Uniform boundary controllability of a discrete 1-D wave equation [1]

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Abstract

A numerical scheme for the controlled discrete 1-D wave equation is considered. We prove the convergence of the boundary controls of the discrete equations to a control of the continuous wave equation when the mesh size tends to zero when time and space steps coincide. This positive result is in contrast with previous negative ones for space semi-discretizations.

Key words:

Dedicated to the memory of Jacques-Louis Lions

1 Introduction and main results

Let us consider the $1-d$ wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1, \quad 0 < t < T, \\ u(0,t) = 0, \quad u(1,t) = v(t), & 0 < t < T, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & 0 < x < 1, \end{cases} \tag{1.1}$$

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where (u_0, u_1) is the initial state and v is the control function that acts on the system through the extreme $x = 1$ of the space interval $(0, 1)$. System (1.1) describes the vibrations of a controlled string and it is well posed in appropriate functional spaces.

It is well known that when $v = 0$, i.e., when no control acts on the boundary $x = 1$, the energy

$$E(t) = \frac{1}{2}\int_0^1 \Big(| \; u_x(x,t) \; |^2 + | \; u_t(x,t) \; |^2\Big)\, dx \tag{1.2}$$

of the solutions satisfies

$$\frac{dE(t)}{dt} = E'(t) = 0, \quad \forall t \in [0, T]$$

and therefore it is conserved in time.

The following exact controllability result for (1.1) is also well known (see [L]): *Given $T \geq 2$ and $(u_0, u_1) \in L^2(0, 1) \times H^{-1}(0, 1)$, there exists a control function $v \in L^2(0, T)$ such that the solution $u = u(x, t)$ of (1.1) satisfies:*

$$u(T) = u_t(T) = 0, \qquad 0 < x < 1. \tag{1.3}$$

Among the controls with the property (1.3) there exists a unique control of minimal L^2-norm. It will be called the HUM control. Note that (1.3) makes sense since, in these conditions, system (1.1) admits a unique solution $u \in \mathcal{C}([0, T]; L^2(0, 1)) \cap \mathcal{C}^1([0, T]; H^{-1}(0, 1))$.

Let us now consider the uncontrolled system:

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0, & 0 < x < 1, \quad 0 < t < T, \\ \phi(0, t) = \phi(1, t) = 0, & 0 < t < T, \\ \phi(x, 0) = \phi_0(x), \qquad \phi_t(x, 0) = \phi_1(x), & 0 < x < 1. \end{cases} \tag{1.4}$$

It admits a unique solution $\phi = \phi(x, t)$ in $\mathcal{C}([0, T]; H_0^1(0, 1)) \cap \mathcal{C}^1([0, T]; L^2(0, 1))$ when $(\phi_0, \phi_1) \in H_0^1(0, 1) \times L^2(0, 1)$. This system is the adjoint of (1.1) up to an inversion of time.

The solutions of (1.4) in terms of the Fourier coefficients are:

$$\phi(x, t) = \sum_{k=1}^{\infty} [\alpha_k \cos(k\pi t) + \frac{\beta_k}{k\pi} \sin(k\pi t)]\varphi_k(x), \tag{1.5}$$

with α_k, $\beta_k \in \mathbb{R}$ and $\varphi_k(x) = \sin(k\pi x)$.

The exact controllability problem (1.3) has been studied in a much more general setting and several approaches for its solutions are now available. In particular, the *Hilbert Uniqueness Method (HUM)* introduced by J.L. Lions in [L] reduces the exact controllability property above to the following equivalent observability property for the adjoint system (1.4): *Given $T \geq 2$ there exists a positive constant $C(T) > 0$ such that*

$$E(0) \leq C(T) \int_0^T \mid \phi_x(1,t) \mid^2 dt. \tag{1.6}$$

holds for every solution of (1.4).

Inequality (1.6) can be easily proved by several methods including *Fourier series, D'Alembert Formula* and *multiplier techniques* (see, for instance, [L]).

In the paper [IZ] the following semi-discrete version of (1.4) was analyzed:

Take $N \in \mathbb{N}$, set $h = 1/(N+1)$ and consider the finite-difference space semi-discretization of (1.4):

$$\begin{cases} \phi_j'' = [\phi_{j+1} + \phi_{j-1} - 2\phi_j] \mid h^2, & t > 0, \quad j = 1, ..., N, \\ \phi_0 = \phi_{N+1} = 0, & t > 0, \\ \phi(0) = \phi_{0,j}, \quad \phi_j'(0) = \phi_{1,j}, & j = 1, ..., N. \end{cases} \tag{1.7}$$

The energy of system (1.7) is given by

$$E_h(t) = \frac{h}{2} \sum_{j=1}^{N} \mid \phi_j(t) \mid^2 + \frac{h}{2} \sum_{j=0}^{N} \frac{\mid \phi_{j+1}(t) - \phi_j(t) \mid^2}{h^2}; \tag{1.8}$$

it is also conserved in time.

The corresponding semi-discrete version of (1.6) is

$$E_h(t) \leq C \int_0^T \mid \frac{\phi_N(t)}{h} \mid^2 dt. \tag{1.9}$$

More precisely, one seeks for a positive constant $C > 0$ such that (1.9) holds.
The corresponding eigenvalue problem is of the form:

$$\begin{cases} -[\varphi_{k+1} + \varphi_{k-1} - 2\varphi_k]/h^2 = \lambda \varphi_k, \;\; k = 1, ..., N, \\ \varphi_0 = \varphi_{N+1} = 0. \end{cases} \tag{1.10}$$

The eigenvalues and eigenvectors of (1.10) may be computed explicitly (see [IK], p. 456). One has:

$$\begin{cases} \lambda_k(h) = \dfrac{4}{h^2}\sin^2\left(\dfrac{\pi k h}{2}\right), & k = 1, ..., N, \\ \bar{\varphi}_k \equiv (\varphi_{k,1}, ..., \varphi_{k,N});\ \varphi_{k,j} = \sin(k\pi j h), & j, k = 1, ..., N. \end{cases} \tag{1.11}$$

The solutions of (1.7) in Fourier series are:

$$\bar{\phi} = \sum_{k=1}^{N} \left(a_k \sin(\sqrt{\lambda_k t}) + b_k \cos(\sqrt{\lambda_k t}) \right) \bar{\varphi}_k. \tag{1.12}$$

As pointed out in [IZ], for any $T > 0$

$$sup_{\bar{\phi}\in S_h} \left[\frac{E_h(0)}{\int_0^T \mid \phi_N(t)/h \mid^2 dt} \right] \to \infty,\ as\ h \to 0, \tag{1.13}$$

where S_h is the set of all solutions of (1.7).

This fact shows that the observability constant in (1.9) may not remain uniformly bounded as $h \to 0$ and it is due to the pathological behavior of the high frequencies.

As a consequence of this, by Hahn-Banach Theorem, one deduces the existence of finite energy initial data for the wave equation (1.1) for which the controls corresponding to the semi-discrete model diverge when $h \to 0$. As it was proved in [IZ], if the high frequencies are conveniently filtered or truncated then it is possible to obtain positive results, i.e., observability inequalities with constants which are independent of h (see [IZ], p. 411, for more details).

Moreover, note that, as proved by S. Micu in [M], when the initial datum contains only a finite number of Fourier components, the controls of the semi-discrete model remain bounded and converge as $h \to 0$ to a control of the wave equation (1.1).

In this paper we analyze the analogue of (1.6) for a complete discrete version (in space-time) of the wave equation.

Given $M, N \in \mathbb{N}$ we set $\Delta x = 1/(N+1)$ and $\Delta t = T/(M+1)$ and introduce the nets

$$x_0 = 0 < x_1 = \Delta x < ... < x_N = N\Delta x < x_{N+1} = 1,$$

$$t_0 = 0 < t_1 = \Delta t < ... < t_M = M\Delta t < t_{M+1} = T$$

with $x_j = j\Delta x$ and $t_n = n\Delta t$, $j = 0, 1, ..., N+1$, $n = 0, ..., M+1$.

