DECAY RATES FOR THE THREE-DIMENSIONAL LINEAR SYSTEM OF THERMOELASTICITY

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Abstract

We consider the two and three-dimensional system of linear thermoelasticity in a bounded smooth domain with Dirichlet boundary conditions. We analyze whether the energy of solutions decays exponentially uniformly to zero as $t \to \infty$. First of all, by a decoupling method, we reduce the problem to an observability inequality for the Lamé system in linear elasticity and more precisely to whether the total energy of the solutions can be estimated in terms of the energy concentrated on its longitudinal component. We show that when the domain is convex the decay rate is never uniform. In fact, the lack of uniform decay holds in a more general class of domains in which there exist rays of geometric optics of arbitrarily large length that are always reflected perpendicularly or almost tangentially on the boundary. We also show that, in three space dimensions, the lack of uniform decay may be also due to a critical polarization of the energy on the transversal component of the displacement. In two space dimensions we prove a sufficient (and almost necessary) condition for the uniform decay to hold in terms of the propagation of the transversal characteristic rays, under the further assumption that the boundary of the domain does not have contacts of infinite order with its tangents. We also give an example, due to D. Hulin, in which these geometric properties hold. In three space dimensions we indicate (without proof) how a careful analysis of the polarization of singularities may led to sharp sufficient conditions for the uniform decay to hold. In two space dimensions we prove that smooth solutions decay polynomially in the energy space to a finite-dimensional subspace of solutions except when the domain is a ball or an annulus. Finally we discuss some closely related controllability and spectral issues.

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Contents

1 Introduction: the non-uniform decay

Let Ω be a bounded smooth domain of $\mathbb{R}^n$ with $n = 2$ or 3 and consider the linear system of thermoelasticity with Dirichlet boundary conditions:

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\nabla\theta = 0 & \text{in} \quad \Omega\times(0,\infty) \\ \theta_t - \Delta\theta + \beta \operatorname{div} u_t = 0 & \text{in} \quad \Omega\times(0,\infty) \\ u = 0, \theta = 0 & \text{on} \quad \partial\Omega\times(0,\infty) \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x), \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

In (1), $\lambda, \mu > 0$ are the Lamé coefficients and $\alpha, \beta > 0$ the coupling parameters. The displacement $u = (u_1, u_2, u_3)$, $u_j = u_j(x,t)$, $x = (x_1, x_2, x_3) \in \Omega$, $t > 0$ is a vector field ($u = (u_1, u_2), u_j = u_j(x,t), x = (x_1, x_2) \in \Omega$ when $n = 2$) and $\theta = \theta(x,t)$, the temperature, is a scalar function.

System (1) is well-posed in $H = \left(H_0^1(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n \times L^2(\Omega)$ and the energy

$$E(t) = \frac{1}{2}\int_\Omega \left[\mu \mid \nabla u(x,t) \mid^2 + (\lambda+\mu) \mid \operatorname{div} u(x,t) \mid^2 + \mid u_t(x,t) \mid^2 + \frac{\alpha}{\beta} \mid \theta(x,t) \mid^2\right] dx \tag{2}$$

decreases along trajectories. More precisely,

$$\frac{dE(t)}{dt} = -\frac{\alpha}{\beta}\int_{\Omega} \mid \nabla\theta(x,t) \mid^2 dx \leq 0. \tag{3}$$

C. Dafermos in [D] studied the problem of whether the energy of every solution of (1) converges to zero as $t \to \infty$, i.e.

$$E(t) \to 0, \quad \text{as} \quad t \to \infty \tag{4}$$

which is equivalent to the convergence of solutions to zero in H. He proved that (4) holds if and only if the domain Ω satisfies the following condition:

$$(C)\left\{\begin{array}{l} \text{If } \varphi \in \left(H_0^1(\Omega)\right)^n \text{ is such that} \\ \qquad \left\{\begin{array}{lll} -\Delta\varphi = \gamma^2\varphi & \text{in} & \Omega \\ \operatorname{div}\varphi = 0 & \text{in} & \Omega \\ \varphi = 0 & \text{on} & \partial\Omega \end{array}\right. \\ \text{for some } \gamma \in \mathbb{R}, \text{ then } \varphi \equiv 0. \end{array}\right. \tag{5}$$

Condition (C) guarantees that the Lamé system has no eigenfunction with null divergence. One can also interpret (C) in terms of Stokes system. Indeed, (C) guarantees that there is no eigenfunction of the Stokes system with constant pressure (we refer to [LiZ] for an application of this property to the approximate controllability of Stokes system).

Condition (C) holds generically with respect to the domain Ω. Indeed, (C) holds as soon as the eigenfunctions of the Dirichlet Laplacian are simple and this is known to be a generic property among smooth domains (see [A], [M] and [U]). On the other hand, condition (C) fails when Ω is a ball of $\mathbb{R}^n$. In this case there exists an infinite number of linearly independent vector fields φ satisfying (5) and for each of them $(u,\theta) = (e^{i\gamma t}\varphi(x), 0)$, with φ solution of (5), is a solution of (1) of constant energy. In two space dimensions C. A. Berenstein [B] proved that the only simply connected $C^{2,\alpha}$ domain in which (1.5) holds for an infinite number of linearly independent eigenfunctions is the ball.

Therefore, roughly speaking, one can say that, generically with respect to Ω, every solution of (1) tends to zero in H as $t \to \infty$.

We are interested on whether the decay rate is uniform or not. In other words, we analyze the existence of positive constants $C, \omega > 0$ such that

$$E(t) \leq Ce^{-\omega t}E(0), \ \forall t > 0 \tag{6}$$

holds for every solution of (1).

It is easy to check that (6) holds if and only if there exists a time $T > 0$ and a constant $C > 0$ such that

$$E(0) \leq C\int_0^T \int_{\Omega} \mid \nabla\theta \mid^2 dxdt, \tag{7}$$

for every solution of (1).

Therefore, in order to prove the uniform decay one has to show that the total energy of solutions can be estimated in terms of the energy concentrated on the heat component.

By using the decoupling method introduced by D. Henry, O. Lopes and A. Perissinotto in [HLP] we reduce this problem to the analysis of the Lamé system:

$$\left\{\begin{array}{lll} \varphi_{tt} - \mu\Delta\varphi - (\lambda+\mu)\nabla\operatorname{div}\varphi = 0 & \text{in} & \Omega\times(0,T) \\ \varphi = 0 & \text{on} & \partial\Omega\times(0,T) \\ \varphi(x,0) = \varphi^0(x), \varphi_t(x,0) = \varphi^1(x) & \text{in} & \Omega. \end{array}\right. \tag{8}$$

More precisely, our first result reads as follows:

Theorem 1.1 *Assume that $n = 2$ or 3. In the class of domains Ω satisfying condition* (C), *the uniform decay property* (6) *for system* (1) *holds if and only if there exists $T > 0$ and $C > 0$ such that*

$$\| \varphi^0 \|^2_{(L^2(\Omega))^n} + \| \varphi^1 \|^2_{(H^{-1}(\Omega))^n} \leq C \int_0^T \| \operatorname{div} \varphi \|^2_{H^{-1}(\Omega)} \, dt \tag{9}$$

holds for every solution of the Lamé system (8).

Therefore the problem is reduced to analyze whether the total energy of the solutions φ of the Lamé system (8) can be estimated uniformly in terms of the energy concentrated on the component $\operatorname{div} \varphi$.

This allows to prove that, for a large class of domains, the decay rate is not uniform:

Theorem 1.2 *Assume that Ω is convex or such that there exists a ray of geometric optics in Ω of arbitrarily large length which is always reflected perpendicularly on the boundary. Then, the observability inequality* (9) *for the Lamé system fails for any $T > 0$ and therefore the decay of solutions of* (1) *is not uniform.*

Remark 1.1

1. In the next section we give a sharper sufficient condition (see (H_T) in (16)) that guarantees that the decay rate is not uniform.

2. In three space dimensions the uniform decay may also fail because of the critical polarization along rays. This will be discussed in detail in section 5.3.

3. When Ω is convex the observability inequality (9) for the Lamé system fails for two different reasons:

   (a) Because of the existence of solutions of (8) exponentially concentrated on an arbitrarily small neighborhood of the boundary for an arbitrarily large time and such that the energy concentrated on the longitudinal component $\operatorname{div} \varphi$ of the solution is arbitrarily small compared to the total energy.

   (b) In every convex smooth domain there exists a ray of infinite length which is always reflected perpendicularly on the boundary (see Figure 1):

Figure 1.

In this case the observability property (9) fails since there are solutions whose energy is concentrated along this ray on the transversal component of the solution, $\mathrm{curl}\,\varphi$, and this for an arbitrarily large time interval.

In both cases the proof of the existence of these solutions requires a geometric optics construction in the spirit of Ralston [R1,2].

4. In view of Theorem 1, convex domains may be classified in two sets:

   (a) Those in which property (C) fails. In this case there are solutions that do not decay as $t \to \infty$.

   (b) Those in which (C) holds. In this case every solution converges to zero but without an uniform decay rate.

   Of course, convex domains are generically in class (b). To our knowledge it is unknown whether the class (a) contains any convex domain other than the ball.

5. As we have stated in Theorem 2, the uniform decay fails in many non-convex domains with the property that there exists a ray of infinite length which is always reflected perpendicularly on the boundary (see Figure 2):

Figure 2.

6. There are domains in which the two geometric properties of convex domains indicated in point 1 above do not hold. The following two-dimensional example is due to D. Hulin [Hu]:

Figure 3.

This domain Ω can be obtained by removing from the half ball of radious one centered at the origin $\{(x_1, x_2) : x_2 > 0, x_1^2 + x_2^2 < 1\}$ the ball of radious $1/3$ centered at $(-2/3, 0)$ and the ball of radious $2/3$ centered at $(1/3, 0)$ and then regularizing the three vertices $(-1, 0)$, $(-1/3, 0)$, $(1, 0)$.

In this case one can check that for any T sufficiently large every ray of geometric optics of length T intersects the boundary at some point with an angle γ between two constants $0 < c_1 < c_2 < \pi/2$ (i.e. $0 < c_1 \leq \gamma \leq c_2 < \pi/2$) which are independent of the ray.

7. We recall that when the energy of every solution converges to zero but there is no uniform decay rate, then the decay may be arbitrarily slow. More precisely, in those cases, for every continuous function $h : [0, \infty) \to [0, \infty)$ such that $h(t) \to 0$ as $t \to \infty$ there exists an initial data $(u^0, u^1, \theta^0) \in H$ and a sequence of times $t_n \to \infty$ such that the energy of the corresponding solution of (1) satisfies $E(t_n) \geq h(t_n)$ for all $n \geq 1$.

■

We shall show however that for most two-dimensional domains for which the energy does not decay uniformly, smooth solutions decay polynomially. This will be discussed in section 7.2. As we shall see, the polynomial decay corresponds to a weaker version of the inequality (1.9), namely,

$$\| \varphi^0 \|^2_{(L^2(\Omega))^n} + \| \varphi^1 \|^2_{(H^{-1}(\Omega))^n} \leq C \int_0^T \| \operatorname{div} \varphi \|^2_{L^2(\Omega)} \, dt. \tag{10}$$

Observe that in the right hand side of (1.10) we use an extra derivative of $\operatorname{div} \varphi$ than in (1.9). The main difficulty for proving the polynomial decay for smooth solutions of (1.1) will be precisely to show that (1.10) holds for most two-dimensional domains.

In order to state the main results that provide sufficient conditions for the uniform and polynomial decay to hold, some notation has to be introduced. This will be done in the following section.

2 Main results on the uniform decay

2.1 Preliminaries

In order to state the sufficient condition for the uniform and polynomial decay of (1) we have to introduce some notation that is valid in dimensions $n = 2$ and $n = 3$.

We assume that Ω is of class C^∞ without contacts of infinite order with its boundary.

We denote by $c_L = \sqrt{\lambda + 2\mu}$ and $c_T = \sqrt{\mu}$ the longitudinal and transversal velocities of propagation. We also set $\nu_L = 1/c_L$ and $\nu_T = 1/c_T$. Considering these two different velocities of propagation is motivated by the fact that, when φ solves the Lamé system (8), its transversal component $w = \operatorname{curl} \varphi$ satisfies

$$w_{tt} - \mu \Delta w = 0 \ \text{ in } \ \Omega \times (0, \infty) \tag{11}$$

and the longitudinal one, $\rho = \operatorname{div} \varphi$, solves

$$\rho_{tt} - (\lambda + 2\mu) \Delta \rho = 0 \ \text{ in } \ \Omega \times (0, \infty). \tag{12}$$

Obviously both equations are coupled on the boundary $\partial\Omega$.

We introduce now the transversal characteristic manifold $\operatorname{Char} \mathcal{T} = (\operatorname{Char} \mathcal{T})_\Omega \cup (\operatorname{Char} \mathcal{T})_{\partial\Omega}$ where

$$(\operatorname{Char} \mathcal{T})_\Omega = \Big\{ (x, t, \xi, \tau) : x \in \Omega, \, t > 0, \, \xi^2 = \tau^2 \nu_T^2, \, \tau \neq 0 \Big\} \tag{13}$$

$$(\operatorname{Char} \mathcal{T})_{\partial\Omega} = \Big\{ (y, t, \eta, \tau) : y \in \partial\Omega, \, t > 0, \, \eta^2 \leq \tau^2 \nu_T^2, \, \tau \neq 0 \Big\} . \tag{14}$$

Char $\mathcal{T}$ is endowed with a generalized hamiltonian flow. For any $\rho_0 \in$ Char $\mathcal{T}$ there exists an unique generalized bicharacteristic $s \in I\!\!R \to \rho(s)$ such that $\rho(0) = \rho_0$. We have $\tau(\rho(s)) = \tau_0 =$ const., $t(\rho(s)) = t(\rho(0)) - 2\tau_0 s\nu_T^2$ and $(x(\rho(s)), \xi(\rho(s)))$ is a generalized geodesic curve in $\bar{\Omega}$. In the interior of Ω, since we are dealing with constant coefficients, $s \to x(\rho(s))$ is a stright segment and it is reflected on the boundary following Descartes' laws when $s \to x(\rho(s))$ intersects transversally $\partial\Omega$. Thus a transversal ray is a continuous parametrized curve $s \to \gamma(s) = (t(s), x(s), \tau(s), \xi(s))$ with values in the transversal characteristic manifold Char$(\mathcal{T})$. We have $x(s) \in \bar{\Omega}, \xi(s) \in I\!\!R^n$ and $c_T \mid \xi(s) \mid = \mid \tau(s) \mid$ while $\tau(s)$ is independent of s and $\tau \neq 0$. When $x(s) = y \in \partial\Omega$ then $\eta(s)$, the tangential component of $\xi(s)$, is well defined and the normal component may take one of the following two values $\xi^{\perp}(s) = \pm\sqrt{\tau^2(s) - c_T^2 \mid \eta(s) \mid^2}$ when $\mid \eta(s) \mid < \nu_T \mid \tau(s) \mid$. Obviously $\xi(s) = 0$ when $\mid \eta(s) \mid = \nu_T \mid \tau(s) \mid$. In this limit case the ray is tangent to the boundary of Ω.

Under the assumption that the domain has not contacts of infinite order with its tangents all rays have the following structure (see L. Hörmander [H], section 24.3). It is constituted by segments in Ω such that

$$\dot{x}(s) = 2\xi c_T\,;\; \dot{t}(s) = -2\tau \tag{15}$$

ξ and τ being constant, that intersect the boundary $\partial\Omega \times I\!\!R$ in one of the following two ways:

- Transversally, i.e. $x(s_i) \in \partial\Omega$, $\mid \eta(s_i) \mid < \nu_T \mid \tau(s_i) \mid$. Those points are isolated, i.e. they do not have accumulation points. At these points $\lim_{s \nearrow s_i} \xi(s) = \big(\eta(s_i), \xi_-^{\perp}(s_i)\big)$, $\lim_{s \searrow s_i} \xi(s) = \big(\eta(s_i), \xi_+^{\perp}(s_i)\big)$ with $\xi_-^{\perp}(s_i) = -\xi_+^{\perp}(s_i)$ and $\xi_+^{\perp}(s_i) \cdot \overrightarrow{n} > 0$, where $\overrightarrow{n}$ denotes the unit inner normal vector to Ω.
- Tangentially at a diffractive point, i.e. $x(s_i) \in \partial\Omega$ and for $\varepsilon > 0$ small enough $x(s) \in \Omega$, for all $0 < \mid s - s_i \mid < \varepsilon$. Moreover, $\xi^{\perp}(s_i) = 0$ and $c_T \mid \eta(s_i) \mid = \tau$.

These segments may be connected to arcs of curves $\tilde{\gamma}(s) = (t(s), x(s), \tau(s), \xi(s))\,, s \in [a,b]$ such that $x(s) \in \partial\Omega, \xi^{\perp}(s) = 0, \tau(s)$ is constant $(\dot{\tau} = 0), \dot{t}(s) = -2\tau$, $\tau^2 = c_T \mid \eta(s) \mid^2$, $\dot{x}(s) = 2c_T\eta(s)$ and $D\eta(s) = 0$, where D denotes the covariant derivative over $\partial\Omega$. In other words $s \in [a,b] \to x(s) \in \partial\Omega$ is a geodesic curve such that $\dot{x}(s) = 2c_T\eta(s)$.

