consideration for the discrete optimal design problem (7) is finite. Thus, the existence of discrete optimal shapes is obvious, and we denote them by $\Omega^*_{h,\triangle t}$.
Now, we prove that any limit in H^c of discrete optimal shapes is an optimal domain for the continuous optimal design problem.

Theorem 2. *Let J be the functional as in (3). Suppose that the discretization $J^{\triangle t}_h$ of J has been chosen such that:*

1. *If* $\Omega, \Omega_{h,\triangle t} \in \mathcal{O}^N$ *are such that* $\Omega_{h,\triangle t} \overset{H^c}{\rightarrow} \Omega$*, then* $J_h^{\triangle t}(\Omega_{h,\triangle t}) \rightarrow J(\Omega)$ *when* $h \rightarrow 0$ *and* $\triangle t \rightarrow 0$.

Then, the discrete optimal design problems (7) converge as $h \rightarrow 0$ *and* $\triangle t \rightarrow 0$ *to the continuous one (5) in the sense that*

(a) $J_h^{\triangle t}$ *reaches its minimum on* $\mathcal{O}_h^N$ *for all* $h > 0$ *and* $\triangle t > 0$.
(b) Any accumulation point as $h \rightarrow 0$, $\triangle t \rightarrow 0$ *in the topology* H^c *of any sequence* $\left(\Omega^*_{h,\triangle t}\right)_{h,\triangle t}$ *of discrete minimizers is a continuous minimizer.*
(c) The whole sequence $\left(\mathcal{I}_h^{\triangle t}\right)_{h,\triangle t}$ *converges to* $\mathcal{I}$.

Remark 3. Similar results hold in the class $\mathcal{O}_\omega^N$ of domains.

Proof (of Theorem 2). Let $(\Omega^*_{h,\triangle t})_{h,\triangle t}$ be a sequence of discrete minimizers for problem (6). Any $\Omega^*_{h,\triangle t}$ belongs to $\mathcal{O}^N$ which is H^c-compact. Let Ω_{ap} be an accumulation point of this sequence. By Lemma 1 $\Omega_{ap} \in \mathcal{O}^N$. From Proposition 1 we have

$$\widetilde{u}^k_{h,\triangle t} \rightarrow \widetilde{u^{\Omega_{ap}}} \text{ strongly in } L^2(0,T; H_0^1(\mathcal{C})),$$

where $u^{\Omega_{ap}}$ is the solution of the continuous problem (1) in Ω_{ap}. Due to the assumption of the Theorem, we obtain

$$\mathcal{I}_h^{\triangle t} = J(\Omega^*_{h,\triangle t}) \rightarrow J(\Omega_{ap}) \quad \text{when } h \rightarrow 0 \text{ and } \triangle t \rightarrow 0. \tag{13}$$

Let us now check that Ω_{ap} is a minimizer for J.
Given $\Omega \in \mathcal{O}^N$, there exist $\Omega_h \in \mathcal{O}_h^N$ such that $\Omega_h \overset{H^c}{\rightarrow} \Omega$ (see Section 4.2.1, [7]). For each h and $\triangle t$, we have

$$\mathcal{I}_h^{\triangle t} \leq J_h^{\triangle t}(\Omega_h).$$

Passing to the limit in this inequality and using (13) and hypothesis 1, we obtain $J(\Omega_{ap}) \leq J(\Omega)$ for all $\Omega \in \mathcal{O}^N$. This proves points a) and b) of the theorem.
Also, we have seen that the only accumulation point of the sequence $\left(\mathcal{I}_h^{\triangle t}\right)_{h,\triangle t}$ is nothing but $\mathcal{I}$. □

Remark 4. We have proved that any limit in H^c of discrete optimal shapes is an optimal domain for the continuous optimal design problem. The obtention of convergence rates would be of interest, but this subject is completely open.