We consider the following finite-difference discretization of (1.1):

$$\begin{cases} \dfrac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}, & j = 1, 2, ..., N;\ n = 1, 2, ..., M, \\ u_0^n = 0, \qquad u_{N+1}^n = v_{\Delta t}^n, & n = 1, 2, ..., M, \\ u_j^0 = u_{0j}, \quad u_j^1 = \Delta t u_{1j} + u_{0j}, & j = 1, 2, ..., N. \end{cases} \tag{1.14}$$

We shall denote by $\bar{u}^n = (u_1^n, ..., u_N^n)$ the solution at the time step n.

As in the context of the continuous wave equation above, we consider the uncontrolled system

$$\begin{cases} \dfrac{\phi_j^{n+1} - 2\phi_j^n + \phi_j^{n-1}}{(\Delta t)^2} = \dfrac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{(\Delta x)^2}, & j = 1, 2, ..., N; \quad n = 1, 2, ..., M, \\ \phi_0^n = \phi_{N+1}^n = 0, & n = 1, 2, ..., M, \\ \phi_j^0 = \phi_{0j}, \quad \phi_j^1 = \phi_{0j} + h\phi_{1j}, & j = 1, 2, ..., N. \end{cases} \tag{1.15}$$

System (1.15) would be the adjoint of (1.14) if the initial conditions were taken at the final steps $n = M, M+1$ instead of $n = 0, 1$.

The numerical schemes (1.14), (1.15) are consistent. Besides, they are stable if and only if $\Delta t \leq \Delta x$. This guarantees that, if the condition $\Delta t \leq \Delta x$ is satisfied, then the solutions of the schemes (1.14), (1.15) converge as $\Delta t \to 0$, $\Delta x \to 0$ to the solutions of (1.1) and (1.4) respectively, in the corresponding norms.

This paper is organized as follows: in section 2 we prove some properties of the solutions of the adjoint system (1.15), which are used in the proof of the main results. In section 3 we prove a uniform discrete version of (1.6) in the particular case $\Delta t = \Delta x = h$. In section 4 we discuss the consequences of this result concerning the controllability of the discrete controlled system (1.14). In section 5 we prove the convergence of the sequence of controls for system (1.14) as the mesh-size h tends to zero. Finally, in section 6 we present some numerical simulations, which confirm the convergence of the scheme we propose.

2 Preliminaries on the discrete system

In this section we analyze in detail some properties of the energy of the homogeneous discrete system (1.15) and we perform a careful analysis of the spectrum. In particular we prove the conservation and non-negativity of the energy.

Following [SV], we define the energy of (1.15)

$$E_n = \frac{\Delta x}{2}\sum_{j=0}^{N}\left[\left(\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}\right)^2 + \left(\frac{\phi_{j+1}^{n+1}-\phi_j^{n+1}}{\Delta x}\right)\left(\frac{\phi_{j+1}^{n}-\phi_j^{n}}{\Delta x}\right)\right]. \qquad (2.1)$$

Proposition 1. *For all $\Delta t,\ \Delta x \in (0,1)$ the energy (2.1) of the solutions of the discrete system (1.15) is conserved in all the time steps.*

Proof. Multiplying the equation (1.15) by $\frac{1}{2}(\phi_j^{n+1}-\phi_j^{n-1})$ and adding in j, we have

$$\frac{1}{2}\sum_{j=1}^{N}\left[\frac{\phi_j^{n+1}-2\phi_j^n+\phi_j^{n-1}}{(\Delta t)^2}(\phi_j^{n+1}-\phi_j^{n-1}) - \frac{\phi_{j+1}^{n}-2\phi_j^n+\phi_{j-1}^{n}}{(\Delta x)^2}(\phi_j^{n+1}-\phi_j^{n-1})\right] = 0. \qquad (2.2)$$

The equation (2.2) may be written as:

$$\frac{1}{2}\sum_{j=1}^{N}\left[\left(\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}\right)^2 + \left(\frac{\phi_{j+1}^{n+1}-\phi_j^{n+1}}{\Delta x}\right)\left(\frac{\phi_{j+1}^{n}-\phi_j^{n}}{\Delta x}\right) - \left(\frac{\phi_j^{n}-\phi_j^{n-1}}{\Delta t}\right)^2 - \left(\frac{\phi_{j+1}^{n}-\phi_j^{n}}{\Delta x}\right)\left(\frac{\phi_{j+1}^{n-1}-\phi_j^{n-1}}{\Delta x}\right)\right]$$

$$+\frac{1}{2(\Delta x)^2}\sum_{j=1}^{N}(\phi_j^n\phi_j^{n+1} - \phi_j^n\phi_j^{n-1} + \phi_j^{n-1}\phi_{j-1}^n - \phi_j^{n+1}\phi_{j-1}^n - \phi_{j+1}^{n+1}\phi_{j+1}^n + \phi_{j+1}^{n+1}\phi_j^n + \phi_{j+1}^n\phi_{j+1}^{n-1} - \phi_{j+1}^{n-1}\phi_j^n) = 0.$$

Denote by S the last sum in the left hand of this equation; we have:

$$S = \sum_{j=1}^{N}\phi_j^n\phi_j^{n+1} - \sum_{j=1}^{N}\phi_j^n\phi_j^{n-1} + \sum_{j=1}^{N}\phi_j^{n-1}\phi_{j-1}^n - \sum_{j=1}^{N}\phi_j^{n+1}\phi_{j-1}^n - \sum_{j=1}^{N}\phi_j^{n+1}\phi_j^n + \sum_{j=1}^{N}\phi_j^{n+1}\phi_{j-1}^n + \sum_{j=1}^{N}\phi_j^n\phi_j^{n-1}$$

$$-\sum_{j=1}^{N}\phi_j^{n-1}\phi_{j-1}^n-\phi_{N+1}^{n+1}\phi_{N+1}^n+\phi_1^{n+1}\phi_1^n+\phi_{N+1}^{n+1}\phi_N^n-\phi_1^{n+1}\phi_0^n+\phi_{N+1}^n\phi_{N+1}^{n-1}-\phi_1^n\phi_1^{n-1}-\phi_{N+1}^{n-1}\phi_N^n+$$
$$\phi_1^{n-1}\phi_0^n.$$

Using the *Dirichlet* boundary conditions $\phi_0^n=\phi_{N+1}^n=0$, for all $n=1,2,...,M$, the previous sum is reduced to:

$$S=\phi_1^{n+1}\phi_1^n-\phi_1^n\phi_1^{n-1}=\left(\phi_1^{n+1}-\phi_0^{n+1}\right)(\phi_1^n-\phi_0^n)-(\phi_1^n-\phi_0^n)\left(\phi_1^{n-1}-\phi_0^{n-1}\right). \tag{2.3}$$

Using (2.3), equation (2.2) may be written as:

$$\frac{1}{2}\sum_{j=0}^{N}\left(\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}\right)^2+\frac{1}{2}\sum_{j=0}^{N}\left(\frac{\phi_{j+1}^{n+1}-\phi_j^{n+1}}{\Delta x}\right)\left(\frac{\phi_{j+1}^n-\phi_j^n}{\Delta x}\right)$$
$$-\frac{1}{2}\sum_{j=0}^{N}\left(\frac{\phi_j^{n}-\phi_j^{n-1}}{\Delta t}\right)^2-\frac{1}{2}\sum_{j=0}^{N}\left(\frac{\phi_{j+1}^{n}-\phi_j^{n}}{\Delta x}\right)\left(\frac{\phi_{j+1}^{n-1}-\phi_j^{n-1}}{\Delta x}\right)=0.$$

From the definition (2.1) of the energy we have:

$$E_n=E_{n-1},\quad\forall\quad n=1,2...,M,$$

therefore, $E_n=E_0$ for all $n=1,...,M$. The proof of Proposition 1 is now complete. □