Let us also recall that the characteristic manifold of the Lamé system is the union of the transversal and longitudinal one. Thus Char $=$ Char$_\Omega \cup$ Char$_{\partial\Omega}$ where

$$\text{Char}_\Omega = \{(x,\xi,t,\tau) : x \in \Omega, \mid \tau \mid = c_L \mid \xi \mid \;\text{ or }\; \mid \tau \mid = c_T \mid \xi \mid\}\,;$$

$$\text{Char}_{\partial\Omega} = \{(y,t,\eta,\tau) : c_T \mid \eta \mid \leq \mid \tau \mid \;\text{ and } y \in \partial\Omega\}\,.$$

In the sequel the generalized bicharacteristics will be referred to as "rays".

The following two basic results on rays will be used (see L. Hörmander [Ho], section 24.3, p. 441):

(a) The uniform limit of rays is a ray;

(b) Every ray $s \in [a,b] \to \rho(s)$ is the limit of rays having only transversal intersections with $\partial\Omega$.

Note that, at this level, we use the fact that Ω has not contacts with infinite order with its boundary.

Let us introduce the set:

$$\mathcal{L} = \{(y,t,\eta,\tau)\;:\; y \in \partial\Omega\,,\, 0 < \mid \eta \mid \leq \mid \tau \mid \nu_L\}\,.$$

It represents the subset of the boundary that "couples strongly" the longitudinal and transversal waves.

In the exterior of $\mathcal{L}$ the following two subregions have to be distinguished:

(i) $\eta = 0$ corresponds to rays that intersect $\partial\Omega$ perpendicularly. At these points, a transversal wave, by reflection, generates a more smooth longitudinal wave.

(ii) $|\tau| \nu_L < |\eta| \leq |\tau| \nu_T$. In this region the transversal wave does not generate longitudinal waves by reflection.

When $|\eta| = |\tau| \nu_L$ the angle is critical.
These situations are illustrated in the following figures:

$0 < |\eta| \leq |\tau| \nu_L$
Figure 4

$|\tau| \nu_L < |\eta| < |\tau| \nu_T$
Figure 5

Observe that we do not discuss the case $|\eta| > |\tau| \nu_T$. Indeed we will never enter the elliptic region for the transversal waves in our arguments below since singularities may not concentrate in that region.

The situation we have considered in section 1 above corresponds to the case in which $T > 0$ and Ω are such that the following assumption holds:

$$(H_T) \left\{ \begin{array}{l} \text{There exists a ray } s \in [a,b] \to \rho(s) \in \text{Char}\,\mathcal{T} \text{ without contacts of infinite} \\ \text{order with } \partial\Omega \text{ such that} \\ |t\,(\rho(b)) - t\,(\rho(a))| > T \text{ and } \rho(s) \notin \mathcal{L},\ \forall s \in [a,b]. \end{array} \right. \tag{16}$$

Indeed, as we will see in section 5, under assumption (H_T) the proof of Theorem 1.2 applies and provides a sequence of solutions φ_k of Lamé's system (8) such that

$$\left. \begin{array}{l} \| \varphi_k^0 \|^2_{(L^2(\Omega))^3} + \| \varphi_k^1 \|^2_{(H^{-1}(\Omega))} = 1\,,\ \forall k \\ \displaystyle\int_0^T \| \operatorname{div} \varphi_k \|^2_{H^{-1}(\Omega)}\, dt \to 0 \text{ as } k \to \infty. \end{array} \right\} \tag{17}$$

In particular, when Ω is a convex smooth domain there are two types of rays satisfying (16) for all $T > 0$:

(a) The rays that always intersect perpendicularly the boundary with $\eta = 0$;

(b) The rays that are sufficiently close to the boundary of the domain so that they enter in the class described in Figure 5 above.

Obviously there are domains in which (16) holds because of the existence of rays that have both perpendicular and almost tangential reflections (see figures 6 and 7 below).

Figure 6

In this Figure 6 the ray starts from $A \in \partial\Omega$ in the direction perpendicular to $\partial\Omega$ and hits again the boundary almost tangentially at B. It then enters the boundary at $C \in \partial\Omega$ and quits the boundary at D to hit the boundary perpendicularly again at E. Then the ray is reflected backwards along the same trajectory. In figure 7 below we exhibit a case of a ray that is always reflected perpendicularly or tangentially on the boundary.

Figure 7

2.2 Uniform decay in two space dimensions

As we mentioned in Remark 1.1 above there are smooth domains in which there is no $T > 0$ such that (H_T) holds. In those cases one cannot exclude the fact that (9) holds for some $T > 0$ and therefore the solutions of the system of thermoelasticity (1) might decay uniformly. In this section we give a sufficient and almost necessary condition for the uniform decay to hold in two space dimensions.

The following is proved:

Theorem 2.1 *Assume that Ω is a bounded C^∞ domain in $\mathbb{R}^2$ such that*

(a) *There is no contact of infinite order between Ω and its tangents;*

(b) *There exists $T_0 > 0$ such that every ray $s \in [a,b] \to \rho(s) \in \mathrm{Char}(\mathcal{T})$ such that $\mid t(\rho(b)) - t(\rho(a)) \mid > T_0$ satisfies that for some $s_0 \in [a,b]$, $\rho(s_0) = (y, s_0, \eta, \tau)$ with $y \in \partial\Omega$ and $0 < \mid \eta \mid < \mid \tau \mid \nu_L$.*

Then, for any $T > T_0$, there exists a constant $C > 0$ such that

$$\| \varphi^0 \|^2_{(L^2(\Omega))^3} + \| \varphi^1 \|^2_{(H^{-1}(\Omega))^3} \leq C \left[\int_0^T \| \operatorname{div} \varphi \|^2_{H^{-1}(\Omega)} \, dt + ||K(\varphi^0, \varphi^1)||^2_{(L^2(\Omega))^3 \times (H^{-1}(\Omega))^3} \right] \tag{18}$$

for every solution φ of the Lamé system (8) with $K : (L^2(\Omega))^3 \times (H^{-1}(\Omega))^3 \to (L^2(\Omega))^3 \times (H^{-1}(\Omega))^3$ a suitable linear and compact map.

Consequently, the set of eigenvalues γ^2 of the Lamé system

$$\begin{cases} -\mu\Delta\varphi - (\lambda + \mu)\nabla \operatorname{div} \varphi = \gamma^2 \varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{19}$$

such that it exists a non-trivial eigenfunction φ with $\operatorname{div} \varphi \equiv 0$ in Ω, is finite

If we further assume that Ω satisfies condition (C), then (9) holds for solutions of Lamé's system (8) and the energy of solutions of the linear system of thermoelasticity (1) decays exponentially uniformly to zero as $t \to \infty$.

Remark 2.1 The assumption (b) of Theorem 2.1 guarantees that every transversal ray of sufficiently large length intersects the boundary in the region $\mathcal{L}$ in which transversal and longitudinal waves are strongly coupled. This guarantees a uniform rate of transmission of energy from the transversal to the longitudinal component of every solution of the Lamé system. This implies that, up to an additive compact perturbation, the longitudinal component suffices to observe the total energy of solutions (see (18)).

From (18) the fact that the number of eigenvalues of the Lamé system such that $\operatorname{div}\varphi \equiv 0$ holds is finite follows by a classical compactness argument (see for instance J. Rauch and M. Taylor [RT] and Appendixes I and II in J.-L. Lions [Li]). Under the assumption (C) this compact perturbation may be removed from (2.7). Note however that, as indicated in the introduction, when Ω is a 2d smooth simply connected domain and it is not a ball, from [B] it is known that the number of eigenvalues of the Lamé system such that $\operatorname{div}\varphi \equiv 0$ is finite.

■

Remark 2.2 Under the assumptions (a) and (b) of Theorem 2.1 and without assuming that (C) holds the following more general result may be proved.

Let us denote by F the finite dimensional subspace of $\left(H_0^1(\Omega)\right)^2 \times \left(L^2(\Omega)\right)^2 \times L^2(\Omega)$ constituted by elements of the form $(\alpha\varphi(x), \beta\varphi(x), 0)$ with $\alpha, \beta \in I\!R$ and φ being an eigenfunction of the Lamé system (18) such that $\operatorname{div}\varphi \equiv 0$ in Ω.

Then, there exist $C > 0$ and $\omega > 0$ such that

$$\operatorname{dist}\left((u(t), u_t(t), \theta(t)), F\right) \leq C e^{-\omega t} E(0)\,,\ \forall t > 0 \tag{20}$$

for every solution of (1).

In (20) dist denotes the distance in the energy space $\left(H_0^1(\Omega)\right)^2 \times \left(L^2(\Omega)\right)^2 \times L^2(\Omega)$. In other words, (20) guarantees the uniform exponential decay of the projection of solutions of (1) into the orthogonal complement of F.

If, as we have done in the statement of Theorem 2.1, we assume also that (C) holds, $F = \{0\}$ and therefore (20) is equivalent to (6).

■

Remark 2.3 Observe that the assumption (b) of Theorem 2.1 is of open nature. Therefore if (b) holds for Ω, it also holds for small perturbations of Ω, for instance, of the form $\Omega + a = \{y = x + a(x) \in I\!R^n : x \in \Omega\}$ with $a \in C^3(\Omega, I\!R^n)$ small enough in the C^3 norm. On the other hand, as we have mentioned in the introduction, (C) holds as soon as the eigenvalues of the Laplacian are simple, and therefore (C) is a generic property of smooth domains.

Consequently, if (a) and (b) hold there exists a domain $\widetilde{\Omega}$, arbitrarily close to Ω, such that the solutions of the linear system of thermoelasticity (1) in $\widetilde{\Omega}$ decay exponentially uniformly to zero as $t \to \infty$.

■

Remark 2.4 Assume that the domain Ω is such that, as in Figure 3, there exist T_0, $\alpha > 0$ and $\beta \in (0,1)$ so that for every transversal ray $s \in [a,b] \to \rho(s) \in \operatorname{Char}(\mathcal{T})$ such that $|t(\rho(b)) - t(\rho(a))| > T_0$, there exists $s_0 \in [a,b]$ such that $\rho(s_0) = (y, t, \eta, \tau)$ with $y \in \partial\Omega$ satisfying

$$0 < \alpha <\mid \eta \mid<\mid \tau \mid (1-\beta)\, \nu_T. \tag{21}$$

Then, if the Lamé constants $\lambda, \mu > 0$ are such that

$$\frac{\sqrt{\lambda + 2\mu}}{\sqrt{\mu}}(1 - \beta) \leq 1 \tag{22}$$

it also follows that

$$0 < \alpha <| \eta |<| \tau | \nu_L$$

and therefore the assumption (b) of Theorem 2.1 is satisfied.

In view of the remarks above this implies that the uniform decay of solutions of (1) holds for the values of the Lamé constants as in (22) by possibly doing an arbitrarily small perturbation in Ω so that (C) holds. Note that condition (22) is equivalent to assuming that λ/μ is sufficiently close to -1. This requires λ to be negative. Along this paper, for simplicity, we have assumed that $\lambda, \mu > 0$. However, all the results hold if $\lambda + \mu > 0$ and $\mu > 0$. In this larger class one can find Lamé coefficients $\mu > 0$ and $\lambda > -\mu$ with $\lambda + \mu$ sufficiently small such that (22) holds. Note that $\lambda + \mu > 0$ and $\mu > 0$ implies that $\lambda + 2\mu > \mu$ or, equivalently, $c_L > c_T$ which is essential for the obtention of our results. ■

Remark 2.5 Condition (b) is not sufficient in three space dimensions. Indeed, as we will see in section 5.2 when $n = 3$ transversal rays that allways remain in the region $0 <| \eta |<| \tau | \nu_L$ may led to a sequence of solutions of the Lamé system satisfying (17) due to critical polarization. A sharp sufficient condition for the uniform decay to hold will be stated without proof in section 6.2 below.

■

The proof of Theorem 2.1 combines the techniques developed in [BLR] for the study of the observability of the classical scalar wave equation and the analysis of the interaction between the transversal and the longitudinal components of solutions of the Lamé system.

The rest of the paper is organized as follows. In section 3 we prove Theorem 1.1. This reduces the decay problem to the analysis of the Lamé system (8). In section 4 we introduce some basic definitions and results related to the propagation of singularities. In section 5 we prove Theorem 1.2 on the non-uniform decay and a refined 3-d version. In section 6 we prove Theorem 2.1 that provides a sufficient condition for the uniform decay in two space dimensions and indicate without proof what the corresponding 3-d result should be. In section 7 we prove the polynomial decay rate in 2-d. Finally, in section 8, we discuss some controllability and spectral problems. We prove in particular that, under the assumptions of Theorem 1.2, the Lamé system is not exactly controllable with curl-free volume forces acting on the whole domain Ω and we derive some results on the finiteness of the set of eigenvalues of the Lamé system with divergence free eigenfunctions in 2-d. In two Appendixes we give the proofs of some technical results stated without proof in previous sections.

3 Reduction to the analysis of the Lamé system

3.1 Uniform decay

First of all, following the decoupling method introduced in [HLP], we introduce the decoupled system of thermoelasticity

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda + \mu)\nabla \operatorname{div} u + \alpha\beta\mathcal{P}u_t = 0 & \text{in} \quad \Omega \times (0, \infty) \\ \theta_t - \Delta\theta + \beta \operatorname{div} u_t = 0 & \text{in} \quad \Omega \times (0, \infty) \\ u = 0,\, \theta = 0 & \text{on} \quad \partial\Omega \times (0, \infty) \\ u(x, 0) = u^0(x),\, u_t(x, 0) = u^1(x),\, \theta(x, 0) = \theta^0(x) & \text{in} \quad \Omega, \end{cases} \tag{23}$$

where $\mathcal{P} \in \mathcal{L}\left(\left(L^2(\Omega)\right)^3, \left(L^2(\Omega)\right)^3\right)$ is the orthogonal projection from $\left(L^2(\Omega)\right)^3$ into the closed subspace $\mathcal{H} = \{\nabla\varphi : \varphi \in H_0^1(\Omega)\}$.

Observe that $\mathcal{P}u = \nabla\varphi$ if and only if

$$\begin{cases} -\Delta\varphi = -\operatorname{div} u & \text{in} \quad \Omega \\ \varphi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{24}$$

It is clear that

$$\int_\Omega \mathcal{P}u \cdot u dx = \int_\Omega \nabla\varphi \cdot u dx = -\int_\Omega \varphi \operatorname{div} u dx = \int_\Omega |\nabla\varphi|^2 \, dx \sim \| \operatorname{div} u \|_{H^{-1}(\Omega)}^2 . \tag{25}$$

Therefore $\mathcal{P}$ in (23) plays the role of a damping term acting on the system of wave equations satisfied by u but, in principle, the damping mechanism is only effective on the longitudinal component $\operatorname{div} u_t$ of the velocity field u_t.

System (23) is decoupled in the sense that the displacement u satisfies a damped Lamé system

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\beta\mathcal{P}u_t = 0 & \text{in} \quad \Omega \times (0,\infty) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{26}$$

Once the solution u of (26) is computed one can obtain the temperature θ by solving the heat equation

$$\begin{cases} \theta_t - \Delta\theta = -\beta \operatorname{div} u_t & \text{in} \quad \Omega \times (0,\infty) \\ \theta = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega. \end{cases} \tag{27}$$

The decoupled system (23) is also well-posed in H. Let us denote by $\{S(t)\}_{t\geq 0}$ and $\{S_d(t)\}_{t\geq 0}$ the semigroups generated by the original system (1) and the decoupled system (23) respectively.

The following result holds (see [HLP] and [Z1] for the details of the proof):

Lemma 3.1 *For any $0 < T < \infty$ the difference of the two semigroups $S(t) - S_d(t)$ is compact from H into $C([0,T];H)$. In order words, for any bounded set B of H the set of trajectories*

$$\left\{ [S(t) - S_d(t)] \, (u^0, u^1, \theta^0) : (u^0, u^1, \theta^0) \in B \right\}$$

is relatively compact in $C([0,T];H)$.

This compact decoupling result plays a crucial role in the proof of Theorem 1.1.

We divide the proof of Theorem 1.1 in two parts.

Part 1. First we prove that (9) guarantees the exponential decay of solutions of (1).

Indeed, from (9) it is immediate to see that

$$\begin{aligned} \| \varphi^0 \|_{\left(H_0^1(\Omega)\right)^n}^2 + \| \varphi^1 \|_{(L^2(\Omega))^n}^2 \quad &\leq \quad C\int_0^T \| \operatorname{div} \varphi_t \|_{H^{-1}(\Omega)}^2 \, dt \\ &\leq \quad C\int_0^T \int_\Omega \mathcal{P}\varphi_t \cdot \varphi_t dx dt \end{aligned} \tag{28}$$

holds true for the same time T and a different positive constant $C > 0$ for every solution φ of the Lamé system (8). To deduce (28) it is sufficient to apply (9) to $\psi = \varphi_t$ that also solves (8) and to observe that

$$\| \psi(0) \|^2_{(L^2(\Omega))^n} + \| \psi_t(0) \|^2_{(H^{-1}(\Omega))^n} = \| \varphi^1 \|^2_{(L^2(\Omega))^n} + \| \mu\Delta\varphi^0 + (\lambda+\mu)\nabla \operatorname{div} \varphi^0 \|^2_{(H^{-1}(\Omega))^n}$$

$$\sim \| \varphi^0 \|^2_{\left(H^1_0(\Omega)\right)^n} + \| \varphi^1 \|^2_{(L^2(\Omega))^n} .$$

From (28) it is easy to see that there exists $C > 0$ such that

$$E_u(0) \le C \int_0^T \int_\Omega \mathcal{P}u_t \cdot u_t dxdt \tag{29}$$

for every solution of the damped Lamé system (26) with

$$E_u(0) = \frac{1}{2} \int_\Omega \left[| u_t(x,t) |^2 + \mu | \nabla u(x,t) |^2 + (\lambda+\mu) | \operatorname{div} u(x,t) |^2 \right] dx. \tag{30}$$

Indeed, the solution u of (26) can be decomposed as $u = \varphi + \eta$ where φ solves (8) with initial data (u^0, u^1) and η solves

$$\begin{cases} \eta_{tt} - \mu\Delta\eta - (\lambda+\mu)\nabla \operatorname{div} \eta = -\alpha\beta\mathcal{P}u_t & \text{in} \quad \Omega \times (0,T) \\ \eta = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ \eta(0) = \eta_t(0) = 0. \end{cases} \tag{31}$$

From (28) it follows that

$$\begin{aligned} E_u(0) &\le C \int_0^T \| \operatorname{div} \varphi_t \|^2_{H^{-1}(\Omega)} dt \le C \int_0^T \left[\| \operatorname{div} u_t \|^2_{H^{-1}(\Omega)} + \| \operatorname{div} \eta_t \|^2_{H^{-1}(\Omega)} \right] dt \\ &\le C \int_0^T \int_\Omega \mathcal{P}u_t \cdot u_t dxdt + C \int_0^T \int_\Omega | \eta_t |^2 dxdt. \end{aligned} \tag{32}$$

On the other hand, classical energy estimates for the Lamé system (31) allow us to show that

$$\| \eta_t \|^2_{L^2(\Omega\times(0,T))} \le C \| \mathcal{P}u_t \|^2_{L^2(\Omega\times(0,T))} .$$

But in view of (25) we also have

$$\| \mathcal{P}u_t \|^2_{L^2(\Omega\times(0,T))} \sim \int_0^T \int_\Omega \mathcal{P}u_t \cdot u_t dxdt.$$

Therefore, from (32) it follows that (29) holds.