5 Gradient calculations: A numerical approach

5.1 Preliminaries

We have proven that the discrete optimal shapes converge in H^c to an optimal shape for the continuous problem. Now we address the problem of efficiently

computing the discrete optimal shapes. Despite of the fact that, for $h > 0$ and $\triangle t > 0$ given, the existence of the discrete optimal shapes is trivial, its computation may be rather complex because of the very large number of existing admissible domains.
The search of the discrete optimal shapes is usually performed by gradient type methods. The main idea of these methods is to iterate in the discrete domain using the information provided by the gradient of the functional with respect to perturbations of the domain in the continuous framework. This gradient can be calculated using classical methods of differentiation with respect to the domain (see [9], [16], [19], [20]).
As far as we know, the convergence of an iterative method based on these ideas is not proved so far. In fact, in principle, taking into account that the information we are using to iterate on the discrete domains comes from the continuous framework, it is not even clear that the discrete functional decreases along the iteration. We refer to [9] and [19] for an analysis of the comparison between discrete and continuous gradients. As we shall see, however, the method turns out to be efficient in practice.
The second drawback of this procedure is that it is based on tools coming from the differentiation with respect to the shape of the domain. This requires a minimal amount of regularity of the domains under consideration and, consequently, can not be applied in the general geometric setting in which our convergence result in Theorem 2 has been established.
The use of differentiation with respect to domain deformations can be fully justified by restricting the class of admissible domains to consider only sufficiently smooth ones (see [22], [23], [24]). In that setting the existence of optimal domains can be proved by classical regularity and compactness results for the solutions of the PDE under consideration both in the elliptic and the parabolic case (see [16], [17]). However, as far as we know, the convergence of these iterative numerical methods is still to be proved in this context too.

Let us now describe how to use differentiation with respect to the domain to build an iterative method for searching optimal shapes.

5.2 A example for the continuous problem

Consider the functional

$$J(\Omega) = \frac{1}{2}\int_0^T \int_\Omega |\nabla(u - \widetilde{u}_E)|^2 dxdt, \tag{14}$$

where u_E is the solution of the problem (1) in the domain E that we want to recover. Obviously the solution of this minimization problem is $\Omega = E$. We use it as a test of the efficiency of our method.
The aim of this section is to obtain an expression for the variation of the functional (14). The main tool is the so-called shape differentiation ([16], [20], [22]). To do this, we consider normal variations of the domain and the new domains of the form

$$\Omega + \alpha = \{x + \alpha(x) : x \in \Omega\},$$

where α represents the variations of Ω, with $\alpha \in C^2$. These variations α are assumed to be small enough and oriented along the normal direction over the boundary $\partial\Omega$. This induces a variation on the solution: $\delta u = u(\Omega+\alpha) - u(\Omega)$. Differentiating in (14) we obtain

$$\langle \delta J(\Omega), \alpha \rangle = \frac{1}{2} \int_0^T \int_\Gamma \alpha |\partial_n (u - \widetilde{u}_E)|^2 d\sigma dt + \int_0^T \int_\Omega \nabla(u - \widetilde{u}_E) \cdot \nabla(\delta u) dx dt \tag{15}$$

where $\Gamma = \partial\Omega$.

On the other hand, differentiating the state equation (1) we have (see [16], [20],[22])

$$\begin{cases} \delta u_t - \triangle(\delta u) = 0 & \Omega \times [0,T], \\ \delta u = -\alpha(\partial_n u) & \Gamma \times [0,T], \\ \delta u(0) = 0 & \Omega. \end{cases} \tag{16}$$

Let $\phi \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$. Multiplying the previous equation by ϕ and integrating by parts, we get

$$0 = \int_0^T \int_\Omega [-\phi_t - \triangle\phi] \delta u dx dt + \int_\Omega \phi(T) \delta u(T) dx - \int_0^T \int_\Gamma \partial_n(\delta u) \phi d\sigma dt + \int_0^T \int_\Gamma (\partial_n \phi) \delta u d\sigma dt.$$