Proposition 2. *If $\Delta t\le\Delta x$, then for all the non-trivial solutions of the discrete system (1.15) and for every $n=1,2,...,M$ it holds*

$$\frac{E_n}{\Delta x}\ge\frac{1}{4(\Delta x)^2}\sum_{j=0}^{N}\left[(\phi_j^{n+1}-\phi_{j+1}^n)^2+(\phi_{j+1}^{n+1}-\phi_j^n)^2\right]. \tag{2.4}$$

Proof. Taking into account that $\Delta t\le\Delta x$, using the Dirichlet conditions $\phi_0^n=\phi_{N+1}^n=0$ and the identities

$$\sum_{j=0}^{N}\phi_{j+1}^{n+1}\phi_{j+1}^n=\sum_{j=0}^{N}\phi_j^{n+1}\phi_j^n,\quad\sum_{j=0}^{N}(\phi_j^{n+1})^2=\sum_{j=0}^{N}(\phi_{j+1}^{n+1})^2,\quad\sum_{j=0}^{N}(\phi_j^n)^2=\sum_{j=0}^{N}(\phi_{j+1}^n)^2,$$

we have:

$$\frac{E_n}{\Delta x}=\frac{1}{2}\sum_{j=0}^{N}\left(\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}\right)^2+\frac{1}{2}\sum_{j=0}^{N}\left(\frac{\phi_{j+1}^{n+1}-\phi_j^{n+1}}{\Delta x}\right)\left(\frac{\phi_{j+1}^n-\phi_j^n}{\Delta x}\right)$$

$$\geq \frac{1}{(\Delta x)^2}\left[\frac{1}{2}\sum_{j=0}^{N}(\phi_j^{n+1}-\phi_j^n)^2+\frac{1}{2}\sum_{j=0}^{N}(\phi_{j+1}^{n+1}-\phi_j^{n+1})(\phi_{j+1}^n-\phi_j^n)\right]$$
$$=\frac{1}{(\Delta x)^2}\frac{1}{2}\sum_{j=0}^{N}\left[(\phi_j^{n+1})^2+(\phi_j^n)^2-\phi_j^{n+1}\phi_{j+1}^n-\phi_{j+1}^{n+1}\phi_j^n\right]$$
$$=\frac{1}{2(\Delta x)^2}\sum_{j=0}^{N}\frac{1}{2}\left\{\left[(\phi_j^{n+1})^2+(\phi_{j+1}^n)^2-2\phi_j^{n+1}\phi_{j+1}^n\right]+\left[(\phi_{j+1}^{n+1})^2+(\phi_j^n)^2-2\phi_{j+1}^{n+1}\phi_j^n\right]\right\}$$
$$=\frac{1}{4(\Delta x)^2}\sum_{j=0}^{N}\left[(\phi_j^{n+1}-\phi_{j+1}^n)^2+(\phi_{j+1}^{n+1}-\phi_j^n)^2\right]\geq 0.$$
□

Corollary 1. *The energy of the solutions of (1.15) satisfies*

(1) $E_n \geq 0$.
(2) $E_n = 0$ *if and only if* $(\bar{\phi}^n) = \bar{0}$.

Proof. 1) is immediate from (2.4).

2) According to (2.4), if the energy vanishes, we have that $\phi_j^{n+1} = \phi_{j+1}^n$ and $\phi_{j+1}^{n+1} = \phi_j^n$ for all $j = 0, 1, \ldots N$ and for all $n = 0, 1, \ldots M$. Then, taking into account that $\phi_0^n = \phi_{N+1}^n = 0$ we get that $\bar{\phi}^n = 0$ for all $n = 0, 1, \ldots M + 1$. □

Now we perform a careful analysis of the spectrum of the adjoint problem (1.15). The discrete system (1.15) is equivalent to:

$$\frac{\bar{\phi}^{n+1}-2\bar{\phi}^n+\bar{\phi}^{n-1}}{(\Delta t)^2}+A_{\Delta x}\bar{\phi}^n=0,\quad n=1,2,...,M, \tag{2.5}$$

together with the initial data $(\bar{\phi}^0, \bar{\phi}^1)$ and the boundary conditions $\phi_0^n = \phi_{N+1}^n = 0$. Here and in the sequel $A_{\Delta x}$ is the tridiagonal matrix:

$$A_{\Delta x}=\frac{1}{(\Delta x)^2}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0\\ -1 & 2 & -1 & \cdots & 0 & 0 & 0\\ 0 & -1 & 2 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0\\ 0 & 0 & 0 & \cdots & -1 & 2 & -1\\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}.$$

Recall that the eigenvalues and the corresponding eigenvectors of $A_{\Delta x}$ are given by formula (1.11) with $h = \Delta x$.

We now look for solutions of (2.5) in separated variables of the form $\bar{\phi}^n = e^{i\bar{\omega}_{\Delta x,k}\Delta t n}\bar{\varphi}_k$ with $\bar{\varphi}_k$ eigenvectors of $A_{\Delta x}$. System (2.5) is then equivalent to

$$\left[\frac{e^{i\bar{\omega}_{\Delta x,k}\Delta t(n+1)} - 2e^{i\bar{\omega}_{\Delta x,k}\Delta tn} + e^{i\bar{\omega}_{\Delta x,k}\Delta t(n-1)}}{(\Delta t)^2}\right]\bar{\varphi}_k + A_{\Delta x}e^{i\bar{\omega}_{\Delta x,k}\Delta tn}\bar{\varphi}_k = 0,$$

which is just

$$e^{i\bar{\omega}_{\Delta x,k}\Delta tn}\left[\frac{e^{i\bar{\omega}_{\Delta x,k}\Delta t} - 2 + e^{-i\bar{\omega}_{\Delta x,k}\Delta t}}{(\Delta t)^2} + \lambda^2_{\Delta x,k}\right]\bar{\varphi}_k = 0.$$

This holds if and only if

$$e^{i\bar{\omega}_{\Delta x,k}\Delta t} - 2 + e^{-i\bar{\omega}_{\Delta x,k}\Delta t} = -(\Delta t)^2\lambda^2_{\Delta x,k}. \tag{2.6}$$

Using the equality

$$e^{i\bar{\omega}_{\Delta x,k}\Delta t} - 2 + e^{-i\bar{\omega}_{\Delta x,k}\Delta t} = 2(\cos(\bar{\omega}_{\Delta x,k}\Delta t) - 1) = -4\sin^2\frac{\bar{\omega}_{\Delta x,k}\Delta t}{2}$$

we obtain that (2.6) is

$$-4\sin^2\frac{\bar{\omega}_{\Delta x,k}\Delta t}{2} = -(\Delta t)^2\lambda^2_{\Delta x,k}.$$

Hence,

$$\bar{\omega}_{\Delta x,k} = \pm\frac{2}{\Delta t}\arcsin\frac{\Delta t \mid \lambda_{\Delta x,k} \mid}{2}.$$

In the particular case $\Delta t = \Delta x = h$ we have

$$\bar{\omega}_{h,k} = \pm\frac{2}{h}\arcsin\left(\sin\frac{k\pi h}{2}\right) = \pm k\pi. \tag{2.7}$$

Decomposing the general solution $\bar{\phi}^n$ of (1.15) on the basis of the eigenvectors of $A_{\Delta x}$ we get

$$\bar{\phi}^n = \sum_{k=-N,k\neq 0}^{N} a_k e^{i\bar{\omega}_{\Delta x,k}n\Delta t}\bar{\varphi}_{|k|}, \tag{2.8}$$

with $a_k \in \mathbb{C}$. The real solutions can be written as follows

$$\bar{\phi}^n = \sum_{k=1}^{N}\left[\alpha_k\cos(\bar{\omega}_{\Delta x,k}n\Delta t) + \frac{\beta_k}{k\pi}\sin(\bar{\omega}_{\Delta x,k}\Delta tn)\right]\bar{\varphi}_k, \tag{2.9}$$

with $\alpha_k, \beta_k \in \mathbb{R}$.