From (29) and the semigroup property applied to the damped Lamé system (14) it follows that there exists $C > 0$ and $\omega > 0$ such that

$$E_u(t) \le Ce^{-\omega t} E_u(0), \ \forall t > 0 \tag{33}$$

for every solution of (26) since

$$\frac{dE_u(t)}{dt} = -\alpha\beta \int_\Omega \mathcal{P}u_t \cdot u_t dx.$$

Let us consider now the energy E_θ corresponding to the temperature:

$$E_\theta(t) = \frac{\alpha}{2\beta} \int_\Omega \theta^2(x,t)dx. \tag{34}$$

We have

$$\begin{aligned} \frac{dE_\theta(t)}{dt} &= -\tfrac{\alpha}{\beta} \int_\Omega \mid \nabla\theta(x,t) \mid^2 dx + \alpha \int_\Omega u_t \cdot \nabla\theta dx \\ &\leq -\frac{\alpha}{2\beta} \int_\Omega \mid \nabla\theta(x,t) \mid^2 dx + \frac{\alpha\beta}{2} \int_\Omega u_t^2 dx \\ &\leq -\frac{\alpha}{2\beta} \int_\Omega \mid \nabla\theta(x,t) \mid^2 dx + CE_u(t). \end{aligned} \tag{35}$$

Applying Poincaré's and Gronwal's inequalities in (35) and the fact that the energy E_u decays exponentially we deduce that

$$E_\theta(t) \leq Ce^{-\omega t} \left[E_u(0) + E_\theta(0)\right] \tag{36}$$

with, possibly, a different decay rate ω.

Combining (35) and (36) we deduce that (6) holds for the solutions of the decoupled system (23). As a consequence of (6) for (23) we deduce the existence of $T > 0$ and $0 < \gamma < 1$ such that

$$E(T) \leq \gamma E(0) \tag{37}$$

for every solution of (23).

In view of Lemma 3.1 this implies the existence of a compact linear map $K : H \to H$ such that

$$E(T) \leq \gamma E(0) + ||K(u^0, u^1, \theta^0)||_H^2 \tag{38}$$

for every solution of the original system (1).

Combining (3) and (38) we deduce the existence of $C > 0$ such that

$$E(0) \leq C \left[\int_0^T \int_\Omega \mid \nabla\theta \mid^2 dxdt + ||K(u^0, u^1, \theta^0)||_H^2 \right]. \tag{39}$$

We claim that there exists $C > 0$ such that

$$||K(u^0, u^1, \theta^0)||_H^2 \leq C \int_0^T \int_\Omega \mid \nabla\theta \mid^2 dxdt \tag{40}$$

for every solution of (1).

Let us assume for the moment that (40) holds. Then, combining (39) and (40) we get

$$E(0) \leq C \int_0^T \int_\Omega \mid \nabla\theta \mid^2 dxdt \tag{41}$$

for every solution of (1). Thus in view of the semigroup property, (3) and (41) it follows that (6) holds for the original system (1).

Let us finally check that (40) holds.

Arguing by contradiction and using the compactness of $K : H \to H$, the proof of (40) can be reduced to show that the unique solution of (1) such that $\nabla\theta = 0$ in $\Omega \times (0,T)$ is identically zero, i.e.

$$\vartheta_T = \Big\{ (u^0, u^1, \theta^0) \in H : \text{ the solution of (1) satisfies } \nabla\theta = 0 \text{ in } \Omega \times (0,T) \Big\} \equiv 0. \tag{42}$$

Indeed, this is a classical argument. Let us recall the main steps. First of all, (39) and the compactness of K allows to show that ϑ_T is finite-dimensional (see Appendixes I and II in [Li]). But ϑ_T decreases with T, so one can select $T_1 > 0$ such that ϑ_T does not depend on $T > T_1$. For $T > T_1$ and $w \in \vartheta_T$, one has $w(t+\varepsilon, \cdot) \in \vartheta_T$ for $\varepsilon > 0$ so that d/dt maps ϑ_T into itself. If $\vartheta_T \neq 0$, by selecting an eigenfunction of d/dt on ϑ_T, we conclude the existence of an element $(u^0, u^1, \theta^0) \in \vartheta$ such that the corresponding solution (u, θ) of (1) is of the form $(u, \theta) = e^{\lambda t}(\varphi(x), \eta(x))$ for some $\lambda \in \mathbb{C}$ and $(\varphi, \eta) \in \left(H_0^1(\Omega)\right)^n \times L^2(\Omega)$.

However, since $\nabla\theta = 0$ in $\Omega \times (0, T)$ and θ vanishes on the lateral boundary we conclude that $\eta \equiv 0$ and then $\operatorname{div}\varphi \equiv 0$ and $\varphi \in \left(H_0^1(\Omega)\right)^n$ solves (5) with $\gamma = -\lambda^2/\mu$. Since Ω has been assumed to be such that the generic property (C) holds, this implies that $\varphi \equiv 0$ too and therefore we are led to a contradiction.

Part 2. Let us see now that the uniform decay of solutions of (1) implies (9).

In view of the uniform decay (6) of the solutions of (1) we deduce the existence of constants $C > 0$ and $T > 0$ such that (41) holds for every solution of (1).

The compact decoupling result of Lemma 2.1 implies then that (38) holds with $0 < \gamma < 1$ and K as above.

Taking $\theta^0 \equiv 0$ we deduce, in particular, that

$$E_u(T) \le \gamma E_u(0) + ||K(u^0, u^1)||_H^2 \tag{43}$$

for every solution u of the damped Lamé system (26) with $K : H = \left(H_0^1(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n \to H$ linear and compact.

On the other hand (43) implies the existence of $T > 0$ and $C > 0$ such that

$$E_u(0) \le C\left[\int_0^T \int_\Omega \mathcal{P}u_t \cdot u_t dxdt + ||K(u^0, u^1)||_H^2\right] \tag{44}$$

for every solution of (26).

Using the decomposition $u = \varphi + \eta$ of Part 1 of the proof we conclude that

$$\parallel (\varphi^0, \varphi^1) \parallel^2_{\left(H_0^1(\Omega)\right)^n \times (L^2(\Omega))^n} \le C\left[\int_0^T \parallel \operatorname{div}\varphi_t \parallel^2_{H^{-1}(\Omega)} dt + ||K(u^0, u^1)||_H^2\right] \tag{45}$$

for every solution of the Lamé system (8).

Finally we show the existence of $C > 0$ such that

$$||K(u^0, u^1)||_H^2 \le C \int_0^T \parallel \operatorname{div}\varphi_t \parallel^2_{H^{-1}(\Omega)} dt \tag{46}$$

by the same argument used in Part 1 to prove (40) and always under the assumption that Ω verifies the condition (C).

This concludes the proof of (28) for solutions of (8). It is then easy to see that (9) holds too. Given φ solution of (8) with data $(\varphi^0, \varphi^1) \in \left(L^2(\Omega)\right)^n \times \left(H^{-1}(\Omega)\right)^n$ we set $\psi(x,t) = \int_0^T \varphi(x,t) + \chi(x)$ with $\chi \in \left(H_0^1(\Omega)\right)^n$ such that

$$\begin{cases} -\mu\Delta\chi - (\lambda+\mu)\nabla \operatorname{div}\chi = \varphi^1 & \text{in} \quad \Omega \\ \chi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{47}$$

Then ψ solves (8) with initial data (χ, φ^0). Applying (28) to ψ and taking into account that $\psi_t = \varphi$ we see that

$$\| \chi \|^2_{\left(H_0^1(\Omega)\right)^n} + \| \varphi^0 \|^2_{(L^2(\Omega))^n} \leq C \int_0^T \| \operatorname{div} \varphi \|^2_{H^{-1}(\Omega)} \, dt$$

and this is equivalent to (9) since the norms $\| \chi \|_{\left(H_0^1(\Omega)\right)^n}$ and $\| \varphi^1 \|_{(H^{-1}(\Omega))^n}$ are equivalent.

■

3.2 Polynomial decay

As we shall see in section 7, for most smooth 2-d domains, although an inequality of the form (9) might possibly not hold, the following holds true

$$\| \varphi^0 \|^2_{(L^2(\Omega))^2} + \| \varphi^1 \|^2_{H^{-1}(\Omega))^2} \leq C \int_0^T \| \operatorname{div} \varphi \|^2_{L^2(\Omega)} \, dt, \tag{48}$$

for all solution of the Lamé system.

Note that on the right hand side of (48) we introduce an extra derivative with respect to (9). This means that the total energy of solutions may be estimated in terms of the energy of solutions of one more degree of regularity concentrated on the longitudinal component.

In fact, as we shall see, the only obstruction for (48) to hold is the fact that (C) has to be assumed.

Roughly speaking, when (48) holds, solutions of the 2-d system of thermoelasticity decay polynomially provided the data are smooth enough.

In addition to the energy space H we introduce the domain D of the generator of the semigroup $S(t)$ associated to (1):

$$D = \left(H^2 \cap H_0^1(\Omega)\right)^2 \times \left(H_0^1(\Omega)\right)^2 \times \left(H^2 \cap H_0^1(\Omega)\right), \tag{49}$$

endowed with the natural norm. We also introduce the space V, dual of D with respect to the pivot space H. It then follows that

$$\left\| S(t) \left(u^0, u^1, \theta^0\right) \right\|_D \leq \left\| \left(u^0, u^1, \theta^0\right) \right\|_D, \ \forall t > 0 \tag{50}$$

for all $(u^0, u^1, \theta^0) \in D$ and also,

$$\left\| \left(u^0, u^1, u^0\right) \right\|^2_H \leq \left\| \left(u^0, u^1, \theta^0\right) \right\|_D \left\| \left(u^0, u^1, \theta^0\right) \right\|_V, \forall \left(u^0, u^1, \theta^0\right) \in D. \tag{51}$$

As we shall see in section 7, for most 2d domains the following holds

$$\left| \begin{array}{l} \text{There exist } T \text{ and } C > 0 \text{ such that} \\ \| S(T) \left(u^0, u^1, \theta^0\right) \|^2_V \leq C \left[\| (u^0, u^1, \theta^0) \|^2_H - \| S(T) \left(u^0, u^1, \theta^0\right) \|^2_H \right] \\ \text{for all } \ (u^0, u^1, \theta^0) \in H. \end{array} \right. \tag{52}$$

The following result shows that (3.30) suffices to obtain an explicit polynomial decay rate for smooth solutions of (1):

Theorem 3.1 *Let Ω be a bounded smooth domain of $I\!R^2$ such that* (3.30) *holds for some $T > 0$.*
Then, there exists $C > 0$ such that

$$E(t) \leq \frac{C}{t} \parallel \left(u^0, u^1, \theta^0\right) \parallel_D^2, \forall t > 0 \tag{53}$$

for every solution of (1) *with initial data in the domain*

$$D = \left(H^2 \cap H_0^1(\Omega)\right)^2 \times \left(H_0^1(\Omega)\right)^2 \times \left(H^2 \cap H_0^1(\Omega)\right).$$

Proof of Theorem 3.1 Let us introduce the sequence of positive numbers

$$\alpha_n = \left\| S(nT)\left(u^0, u^1, \theta^0\right)\right\|_H^2.$$

In view of (3.30) we have

$$\alpha_n - \alpha_{n+1} \geq \frac{1}{C}\left\| S\left((n+1)T\right)\left(u^0, u^1, \theta^0\right)\right\|_V^2$$

which combined with (3.28)-(3.29) gives

$$\alpha_n - \alpha_{n+1} \geq \frac{\alpha_{n+1}^2}{C\left\|S\left((n+1)T\right)\left(u^0, u^1, \theta^0\right)\right\|_D^2} \geq \frac{\alpha_{n+1}^2}{C\left\|\left(u^0, u^1, \theta^0\right)\right\|_D^2}, \forall n \geq 1. \tag{54}$$

Without loss of generality we may assume that $\left\|\left(u^0, u^1, \theta^0\right)\right\|_D = 1$. Then, (3.32) becomes

$$\begin{cases} \alpha_{n+1} + \frac{\alpha_{n+1}^2}{C} \leq \alpha_n, \forall n \geq 1 \\ \alpha_1 \leq 1. \end{cases}$$

It is then easy to see that $\alpha_n \leq \beta_n$ for all $n \geq 1$ where β_n solves

$$\begin{cases} \beta_{n+1} + \frac{\beta_{n+1}^2}{C} = \beta_n, \forall n \geq 1 \\ \beta_1 = 1 \end{cases}$$

that clearly verifies $\beta_n \leq C'/n$, for some $C' > 0$ and all $n \geq 1$.

■

Let us now analyze condition (3.30). The following holds:

Lemma 3.2 *Assume that Ω satisfies condition (C) and that there exists $T > 0$ such that*

$$\begin{cases} \textit{If } (u, \theta) \textit{ solves } (1) \textit{ and } \theta \in L^2\left(0, T; H_0^1(\Omega)\right), \textit{ then} \\ u \in L^2\left(\Omega \times (0, T)\right). \end{cases} \tag{55}$$

Then, (3.30) *holds.*

Proof. We introduce the following Hilbert spaces of solutions $(u,\theta) \in (\mathcal{D}'(\Omega \times (0,T)))^3$ of (1):

$$F_0^T = \Big\{(u,\theta) \in (\mathcal{D}'(\Omega \times (0,T)))^3 \quad : \quad (u,\theta) \text{ satisfy (1) and } (u,\theta) \in \left(H^{-1}(\Omega \times (0,T))\right)^3\Big\}; \tag{56}$$

$$F_1^T = \Big\{(u,\theta) \in F_0^T : \nabla\theta \in L^2\left(\Omega \times (0,T)\right)\Big\}; \tag{57}$$

$$G^T = F_1^T \cap \Big\{u \in \left(L^2(\Omega \times (0,T))\right)^3\Big\}, \tag{58}$$

endowed with their natural norms.

In view of the assumption (3.33) the range of the embedding $F_1^T \hookrightarrow F_0^T$ is included in G^T. Thus $G^T = F_1^T$ and by the closed graph Theorem there exists $C > 0$ such that

$$\|(u,\theta)\|_{G^T} \leq C \|(u,\theta)\|_{F_1^T}, \forall (u,\theta) \in F_1^T. \tag{59}$$

Let us now introduce the following subspace of G^T:

$$N^T = \Big\{(u,\theta) \in G^T : \theta \equiv 0\Big\}, \tag{60}$$

i.e. the subspace of solutions of (1) with null temperature.

Inequality (3.37) and the compactness of the embedding $L^2 \hookrightarrow H^{-1}$ imply that N^T is finite-dimensional. Since N^T decreases as T increases we deduce the existence of some $T_0 > 0$ such that N^T is independent of T for all $T \geq T_0$, i.e. $N^T = N$ for all $T > T_0$. Clearly

$$N = \Big\{(u,\theta) : (u,\theta) \text{ solves (1), } \theta \equiv 0 \text{ and } u \in L^2_{loc}\left(0,\infty; L^2(\Omega)\right)\Big\}.$$

Developping solutions $(u,\theta) \in N$ of (1) in Fourier series and using the fact that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{i\lambda t} e^{-i\mu t} dt = 0 \text{ if } \lambda \neq \mu$$

it is then easy to see that condition (C) implies that $N = \{0\}$.

Therefore, for $T > 0$ large enough $N^T = \{0\}$. This fact combined with (3.36) allows us to prove by a classical compactness-uniqueness argument that for $T > 0$ large enough there exists $C > 0$ such that

$$\| u \|^2_{(L^2(\Omega\times(0,T)))^2} \leq C \int_0^T \int_\Omega |\nabla\theta|^2 \, dxdt,$$

for all solution (u,θ) of (1).

Taking into account that

$$\left\|\left(u^0,u^1,\theta^0\right)\right\|_H^2 - \left\|S(T)\left(u^0,u^1,\theta^0\right)\right\|_H^2 = \frac{\alpha}{\beta}\int_0^T \int_\Omega |\nabla\theta|^2 \, dxdt$$

and that the projection of V over the first two components (u, u_t) of the unknown coincides with $\left(L^2(\Omega)\right)^2 \times \left(H^{-1}(\Omega)\right)^2$ we deduce that (3.30) holds.