Let us choose ϕ as the solution of the adjoint problem

$$\begin{cases} -\phi_t - \triangle\phi = -\triangle(u - \widetilde{u}_E) & \Omega \times [0,T], \\ \phi = 0 & \Gamma \times [0,T], \\ \phi(T) = 0 & \Omega. \end{cases} \tag{17}$$

Multiplying this equation (17) by δu and integrating by parts we get

$$\begin{aligned} \int_0^T \int_\Omega \nabla(u - \widetilde{u}_E) \cdot \nabla(\delta u) dx dt - \int_0^T \int_\Gamma \partial_n(u - \widetilde{u}_E) \delta u d\sigma dt \\ = - \int_0^T \int_\Omega \phi_t \delta u dx dt - \int_0^T \int_\Omega \triangle\phi \delta u dx dt \\ = \int_0^T \int_\Omega \phi[\delta u_t - \triangle(\delta u)] dx dt - \int_0^T \int_\Gamma (\partial_n \phi) \delta u d\sigma dt. \end{aligned}$$

Therefore,

$$\int_0^T \int_\Omega \nabla(u - \widetilde{u}_E) \cdot \nabla(\delta u) dx dt = \int_0^T \int_\Gamma \alpha(\partial_n u)(\partial_n \phi) d\sigma dt - \int_0^T \int_\Gamma \alpha \partial_n(u - \widetilde{u}_E) \partial_n u d\sigma dt. \tag{18}$$

Taking (18) in (15) we obtain

$$\begin{aligned}\langle \delta J(\Omega), \alpha\rangle &= \frac{1}{2}\int_0^T \int_\Gamma \alpha|\partial_n(u-\widetilde{u}_E)|^2 d\sigma dt + \int_0^T \int_\Gamma \alpha(\partial_n u)(\partial_n \phi) d\sigma dt \\ &\qquad - \int_0^T \int_\Gamma \alpha \partial_n(u-\widetilde{u}_E)\partial_n u d\sigma dt \\ &= \int_0^T \int_\Gamma \alpha\Big(\partial_n(u-\widetilde{u}_E)\Big(\frac{1}{2}\partial_n(u-\widetilde{u}_E) - \partial_n u\Big) + \partial_n u \partial_n \phi\Big) d\sigma dt.\end{aligned}$$

In this way we obtain the following expression for the variation of J:

$$\langle \delta J(\Omega), \alpha\rangle = \int_0^T \int_\Gamma \alpha\Big(-\frac{1}{2}\Big((\partial_n u)^2 - (\partial_n \widetilde{u}_E)^2\Big) + \partial_n u \partial_n \phi\Big) d\sigma dt. \tag{19}$$

Note that using the adjoint state, the expression of $\langle \delta J(\Omega), \alpha\rangle$ in (15) has been simplified. Indeed, in the final one (19), the variation of the state δu does not enter. This is a significant improvement since, according to (16), computing δu would require solving an initial boundary value problem for each α. In view of (19), it is sufficient to compute the adjoint solution ϕ and then an integral for each α.

5.3 Optimization algorithm

We introduce a full-discretization of the functional (14):

$$J_h^{\triangle t}(\Omega_{h,\triangle t}) = \frac{\triangle t}{2}\sum_{k=1}^{M}\int_{\Omega_{h,\triangle t}} |\nabla(u_{h,\triangle t}^k - \widetilde{u}_E)|^2 dx. \tag{20}$$

We discretize the adjoint problem (17) in the same way as the state equation, i.e. let $\phi_{h,\triangle t}^k \in X_h$ be the solution of

$$\begin{cases} \displaystyle\int_{\Omega_{h,\triangle t}} \frac{\phi_{h,\triangle t}^k - \phi_{h,\triangle t}^{k+1}}{\triangle t}\varphi dx + \int_{\Omega_{h,\triangle t}} (\nabla \phi_{h,\triangle t}^k)\cdot(\nabla\varphi) dx \\ \qquad\qquad \displaystyle = \int_{\Omega_{h,\triangle t}} \nabla(u_{h,\triangle t}^k - \widetilde{u}_E)\cdot\nabla\varphi dx \quad \forall \varphi \in X_h, k=1,\dots,M, \\ \phi_{h,\triangle t}^{M+1} = 0. \end{cases} \tag{21}$$