Remark 1. *In view of (1.5) and (2.9), we see that, when $\Delta t = \Delta x$, solutions of the discrete system (1.15) are in fact the restriction to the mesh of solutions of the continuous system (1.4) involving N Fourier components.*

3 Uniform observability of the discrete system

In this section we study in detail the observability problem for the finite-difference discrete version (1.15) of the wave equation (1.4) mentioned in the introduction.

Recall that here and in the sequel $\Delta t = \Delta x$, in which case the first equation in (1.15) reduces to

$$\phi_j^{n+1} + \phi_j^{n-1} = \phi_{j+1}^n + \phi_{j-1}^n. \tag{3.1}$$

The following holds:

Theorem 1. Let $T = 2$. Then,

$$E_0^h = \frac{1}{4}\left[h \sum_{n=0}^{M} \left| \frac{\phi_N^n}{h} \right|^2 \right], \tag{3.2}$$

for all $h = 1/p,\ p \in \mathbb{N}$ and for every initial data (ϕ_j^0, ϕ_j^1) of the adjoint problem (1.15) with M and N given by

$$M = 2p - 1, \qquad N = p - 1. \tag{3.3}$$

Remark 2. *Observe that, according to (3.2), in Theorem 1 it is implicitly assumed that $\Delta t = \Delta x = h$. Note that (3.2) provides not only an observability inequality, but also an identity with a multiplicative constant $\frac{1}{4}$. A similar identity holds for the wave equation (1.4) in the minimal observability time $T = 2$. Namely,*

$$E = \frac{1}{4}\int_0^T \mid \phi_x(1,t) \mid^2 dt, \tag{3.4}$$

for every solution ϕ of (1.4), where E is the energy given by (1.2).

Observe, that unlike it happens in the semi-discrete case, this theorem implies that *the observability constant remains bounded as the mesh-size tends to zero*. Note also, that as a consequence of Remark 2, the uniform observability property holds for all $T \geq 2$.

Remark 3. *Let us discuss the choice of the approximation $-\phi_N^n/h$ for the normal derivative $\phi_x(1,t)$. Needless to say, Taylor's expansion suggests that the simplest approximation for $\phi_x(1,t)$ is $(\phi(1,t)-\phi(1-h,t))/h$ or, with the notation above,*

$$\phi_x(1,t) \simeq \frac{\phi_{N+1}^n - \phi_N^n}{h}.$$

Taking into account that, due to the Dirichlet boundary conditions, $\phi_{N+1}^n = 0$, we deduce that

$$\phi_x(1,t) \simeq \frac{-\phi_N^n}{h}.$$

On the other hand, as we shall see in section 5, it follows that $(-\phi_N^n)/h \to \phi_x(1,t)$ as $h \to 0$ in a suitable sense under natural assumptions on the initial data.

It is possible to prove **Theorem 1** using two different methods: resorting to *Fourier series* decomposition of the solutions and with the aid of *discrete multiplier's techniques*. Here, for simplicity, we present a proof based on Fourier analysis.

Proof of Theorem 1.

We take into account the following orthogonality properties of the eigenvectors of the matrix $A_{\Delta x}$ in the discrete space (for more details, see [IZ]):

$$h\sum_{j=1}^{N}\sin(jk\pi h)\sin(jl\pi h) = \frac{h}{2}\sum_{j=1}^{N}[\cos(k-l)\pi hj - \cos(k+l)\pi hj] = \frac{(N+1)h}{2}\delta_{k,l}, \tag{3.5}$$

where $\delta_{k,l}$ is *Kronecker's delta* and

$$\sum_{j=0}^{N}(\sin((j+1)k\pi h) - \sin(jk\pi h)(\sin((j+1)l\pi h) - \sin(jl\pi h))) = 0, \quad \text{if } k \neq l. \tag{3.6}$$

In order to obtain the identity (3.2), it is useful to use the following calculus identities, which provide suitable orthogonality properties in the discrete time

$$\sum_{n=0}^{M}\sin^2(nk\pi h) = \sum_{n=0}^{M}\cos^2(nk\pi h) = \frac{M+1}{2}, \tag{3.7}$$

$$\sum_{n=0}^{M}\sin(2nk\pi h) = 0, \quad \sin^2(Nk\pi h) = \sin^2(k\pi h), \tag{3.8}$$

$$\frac{1}{M+1}\sum_{k=0}^{M} e^{\frac{2\pi i}{M+1}kq} = \begin{cases} 1, & \text{if } q=0, \\ 0, & \text{if } q\neq 0, \ \mid q \mid < M+1. \end{cases} \tag{3.9}$$

From (3.9) it follows that

$$\sum_{n=0}^{M} e^{i\pi h(k\pm l)n} = \sum_{n=0}^{M} e^{\frac{2\pi i}{M+1}\frac{M+1}{2}hn(k\pm l)} = \sum_{n=0}^{M} e^{\frac{2\pi i}{M+1}n(k\pm l)}. \tag{3.10}$$

Using (3.10) and denoting by $q=(k\pm l)\in\mathbb{Z}$, we have

$$\sum_{n=0}^{M}\sin nh(k\pm l)\pi = \sum_{n=0}^{M}\cos nh(k\pm l)\pi = 0. \tag{3.11}$$

The following formula expresses the energy of solutions in terms of the Fourier coefficients of their initial data:

Lemma 1. *The following identity holds for every solution ϕ of (1.15):*

$$E_0^h = \frac{N+1}{4h}\sum_{k=1}^{n}\sin^2(k\pi h)\left[\alpha_k^2 + \left(\frac{\beta_k}{k\pi}\right)^2\right]. \tag{3.12}$$

Proof of Lemma 1. We have,

$$\phi_j^1 - \phi_j^0 = \sum_{k=1}^{N}\sin(jk\pi h)\left[\alpha_k(\cos(k\pi h)-1) + \frac{\beta_k}{k\pi}\sin(k\pi h)\right]$$

and

$$(\phi_j^1-\phi_j^0)^2 = \sum_{k=1}^{N}\sin^2(jk\pi h)\left\{\alpha_k[\cos(k\pi h)-1] + \frac{\beta_k}{k\pi}\sin(k\pi h)\right\}^2 +$$

$$+\sum_{k,l=1;k\neq l}^{N}\sin(jk\pi h)\sin(jl\pi h)\left\{\alpha_k[\cos(k\pi h)-1]+\frac{\beta_k}{k\pi}\sin(k\pi h)\right\}\left\{\alpha_l[\cos(l\pi h)-1]+\frac{\beta_l}{l\pi}\sin(l\pi h)\right\}.$$

Adding in j on the second term and taking into account the orthogonality property (3.5), it follows that

$$\sum_{j=0}^{N}(\phi_j^1-\phi_j^0)^2 = \frac{N+1}{2}\sum_{k=1}^{N}\alpha_k^2[\cos(k\pi h)-1]^2+(\frac{\beta_k}{k\pi})^2\sin^2(k\pi h)+2\alpha_k\frac{\beta_k}{k\pi}\sin(k\pi h)[\cos(k\pi h)-1]$$
$$= \frac{N+1}{2}\sum_{k=1}^{N}\left\{4(\alpha_k)^2\sin^4(\frac{k\pi h}{2})+4(\frac{\beta_k}{k\pi})^2\sin^2(\frac{k\pi h}{2})\cos^2(\frac{k\pi h}{2})-4\alpha_k\frac{\beta_k}{k\pi}\sin^2(\frac{k\pi h}{2})\sin(k\pi h)\right\}$$

$$= 2(N+1)\sum_{k=1}^{N}\sin^2(\frac{k\pi h}{2})\left[\alpha_k^2\sin^2(\frac{k\pi h}{2})+(\frac{\beta_k}{k\pi})^2\cos^2(\frac{k\pi h}{2})-2\alpha_k\frac{\beta_k}{k\pi}\sin(\frac{k\pi h}{2})\cos(\frac{k\pi h}{2})\right].$$