■

The following Lemma provides a sufficient condition for (3.33) to hold:

Lemma 3.3 *Let us assume that* Ω *is such that there exists* $T > 0$ *such that*

$$\left\{ \begin{array}{l} \textit{Every solution } \varphi \textit{ of the Lamé system } (8) \textit{ such that} \\ \operatorname{div} \varphi \in L^2\left(\Omega \times (0,T)\right) \textit{ satisfies } \varphi \in \left(L^2(\Omega \times (0,T))\right)^2. \end{array} \right. \tag{61}$$

Then (3.33) *holds.*

Remark 3.1 Lemmas 3.2 and 3.3 show that in order to obtain the polynomial decay rate (3.31) it suffices to check that (3.39) holds for the Lamé system and to assume that the spectral condition (C) holds.

As we have seen in the proof of Lemma 3.2, according to Lemma 3.3 and even if (C) does not hold, one can prove that the number of eigenvalues for which (C) fails is finite as soon as (3.39) holds. Consequently one deduces that

$$\operatorname{dist}\left(\left(u(t), u_t(t), \theta(t)\right), F\right) \leq \frac{C}{t} \left\| \left(u^0, u^1, \theta^0\right) \right\|_D^2$$

for all $\left(u^0, u^1, \theta^0\right) \in D$, where F denotes the subspace of solutions $(u, \theta) = (u, 0)$ of (1) such that u belongs to the subspace generated by the eigenfunctions that do not fulfill (C).

■

Proof of Lemma 3.3.

We shall prove that if (3.39) holds true with $T - \varepsilon$ ($\varepsilon > 0$), (3.33) holds trues for T.

We decompose the elastic component u of the solution (u, θ) of (1) as $u = v + w$ where

$$\left\{ \begin{array}{lll} w_{tt} - \mu \Delta w - (\lambda + \mu) \nabla \operatorname{div} w = -\alpha \nabla \theta & \text{in} & \Omega \times (0,T) \\ w = 0 & \text{on} & \partial\Omega \times (0,T) \\ w(0) = w_t(0) = 0 & \text{in} & \Omega \end{array} \right. \tag{62}$$

and

$$\left\{ \begin{array}{lll} v_{tt} - \mu \Delta v - (\lambda + \mu) \nabla \operatorname{div} v = 0 & \text{in} & \Omega \times (0,T) \\ v = 0 & \text{on} & \partial\Omega \times (0,T) \\ v(0) = u^0, v_t(0) = u^1 & \text{in} & \Omega. \end{array} \right. \tag{63}$$

Observe that the heat equation satisfied by θ can also be written as

$$\partial_t \left[\operatorname{div} v + \theta/\beta + \operatorname{div} w\right] = \Delta\theta/\beta \text{ in } \Omega \times (0,T). \tag{64}$$

First we note that $\nabla\theta \in \left(L^2(\Omega \times (0,T))\right)^2$ automatically implies that $w \in \left(H^1(\Omega \times (0,T))\right)^2$.

In view of the structure of the system (3.41) we deduce that $WF_b(v) \subset$ Char, where Char denotes the characteristic manifold of the Lamé system (see section 4 below for the definition and basic properties). Therefore $WF_b(\operatorname{div} v) \subset$ Char as well. Notice that $\operatorname{div} v$ solves the wave equation $\left(\partial_t^2 - c_L^2 \Delta\right) (\operatorname{div} v) = 0$. Therefore, the fact that $WF_b(\operatorname{div}) \subset$ Char when $x \in \Omega$ is trivial. However, since the boundary conditions that $\operatorname{div} v$ satisfies are unknown, the fact that $WF_b(\operatorname{div}) \subset$ Char, does indeed provide some information when $y \in \partial\Omega$.

Let $J = [\varepsilon/2, T - \varepsilon/2]$, $\text{Char}_J = \text{Char} \cap \{t \in J\}$. In view of the fact that $WF_b(\operatorname{div}(v)) \subset$ Char,one has $\operatorname{div}(v) \in L^2(\Omega \times (0,T))$ if $\operatorname{div}(v) \in L^2_\rho$ microlocally for every $\rho \in \text{Char}_J$. Taking into account that

∂_t is elliptic over Char the latter is equivalent to $\partial_t(\operatorname{div}(v)) \in H^{-1}_\rho$ for all $\rho \in \mathrm{Char}_J$. But this can be easily derived from (3.42) taking into account that $\theta \in L^2\left(0,T; H^1_0(\Omega)\right)$ and $w \in H^1(\Omega \times (0,T))$.

We have proved that $\operatorname{div} v \in L^2(\Omega \times (0,T))$. In view of (3.39) we deduce that $v \in \left(L^2(\Omega \times J)\right)^2$ which implies by equation (3.41) $v \in \left(L^2(\Omega \times (0,T))\right)^2$. Since on the other hand $w \in \left(H^1(\Omega \times (0,T))\right)^2$, we deduce that $u = v + w \in \left(L^2(\Omega \times (0,T))\right)^2$. Thus (3.33) holds.

■

4 Preliminaries on geometry and the propagation of singularities

In this section we recall concepts and properties related to the propagation of singularities. We also state the basic result (Theorem 4.1) that will be used in section 6. We refer to Appendix B for a proof. In order to state a sharp result in dimension $n = 3$ (see Theorem 5.1 and Conjecture 6.2), we also introduce the notion of polarization in the 3-d system of elasticity.

Let u be a solution of

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u = 0 & \text{in} \quad \Omega \times I\!R \\ u = 0 & \text{on} \quad \partial\Omega \times I\!R. \end{cases} \tag{65}$$

Let us first recall that given a point $\rho \in T^*(\Omega \times I\!R)$ we say that $u \in H^s_\rho$ if and only if $Au \in L^2$ for a pseudo-differential operator of order s which is elliptic at ρ. When $\rho \in T^*(\partial\Omega \times I\!R)$ lies on the boundary, A has to be replaced by a pseudo-differential operator in the tangential directions. Similarly $WF_b(u)$ is defined by the condition: $\rho \notin WF_b(u)$ if and only if there exists A elliptic at ρ such that $Au \in C^\infty$. By classical results on boundary value problems, we have for any solution of (4.1):

$$WF_b(u) \subset \mathrm{Char} \tag{66}$$

where Char is the characteristic manifold of the Lamé system defined in section 2.1.

If moreover, u satisfies the additional condition

$$\operatorname{div} u \in L^2_{loc}(I\!R; L^2(\Omega)) \tag{67}$$

then one has

$$u \in H^1_\rho \text{ for any } \rho \notin \mathrm{Char}\mathcal{T} \tag{68}$$

where Char$\mathcal{T}$ is the transversal characteristic manifold defined in section 2.1.

In view of (4.4), the study of the H^1-regularity of solutions of equations (4.1)-(4.3) is reduced to the H^1_ρ-regularity on Char$(\mathcal{T})$. The following propagation Theorem holds. It will be proved in Appendix B:

Theorem 4.1 *Assume that Ω is a C^∞ domain of $I\!R^n$ with $n = 2$ or 3 which does not have contacts of infinite order with its boundary.*

Let u be a solution of (4.1),(4.3) and let $s \to \gamma(s)$ be a transversal bicharacteristic ray.

Then $u \in H^1_{\rho(s_1)}$ if and only if $u \in H^1_{\rho(s_2)}$ for any s_1, s_2.

Let us recall that the set $\mathcal{L}$ that couples strongly the longitudinal and transversal waves is defined by

$$\mathcal{L} = \{(y,t,\eta,\tau) \,:\, y \in \partial\Omega\,,\, 0 <\mid \eta \mid \le \mid \tau \mid \nu_L\}\,.$$

As we shall see in the proof of Theorem 6.1 of uniform decay in two space dimensions, one has $u \in H^1_\rho$ for any ρ in the interior of $\mathcal{L}$ for any solution of (4.1),(4.3) in two space dimension, and so in that case, Theorem 4.1 is sufficient to understand the global H^1 regularity of u. However when $n = 3$, the scalar condition div$u \in L^2$ gives only an information on the polarization of the vector field u at points ρ in the interior of $\mathcal{L}$. We recall that the propagation and reflection of C^∞-polarization for solutions of systems with scalar principal part has been studied by N. Dencker [De] and C. Gérard [G], but to our knowledge, no general result extending Theorem 4.1 on propagation of the polarization along bicharacteristic rays is known near points where the ray is tangent to the boundary. Thus, we restrict ourself to the description of the geometry of the propagation in order to state Theorem 5.2 and Conjecture 6.2.

Given $\rho \in \mathrm{Char}(\mathcal{T})$ such that $x(\rho) \in \Omega$ and $e \in \mathbb{C}^3$, we say that $u \in H^1_{\rho,e}$ if there exists a vectorial pseudo-differential operator $A = (A_1, A_2, A_3)$ of order 1 with symbol $\sigma(A)$ such that $A \cdot u \in L^2_\rho$ and $e \in (\ker(\sigma(A)(\rho)))^\perp = \mathbb{C}\sigma(A)(\rho)$. In this case we say that the vector field u is polarized perpendicularly with respect to e.

For example the condition (4.3) implies that $u \in H^1_{\rho,\xi(\rho)}$.

When $\rho = (y, t, \eta, \tau)$, $y \in \partial\Omega$, $|\eta| < \nu_T|\tau|$ is an hyperbolic point of the boundary, and under the condition $Au \in L^2_\rho$, with $A = A^1 + A^0\partial_n$ where the A^j are tangential pseudodifferential operators of order j, we will write $u \in H^1_{\rho^\pm,e^\pm}$ with $\rho^\pm = (y, t, \xi^\pm, \tau)$, $\xi^\pm = (\eta, \pm\sqrt{\nu_T\tau^2 - \eta^2})$ and $e^\pm \in (\mathrm{Ker}\sigma(A)(\rho^\pm))^\perp$. Due to the fact that we always have $u \in H^1_{\rho,\xi(\rho)}$ by condition (4.3), we restrict ourself to the study of the polarization in directions lying in the plane π_ρ orthogonal to $\xi(\rho)$.

When $\rho = (y, t, \eta, \tau)$ is in the interior of $\mathcal{L}$, and for u solution of (4.1), (4.3), the algebraic study of the system (5.26) gives the property

$$u \in H^1_{\rho^\pm,e^\pm} \tag{69}$$

where $e^\pm$ are the directions perpendicular to $\xi^\pm(\rho)$ in the plane $(\xi^\pm, \vec{n}(\rho))$. In other words, for ρ in the interior of $\mathcal{L}$ the condition (4.3) implies that the vector field u is polarized in the direction tangent to the boundary and perpendicular to the propagation, i. e. in the direction $\pi^+_\rho \cup \pi^-_\rho$, where $\pi^\pm_\rho$ are the planes orthogonal to $\xi^\pm(\rho)$.

We shall now describe the geometric law of evolution of the polarization along a characteristic ray $s \to \rho(s)$ which is used in the statement of Theorem 5.2 and Conjecture 6.2.

1.- When $x(\rho(s)) \in \Omega$, then the polarization $e(s)$ remains constant, due to the fact that (4.1) is a constant coefficient equation.

2.- When $\rho(s) = (y, t, 0, \tau)$, with $y \in \partial\Omega$ is a point of perpendicular reflection, the polarization e is preserved by reflection.

3.- When $\rho(s) = (y, t, 0, \tau)$, with $\nu_L|\tau| < |\eta| < \nu_T|\tau|$ is a point of transversal reflection with $\rho(s)$ outside the interior of $\mathcal{L}$, let $\gamma^\pm$ be the two half-bicharacteristics issued from $\rho^\pm$. We can choose coordinates so that

$$\eta = \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix}, \quad \xi^\pm = \begin{pmatrix} \xi_1 \\ 0 \\ \pm\xi_3 \end{pmatrix}$$

with $\xi_3^2 + \xi_1^2 = \nu_T^2\tau^2$. Let δ be the solution of $\delta^2 + \eta^2 = \nu_L^2\tau^2$ with $\mathrm{Im}\delta \geq 0$. Then if u is polarized perpendicularly to e^- on γ^-, u is polarized perpendicularly to e^+ on γ^+ where $e^\pm$ are perpendicular to the planes $(\xi^\pm, v^\pm)$ with

$$v^- = \begin{pmatrix} \xi_3 a^- \\ b \\ \xi_1 a^- \end{pmatrix}, \quad v^+ = \begin{pmatrix} -\xi_3 a^+ \\ b \\ \xi_1 a^+ \end{pmatrix}$$

are related by the equation $a^+(\xi_1^2+\xi_3\delta) = a^-(\xi_1^2-\xi_3\delta)$ (this law of reflection is a consequence of formula (5.30)).

4.- When $\rho(s)$ is a tangent point of the boundary, the polarization is continuous at $\rho(s)$.

5.- Finally on intervals where the ray lives on the boundary, the polarization direction evolves as follows

$$e(s) = \alpha \overrightarrow{n}(s) + \beta \overrightarrow{b}(s)$$

where α evolves according to the equation $\dot{\alpha} = ik\sqrt{1-\nu_L^2/\nu_T^2}$, k being the geodesic curvature, β is constant and $\overrightarrow{b}(s) = \overrightarrow{t}(s) \wedge \overrightarrow{n}(s)$, $\overrightarrow{t}(s)$ being the tangent vector to the geodesic in the boundary $s \to x(s) \in \partial\Omega$.

Remark 4.1 In Theorem 5.2, we will only use rays which intersect the boundary at points $\rho = (y,t,\eta,\tau)$ where $|\eta|^2 \neq \nu_L|\tau|$ and $|\eta| \neq \nu_T|\tau|$, so the analysis of system (5.26) will be sufficient to take care of the polarization. Notice however that to prove the Conjecture 6.2 one needs a suitable definition of the polarization at points tangent to the boundary and a propagation statement compatible with the law given by the description above.

■

5 Proof of the non-uniform decay

This section is devoted to prove Theorem 1.2. In view of Theorem 1.1 it is sufficient to consider the Lamé system

$$\begin{cases} \varphi_{tt} - \mu\Delta\varphi - (\lambda+\mu)\nabla \operatorname{div}\varphi = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \varphi = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ \varphi(x,0) = \varphi^0(x), \varphi_t(x,0) = \varphi^1(x) & \text{in} \quad \Omega \end{cases} \tag{70}$$

and to show that there is no $C>0$ and $T>0$ such that

$$\| \varphi^0 \|^2_{(L^2(\Omega))^3} + \| \varphi^1 \|^2_{(H^{-1}(\Omega))^3} \leq C \int_0^T \| \operatorname{div}\varphi \|^2_{H^{-1}(\Omega)} \, dt \tag{71}$$

holds for every solution of (70).

The proof of the fact that (71) does not hold is inspired by the, by now classical, construction of gaussian beams by J. Ralston [R1,2]. Roughly speaking, by a geometric optics construction we will exhibit the existence of a family of solutions of Lamé's system (70) with most of its energy concentrated on its transversal component.

First of all we recall some basic issues related to this construction of geometric optics.

5.1 Preliminaries on geometric optics with complex phase

Let Ω be a bounded smooth domain of $I\!R^n$ with boundary $\mathcal{S} = \partial\Omega$ of class C^∞. In a neighborhood of a given $x_0 \in \mathcal{S}$ we denote by $x = (x', x_n)$ the system of normal geodesic coordinates where $x' \in \mathcal{S}$ and $x_n \in I\!R$ are characterized by

$$| x_n | = \operatorname{dist}(x,\mathcal{S}); \Omega = \{x_n > 0\}\, ; \operatorname{dist}(x',x) = \operatorname{dist}(x,\mathcal{S}).$$

In this system of coordinates the metric on $T^* I\!R^n$ takes the form $\xi_n^2 + g(x_n, x', \xi')$ where $g(0,x',\xi') = \| \xi' \|^2_{X'}$ is, by definition, the induced metric over $\mathcal{S}$.

Let $\varphi = \varphi(t, x')$ be a C^∞ function with values in $\mathbb{C}$, defined near $(t_0, x'_0) \in \mathbb{R} \times \mathcal{S}$ and satisfying

$$\begin{cases} \text{Im } \varphi \geq 0, \ \text{Im } \varphi(t_0, x'_0) = 0 \\ \text{Im } \nabla^2 \varphi(t_0, x'_0) > 0; \, d\varphi(t_0, x'_0) \neq 0 \end{cases} \tag{72}$$

where ∇^2 and d denote the Hessian and the gradient in the coordinates $(t, x') \in \mathbb{R} \times \mathcal{S}$. In (4.3) by $\text{Im}\nabla^2\varphi > 0$ we mean that the Hessian matrix is positive definite.

In view of (5.3) it follows that $\text{Im} d\varphi(t_0, x'_0) = 0$. Thus, we set

$$\text{Re}\, d\varphi(t_0, x'_0) = d\varphi(t_0, x'_0) = (\tau_0, \xi'_0) \neq 0. \tag{73}$$

When F is a closed set we denote by $\mathcal{S}(F)$ the set of symbols $a(y,k) = \sum_{j=1}^{j_0} a_j(y) k^{-j}$ with $j_0 < \infty$ such that a_j is of $C^\infty-$class in a neighborhood of F for each $j = 1, \cdots, j_0$.

We want to solve, in an approximate way, the system

$$\begin{cases} (\partial_t^2 - c^2\Delta) \left(e^{ik\psi}\sigma \right) = 0 & \text{in} \quad \Omega \times \mathbb{R} \\ \psi = \varphi, \, \sigma = a \in \mathcal{S}(t_0, x'_0) & \text{on} \quad \mathcal{S} \times \mathbb{R} \end{cases} \tag{74}$$

in the following cases:

(i) The hyperbolic case :$c \parallel \xi'_0 \parallel_{X'} <\mid \tau_0 \mid$;

(ii) The elliptic case : $c \parallel \xi'_0 \parallel_{X'} >\mid \tau_0 \mid$.