We discretize (19) to get an approximate estimate of the variation of the discrete functional (20):

$$\begin{aligned}\langle \delta J_h^{\triangle t}(\Omega_{h,\triangle t}), \alpha\rangle \sim \triangle t \sum_{k=1}^{M}\int_{\Gamma_{h,\triangle t}} \alpha\Big(-\frac{1}{2}\Big((\partial_n u_{h,\triangle t}^k)^2 - (\partial_n \widetilde{u}_E)^2\Big) \\ + (\partial_n u_{h,\triangle t}^k)(\partial_n \phi_{h,\triangle t}^k)\Big) d\sigma.\end{aligned} \tag{22}$$

Note that both in the continuous and the discrete case $\partial_n u = \nabla u \cdot \mathbf{n}$.
We denote by Γ_{int} the interior boundary of the domain, and Γ_{out} the outer one, and by $\tau_i^{h^-}$ the triangles belonging to in Γ_{int} and $\tau_i^{h^+}$ those in Γ_{out}. The inner $\tau_i^{h^-}$ and outer triangles $\tau_i^{h^+}$ are linked by the fact that they have a common edge on the boundary Γ_h, and $\mathcal{F}_h$ is the set of nodes of the boundary (see Fig. 2).
As we mentioned above, in the continuous case the deformations α considered are oriented in the normal direction along the boundary. In the discrete setting it is natural to interprete this fact by considering perturbations in which one adds triangles $\tau_i^{h^+}$ or drops $\tau_i^{h^-}$ depending how they contribute to the decrease of the functional.

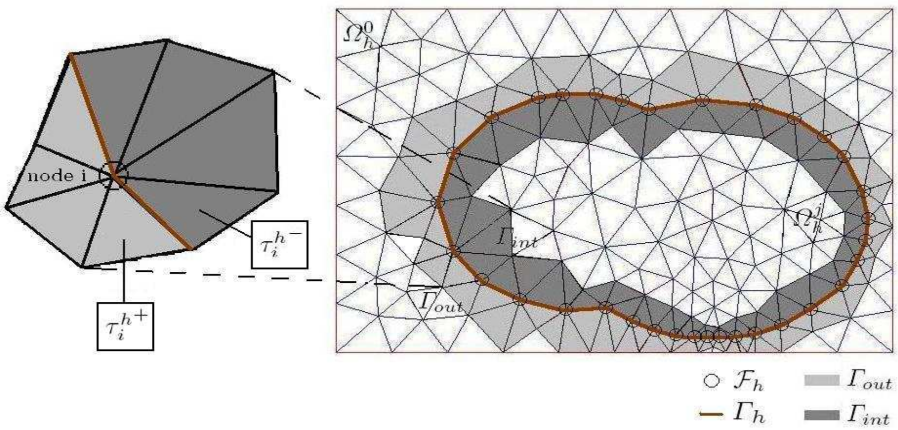


Fig. 2. Outer and inner boundaries of the domain, Γ_{out} and Γ_{int} respectively.

To do this we compute the contribution of each edge of the boundary to the gradient of the discrete functional as follows:

$$\delta J_h^{\triangle t}(\Omega_{h,\triangle t}^j)\Big|_{\Gamma_{h,\triangle t}^j \cap \tau_i^{h^-}} := \triangle t \sum_{k=1}^{M} \int_{\Gamma_{h,\triangle t}^j \cap \tau_i^{h^-}} \Big(-\frac{1}{2}\Big((\partial_n u_{h,\triangle t}^k)^2 - (\partial_n \widetilde{u}_E)^2\Big) + (\partial_n u_{h,\triangle t}^k)(\partial_n \phi_{h,\triangle t}^k)\Big) d\sigma, \quad (23)$$