Finally, for the first term of the energy (2.1) we obtain the expression

$$\sum_{j=0}^{N}(\phi_j^1-\phi_j^0)^2 = 2(N+1)\sum_{k=1}^{N}\sin^2(\frac{k\pi h}{2})\left[\alpha_k\sin(\frac{k\pi h}{2})-\frac{\beta_k}{k\pi}\cos(\frac{k\pi h}{2})\right]^2. \tag{3.13}$$

For the second term entering in the energy (2.1) we have

$$\phi_{j+1}^1-\phi_j^1 = \sum_{k=1}^{N}\left\{[\sin((j+1)k\pi h)-\sin(jk\pi h)][\alpha_k\cos(k\pi h)+\frac{\beta_k}{k\pi}\sin(k\pi h)]\right\},$$

$$\phi_{j+1}^0-\phi_j^0 = \sum_{k=1}^{N}\alpha_k[\sin((j+1)k\pi h)-\sin(jk\pi h)].$$

Hence,

$$(\phi_{j+1}^1-\phi_j^1)(\phi_{j+1}^0-\phi_j^0) = \sum_{k=1}^{N}[\sin((j+1)k\pi h)-\sin(jk\pi h)]^2[\alpha_k^2\cos(k\pi h)+\alpha_k\frac{\beta_k}{k\pi}\sin(k\pi h)]$$
$$+\sum_{k,l=1;k\neq l}^{N}[\sin((j+1)k\pi h)-\sin(jk\pi h)][\sin((j+1)l\pi h)-\sin(jl\pi h)][\alpha_k\cos(k\pi h)+\frac{\beta_k}{k\pi}\sin(k\pi h)]\alpha_l.$$

Adding in j and using (3.6)

$$\sum_{j=0}^{N}(\phi_{j+1}^1-\phi_j^1)(\phi_{j+1}^0-\phi_j^0) = \sum_{k=1}^{N}\left\{\left[\alpha_k^2\cos(k\pi h)+\alpha_k\frac{\beta_k}{k\pi}\sin(k\pi h)\right]\right\}\left\{2\sin\frac{k\pi h}{2}\cos(\frac{(2j+1)k\pi h}{2})\right\}^2$$
$$= 4\sum_{j=0}^{N}\sin^2\frac{k\pi h}{2}\cos^2(\frac{(2j+1)k\pi h}{2}\alpha_k[\alpha_k\cos(k\pi h)+\frac{\beta_k}{k\pi}\sin(k\pi h)].$$

Using the identity

$$2\sum_{j=0}^{N}\cos^2\left(\frac{(2j+1)k\pi h}{2}\right) = \sum_{j=0}^{N}[\cos((2j+1)k\pi h)+1] = N+1,$$

we obtain

$$\sum_{j=0}^{N}(\phi_{j+1}^1-\phi_j^1)(\phi_{j+1}^0-\phi_j^0)=2(N+1)\sum_{k=1}^{N}\sin^2\frac{k\pi h}{2}\alpha_k[\alpha_k\cos(k\pi h)+\frac{\beta_k}{k\pi}\sin(k\pi h)]. \quad (3.14)$$

Adding the relations (3.13), (3.14) and taking into account the definition (2.1) we obtain

$$2hE_0^h=2(N+1)\sum_{k=1}^{N}\sin^2\frac{k\pi h}{2}\left\{\left[\alpha_k\sin\frac{k\pi h}{2}-\frac{\beta_k}{k\pi}\cos\frac{k\pi h}{2}\right]^2+\left[\alpha_k\frac{\beta_k}{k\pi}\sin(k\pi h)+\alpha_k^2\cos(k\pi h)\right]\right\}$$

$$=\frac{N+1}{2}\sum_{k=1}^{N}\sin^2(k\pi h)(\alpha_k^2+(\frac{\beta_k}{k\pi})^2).$$

This concludes the proof of (3.12).

□

The following Lemma gives a similar identity for the energy concentrated on the boundary appearing in the right hand side of the observability identity (3.2).

Lemma 2. *The following identity holds for all solutions ϕ of the adjoint system (1.15)*

$$h\sum_{n=0}^{M}\left|\frac{\phi_N^n}{h}\right|^2=\frac{M+1}{2h}\sum_{k=1}^{N}\sin^2(k\pi h)[\alpha_k^2+(\frac{\beta_k}{k\pi})^2], \quad (3.15)$$

with $M,\ N$ and h as in Theorem 1.

Proof of Lemma 2. We evaluate ϕ_N^n:

$$\phi_N^n=\sum_{k=1}^{N}[\alpha_k\cos(nk\pi h)+\frac{\beta_k}{k\pi}\sin(nk\pi h)]\sin(Nk\pi h).$$

Thus

$$|\ \phi_N^n\ |^2=\sum_{k=1}^{N}[\alpha_k\cos(nk\pi h)+\frac{\beta_k}{k\pi}\sin(nk\pi h)]^2\sin^2(Nk\pi h)+$$

$$+\sum_{k,l=1;k\neq l}^{N}[\alpha_k\cos(nk\pi h)+\frac{\beta_k}{k\pi}\sin(nk\pi h)][\alpha_l\cos(nl\pi h)+\frac{\beta_l}{l\pi}\sin(nl\pi h)]\sin(Nk\pi h)\sin(Nl\pi h).$$

Adding on n in the last relation, using (3.6), (3.9) and the hypothesis on M and N, ($M+1=2(N+1)$), we obtain:

$$\sum_{n=0}^{M}|\ \phi_N^n\ |^2=\sum_{n=0}^{M}\sum_{k=1}^{N}[\alpha_k^2\cos^2(nk\pi h)+(\frac{\beta_k}{k\pi})^2\sin^2(nk\pi h)+\alpha_k\frac{\beta_k}{k\pi}\sin(2nk\pi h)]\sin^2(Nk\pi h). \quad (3.16)$$

From (3.16) taking into account the orthogonality properties (3.7) and (3.8), identity (3.15) follows. □

Relations (3.12) and (3.15) from Lemma 1 and Lemma 2 imply (3.2) for $T = 2$. This concludes the proof of Theorem 1. □

4 Exact controllability

This section is devoted to analyze the exact controllability property of the discrete system (1.14).

Initial data $(\bar{u}^0, \bar{u}^1) \in (\mathbb{R}^N)^2$ are said to be *exactly controllable in time* $T = (M+1)h$ for the discrete system (1.14) if there exists a control function $\bar{v}_h \in \mathbb{R}^M$ such that the solution of (1.14) with those initial data satisfies

$$\bar{u}^M = \bar{u}^{M+1} = \bar{0}. \tag{4.1}$$

The following lemma provides a characterization of the property of exact controllability:

Lemma 3. *The initial data $(\bar{u}^0, \bar{u}^1)$ are exactly controllable if and only if there exists a function $\bar{v}_h \in \mathbb{R}^M$ such that:*

$$\sum_{n=1}^{M} v_h^n \phi_N^n = \sum_{j=1}^{N} [u_j^0 \phi_j^1 - u_j^1 \phi_j^0], \tag{4.2}$$

for every $(\bar{\phi}^0, \bar{\phi}^1)$ initial data of the adjoint system (1.15).