Given N_0 large enough (it will be clear from the construction that $N_0 = 2$ suffices), given a closed set F and a C^∞ function f on a neighborhood of F, we say that $f \sim 0$ in F if f vanishes at order $2N_0$ in F. On the other hand, given a symbol $b = \sum_{j=0}^{j_0} b_j k^{-j}$ we write $b \sim 0$ in F if b_j vanishes at order $2(N_0 - j)$ in F for each $j = 1, \cdots, j_0$.

Hyperbolic case

In this case the equation $\xi_n^2 + g\,(0, x'_0, \xi'_0) = \tau_0^2/c^2$ has two real distinct roots. Let us choose one of them, ξ_n, and denote $x_0 = (x'_0, 0) \in \mathbb{R}^n$ and $\xi_0 = (\xi'_0, \xi_n) \in T^*_{x_0}\mathbb{R}^n$.

The null bicharacteristic of the operator $\partial_t^2 - c^2\Delta$ passing through $(t_0, x_0, \tau_0, \xi_0)$ is given by

$$\left\{ (t, x(t), \tau_0, \xi_0)\,;\, x(t) = x_0 - (t - t_0) \frac{\tau_0 \xi_0}{\mid \xi_0 \mid^2} \right\}. \tag{75}$$

We set $F = \{(t, x(t))\,; t \in \mathbb{R}\}$.

The following holds:

Proposition 5.1 *There exists a C^∞ function $\psi = \psi(t, x)$ in a neighborhood of F such that*

$$\begin{cases} \psi \Big|_{\mathbb{R}\times\mathcal{S}} - \varphi \sim 0 \ \textit{in} \ (t_0, x'_0) \\ (\psi')^2 - c^2(\nabla\psi)^2 \sim 0 \ \textit{in} \ F. \end{cases} \tag{76}$$

The development of ψ over F is given by

$$\begin{cases} \psi(t, x) = \varphi(t_0, x'_0) + (x - x(t)) \cdot \xi_0 + q(t, x) \\ q(t, x(t)) \equiv 0, \nabla_x q(t, x(t)) \equiv 0; \\ \text{Im } \nabla_x^2 q(t, x(t)) > 0 \end{cases} \tag{77}$$

and

$$\left[\det\ \mathrm{Im}\left(\nabla^2_{t,x'}\varphi\right)(t_0,x'_0)\right]^{1/2} = c\cos\theta\left(\det\ \mathrm{Im}\left(\nabla^2_x\psi\right)(t_0,x_0)\right)^{1/2} \tag{78}$$

where θ denotes the angle between ξ_0 and the normal to $\mathcal{S}$ at x_0.

On the other hand, for every symbol $a \in \mathcal{S}(t_0, x'_0)$, there exists $\sigma \in \mathcal{S}(F)$ such that

$$\begin{cases} \sigma\Big|_{\mathcal{S}\times I\!R} - a \sim 0 \ \textit{in}\ (t_0,x'_0) \\ \left(\partial_t^2 - c^2\Delta\right)\left(e^{ik\psi}\sigma\right) = k^2 e^{ik\psi} r, \ \textit{with}\ r \sim 0\ \textit{in}\ F. \end{cases} \tag{79}$$

Moreover, given a C^∞ function $\chi = \chi(t,x)$ such that $\chi = 1$ near F and with $\operatorname{supp}\chi$ *contained in a small neighborhood of F, the sequence of functions*

$$u_k = \chi(t,x)k^{-1+n/4}e^{ik\psi}\sigma \tag{80}$$

is exponentially concentrated near F and

$$\begin{cases} \left\|\left(\partial_t^2 - c^2\Delta\right)u_k\right\|_{H^1\left([-T,T]\times I\!R^n\right)} \to 0 \ \textit{as}\ k\to\infty \\ \displaystyle\int_{I\!R^n}\left[\left|\frac{\partial u_k}{\partial t}\right|^2 + c^2\left|\nabla u_k\right|^2\right]dx \to 2\pi^{n/2} \mid \tau_0 \mid^2 \left|\sigma_0\left(t,x(t)\right)\right|^2 \left[\det\ \mathrm{Im}\,\nabla^2_x q\left(t,x(t)\right)\right]^{-1/2}. \end{cases} \tag{81}$$

By the energy identity it follows that

$$\frac{d}{dt}\left[\mid \sigma_0 \mid^2 \left[\det\left(\mathrm{Im}\,\nabla^2_x q\right)\right]^{-1/2}(t,x(t))\right] = 0 \tag{82}$$

where σ_0 is the first term in the development of σ : $\sigma = \Sigma\sigma_j k^{-j}$

The elliptic case

In this case the equation $\xi_n^2 + g\left(0,x'_0,\xi'_0\right) = \frac{\tau_0^2}{c^2}$ has two complex roots $\xi_n^\pm$:

$$\xi_n^+ = -\xi_n^- = i\sqrt{\parallel \xi'_0 \parallel^2_{X'_0} - \frac{\tau_0^2}{c^2}}.$$

We set $x_0 = (x'_0, 0)$ and $F = \{(t_0, x_0)\}$.

The following holds:

Proposition 5.2 *There exists a C^∞ function $\psi = \psi(t,x)$ in a neighborhood of (t_0,x_0) such that*

$$\begin{cases} \psi\Big|_{I\!R\times\mathcal{S}} - \varphi \sim 0\ \textit{in}\ (t_0,x_0) \\ (\psi_t)^2 - c^2(\nabla_x\psi)^2 \sim 0\ \textit{in}\ (t_0,x_0). \end{cases} \tag{83}$$

In the system of normal geodesic coordinates we have

$$\psi(t,x',x_n) = \varphi(t,x') + \xi_n^+ x_n + O\left(|x_n x'| + |x_n t| + x_n^2 + \mid x' \mid^3\right). \tag{84}$$

On the other hand, for every symbol $a \in \mathcal{S}(t_0,x'_0)$ there exists $\sigma \in \mathcal{S}(t_0,x_0)$ such that

$$\begin{cases} \sigma\Big|_{I\!R\times\mathcal{S}} - a \sim 0\ \textit{in}\ (t_0,x'_0) \\ \left(\partial_t^2 - c^2\Delta\right)\left(e^{ik\psi}\sigma\right) = k^2 e^{ik\psi} r, \ \textit{with}\ r\sim 0\ \textit{in}\ (t_0,x_0). \end{cases} \tag{85}$$

Given $\chi = \chi(t,x)$ a C_c^∞ function such that $\chi \equiv 1$ near (t_0, x'_0), the sequence

$$u_k = \chi(t,x) k^{-1+n/4} e^{ik\psi} \sigma \tag{86}$$

in view of (84) *satisfies*

$$\begin{cases} \| (\partial_t^2 - c^2 \Delta) \, u_k \, |_{x_n > 0} \|_{H^1} \to 0 \ \text{as} \ k \to \infty, \\ \sup_t \int_{x_n \geq 0} \left[\left| \dfrac{\partial u_k}{\partial t} \right|^2 + c^2 \left| \nabla_x u_k \right|^2 \right] dx \leq C k^{-1} \ \text{as} \ k \to \infty. \end{cases} \tag{87}$$

Remark 5.1 Note that, in contrast with (81), (87) provides a polynomial decay rate for the energy.

■

For the proof of Propositions 5.1 and 5.2 we refer to J. Ralston [R2]. We recall that the construction of the phase ψ and the symbol σ only involves the Taylor expansion on F and also that $\mathrm{Im}^2 \nabla^2 \varphi > 0$ implies that there is no caustics in ψ in Proposition 5.1.

5.2 Non-uniform decay

Let us recall the definition given in section 2 of the region $\mathcal{L} \subset \mathrm{Char}(\mathcal{T})$ that couples strongly the longitudinal and the transversal waves:

$$\mathcal{L} = \{ (y,t,\eta,\tau) : y \in \partial\Omega, \, 0 <| \eta | \leq | \tau | \nu_L \} \, . \tag{88}$$

Assume that Ω is as in Theorem 1.2 or, more generally, as described in section 2, let us assume that for any $T > 0$ the assumption (H_T) holds, i.e. there exists a ray $s \in [a,b] \to \rho(s) \in \mathrm{Char}(\mathcal{T})$ without contacts of infinite order with $\partial\Omega$ such that $t(\rho(b)) - t(\rho(a)) > T$ and $\rho(s) \notin \mathcal{L}$ for any $s \in [a,b]$. Under this assumption we are going to prove the existence of a family of solutions $\{\varphi_k\}$ of the Lamé system (5.1) such that

$$\left\| \left(\varphi_k^0, \varphi_k^1 \right) \right\|_{(L^2(\Omega))^n \times (H^{-1}(\Omega))^n} = 1 \, , \, \forall k \in I\!N \tag{89}$$

$$\int_0^T \| \, \mathrm{div} \, \varphi_k \, \|_{H^{-1}(\Omega)}^2 \, dt \to 0 \, , \ \text{as} \ k \to \infty. \tag{90}$$

More precisely, the following holds:

Theorem 5.1 *Assume that Ω is a C^∞ domain of $I\!R^n$ with $n = 2$ or 3. Assume also that, given $T > 0$, the condition (H_T) holds.*

Then, there exists a sequence of solutions φ_k of (5.1) *such that* (89)-(90) *holds.*

Proof of Theorem 5.1.

To fix ideas we assume that $n = 3$.

First of all we observe that it suffices to construct, for any $\varepsilon > 0$, a C^∞−function $\phi = \phi_\varepsilon$ such that

$$\begin{cases} \| \, \phi \, \|_{L^2(\Omega \times (0,T))} \geq 1; \left\| (\partial_t^2 - \mu\Delta - (\lambda + \mu) \nabla \, \mathrm{div}) \, \phi \right\|_{L^2(0,T; H^{-1}(\Omega))} \leq \varepsilon, \\ \| \, \phi \, \|_{L^2(\partial\Omega \times (0,T))} \leq \varepsilon; \displaystyle\int_0^T \| \, \mathrm{div} \, \phi \, \|_{H^{-1}(\Omega)}^2 \, dt \leq \varepsilon^2. \end{cases} \tag{91}$$

Indeed, if the sequence $\{\phi_\varepsilon\}$ satisfying (91) exists it suffices to take as initial data for the Lamé system (5.1):

$$\left(\varphi_k^0, \varphi_k^1\right) = (\phi_\varepsilon(0), \phi_{\varepsilon,t}(0)) \Big/ \alpha_\varepsilon$$

with $\alpha_\varepsilon = \|(\phi_\varepsilon(0), \phi_{\varepsilon,t}(0))\|_{(L^2(\Omega))^3 \times (H^{-1}(\Omega))^3}$ and $\varepsilon = 1/k$ and φ_k the corresponding solution of (5.1). One has $\alpha_\varepsilon \geq c_0 > 0$, otherwise we would have that $||\phi_\varepsilon||_{L^2(\Omega\times(0,T))} \to 0$ and this would contradict the first statement in (5.22). It is then easy to see that (89)-(90) holds.

Indeed,

$$\begin{aligned} \int_0^T \| \operatorname{div} \varphi_k \|_{H^{-1}(\Omega)}^2 dt &\leq 2\left[\frac{1}{\alpha_{1/k}^2 k^2} + \int_0^T \| \operatorname{div} w_k \|_{H^{-1}(\Omega)}^2\right] \\ &\leq \frac{C}{k^2} + C \| w_k \|_{L^2(\Omega\times(0,T))}^2 \end{aligned}$$

where $w_k = \varphi_k - \frac{\phi_{1/k}}{\alpha_{1/k}}$ solves

$$\begin{cases} \left[\partial_t^2 - \mu\Delta - (\lambda+\mu)\nabla \operatorname{div}\right] w_k = -\frac{1}{\alpha_{1/k}}\left[\partial_t^2 - \mu\Delta - (\lambda+\mu)\nabla \operatorname{div}\right]\phi_{1/k} = f_k & \text{in} \quad \Omega\times(0,T) \\ w_k = -\frac{\phi_{1/k}}{\alpha_{1/k}} = g_k & \text{on} \quad \partial\Omega\times(0,T) \\ w_k(0) = \partial_t w_k(0) = 0 & \text{in} \quad \Omega. \end{cases}$$

In view of the results of [Li] we know that

$$\| w_k \|_{L^\infty(0,T;L^2(\Omega))} \leq C\left[\| f_k \|_{L^1(0,T;H^{-1}(\Omega))} + \| g_k \|_{L^2(\partial\Omega\times(0,T))}\right]$$

and the right hand side of this equality tends to zero as $k \to \infty$ due to (91).

Thus, let us focus on the construction of the sequence $\{\phi_\varepsilon\}_{\varepsilon>0}$ satisfying (91).

Let $s \in [a,b] \to \rho(s) \in \mathrm{Char}(\mathcal{T})$ be a ray satisfying the hypothesis (H_T) above. Let $J = \{s \in [a,b] : \rho(s) \in \partial\Omega\}$. Clearly J is a closed set that can be split $J = J_\perp \cup J_{//}$ where

$$\begin{aligned} J_\perp &= \{s : \rho(s) = (y,t,\eta,\tau),\, y \in \partial\Omega,\, \eta = 0\}\,; \\ J_{//} &= \{s : \rho(s) = (y,t,\eta,\tau),\, y \in \partial\Omega,\, |\,\eta\,| > |\,\tau\,|\,\nu_L\}\,. \end{aligned}$$

The set $J_\perp$ is finite. Indeed it corresponds to the points where the ray intersects the boundary perpendicularly and the distance between two perpendicular reflections is bounded below by a positive constant depending only on the geometry of Ω. Therefore $J_{//}$ is closed as well and there exists $\delta > 0$ such that $|\,\eta\,| \geq |\,\tau\,| (\nu_L + \delta)$ for every $s \in J_{//}$.

Given $\varepsilon > 0$ and taking into account that every ray is the uniform limit of rays having only transversal intersections with $\partial\Omega$, there exists a ray $\widetilde{\rho} : [a,b] \to \widetilde{\rho}(s)$ such that $\widetilde{J} = \{s \in [a,b] : \widetilde{\rho}(s) \in \partial\Omega\}$ is finite, $\widetilde{J} = \widetilde{J}_\perp \cup \widetilde{J}_{//}$ with $\#\left(\widetilde{J}_\perp\right) \leq \#\left(J_\perp\right)$ and moreover:

$$\begin{cases} t(\widetilde{\rho}(a)) < 0; t(\widetilde{\rho}(b)) > T\,,\ a,b \notin \widetilde{J}; \\ s \in \widetilde{J}_\perp \Rightarrow |\eta\left(\widetilde{\rho}(s)\right)| \leq \varepsilon^2 \\ s \in \widetilde{J}_{//} \Rightarrow |\tau\left(\widetilde{\rho}(s)\right)|\left(\nu_L + \delta/2\right) \leq |\eta\left(\widetilde{\rho}(s)\right)| \leq |\tau\left(\widetilde{\rho}(s)\right)|\left(\nu_T - \varepsilon^2\right) \\ s \in \widetilde{J}_\perp, s' \in \widetilde{J}_{//} \Rightarrow |\, s - s'\,| \geq \frac{1}{2}\operatorname{dist}\left(J_\perp, J_{//}\right) > 0. \end{cases} \tag{92}$$

Note that, at this level we have used the fact that the ray has not contacts of infinite order with $\partial\Omega$.

Let $N = N(\varepsilon) = \#\left(\widetilde{J}\right)$. Then $\widetilde{J} = \{s_1, \cdots, s_N\}$ with $a = s_0 < s_1 < \cdots < s_N < b = s_{N+1}$ and let F_j be the segment in $\Omega \times I\!R$ projection over (t, x) of $\{\widetilde{\rho}(s) : s \in [s_j, s_{j+1}]\}$. We can assume that $\widetilde{\rho}(s_j) = (t_j, x_j, \tau_j, \xi_j)$ with $\tau_j \equiv -1/2$ and $t_j = s_j$.

We look for $\phi = \phi_\varepsilon$ solution of (91) of the form

$$\left.\begin{array}{l} \phi = \sum_{j=0}^{N} \phi_j \,;\, \phi_j = k^{-1/4}\left(\nabla v_j + \operatorname{curl} w_j\right) \\ v_j = \chi_{j,L} e^{ik\psi_{j,L}} \sigma_L^j \,;\, w_j = \ \chi_j e^{ik\psi_j} \sigma^j \end{array}\right\} \tag{93}$$

with $k > 0$ large and $\chi_j(t, x) \in C_c^\infty$ such that $\chi_j = 1$ near F_j and with support in a small neighborhood of F_j. We choose $\chi_{0,L} \equiv 0$ and therefore $v_0 \equiv 0$. On the other hand, $\chi_{j,L} \in C_c^\infty$ is such that $\chi_{j,L} \equiv 1$ in a neighborhood of (t_j, x_j) and has its support in a neighborhood of (t_j, x_j), for $j = 1, \cdots, N$.

The first term $e^{ik\psi_0}\sigma_0$ is constructed with the help of Proposition 5.1 with speed of propagation $c = c_T$. Notice however that x_0 has been assumed to belong to the interior of Ω. Thus, in this case, the hypersurface $\mathcal{S}$ is taken to be the hyperplane perpendicular to ξ_0 at x_0 and φ any function satisfying the conditions (77). Obviously, (τ_0, ξ_0) is determined by the segment F_0 mentioned above with $\tau_0 = c_T \mid \xi_0 \mid$ and $\tau_0 = -1/2$.

Thus $\phi_0 = k^{-1/4} \operatorname{curl}\left(e^{ik\psi_0}\sigma^0\right)$.