the contribution of the edge $\Gamma_{h,\triangle t}^j \cap \tau_i^{h^-}$ to this approximation of the variation of the functional $J_h^{\triangle t}$.
The functional $J_h^{\triangle t}(\Omega_{h,\triangle t})$ being defined on a finite number of polygonal domains its continuous derivative is not well defined. But (23) provides an approximation to its change rate locally on each edge of the boundary. However one has to interprete the estimated variation in (23) in the context of the

given triangulation and the possible polygonal configurations.
To do this, given a discrete domain, in view of (23), we analyze the contribution of each one of its boundary triangles, both inner and outer ones, and we obtain the new domain adding or cutting triangles based on their contribution to decreasing the value of $J_h^{\triangle t}$ (see [9], [16], [17], [19], [20]).
To compute $\delta J_h^{\triangle t}(\Omega_{h,\triangle t}^j)\Big|_{\Gamma_{h,\triangle t}^j \cap \tau_i^{h-}}$, according to (23), we need to solve the discrete state equation (6) and the discretization of the adjoint problem (21) with $\Omega_{h,\triangle t} = \Omega_{h,\triangle t}^j$.
For each node of the boundary, $\ell \in \mathcal{F}_h$, we compute the variation of the functional at this node as the average of the variation of the funcional in the edges $\Gamma_{h,\triangle t}^j \cap \tau_i^{h-}$ containing the node ℓ. We denote by $\delta J_h^{\triangle t}(\Omega_{h,\triangle t}^j)\Big|_\ell$ the variation of the functional at the node ℓ.
Following this procedure, the new domain $\Omega_{h,\triangle t}^j$ is obtained from the previous one $\Omega_{h,\triangle t}^{j-1}$ adding the triangles containing the node ℓ where the contribution of $\delta J_h^{\triangle t}(\Omega_{h,\triangle t}^j)\Big|_\ell$ is negative, and cutting ones where its contribution is positive.
We explain this procedure in more detail below.

Let us now describe the algorithm. We fix a tolerance $TOL > 0$.

1. We choose $h > 0$ and $\triangle t > 0$ and construct the mesh $\mathcal{T}_h$ of $\mathcal{C}$.
2. Consider the initial guess $\Omega_{h,\triangle t}^0 = \mathcal{C}$.
3. Iteration scheme, $j \geq 0$. It is applied while $|J_h^{\triangle t}(\Omega_{h,\triangle t}^j)| > TOL$:
   a) Solve the discrete state problem (6) with $\Omega_{h,\triangle t} = \Omega_{h,\triangle t}^j$.
   b) Solve the adjoint discrete problem (21) with $\Omega_{h,\triangle t} = \Omega_{h,\triangle t}^j$.
   c) Compute $\delta J_h^{\triangle t}(\Omega_{h,\triangle t}^j)\Big|_{\Gamma_{h,\triangle t}^j \cap \tau_i^{h-}}$ as in (23).
   d) Compute $\delta J_h^{\triangle t}(\Omega_{h,\triangle t}^j)\Big|_\ell$.
   e) Deformation of the domain.
   We build the new domain $\Omega_{h,\triangle t}^{j+1}$ as follows:
   $$\Omega_{h,\triangle t}^{j+1} = \Omega_{h,\triangle t}^j \bigcup \{\tau_\ell^h : \delta J_h^{\triangle t}(\Omega_{h,\triangle t}^j)\Big|_\ell < 0\} \setminus \{\tau_\ell^h : \delta J_h^{\triangle t}(\Omega_{h,\triangle t}^j)\Big|_\ell > 0\},$$
   where τ_ℓ^h are the triangles that contain the node ℓ.
   f) Compute the functional (20) in the new domain $\Omega_{h,\triangle t}^{j+1}$.
   g) We take $\Omega_{h,\triangle t}^j = \Omega_{h,\triangle t}^{j+1}$ and go back to the beginning of this iteration scheme.