Proof. Multiplying the first equation of (1.14) by the solution $(\bar{\phi}^n)$ of the adjoint problem and adding in j and n, we deduce that

$$\sum_{n=1}^{M} v_h^n \phi_N^n + \sum_{j=0}^{N} [u_j^1 \phi_j^0 - u_j^0 \phi_j^1] = \sum_{j=0}^{N} [u_j^{M+1} \phi_j^M - u_j^M \phi_j^{M+1}]. \tag{4.3}$$

In view of this identity, it is immediate to see that the control v_h such that (4.1) holds is characterized by the property (4.2). □

Remark 4. *In the continuous case we have the following characterization of the property of exact controllability: the control* v *steers the initial data* (u^0, u^1) *of problem (1.1) to* $(0,0)$ *if and only if*

$$\int_0^T v(t)\phi_x(1,t)dt = \int_0^1 (u^1\phi^0 - u^0\phi^1)dx,$$

for every solution ϕ *of the homogeneous problem (1.4) with initial data* (ϕ^0, ϕ^1).

Let $(\bar{u}^0, \bar{u}^1) \in \mathbb{R}^N \times \mathbb{R}^N$ with the Fourier decomposition

$$\begin{cases} u_j^0 = \sum_{k=1}^N a_k^h \sin(jk\pi h), \quad j = 1, 2, ..., N, \\ \\ u_j^1 = \sum_{k=1}^N (a_k^h \cos(k\pi h) + \frac{b_k^h}{k\pi} \sin(k\pi h)) \sin(jk\pi h), \quad j = 1, 2, ..., N. \end{cases} \tag{4.4}$$

We define the following norm $||.||_*$

$$||(\bar{u}^0, \bar{u}^1)||_*^2 = \sum_{k=1}^N \left[(a_k^h)^2 + \left(\frac{b_k^h}{k\pi} \right)^2 \right]. \tag{4.5}$$

We can prove now the existence of a bounded sequence of HUM controls for the discrete problem (1.14).

Theorem 2. Let $T = 2$ and $h = \frac{1}{p}, \ p \in \mathbb{N}$. We set M and N as in (3.3). Then, system (1.14) is exactly controllable in time $T = 2$. Moreover, the controls of minimal norm are uniformly bounded and, more precisely,

$$h \sum_{n=0}^M |v_h^n|^2 \le ||(\bar{u}^0, \bar{u}^1)||_*^2 \tag{4.6}$$

for all $h = \frac{1}{p}, \ p \in \mathbb{N}$ and for all initial data $(\bar{u}^0, \bar{u}^1)$ of (1.14) given by (4.4), $\{v_h^n\}$ being the corresponding control of minimal norm.

Sketch of the proof.. Let us consider the coercive, continuous and convex functional $J_h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ given by

$$J_h(\bar{\phi}^0, \bar{\phi}^1) = \frac{h}{2} \sum_{n=0}^M \mid \frac{\phi_N^n}{h} \mid^2 - h \sum_{j=0}^N (u_j^0 \frac{\phi_j^1 - \phi_j^0}{h} - \frac{u_j^1 - u_j^0}{h} \phi_j^0), \tag{4.7}$$

where $(\bar{\phi}^n)$ is the solution of the adjoint problem (1.15) with initial data $(\bar{\phi}^0, \bar{\phi}^1) \in \mathbb{R}^{2N}$ and $(\bar{u}^0, \bar{u}^1) \in \mathbb{R}^{2N}$ are the initial data to be controlled.
If we denote by $(\hat{\phi}^0, \hat{\phi}^1)$ the minimizer of J_h, then

$$J_h(\hat{\phi}^0, \hat{\phi}^1) \leq J_h(0,0) = 0. \tag{4.8}$$

Let $(\hat{\phi}^n)$ be the solution of the adjoint problem (1.15) with initial data $(\hat{\phi}^0, \hat{\phi}^1)$ that, in view of (2.9), may be written as

$$\begin{cases} \hat{\phi}_j^0 = \sum_{k=1}^N \hat{\alpha}_k^h \sin(jk\pi h), \quad j = 1, 2, ..., N, \\ \\ \hat{\phi}_j^1 = \sum_{k=1}^N (\hat{\alpha}_k^h \cos(k\pi h) + \frac{\hat{\beta}_k^h}{k\pi} \sin(k\pi h)) \sin(jk\pi h), \quad j = 1, 2, ..., N. \end{cases} \tag{4.9}$$

Using (3.15), the first term in the definition (4.7) of $J_h(\hat{\phi}^0, \hat{\phi}^1)$ is

$$\frac{1}{2h} \sum_{n=0}^{M} \mid \hat{\phi}_N^n \mid^2 = \frac{M+1}{4h} \sum_{k=1}^{N} \sin^2(k\pi h) \left[(\hat{\alpha}_h^h)^2 + (\frac{\hat{\beta}_k^h}{k\pi})^2 \right]. \tag{4.10}$$

In view of (4.4) and (4.9) we obtain the following expression for the second term of $J_h(\hat{\phi}^0, \hat{\phi}^1)$

$$\sum_{j=0}^{N} (u_j^0 \ \hat{\phi}_j^1 - u_j^1 \ \hat{\phi}_j^0) = \frac{N+1}{2} \sum_{k=1}^{N} \frac{\sin(k\pi h)}{k\pi} [a_k^h \ \hat{\beta}_k^h - \hat{\alpha}_k^h \ b_k^h]. \tag{4.11}$$

Applying Cauchy-Schwarz inequality and taking into account that $N+1 = 1/h$, we have

$$\mid \sum_{j=0}^{N} (u_j^0 \ \hat{\phi}_j^1 - u_j^1 \ \hat{\phi}_j^0) \mid \leq \left[\sum_{k=1}^{N} [(a_k^h)^2 + (\frac{b_k^h}{k\pi})^2] \right]^{\frac{1}{2}} \left[\frac{N+1}{4h} \sum_{k=1}^{N} \sin^2(k\pi h) [(\hat{\alpha}_k^h)^2 + (\frac{\hat{\beta}_k^h}{k\pi})^2] \right]^{\frac{1}{2}}.$$

Using now Lemma 1 and the observability identity (3.2), we deduce that

$$\mid \sum_{j=0}^{N} (u_j^0 \ \hat{\phi}_j^1 - u_j^1 \ \hat{\phi}_j^0) \mid \leq \parallel (\bar{u}^0, \bar{u}^1) \parallel_*^{\frac{1}{2}} \ (E_0^h)^{\frac{1}{2}} = \parallel (\bar{u}^0, \bar{u}^1) \parallel_*^{\frac{1}{2}} \ \left[\frac{1}{4} h \sum_{n=0}^{M} \mid \frac{\hat{\phi}_N^n}{h} \mid^2 \right]^{\frac{1}{2}},$$

where E_0^h is the energy (2.1) of the solution $(\hat{\phi}^n)$ with initial data $(\hat{\phi}^0, \hat{\phi}^1)$ and $\| \cdot \|_*$ is the norm defined by (4.5).

This fact together with (4.8) implies

$$\left[h \sum_{n=0}^{M} \mid \frac{\hat{\phi}_N^n}{h} \mid^2 \right]^{\frac{1}{2}} \leq \parallel (\bar{u}^0, \bar{u}^1) \parallel_* . \tag{4.12}$$

Moreover, it is easy to see that the control $v_h^n = \hat{\phi}_N^n/h$ satisfies (4.2). Indeed, using the fact that the gradient of J_h vanishes at the minimizer, it is immediate to see that (4.12) holds for this control.

This concludes the proof of Theorem 2. ☐

Remark 5. *Note, that with the notations used in the proof of Theorem 2, the controls (v_h^n) corresponding to a fixed value of h are given by:*

$$v_h^n = -\frac{1}{h}\sum_{k=1}^{N}\cos(k\pi)\sin(k\pi h)[\hat{\alpha}_h^k\cos(nk\pi h) + \frac{\hat{\beta}_k^h}{k\pi}\sin(nk\pi h)]. \tag{4.13}$$

These controls are of minimal norm since they are obtained minimizing a functional of the form (4.7).

5 Convergence of the controls

The aim of this section is to obtain controls for the continuous system (1.1) as limits as $h \to 0$ $(\Delta t = \Delta x = h)$ of sequences of controls for the corresponding discrete problems (1.14). We proceed as follows.