We then construct u_j for $j \geq 1$ by recurrence. We set $\varphi = \psi_{j-1}|_{\partial\Omega\times I\!R_t}$ that is well defined near (t_j, x_j). On the system of normal geodesic coordinates (x', x^3) near x_j we set $d_{x'}\varphi\,(t_j, x_j) = \eta_j$, $d_x\psi_{j-1}\,(t_j, x_j) = \left(\eta_j, -\xi_j^{(3)}\right)$. In view of the third statement of (92) we have $\xi_j^{(3)} \neq 0$. We denote by ψ_j the function given by Proposition 5.1 with speed of propagation c_T and such that $d_x\psi_j\,(t_j, x_j) = \left(\eta_j, \xi_j^{(3)}\right)$. In what concerns the construction of $\psi_{j,L}$ we distinguish two cases:

- If $t_j \in \widetilde{J}_{//}$ we have $\mid \eta_j \mid > \nu_L \mid \tau_0 \mid$ and we apply Proposition 5.2 with speed c_L that provides $\psi_{j,L}$ in a neighborhood of (t_j, x_j).
- If $t_j \in \widetilde{J}_\perp$ we have $\mid \eta_j \mid \leq \varepsilon^2 << \nu_L \mid \tau_0 \mid$ and we apply Proposition 5.1 with speed c_L and the choice of the ray such that $d_x\psi_{j,L}\,(t_j, x_j) = \left(\eta_j, \xi_{j,L}^{(3)}\right) = \xi_{j,L}$ with $\operatorname{sign}\left(\xi_{j,L}^{(3)}\right) = \operatorname{sign}\left(\xi_j^{(3)}\right)$ that defines $\psi_{j,L}$ in a neighborhood of the segment

$$F_{j,L} = \left\{(t, x(t)) : x(t) = x_j - (t - t_j)\,\tau_0 \frac{\xi_{j,L}}{\mid \xi_{j,L} \mid^2}\right\}. \tag{94}$$

The symbols are built so to satisfy the boundary condition $\phi|_{\partial\Omega} \sim 0$ and therefore are determined by recurrence from Propositions 5.1 and 5.2 and the choice of the traces such that:

$$\begin{array}{l} ik\left[\nabla\psi_{j,L}\sigma_L^j + \nabla\psi_j \wedge \sigma^j\right] + \nabla\sigma_L^j + \operatorname{curl}\sigma^j \\ +(ik\left(\nabla\psi_{j-1} \wedge \sigma_{j-1}\right) + \operatorname{curl}\sigma^{j-1})|_{\partial\Omega} \sim 0 \text{ on } \ (t_j, x_j)\,. \end{array} \tag{95}$$

System (95) is solvable due to the fact that

$$\nabla\psi_{j,L} \cdot \nabla\psi_j\,(t_j, q_j) = \eta_j^2 + \xi_j^{(3)}\xi_{j,L}^{(3)} \neq 0. \tag{96}$$

Obviously (95) has not to be understood as a pointwise identity but rather in the sense of the definition of $f \sim 0$ given in Section 5.1.

The solution of (95) is not unique. We impose to it the compatibility condition

$$\nabla\psi_j \cdot \sigma_0^j\,(t_j, x_j) = 0,\ \forall j \geq 0. \tag{97}$$

This guarantees that

$$\nabla\psi_j \cdot \sigma_0^j = 0 \text{ on } F_j,\ \forall j \geq 0. \tag{98}$$

Let us compute the solution of (95) at first order approximation. We have

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \delta \end{pmatrix} \sigma_{0,L}^j + \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \wedge \sigma_0^j = -\begin{pmatrix} \xi_1 \\ \xi_2 \\ -\xi_3 \end{pmatrix} \wedge \sigma_0^{j-1}.$$

By (ξ_1, ξ_2, ξ_3) we denote the three components of the vector $\left(\eta_j, \xi_j^{(3)}\right)$. Notice that the third component δ of the vector multiplying $\sigma_{0,L}$ may be purely imaginary when $t_j \in \widetilde{J}_{//}$.

Modulo a rotation and without loss of generality we can assume that $\xi_2 = 0$. By the normalization (97) we can write

$$\sigma_0^j = \alpha_+ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta_+ \begin{pmatrix} \xi_3 \\ 0 \\ -\xi_1 \end{pmatrix};\sigma_0^{j-1} = \alpha_- \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta_- \begin{pmatrix} \xi_3 \\ 0 \\ \xi_1 \end{pmatrix}$$

The system reduces to

$$\begin{pmatrix} -\alpha_+\xi_3 \\ \beta_+ \parallel \xi \parallel^2 \\ \alpha_+\xi_1 \end{pmatrix} + \sigma_{0,L}^j \begin{pmatrix} \xi_1 \\ 0 \\ \delta \end{pmatrix} = -\begin{pmatrix} \alpha_-\xi_3 \\ -\beta_- \parallel \xi \parallel^2 \\ \alpha_-\xi_1 \end{pmatrix}$$

or equivalently to

$$\begin{cases} \sigma_{0,L}^j\xi_1 - \alpha_+\xi_3 = -\alpha_-\xi_3 \\ \beta_+ = \beta_- \\ \alpha_+\xi_1 + \sigma_{0,L}^j\delta = -\alpha_-\xi_1. \end{cases}$$

We get

$$\beta_+ = \beta_-; \sigma_{0,L}^j = (\alpha_+ - \alpha_-)\,\frac{\xi_3}{\xi_1} = -\frac{2\alpha_-\xi_1\xi_3}{\xi_3\delta + \xi_1^2};\ \alpha_+ = \alpha_- \left(\frac{\xi_3\delta - \xi_1^2}{\xi_3\delta + \xi_1^2}\right). \tag{99}$$

It remains to build the cut-off functions.

The functions χ_j are chosen to be in C_0^∞ and equal to one near F_j. For $t_j \in \widetilde{J}_{//}$, $\chi_{j,L}$ is chosen to be in C_0^∞ and equal to one in a small neighborhood of (t_j, x_j). For $t_j \in \widetilde{J}_\perp$ we choose $\psi_{j,L} \in C_0^\infty$ such that $\chi_{j,L}\,(t, x(t)) = \theta\,(t - t_j)$ for $(t, x(t)) \in F_{j,L}$ with $\theta(t) \equiv 1$ for $\mid t \mid \leq a$ and $\theta(t) \equiv 0$ for $\mid t \mid \geq 2a$ where θ is a fixed function (independent of j and ε) and $a > 0$ is small enough such that when $\mid t - t_j \mid \leq 2a, \operatorname{dist}(x(t), \partial\Omega) \geq C|t - t_j|$.

Let us show that for k large enough ($k = k(\varepsilon)$) the function ϕ given in (93) satisfies (91) provided the integer N_0 that enters in the identities ~ 0 is large enough:

- The traces on the boundary are concentrated near the points (t_j, x_j). System (95) and the conditions (76)(1) and (83)(1) imply

$$\lim_{k\to\infty} \parallel \phi \parallel_{L^2(\partial\Omega\times(0,T))} = 0. \tag{100}$$

- For $t_j < t < t_{j+1}$, let e_j be the energy density (see (82)) (which is a constant independent of t):

$$e_j = \pi^{3/2}\tau_0^2 \mid \sigma_0^j \mid^2 \left(\det \ \mathrm{Im}\, \nabla_x^2 \psi_j\right)^{-1/2} (t, x(t)) \, , (t, x(t)) \in F_j \tag{101}$$

and for $t_j \in \widetilde{J}_\perp$, $t_j < t < t_{j+1}$,

$$e_{j,L} = \pi^{3/2}\tau_0^2 \nu_L^2 \left|\sigma_{0,L}^j\right|^2 \left(\det \ \mathrm{Im}\, \nabla_x^2 \psi_{j,L}\right)^{-1/2} (t, x(t)), \ (t, x(t)) \in F_{j,L}. \tag{102}$$

By an appropriate choice of $\sigma_0^0\,(t_0, x_0)$ we can suppose that $e_0 = 1$. We have

$$\begin{cases} \lim\limits_{k\to\infty} \parallel k^{-1/4}\, \mathrm{curl}\, w_j \parallel^2_{L^2(\Omega)} (t) = e_j \text{ for } t_j < t < t_{j+1}, \\ \lim\limits_{k\to\infty} \parallel k^{-1/4} \nabla v_j \parallel^2_{L^2(\Omega)} (t) = \left|\theta\,(t - t_j)\right|^2 e_{j,L} \text{ for } t_j < t, t_j \in \widetilde{J}_\perp. \end{cases} \tag{103}$$

In view of identity (99) it is easy to see that

$$\mid \nabla\psi_j \wedge \sigma_{0j} \mid = \mid \nabla\psi_{j-1} \wedge \sigma_0^{j-1} \mid \ \text{ on } \ (t_j, x_j) \tag{104}$$

for $t_j \in \widetilde{J}_{//}$ since δ is purely imaginary in this case.
On the other hand, for $t_j \in \widetilde{J}_\perp$, in view of (99) and (92)(2), we have

$$\left|(\nabla\psi_j + \nabla\psi_{j-1})\,(t_j, x_j)\right| \leq 2\varepsilon^2$$

and therefore there exists $C_0 > 0$ such that

$$\begin{cases} \left|\nabla\psi_j \wedge \sigma_0^j\right| (t_j, x_j) \geq (1 - C_0\varepsilon^2) \left|\nabla\psi_{j-1} \wedge \sigma_0^{j-1}\right| (t_j, x_j) \\ \left|\nabla\psi_{j,L}\sigma_{0,L}^j\right| (t_j, x_j) \leq C_0\varepsilon^2 \left|\nabla\psi_{j-1} \wedge \sigma_0^{j-1}\right| (t_j, x_j)\, , \ \forall t_j \in \widetilde{J}_\perp. \end{cases}$$

In view of (78) and taking into account that $\widetilde{J}_\perp$ is a finite set, we deduce the existence of a constant C_1 that only depends on $\# J_1$ such that

$$\forall j, e_j \geq 1 - C_1\varepsilon^2\, ; \, \forall t_j \in \widetilde{J}_\perp, \, e_{j,L} \leq C_1\varepsilon^2.$$

Therefore

$$\lim_{k\to\infty} \parallel \phi \parallel^2_{L^2(\Omega\times(0,T))} \geq T\left(1 - O(\varepsilon^2)\right).$$

On the other hand, in view of (87)(2) and (102)(2) we deduce that

$$\limsup_{k\to\infty} \int_0^T \parallel \mathrm{div}\, \phi \parallel^2_{H^{-1}(\Omega)} dt \leq \limsup_{k\to\infty} \sum_{\widetilde{J}_\perp} \int_0^T k^{-1/2} \parallel \nabla v_j \parallel^2_{L^2(\Omega)} = O(\varepsilon^4).$$

- Finally, we have

$$\limsup_{k\to\infty} \left\| \left(\partial_t^2 - \mu\Delta - (\lambda+\mu)\nabla \operatorname{div}\right) \phi \right\|^2_{L^2(0,T;(H^{-1}(\Omega))^3)}$$
$$\leq \limsup_{k\to\infty} \sum_{\widetilde{J}_\perp} \int_0^T k^{-1/2} \left\| \left(\partial_t^2 - c_L^2\Delta\right) v_j \right\|^2_{L^2(\Omega)} dt$$
$$= \limsup_{k\to\infty} \sum_{\widetilde{J}_\perp} \int_0^T k^{-1/2} \left\| \left(\partial_t^2 - c_L^2\Delta\right) \chi_{j,L} e^{ik\psi_{j,L}} \sigma_L^j \right\|^2_{L^2(\Omega)} dt$$
$$\leq C \sum_{\widetilde{J}_\perp} e_L^j \int_{I\!R} \mid \theta'(s) \mid^2 ds = O(\varepsilon^2).$$

This concludes the proof of the non-uniform decay under the hypothesis (H_T).

■

Remark 5.2 Note that part of the technical difficulties of the proof are related to the fact the ray provided by (H_T) has to be slightly modified to guarantee that only transversal intersections with the boundary arise. Then we have to make sure that the estimates do not blow up as $\varepsilon \to 0$. This is done by taking limits as $k \to \infty$ and making use of the conservation of energy.

■

Remark 5.3 The C^∞ assumption on the regularity of the domain Ω is unnecessary. It is easy to see that the construction above applies when Ω is of class C^3 along rays that only have transversal intersections with the boundary. Thus, Theorem 5.1 applies for domains of class C^3 provided we assume the existence of a family of rays ρ_ε satisfying (5.1) for $0 < \varepsilon < \varepsilon_0$.

■

Remark 5.4 In view of the construction above it is natural to consider the problem of whether there exists a family of solutions of the Lamé system (5.1) such that

$$\left\| \left(\varphi_k^0, \varphi_k^1\right) \right\|_{(L^2(\Omega))^3 \times (H^{-1}(\Omega))^3} = 1,\ \forall k \in I\!N \tag{105}$$

and

$$\int_0^T \|\operatorname{curl} \varphi_k\|^2_{(H^{-1}(\Omega))^3} dt \longrightarrow 0,\ \text{ as } k \to \infty. \tag{106}$$

It is easy to check that our construction can be done at points of perpendicular reflection but it does not apply when a longitudinal ray intersects the boundary almost tangentially since a transversal wave is reflected provided $\lambda + 2\mu > \mu$.

Thus, in the range $\lambda+2\mu > \mu$, a sequence of solutions $\{\varphi_k\}$ of (5.1) can be built so that (105)-(106) holds provided there is a longitudinal ray in Ω along the time interval $(0, T)$ that is always reflected perpendicularly on the boundary. Obviously, when Ω is a smooth convex domain this holds for all $T > 0$.

■

Remark 5.5 Similar problems can be considered when $0 < \lambda + 2\mu < \mu$. In this range of Lamé coefficients $\operatorname{div} \varphi$ and $\operatorname{curl} \varphi$ play reverse roles. It is then easier to build sequences of solutions satisfying (105)-(106) than those verifying (89)-(90).

■

5.3 A refined result in three space dimensions

A careful analysis of the construction above yields to the following remark showing that the assumption (H_T) is not sharp in three space dimensions when constructing the sequence of solutions $\{\varphi_k\}$ of the Lamé system (5.1) satisfying (89)-(90).

Indeed, let us analize the solutions (99) of system (95). Clearly,

$$\begin{cases} \alpha_+ = \alpha_- = 0 \\ \beta_+ = \beta_- \end{cases}$$

is always a solution, independently of the nature of the ray. This corresponds to a purely transversal wave that is polarized in the "critical direction" $\pi_\rho^+ \cap \pi_\rho^-$. If this polarization propagates along the ray so that it coincides with the critical polarization direction at every reflection in which the construction of section 5.2 fails, i.e. essentially when $0 <\mid \eta \mid < \nu_L \mid \tau \mid$, then the same construction with this choice of the polarization direction yields the desired sequence of solutions.

Thus, we introduce the following refined version of (H_T) in three space dimensions:

$$(H_T^3) \qquad \begin{cases} \text{There exists a transversal bicharacteristic ray of length } T \text{ such that} \\ \text{(a) When } \gamma(s) \in \partial\Omega, \mid \eta \mid \neq \nu_L \mid \tau \mid \text{ and } \mid \eta \mid \neq \nu_T \mid \tau \mid; \\ \text{(b) Let } \mathcal{G} = \{s : \gamma(s) \in \partial\Omega \text{ and } 0 <\mid \eta(s) \mid < \nu_L \mid \tau \mid\}. \\ \quad \text{Clearly } \mathcal{G} = \{s_1, \cdots, s_N\} \text{ with } N = 0 \text{ if } \mathcal{G} = \emptyset. \\ \quad \text{Then the polarization directions } e_i \in \pi^+_{\gamma(s_i)} \cap \pi^-_{\gamma(s_i)} \\ \quad \text{are connected by the propagation .} \end{cases}$$

Obviously (H_T^3) is less restrictive than (H_T) since it allows the ray to enter the region $0 <\mid \eta \mid < \nu_L \mid \tau \mid$. However at those points we need to impose the polarization directions $\pi_\gamma^+ \cap \pi_\gamma^-$ to be connected by the propagation to be able to proceed as mentioned above. Notice that in (a) we have imposed the condition $\mid \eta(s) \mid \neq \nu_T \mid \tau(s) \mid$ when $\gamma(s) \in \partial\Omega$. Thus the ray never meets tangentially the boundary. We did not do that in (H_T) but in that case using the fact that the ray does not have contacts of infinite order with $\partial\Omega$, the ray γ was approximated by rays γ_ε fulfilling this condition and preserving the other essential properties. In this case, by imposing this condition (a), we avoid the analysis of the stability of condition (b) with respect to perturbations of the ray.

It is straightforward to see that if we proceed as in section 5.2 by choosing the polarization direction given by $e_i \in \pi^+_{\gamma(s_i)} \cap \pi^-_{\gamma(s_i)}$ for any $s_i \in \mathcal{G}$ the following holds:

Theorem 5.2 *Let Ω be a bounded domain of class C^∞ of $\mathbb{R}^3$ such that (H_T^3) holds. Then there exists a sequence of solutions $\{\varphi_k\}$ of the Lamé system* (5.1) *satisfying* (89)-(90) *in the time interval* $(0, T)$.