5.4 Numerical results

All the numerical experiments we present here have been performed with a Pentium M 715 processor and 512 MB RAM.

Let the set $\mathcal{C}$ be the rectangle $(-1.5, 1.5) \times (-1, 1)$. We consider problems (1) and (5), with force term $f = 1$, initial data $\psi_0(x, y) = \sin(2\pi x)$ and where the functional to be minimized is

$$J(\Omega) = \frac{1}{2}\int_0^T \int_\Omega |\nabla(u - \widetilde{u}_E)|^2 dxdt, \tag{24}$$

where u_E is the solution of the problem (1) in the domain E that we want to recover, and u is the solution in Ω.

Numerical experiment # 1

We take $T = 20$, $\triangle t = 0.1$ and $h = 0.19197$. This time the computation is done over a mesh of 206 nodes and 372 triangles. Our goal is to recover the circle E (see Fig 3).
In order to do this, we compute u_E, the solution of the problem (6) in the domain E, then we minimize the functional (24) by the algorithm that we have described in the previous section.

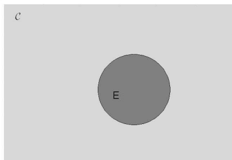
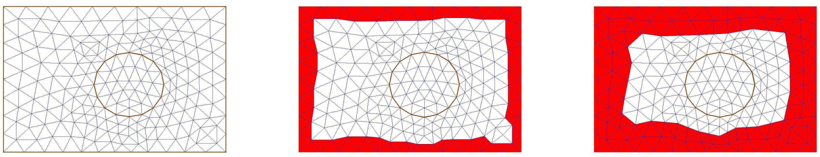


Fig. 3. The unknown body

Fig. 4 depicts the evolution of the domain with respect to the iteration j. We find the circle E in 6 steps and 3621 seconds (CPU time). Fig. 5 depicts the evolution of the cost function. As expected, the limit of the sequence Ω_j is close to the circle E.



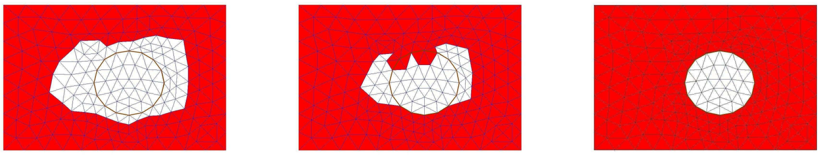


Fig. 4. Evolution of the domain converging to the circle E

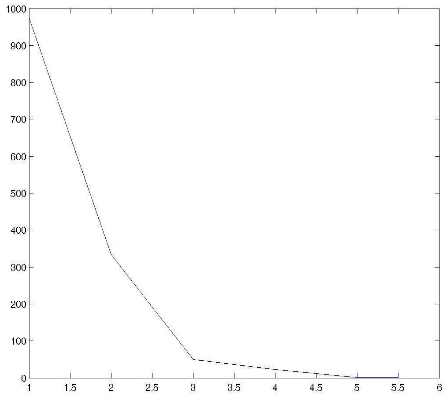


Fig. 5. Evolution of the functional (24)

Numerical experiment # 2

We take $T = 20$, $\triangle t = 0.1$ and $h = 0.31062$. This time the computation is done over a mesh of 118 nodes and 213 triangles. Now, our goal is to recover E as in Fig. 6 bellow:

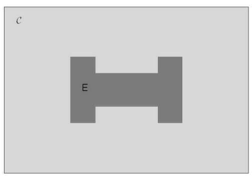


Fig. 6. The unknown body

Fig. 7 depicts the evolution of the domain with respect to the iteration j. In this case, we find E in 5 steps and 1768 seconds (CPU time). Fig. 8 depicts the evolution of the cost function.