Given an initial state $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ of the continuous system (1.1), we develop it in Fourier series

$$(u^0, u^1) = \sum_{k=1}^{\infty}(c_k, d_k)\varphi_k(x), \tag{5.1}$$

with

$$\sum_{k\in\mathbb{N}}\left[\mid c_k\mid^2 + \mid \frac{d_k}{k\pi}\mid^2\right] < \infty. \tag{5.2}$$

Further, we construct the sequence of discrete initial states:

$$(\bar{u}_h^0, \bar{u}_h^1) = \sum_{k=1}^{N}(c_k, c_k\cos(k\pi h) + \frac{d_k}{k\pi}\sin(k\pi h))\bar{\varphi}_k, \tag{5.3}$$

with $h = 1/(N+1)$, $N \in \mathbb{N}$.

In view of Theorem 2, for every $h \in (0,1)$ there exists a HUM control (v_h^n) for discrete system (1.14) with initial data (5.3). Then, we prove in Theorem 3 that the sequence (v_h^n) converges (in a sense defined below) to a function $v \in L^2(0,T)$ which is the HUM control for system (1.1) with initial data (5.1).

Let us define the Hilbert spaces of square summable sequences ℓ^2 and $\hbar^{-1}$ as follows

$$\ell^2 = \left\{ \{a_k\} \ : \ \| a_k \|_{\ell^2}^2 = \sum_{k\in\mathbb{N}} | a_k |^2 < \infty \right\}, \tag{5.4}$$

$$\hbar^{-1} = \left\{ \{a_k\} \in \ell^2 \ : \ \| a_k \|_{\hbar^{-1}}^2 = \sum_{k\in\mathbb{N}} | \frac{a_k}{k\pi} |^2 < \infty \right\}. \tag{5.5}$$

The initial data (5.3) may be written as

$$(\bar{u}_h^0, \bar{u}_h^1) = \sum_{k=1}^{\infty} (c_k^N, c_k^N \cos(k\pi h) + \frac{d_k^N}{k\pi} \sin(k\pi h)) \bar{\varphi}_k, \tag{5.6}$$

(recall that $\bar{\varphi}_k$ are the eigenvectors of $A_{\Delta x}$ defined by (1.11)) where

$$c_k^N = c_k \chi_N(k), \qquad d_k^N = d_k \chi_N(k),$$

χ_N being the characteristic function of the set $\{1, ..., N\}$.

In view of (5.6) and the definition of the norm $\| \cdot \|_*$, we have

$$\| (\bar{u}_h^0, \bar{u}_h^1) \|_* = \| (\{c_k^N\}, \ \{d_k^N\}) \|_{\ell^2 \times \hbar^{-1}}, \tag{5.7}$$

which is uniformly bounded as $h \to 0$ (i.e., $N \to \infty$) because of (5.2).

As the solution u of (1.1) with initial data $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ and control $v \in L^2(0,T)$ satisfies $u \in \mathcal{C}([0,T]; L^2(0,1)) \cap \mathcal{C}^1([0,T]; H^{-1}(0,1))$, then, there exist functions $c_k(t)$, $k \in \mathbb{N}$ such that

$$u(x,t) = \sum_{k\in\mathbb{N}} c_k(t) \varphi_k(x), \tag{5.8}$$

with

$$\sup_{t\in[0,T]} [\| c_k(t) \|_{\ell^2} + \| c_k'(t) \|_{\hbar^{-1}}] < \infty. \tag{5.9}$$

The control $v \in L^2(0,T)$ of minimal L^2-norm (HUM control) verifies

$$v(t) = -\partial_x \hat{\phi}(1,t), \tag{5.10}$$

where $\hat{\phi}$ is the solution of the adjoint problem (1.4) with initial data $(\hat{\phi}^0, \hat{\phi}^1) \in H_0^1(0,1) \times L^2(0,1)$, the unique minimizer of the functional

$$J(\phi^0, \phi^1) = \frac{1}{2} \int_0^T | \partial_x \phi(1,t) |^2 \, dt - \int_0^1 u^0 \phi^1 - < u^1, \phi^0 >_{-1,1}$$

in that space $H_0^1(0,1) \times L^2(0,1)$.

Further, let us consider the family $(\bar{u}_h^n)$ of solutions of (1.14) with initial data $(\bar{u}_h^0, \bar{u}_h^1)$ as in (5.3) and HUM controls (v_h^n) given by Theorem 2. In view of Remark 5 it follows that the solution $(\bar{u}_h^n)$ which is obtained solving (1.15) may be written as

$$\bar{u}_h^n = \sum_{k=1}^{N} c_{k,h}^n \bar{\varphi}_k, \tag{5.11}$$

with $c_{k,h}^n \in \mathbb{R}$.

For every $n = 0, ..., M+1$, we extend the sequence $(c_{k,h}^n)_{k=1}^N$ to a sequence $(c_{k,h}^n)_{k\in\mathbb{N}}$ by puting $c_{k,h}^n = 0$ for $k > N$ and define

$$d_{k,h}^n = \frac{c_{k,h}^n - c_{k,h}^{n-1}}{h}.$$

Then, the solution of the discrete system (1.14) may be written as follows

$$\bar{u}_h^n = \sum_{k\in\mathbb{N}} c_{k,h}^n \bar{\varphi}_k. \tag{5.12}$$

Let $c_{k,h},\ d_{k,h} : [0,T] \to \mathbb{R}$ be continuous functions such that:

$$c_{k,h}(t_n) = c_{k,h}^n, \quad d_{k,h}(t_n) = d_{k,h}^n,$$

i. e., they coincide with the discrete coefficients at the points of the time-mesh, and

$$\sup_{t\in[0,T]} [\| c_{k,h}(t) \|_{\ell^2} + \| d_{k,h}(t) \|_{\hbar^{-1}}] < \infty. \tag{5.13}$$

This is true, for instance, when $c_{k,h},\ d_{k,h}$ are first order splines corresponding to the partition $\{0 = t_0 < t_1 = h < ... < t_M = Mh < t_{M+1} = T\}$.

Besides, we introduce the function

$$v_h(t) = -\frac{1}{h} \sum_{k\in\mathbb{N}} \cos(k\pi)\sin(k\pi h)[\hat{\alpha}_k^h \cos(k\pi t) + \frac{\hat{\beta}_k^h}{k\pi}\sin(k\pi t)]$$

where $\hat{\alpha}_k^h = \hat{\beta}_k^h$ are taken to be zero for $k > N+1$. This function, when restricted to the mesh, coincides with (v_h^n) (recall that v_h^n is given by (4.13)).

The following convergence result holds:

Theorem 3. Let $\bar{u}_h^n$, u be as above with $\Delta t = \Delta x = h$. Consider M, N and h as in Theorem 1. Then,

$$\{c_{k,h}(\cdot)\}_{k\in\mathbb{N}} \to \{c_k(\cdot)\}_{k\in\mathbb{N}} \quad strongly \quad \text{in} \quad L^\infty(0,T;\ell^2), \quad \text{as } h \to 0, \tag{5.14}$$

$$\{c'_{k,h}(\cdot)\}_{k\in\mathbb{N}} \to \{c'_k(\cdot)\}_{k\in\mathbb{N}} \quad strongly \quad \text{in} \quad L^\infty(0,T;\hbar^{-1}), \quad \text{as } h \to 0. \tag{5.15}$$

Moreover

$$v_h(\cdot) \to v(\cdot) \;\; strongly \text{ in } \;\; L^2(0,T) \;\; \text{as} \;\; h \to 0, \tag{5.16}$$

where v is the HUM control (5.10) for the continuous wave equation (1.1) with initial data (5.1).