6 Uniform decay

6.1 Uniform decay in two space dimensions

In this section Ω denotes a bounded, smooth, open set of $I\!R^2$ without contacts of infinite order with its tangents. We also assume that

$$\begin{cases} \text{(i)} \quad \Omega \text{ satisfies the spectral condition } (C); \\ \text{(ii) There exists } T_0 > 0 \text{ such that every transversal ray of length } T_0 \text{ intersects} \\ \qquad \overset{\circ}{\mathcal{L}} = \{(y,t,\eta,s) : y \in \partial\Omega, 0 < c_L \mid \eta \mid < \mid \tau \mid\}. \end{cases} \tag{107}$$

The main result of this section is as follows:

Theorem 6.1 *Under these assumptions, there exist $C, \omega > 0$ such that every solution (u,θ) of the system of thermoelasticity* (1) *of finite energy satisfies*

$$E(t) \leq Ce^{-\omega t} E(0), \forall t > 0. \tag{108}$$

Theorem 6.1 is a consequence of the following result:

Theorem 6.2 *Assume that Ω satisfies* (6.1)*. Then, if $T > T_0$ for any solution (u,θ) of* (1) *such that $(u,\theta) \in \left(L^2(\Omega \times (0,T))\right)^3$ and satisfying*

$$\theta \in L^2\left(0,T; H_0^1(\Omega)\right) \tag{109}$$

it follows that

$$u \in \left(H^1(\Omega \times (0,T))\right)^2. \tag{110}$$

Assuming for the moment that Theorem 6.2 holds true let us see that Theorem 6.1 holds true as well. The proof is similar to that of Lemma 3.3.

We introduce the following Hilbert spaces of solutions $(u,\theta) \in \left(\mathcal{D}'(\Omega \times (0,T))\right)^3$ of (1):

$$\begin{aligned} F_0^T &= \left\{(u,\theta) \in \left(L^2(\Omega \times (0,T))\right)^3\right\}; \\ F_1^T &= \left\{(u,\theta) \in F_0^T : \int_0^T \int_\Omega \mid \nabla\theta \mid^2 dxdt < \infty\right\}; \\ G^T &= \left\{(u,\theta) \in F_1^T : u \in \left(H^1(\Omega \times (0,T))\right)^2\right\}. \end{aligned}$$

In view of Theorem 6.2 the image of the imbedding $F_1^T \hookrightarrow F_0^T$ is contained in G^T. Then $G^T = F_1^T$ and by the closed graph Theorem the existence of $C > 0$ such that

$$\| (u,\theta) \|_{G^T} \leq C \| (u,\theta) \|_{F_1^T}, \forall (u,\theta) \in F_1^T$$

follows. Since Ω satisfies (C), by the compactness of the embedding $H^1(\Omega \times (0,T)) \hookrightarrow L^2(\Omega \times (0,T))$ it follows that, for $T > 0$ large enough, there exists $C > 0$ such that

$$\| u \|^2_{(H^1(\Omega\times(0,T)))^2} \leq C \int_0^T \int_\Omega \mid \nabla\theta \mid^2 dxdt$$

for every solution of (1).

Thus

$$E(0)-E(T)=\frac{\alpha}{\beta}\int_0^T\int_\Omega \mid \nabla\theta \mid^2 dxdt \geq C\int_0^T E(t)dt \geq CTE(T).$$

This inequality implies the exponential decay property (108).

■

In order to prove Theorem 6.2 we need the following auxiliary result:

Theorem 6.3 *Assume that* Ω *satisfies* (6.1). *Let* $T > T_0$. *Then, if* u *is a solution of the Lamé system*

$$\begin{cases} u_{tt}-\mu\Delta u-(\lambda+\mu)\nabla \operatorname{div} u=0 & in \quad \Omega\times(0,T), \\ u=0 & on \quad \partial\Omega\times(0,T) \end{cases} \tag{111}$$

such that

$$\operatorname{div} u \in L^2(\Omega\times(0,T)) \tag{112}$$

it follows that

$$u\in\Big(H^1(\Omega\times(0,T))\Big)^2. \tag{113}$$

The proof of the fact that Theorem 6.3 implies Theorem 6.2 is analogous to the one of Lemma 3.3.

Let us finally prove Theorem 6.3:

Proof of Theorem 6.3.

Let $I=(a,b)$ be an open interval such that $[a,b]\subset(0,T)$. It is easy to see that (113) follows from the fact that $u\in\left(H^1(\Omega\times I)\right)^2$.

To prove this we first observe that, as a consequence of (111), $WF_b(u)\subset \text{Char}$. (Here and in the sequel we keep the same notations of previous sections). Thus, it is sufficient to check that

$$u\in H^1_\rho \text{ for all } \rho\in \text{Char such that } t(\rho)\in[a,b]. \tag{114}$$

When $x(\rho)\in\Omega$ and $\mid\tau(\rho)\mid=c_L\mid\xi(\rho)\mid$, (114) is a consequence of (112). Thus, we may assume that $\rho\in \text{Char}(\mathcal{T})$. Let γ be the transversal ray passing through ρ. In view of assumption (115) (ii), there exists a point ρ_1 of γ such that $\rho_1\in\overset{\circ}{\mathcal{L}}$ and $t(\rho_1)\in(0,T)$. From Appendix A, u can be decomposed as $u=\nabla u+\operatorname{curl} w$ near $x(\rho_1)$. Since $\operatorname{div} u\in L^2$ we deduce that $\Delta v\in L^2$ and therefore $v\in H^2_{\rho_1}$ since $\tau(\rho_1)\neq 0$ and $\left(\partial_t^2-c_L^2\Delta\right)v=0$. Since ρ_1 is of hyperbolic type for $\partial_t^2-c_L^2\Delta$ we have $\nabla v|_{\partial\Omega\times(0,T)}\in H^1_{\rho_1}$. Therefore, since $u=0$ on $\partial\Omega\times(0,T)$, we also have that $\operatorname{curl} w|_{\partial\Omega\times(0,T)}\in H^1_{\rho_1}$. Since $\mid\eta(\rho_1)\mid>0$ we deduce that $\frac{\partial w}{\partial n}\Big|_{\partial\Omega\times(0,T)}\in H^1_{\rho_1}$ and $w|_{\partial\Omega\times(0,T)}\in H^2_{\rho_1}$. Since ρ_1 is of hyperbolic type for $\partial_t^2-c_T^2\Delta$ we deduce that $w\in H^2_{\rho_1}$ and therefore $u\in H^1_{\rho_1}$. The propagation Theorem of Appendix B allows us to conclude that $u\in H^1_\rho$.

■

6.2 Uniform decay in three space dimensions: A conjecture

In order to extend the results of section 6.1 above to three space-dimensions the polarization phenomena has to be taken into account. We conjecture that the following is true:

Conjecture:*Assume that* Ω *is a bounded, smooth, open set of* $\mathbb{R}^3$ *without contacts of infinite order with its tangents. Assume also that*

$$\begin{cases} (i) \quad \Omega \text{ satisfies the spectral condition } (C); \\ (ii) \text{ There exists } T_0 > 0 \text{ such that every transversal ray of length } T_0 \text{ intersects} \\ \qquad \overset{\circ}{\mathcal{L}} = \{(y,t,\eta,s) : y \in \partial\Omega, 0 < c_L \mid \eta \mid < \mid \tau \mid\} \\ \quad \text{twice at points where the critical polarization directions } \pi_\rho^+ \cup \pi_\rho^- \\ \quad \text{are not connected by the propagation.} \end{cases} \tag{115}$$

Then, solutions of the 3d system of thermoelasticity (1.1) decay uniformly as $t \to \infty$.

7 Polynomial decay in two space dimensions

All along this section we assume that Ω is a bounded smooth domain of $\mathbb{R}^2$ without contacts of infinite order with its tangents.

We also assume that

$$\Omega \text{ is not a ball neither an annulus of the form } \mathcal{O}/\lambda\mathcal{O} \text{ with } \mathcal{O} \text{ a ball and } 0 < \lambda < 1. \tag{116}$$

This section is divided in two paragraphs. In the first one we derive some estimates of the total energy of solutions of the Lamé system in terms of its longitudinal component but with a loss of one derivative. In the second one, combining these results and those of section 3.2 we deduce some polynomial decay rates.

7.1 Inequalities with deffect

The main result of this paragraph is as follows:

Theorem 7.1 *Let* Ω *satisfy assumption* (116) *above. Let* T_c *be the geometric control time in* Ω *from the boundary with velocity* c_T. *Assume that* $T > 2T_c$.

Then, if $u \in (\mathcal{D}'(\Omega \times (0,T)))^2$ *solves*

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u = 0 & \text{in} \quad \Omega \times (0,T), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T), \end{cases} \tag{117}$$

and

$$\operatorname{div} u \in L^2(\Omega \times (0,T)) \tag{118}$$

it follows that

$$u \in L^2(\Omega \times (0,T)). \tag{119}$$

Remark 7.1 T_c is the supremum of the lengths of the characteristic rays for $\partial_t^2 - c_T^2\Delta$ in $\Omega \times \mathbb{R}_t$ before they intersect the boundary at a non-diffractive point. From [BLR] we know, for instance, that T_c is the minimal controllability time for the scalar wave equation $(\partial_t^2 - c_T^2\Delta)\, u = 0$ in $H_0^1(\Omega) \times L^2(\Omega)$ with $L^2(\partial\Omega \times (0,T))$ Dirichlet boundary controls.

■

Proof of Theorem 7.1.

Recall that Char = $\mathrm{Char}_\Omega \cup \mathrm{Char}_{\partial\Omega}$ and let us decompose $\mathrm{Char}_{\partial\Omega}$ in the following disjoint subsets:

$$\mathrm{Char}_{\partial\Omega} = \mathrm{Char}^{\perp}_{\partial\Omega} \cup \overset{\circ}{\mathcal{L}} \cup Z$$

where

$$\begin{aligned}
&\mathrm{Char}^{\perp}_{\partial\Omega} = \{(y,t,\eta,\tau) \in \mathrm{Char}_{\partial\Omega} : \eta = 0\}\,;\\
&\overset{\circ}{\mathcal{L}} = \left\{(y,t,\eta,\tau) \in \mathrm{Char}_{\partial\Omega} : 0 <\mid \eta \mid < c_L^{-1} \mid \tau \mid\right\};\\
&Z = \left\{(y,t,\eta,\tau) \in \mathrm{Char}_{\partial\Omega} : c_L^{-1} \mid \tau \mid \leq \mid \eta \mid \leq c_T^{-1} \mid \tau \mid\right\}.
\end{aligned}$$

Let us also recall that the transversal characteristic manifold is given by $\mathrm{Char}(\mathcal{T}) = \mathrm{Char}_\Omega(\mathcal{T}) \cup \mathrm{Char}_{\partial\Omega}$ with $\mathrm{Char}_\Omega(\mathcal{T}) = \{x,t,\xi,\tau) : x \in \Omega, t \in (0,T), \mid \tau \mid = c_T \mid \xi \mid\}$.

In view of (117) we have $WF_b(u) \subset \mathrm{Char}$. Thus (119) is a consequence of the assertion

$$u \in \left(L^2_\rho\right)^2, \forall \rho \in \mathrm{Char}\,. \tag{120}$$

By virtue of the decomposition Lemma of Appendix A, we can decompose u as

$$u = \nabla v + \mathrm{curl}\, w; \left(\partial_t^2 - c_L^2 \Delta\right) v = 0; \left(\partial_t^2 - c_T^2 \Delta\right) w = 0 \tag{121}$$

near any point (x_0, t_0).

We have $\mathrm{div}\, u = \Delta v \in L^2(\Omega \times (0,T))$. Therefore, we also have $\partial_t^2 v \in L^2(\Omega \times (0,T))$. When $x_0 \in \Omega$ and $\rho \in \mathrm{Char}$ is such that $x(\rho) = x_0$ we have $v \in H^2_\rho$. Thus (117) is immediately true for $\rho \in \mathrm{Char} \setminus \mathrm{Char}(\mathcal{T})$.

When $x_0 \in \partial\Omega$ and $\rho \in \mathrm{Char}_{\partial\Omega}$ is such that $x(\rho) = x_0$, taking into account that ∂_t is a tangential elliptic operator of order 1 in ρ we have $v \in H^2_\rho$. We deduce that

$$\left\{\begin{array}{l} \rho \in \mathrm{Char}^{\perp}_{\partial\Omega} \cup \overset{\circ}{\mathcal{L}} \Rightarrow \nabla v\Big|_{\partial\Omega\times(0,T)} \in \left(H^1_\rho\right)^2; \\ \rho \in Z \Rightarrow \nabla v|_{\partial\Omega\times(0,T)} \in \left(H^{1/2}_\rho\right)^2. \end{array}\right. \tag{122}$$

The first assertion in (122) is due to the fact that the points ρ under consideration are hyperbolic with respect to $\partial_t^2 - c_L^2 \Delta$. The second one is a trace result.

Taking into account that u vanishes on the boundary we deduce that (122) implies

$$\left\{\begin{array}{l} \rho \in \mathrm{Char}^{\perp}_{\partial\Omega} \cup \overset{\circ}{\mathcal{L}} \Rightarrow \mathrm{curl}\, w\Big|_{\partial\Omega\times(0,T)} \in \left(H^1_\rho\right)^2; \\ \rho \in Z \Rightarrow \mathrm{curl}\, w|_{\partial\Omega\times(0,T)} \in \left(H^{1/2}_\rho\right)^2. \end{array}\right. \tag{123}$$

This is true for every decomposition (121).

Therefore it is sufficient to show that

$$w \in H^1_\rho, \forall \rho \in \mathrm{Char}(\mathcal{T}). \tag{124}$$

In view of previous results, assertion (124) is independent of the decomposition (121).

We have $(\partial_t^2 - c_T^2\Delta)w = 0$ and in view of (123) it follows that $w|_{\partial\Omega\times(0,T)} \in H_\rho^{3/2}$ when $\rho \notin \mathrm{Char}_{\partial\Omega}^{\perp}$ and $\partial w/\partial n|_{\partial\Omega\times(0,T)} \in H_\rho^1$ when $\rho \in \mathrm{Char}_{\partial\Omega}^{\perp}$ and these boundary conditions guarantee that the H^1 regularity of w propagates along transversal rays.

Let $\rho \in \mathrm{Char}(\mathcal{T})$ and let $\gamma : (0,T) \to \mathrm{Char}(\mathcal{T})$ be the unique transversal ray (time t being its parameter) satisfying $\gamma(t(\rho)) = \rho$. Since, by hypothesis, $T > T_c$, there exists $t_1 \in (0,T)$ such that $\gamma(t_1) = \rho_1 \in \mathrm{Char}_{\partial\Omega}$ is a non-diffractive point of the boundary. We now distinguish three cases:

Case 1. If $\rho_1 \notin \mathrm{Char}_{\partial\Omega}^{\perp}$, by (123) we have $\mathrm{curl}\, w|_{\partial\Omega\times(0,T)} \in H_{\rho_1}^{1/2}$ and this is equivalent to

$$\partial w/\partial n|_{\partial\Omega\times(0,T)} \in H_{\rho_1}^{1/2}$$

and $w|_{\partial\Omega\times(0,T)} \in H_{\rho_1}^{3/2}$. Thus, since ρ_1 is non-diffractive, by the lifting Lemma of [BLR], we have $w|_{\partial\Omega\times(0,T)} \in H_{\rho_1}^{3/2}$ and by propagation $w \in H_\rho^1$.

Case 2. If $\rho_1 \in \mathrm{Char}_{\partial\Omega}^{\perp}$, we denote by Γ the connected component of $\partial\Omega$ containing $x(\rho_1)$ and by ∂_y the tangential derivative along Γ. In view of (123) we have $\mathrm{curl}\, w|_{\partial\Omega\times(0,T)} \in H_\Gamma^1$ and therefore $\partial_y w|_{\partial\Omega\times(0,T)} \in H_\sigma^1$ for all σ in a neighborhood of $\mathrm{Char}_{\partial\Omega}^{\perp}$. Then the H^1 regularity of $g = w|_{\partial\Omega\times(0,T)}$ propagates along the bicharacteristics of ∂_y (i. e. along the curves $(y, t = t_1, \eta = 0, \tau = \tau_1)$). We set $\rho_1 = (y_1, t = t_1, \eta = 0, \tau = \tau_1)$. If there exists a point $\sigma = (y, t = t_1, 0, \tau_1)$ with $y \in \Gamma$ where the argument of Case 1 may be applied we will have $w \in H_\sigma^1$ and therefore $g = w|_{\partial\Omega\times(0,T)} \in H_\sigma^1$ as well (since σ is of hyperbolic type). Thus $w|_{\partial\Omega\times(0,T)} \in H_{\rho_1}^1$ and since $\mathrm{curl}\, w|_{\partial\Omega\times(0,T)} \in H_{\rho_1}^1$ we deduce that $\partial w/\partial n|_{\partial\Omega\times(0,T)} \in L_{\rho_1}^2$. Since ρ_1 is non-diffractive, by the lifting Lemma of [BLR], we deduce that $w \in H_{\rho_1}^1$ and by propagation $w \in H_\rho^1$.

Since $t_1 \in (0,T)$ and $T > 2T_c$, there exists a sign $\varepsilon = \pm 1$ such that $t_1 + \varepsilon s \in (0,T)$ for all $s \in [0, T_c)$. If there is no point $\sigma = (y, t_1, 0, \tau_1)$, $y \in \gamma$ where the argument of Case 1 can be applied, the first point of intersection of the half-line $\{y + l n_y; l > 0\}$ with $\partial\Omega$ has to be a point of perpendicular intersection for all $y \in \Gamma$. Since Ω is not a ball, Ω is then necessarily an annular domain

$$\Omega = \{x \in \mathbb{R}^2 : x = y + l n_y : y \in \Gamma, 0 < l < l_0\}. \tag{125}$$

Case 3. To finish the proof of Theorem 7.1 we may suppose that Ω is of the form (125) and that the curvature of Γ is not constant. We may also assume that Γ is the interior boundary of Ω and we denote by s the curvilinear abscissa of Γ. Let $y_0(s = 0) \in \Gamma$ and ω a cylindrical neighborhood of $\{y_0 + l n_{y_0} : 0 < l < l_0\}$ in Ω. Let $\rho_0 = (y_0, t_0; \eta = 0, \tau_0)$, $t_0 \in (0,T)$. We denote by $x(s)$ the curvature of Γ at s with the convention that $x < 0$ when Γ is concave with respect to Ω as in Figure 8 below.