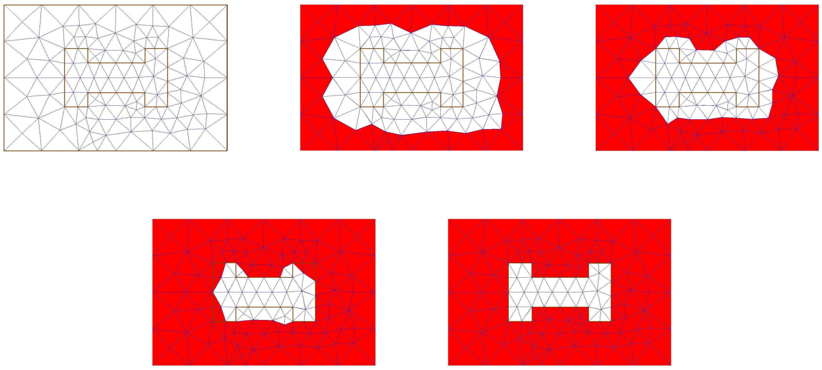


Fig. 7. Evolution of the domain converging to E

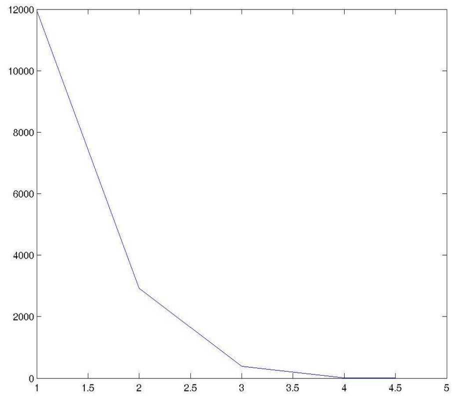


Fig. 8. Evolution of the functional (24)

Numerical experiment # 3

We take $T = 20$, $\triangle t = 0.1$ and $h = 0.14509$. This time the computation is done over a mesh of 281 nodes and 506 triangles. Now, our goal is to recover E as in Fig. 9 bellow:

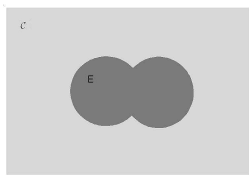


Fig. 9. The unknown body

Fig. 10 depicts the evolution of the domain with respect to the iteration j. In this case, we find E in 6 steps and 6866 seconds (CPU time). Fig. 11 depicts the evolution of the cost function.

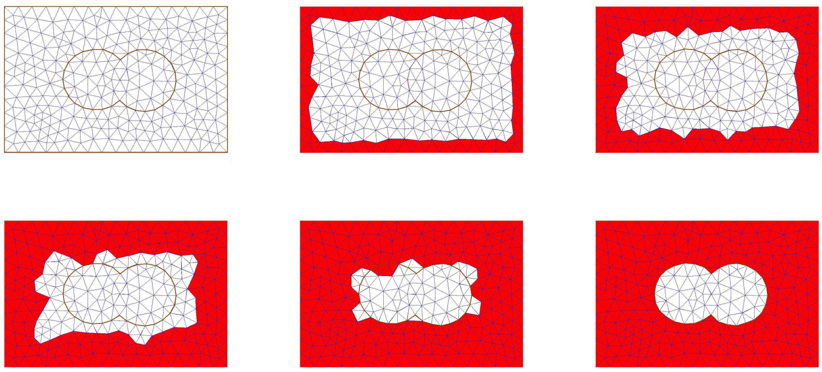


Fig. 10. Evolution of the domain converging to E

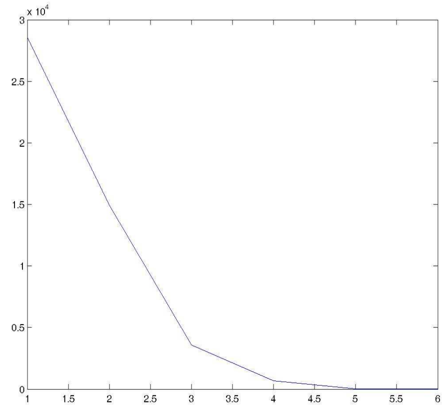


Fig. 11. Evolution of the functional (24)