Remark 6. *a) This result guarantees that the control of minimal norm for the wave equation (1.1) may be obtained as limit when $h \to 0$ of the controls of the discrete system (1.14) when $\Delta t = \Delta x = h$, provided the initial data of the discrete system are taken conveniently. As a consequence of this result, one deduces that, if the control v_h of the discrete model (1.14) is employed to control the continuous one (1.1), one gets solutions of the wave equation (1.1) whose traces at $n = 0$ tend to zero. This confirms that computing the control v_h of the discrete model (1.14) is an efficient way of finding controls for the wave equation. The efficiency of the method has been tested and confirmed in the numerical simulations, which we include in the following section.*

b) Concerning the choice (5.3) of the initial data of the discrete problem (1.14), we have taken

$$\bar{u}_h^1 = \sum_{k=1}^{N} [c_k \cos(k\pi h) + \frac{d_k}{k\pi} \sin(k\pi h)] \bar{\varphi}_k \tag{5.17}$$

instead of

$$\bar{u}_h^1 = \sum_{k=1}^{N} [c_k + h d_k] \bar{\varphi}_k, \tag{5.18}$$

as initial data for the discrete problem. Note that the latter would be a simpler and more natural approximation according to Taylor's expansion and the initial data of the continuous problem.

Note however that, for every fixed k,

$$c_k \cos(k\pi h) + \frac{d_k}{k\pi} \sin(k\pi h) \sim c_k + h d_k, \quad \text{as} \quad h \to 0.$$

Thus (5.17) may be viewed as an approximation of (5.18).

Moreover, the choice (5.17) guarantees that the sequence $(\bar{u}_h^0, \bar{u}_h^1)$ remains uniformly bounded in the norm $\|\cdot\|_*$ as $h \to 0$ and so does the sequence of controls (v_h^n).

c) The choice $\Delta x = \Delta t$ as space and time steps is not a technical matter. In fact, when $\Delta t/\Delta x < 1$ some filtering of the high frequencies is needed to pass to the limit as in the semi-discrete case. This issue will be studied elsewhere.

6 Numerical simulations

In this section we test numerically the efficiency of using the controls of the discrete model (1.14) to control the wave equation (1.1).

We first observe that, the solution of the wave equation (1.1) may be computed explicitly by means of D'Alembert formula. Thus, the efficiency of a control v may be tested by visualizing and checking how far $u(T)$ and $u_t(T)$ are from $(0,0)$. As we shall see, when using the control v_h of (1.14) in (1.1) the solution u of the wave equation (1.1) becomes completely flat around $u = 0$, when approaching $T = 2$, thus confirming the results of our theoretical study.

Let us now explain how the numerical simulations were developed. As in the previous analysis, $T = 2$ and $\Delta t = \Delta x$.

The efficiency of the numerical control on the continuous system is related to the fact that the solutions of the 1-d wave equation solve also the discrete equation (3.1) when $\Delta t = \Delta x$. Thus, the only source of divergence between the solutions of (1.1) and (1.14) along the mesh is that the initial data in (1.14) are an approximation of those in (1.1).

Using MatLab we apply the *conjugate gradient* algorithm to find the minimizer of the functional (4.7) over the set of the solutions of (1.15). The discrete control may be obtained as the discrete normal derivative of the corresponding solution of the adjoint system (1.15).

In Figures 1 and 2 we present two examples corresponding to $N = 70$ (69 interior points) and $N = 46$ (45 interior points), respectively. In each example the first two figures describe the initial data u_0 and u_1 to be controlled along the mesh. The third one presents the solution of the continuous wave equation with the discrete control. This figure confirms the efficiency of the discrete control in the continuous wave model (1.1). In the last figure we draw the graphic of the discrete control.

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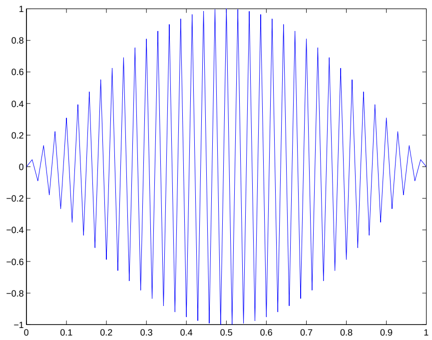
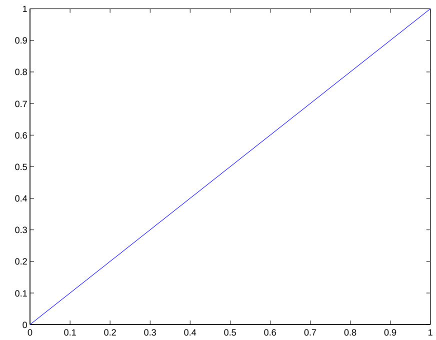
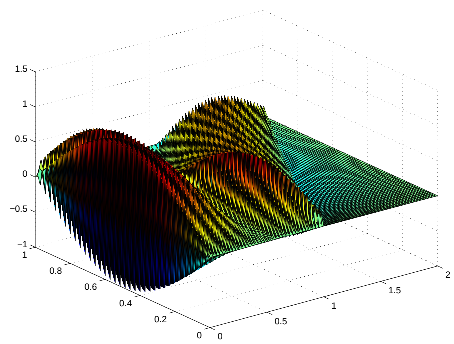


1.1. Initial position

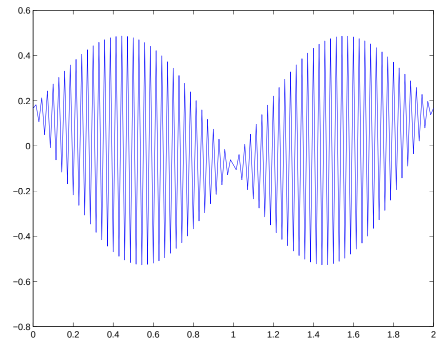


1.2. Initial velocity



1.3. Exact solution

with numerical control



1.4. Numerical control

Fig. 1.

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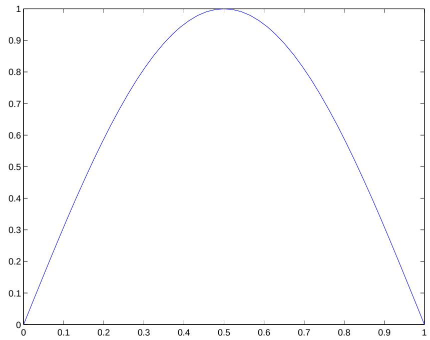
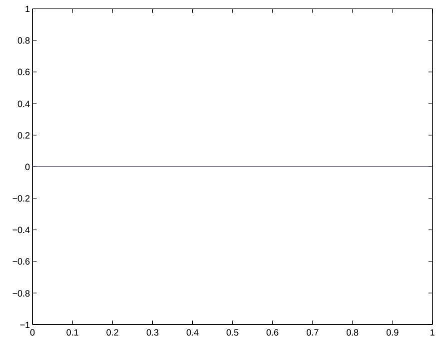
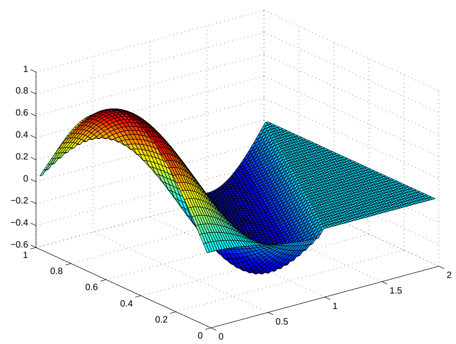


2.1. Initial position

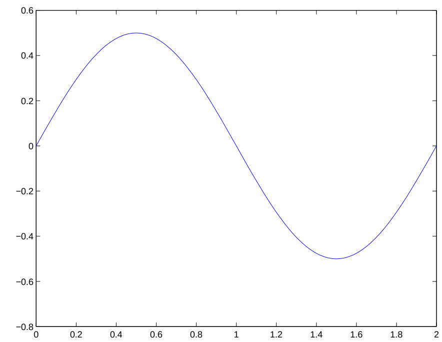


2.2. Initial velocity



2.3. Exact solution

with numerical control



2.4. Numerical control

Fig. 2.

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