Numerical experiment # 4

We take $T = 20$, $\triangle t = 0.1$ and $h = 0.13798$. This time the computation is done over a mesh of 284 nodes and 516 triangles. Now, our goal is to recover E as in Fig. 12 bellow:

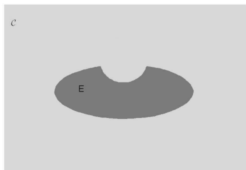


Fig. 12. The unknown body

Fig. 13 depicts the evolution of the domain with respect to the iteration j. In this case, we find E in 6 steps and 10372 seconds (CPU time). Fig. 14 depicts the evolution of the cost function.

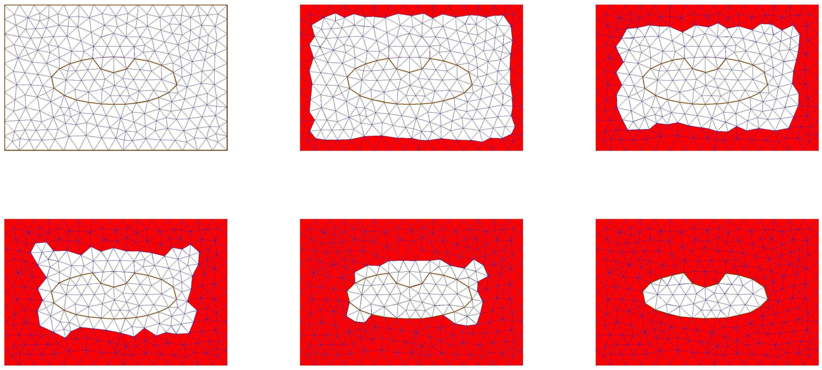


Fig. 13. Evolution of the domain converging to E

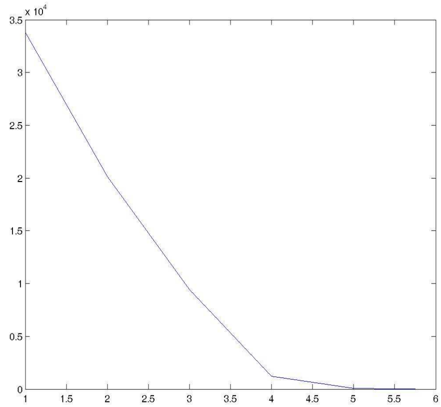


Fig. 14. Evolution of the functional (24)

6 Conclusions

We have considered the problem of numerically approximating optimal shapes in the context of the 2D linear heat equation with Dirichlet boundary conditions. We have addressed the issue of whether discrete optimal shapes for a

suitable discretization of the original continuous optimal design problem provide an approximation of the continuous optimal shape.
We have developed a P1 finite-element approximation in space and an implicit discretization in time for which this convergence result holds in the 2D case, in the class of domains with an a priori bounded number of holes, introduced by V. Šverák ([25]). According to our results convergence holds in the complementary-Hausdorff topology.
Our results can be extended to a more general framework of evolution problems provided a number of properties are guaranteed: (a) the continuous dependence of the solution of the PDE with respect to the domain on which it is posed, and (b) the H^c-compactness of the set of admissible continuous domains. These continuity properties, and the convergence properties of the numerical scheme under consideration, allow proving sufficient continuity conditions of numerical schemes with respect to the numerical mesh, to guarantee the convergence of the optimal shapes.
These results extend to the evolution framework those previously developed in [6] and [7] in the elliptic case.
Then, we use a classical iterative optimization algorithm to obtain a numerical approximation of the discrete optimal domains. Using differentiation with respect to the domain, we can find explicit formulas of the approximate variation of the discrete functional to build numerical methods for the search of the discrete optimal shape, by means of the solution of the discrete adjoint problem. The convergence of the iterative numerical methods we obtain by this procedure is not proved but its efficiency is illustrated by various experiments.

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