Figure 8

The proof of Theorem 7.1 will be complete once the following Lemma is proved:

Lemma 7.1 *If $dx(0)/ds \neq 0$ then $w \in H^1_{\rho_0}$*

Indeed in view of this Lemma, as soon as the curvature x varies $w \in H^1_{\rho_0}$. We have also shown that the H^1 regularity propagates along transversal rays. This shows that $w \in H^1_\rho$ unless Ω is a spherical annulus.

Proof of Lemma 7.1. We may suppose that $t_0 \in (0, T/2)$ (otherwise it is sufficient to change the time variable $t \to T - t$). Let $u = \nabla v + \operatorname{curl} w$ a decomposition of u in $(t_0 - \delta, t_0 + lc_T^{-1} + \delta) \times \omega$, with $\delta > 0$ small enough. We have $(\partial_t^2 - c_T^2 \Delta) w = 0$ and near $\mathrm{Char}^{\perp}_{\partial\Omega}$, $\mathrm{curl} w|_{\partial\Omega \times (0,T)} \in H^1$ (in view of (123)). On the normal geodesic system, the Laplacian over ω can be written as:

$$\Delta = \frac{1}{a}\partial_l(a\partial_l) + \frac{1}{a}\partial_s(\frac{1}{a}\partial_s), \quad a = 1 - lx(s). \tag{126}$$

Since ρ_0 is of hyperbolic type, the solution w of $(\partial_t^2 - c_T^2\Delta)w = 0$ near ρ_0 can be decomposed into an incoming wave w_-, an outoming wave w_+ and a smooth function r:

$$\begin{cases} w = w_- + w_+ + r \\ \rho_0 \notin WF_b((\partial_t^2 - c_T^2\Delta)w_\pm) \cup WF_b(r) \\ w_\pm = \frac{1}{4\pi^2}\int e^{i\varphi_\pm}\sigma_\pm \hat{g}_\pm(\eta,\tau)d\eta d\tau \end{cases} \tag{127}$$

where the phases $\varphi_\pm(l,t,s;\eta,\tau) = \tau t + |\tau|\psi_\pm(l,s,\eta/\tau)$ satisfy the eikonal equation

$$c_T^2\left[(\partial\psi/\partial l)^2 + \frac{1}{a^2}(\partial\psi/\partial s)^2\right] = 1; \quad \psi_\pm|_{l=0} = s\eta/|\tau|; \quad \pm\partial\psi_\pm/\partial l|_{l=0} > 0 \tag{128}$$

and are defined for $\alpha = \eta/|\tau|$ near 0, and where the symbols $\sigma = \sum_{n \geq 0} \sigma^n(l,s,\alpha)(i|\tau|)^{-n}$ satisfy the transport equations

$$\begin{cases} i\partial\varphi/\partial l\partial\sigma/\partial l + \frac{1}{a}\partial_l(a\partial\sigma/\partial l + ia\sigma\partial\varphi/\partial l) + \frac{i}{a^2}\partial\varphi/\partial s\partial\sigma/\partial s + \frac{1}{a}\partial_s(\frac{1}{a}\partial\sigma/\partial s + \frac{i}{a}\partial\varphi/\partial s\sigma) \sim 0 \\ \sigma|_{l=0} = 1 \end{cases} \tag{129}$$

with $\varphi = \varphi_\pm$ for $\sigma = \sigma_\pm$.

In view of (128) we have

$$\partial\varphi_\pm/\partial l = \partial\psi_\pm/\partial l|_{l=0} = \pm(1/c_T^2 - \eta^2/\tau^2)^{1/2} \tag{130}$$

$$\varphi_\pm|_{\eta=0} = \tau t \pm |\tau| l/c_T \tag{131}$$

and using (129) and (131) the leading terms $\sigma^0_\pm$ of the symbols $\sigma_\pm$ satisfy

$$\partial h_\pm/\partial l + \frac{1}{a}\partial(ah_\pm)/\partial l = 0; \quad h_\pm|_{l=0} = 1$$

with $h_\pm = \sigma^0_\pm|_{\eta=0}$. Thus

$$\sigma^0_\pm|_{\eta=0} = h(l,s) = (1 - lx(s))^{-1/2} = a^{-1/2}. \tag{132}$$

We have $w_\pm|_{l=0} = g_\pm$, $\partial w_\pm/\partial l|_{l=0} = E^\pm(g_\pm)$ where $E^\pm$ is a pseudodifferential operator of degree 1 of principal symbol $\sigma_1(E^\pm) = i|\tau|\partial\psi_\pm/\partial l|_{l=0} = \pm i(|\tau|^2/c_T^2 - \eta^2)^{1/2}$. Thus $-(E^-)^{-1}E^+ = Id + R_{-1}$, where R_{-1} denotes a pseudodifferential operator of order -1.

Condition $\mathrm{curl}\, w|_{l=0} \in H^1_{\rho_0}$ is equivalent to

$$\partial(g_+ + g_-)/\partial s \in H^1_{\rho_0}, \quad E^+(g_+) + E^-(g_-) \in H^1_{\rho_0} \tag{133}$$

and this implies the existence of pseudodifferential operators of order -1 such that

$$(\partial_s + R_{-1})g_+ \in H^1_{\rho_0}, \quad (\partial_s + R_{-1})g_- \in H^1_{\rho_0}. \tag{134}$$

When $\alpha = \eta/|\tau|$ is close to zero the phases and symbols $\psi_\pm$, $\sigma_\pm$ are defined in a neighborhood of $l \in [0, l_0]$ so that w_+ will be the incoming part of w at y_1 at time $t_1 = t_0 - \mathrm{sign}(\tau_0)l_0$ and w_- the outcoming part of w at y_1 at time $t_0 + \mathrm{sign}(\tau_0)l_0$. If $\mathrm{sign}(\tau_0) < 0$ (the other case can be treated in a similar way), using (134) in y_1, in a neighborhhod of $\rho_1 = (y_1, t_1, 0, \tau_0)$ there exists a pseudodifferential operator R_{-1} such that $(\partial_s + R_{-1})[w_+|_{l=l_0} \in H^1_{\rho_1}$.

Let us denote $\varphi_0 = \varphi_+|_{l=l_0}$. Given a symbol $q = \sum_{n\geq 0} q^{(n)}(s, \eta/|\tau|)(i|\tau|)^{d-n}$ of degree d defined near $s = 0, \eta/|\tau| = 0$, the operator $g \to A_q(g) = \frac{1}{4\pi^2}\int e^{i\varphi_0} q\hat{g}\, d\eta d\tau$ is a Fourier integral operator of degree d and we have (see [Ho], vol. 3)

$$A_q(g) \in H^{s-d}_{\rho_1}, \quad \forall g \in H^s_{\rho_0}. \tag{135}$$

Moreover, if q is elliptic, i. e. if $q^0(0,0) \neq 0$, we have

$$A_q(g) \in H^{s-d}_{\rho_1}, \quad \text{if and only if} \quad g \in H^s_{\rho_0}. \tag{136}$$

On the other hand, if R is a pseudodifferential operator of degree d defined near ρ_1 there exists a pseudodifferential operator S defined near ρ_0 of degree d such that

$$RA_q(g) = A_q(Sg) \tag{137}$$

and $\sigma_d(R)(\rho_1) = \sigma_d(S)(\rho_0)$.

Thus, for $q_0 = \sigma_+|_{l=l_0}$ we have

$$(\partial_s + R_-)g_+ \in H^1_{\rho_0}; \quad (\partial_s + R_+)A_{q_0}(g_+) \in H^1_{\rho_1}. \tag{138}$$

On the other hand, $\partial_s A_{q_0} = A_p$ with $p = i\partial\varphi_+/\partial s|_{l=l_0} q_0 + \partial q_0/\partial s = q_1 + q_2$. In view of (131) we have $\partial\varphi_+/\partial s|_{\eta=0} = 0$. Thus $q_1 = iq\eta$ where q is a symbol of degree 0. The second statement of (138) and (137) imply that

$$A_q(\partial_s g_+) + A_{q_2}(g_+) A_{q_0}(R_- g_+) \in H^1_{\rho_1}.$$

We now denote by R any pseudodifferential operator of order -1. Thus using the first statement of (138) and (135), we obtain

$$A_{q_2}(g_+) + A_{q_0} R g_+ + A_q R g_+ \in H^1_{\rho_1}. \tag{139}$$

From the fact that $dx(0)/ds \neq 0$ we deduce that $q_2 = \partial q_0/\partial s = \partial\sigma_+/\partial s|_{l=l_0}$ satisfies $q_2^0|_{\eta=0,s=0} \neq 0$ in view of (132). Taking into account that q_2 is elliptic in ρ_0 and (135), (136) and (139) we deduce that

$$g_+ \in H^1_{\rho_0}. \tag{140}$$

Indeed, this can be proved by a classical bootstrap argument: The fact that $g_+ \in H^s_{\rho_0}$, by (135) implies that $A_q R g_+ \in H^{s+1}_{\rho_1}$ and then by (139) $A_{q_2}(g_+) \in H^{\min(1,s+1)}_{\rho_1}$, which, in view of (136), implies that $g_+ \in H^{\min(1,s+1)}_{\rho_0}$.

As a consequence of (140) we obtain $E^+(g_+) \in L^2_{\rho_0}$ and therefore, by (133), $g_- \in H^1_{\rho_0}$ which implies that $w|_{\partial\Omega\times(0,T)} \in H^1_{\rho_0}$ and $\partial w/\partial n|_{\partial\Omega\times(0,T)} \in L^2_{\rho_0}$, Thus, by the lifting Lemma of [BLR], $w \in H^1_{\rho_0}$.

■

7.2 Polynomial decay rates

As an inmediate consequence of Theorem 3.1, Lemmas 3.2 and 3.3 and Theorem 7.1 the following holds:

Theorem 7.2 *Let Ω be a bounded smooth domain of $I\!R^2$ without contacts of infinite order with its tangents. Assume that Ω satisfies the spectral condition (C) and (7.1).*

Then, there exists $C>0$ such that

$$E(t)\leq\frac{C}{t}\parallel\left(u^0,u^1,\theta^0\right)\parallel_D^2,\forall t>0 \tag{141}$$

for every solution of (1) *with initial data in the domain* $D=\left(H^2\cap H_0^1(\Omega)\right)^2\times\left(H_0^1(\Omega)\right)^2\times\left(H^2\cap H_0^1(\Omega)\right)$.

8 Related controllability results and spectral properties

In this section we consider two controllability problems that can be reduced to the observability inequality (9) for the Lamé system (8). First we address the controllability of Lamé's system in elasticity with potential controls. Then we address the null-controllability of the linear system of thermoelasticity with a scalar control acting on the heat component of the system. Finally we discuss the existence of divergence-free eigenfunctions of the Lamé system in two space dimensions.

8.1 Exact controllability of the Lamé system

Let us consider the controlled Lamé system

$$\begin{cases} u_{tt}-\mu\Delta u-(\lambda+\mu)\nabla\operatorname{div}u=f & \text{in} \quad \Omega\times(0,T)\\ u=0 & \text{on} \quad \partial\Omega\times(0,T)\\ u(x,0)=u^0(x),\ u_t(x,0)=u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{142}$$

We assume that $(u^0,u^1)\in\left(H_0^1(\Omega)\right)^n\times\left(L^2(\Omega)\right)^n$ and $f\in\left(L^2\left(\Omega\times(0,T)\right)\right)^n$ with $n=2$ or 3. Then the system (142) admits an unique solution $u\in C\left([0,T];\left(H_0^1(\Omega)\right)^n\right)\cap C^1\left([0,T];\left(L^2(\Omega)\right)^n\right)$.

The exact controllability problem can be stated roughly as follows: *Given* $(u^0,u^1)\in\left(H_0^1(\Omega)\right)^n\times\left(L^2(\Omega)\right)^n$ *find* $f\in\left(L^2\left(\Omega\times(0,T)\right)\right)^n$ *such that the solution of* (142) *satisfies*

$$u(T)\equiv u_t(T)\equiv 0. \tag{143}$$

If do not impose any restriction on the control f it is easy to see that this controllability property holds for any $T>0$.

There are several results in the literature for the case in which some restrictions are imposed on the control:

(a) When the support of f is assumed to be contained in a neighborhood ω of the boundary $\partial\Omega$ the methods of [Li] allow to show that the controllability holds for any $T>\operatorname{diam}(\Omega\backslash\omega)/\sqrt{\mu}$.

(b) More general subsets ω of Ω and sharper estimates on the control time can be obtained by microlocal methods in the spirit of [BLR] and [Ma].

(c) The case in which $f_n \equiv 0$ has been considered in [Z2] and it has been shown that, generically with respect to Ω, the system is approximately controllable if $T > 0$ is large enough, i.e. for any $\varepsilon > 0$ there exists a control such that $\| (u(T), u_t(T)) \|_{(H_0^1(\Omega))^n \times (L^2(\Omega))^n} \leq \varepsilon$.

In this section we address the problem under the assumption that

$$f = \nabla p,\ p \in L^2 \left(0, T;\ H_0^1(\Omega)\right), \tag{144}$$

i.e. we consider the case in which the control is a potential vector field acting everywhere in Ω.

In view of (144) we write the system (142) under the form

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda + \mu)\nabla \operatorname{div} u = \nabla p & \text{in} \quad \Omega \times (0, T) \\ u = 0 & \text{on} \quad \partial\Omega \times (0, T) \\ u(x, 0) = u^0(x),\ u_t(x, 0) = u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{145}$$

We say that the system (145) is *exactly controllable at time* $T > 0$ *if for every* $(u^0, u^1) \in \left(H_0^1(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n$ *there exists* $p \in L^2\left(0, T; H_0^1(\Omega)\right)$ *such that the solution of* (145) *satisfies* (143).

Using J.-L. Lions' HUM (see [Li]) it is easy to prove that system (145) is exactly controllable if and only if there exists $C > 0$ such that inequality (9) holds for every solution of the Lamé system (8).

Observe that this is true for every domain Ω independently of the fact that it satisfies the generic condition (C) or not. Thus the controllability problem above for the Lamé system, as the uniform decay problem for the system of thermoelasticity, can be reduced to the observability inequality (9) for system (8). Therefore the results of Theorems 5.1, 5.2 and 2.1 apply and provide both necessary and sufficient conditions for the exact controllability of (145) to hold.

Note also that the results of section 7 allow us to deduce that for most two-dimensional domains excat controllability does hold with controls $p \in L^2(\Omega \times (0, T))$ instead of $p \in L^2(0, T; H_0^1(\Omega))$.

Similar questions may be formulated when the control is a divergence free vector-field. Indeed, let us consider the system

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda + \mu)\nabla \operatorname{div} u = \operatorname{curl} p & \text{in} \quad \Omega \times (0, T) \\ u = 0 & \text{on} \quad \partial\Omega \times (0, T) \\ u(x, 0) = u^0(x),\ u_t(x, 0) = u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{146}$$

In this case, system (146) *is said to be exactly controllable in time* $T > 0$ *if for any* $(u^0, u^1) \in \left(H_0^1(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n$ *there exists* $p \in L^2\left(0, T; \left(H_0^1(\Omega)\right)^n\right)$ *such that the solution of* (146) *satisfies* (143).

By HUM it is easy to see that the exact controllability property above is equivalent to the following observability inequality for the uncontrolled Lamé system (8):

$$\| \varphi^0 \|^2_{H_0^1(\Omega))^n} + \| \varphi^1 \|^2_{(L^2(\Omega))^n} \leq C \int_0^T \int_\Omega | \operatorname{curl} \varphi |^2 \, dxdt. \tag{147}$$

As we pointed out in Remark 5.4, (147) does not hold if there exists a transversal ray with only perpendicular intersections with $\partial\Omega$ during a time interval of length greater than T.

8.2 Null controllability of the linear system of thermoelasticity

We consider the controlled linear system of three-dimensional thermoelasticity:

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\nabla\theta = 0 & \text{in} \quad \Omega\times(0,T) \\ \theta_t - \Delta\theta + \beta \operatorname{div} u_t = \partial f/\partial t & \text{in} \quad \Omega\times(0,T) \\ u = 0,\ \theta = 0 & \text{on} \quad \partial\Omega\times(0,T) \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x), \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega. \end{cases} \tag{148}$$

In (148) the control $f = f(x,t) \in L^2(0,T;\ H^{-1}(\Omega))$ acts on the system as a heat source. We assume that f is of compact support with respect to time in $(0,T)$.

We say that system (148) is *null-controllable in time* T *if and only if for every* $(u^0, u^1, \theta^0) \in (L^2(\Omega))^n \times (H^{-1}(\Omega))^n \times (H^2\cap H_0^1(\Omega))'$ $(n = 2$ or $3)$ *there exists* $f \in L^2(0,T;\ H^{-1}(\Omega))$ *such that the solution of* (148) *satisfies*

$$u(T) \equiv u_t(T) \equiv 0,\ \theta(T) \equiv 0 \text{ in } \Omega \tag{149}$$

and

$$\| f \|_{L^2(0,T;H^{-1}(\Omega))} \leq C \left\| \left(u^0, u^1, \theta^0\right) \right\|_{(L^2(\Omega))^3\times(H^{-1}(\Omega))^3\times\left(H^2\cap H_0^1(\Omega)\right)'}, \forall \left(u^0, u^1, \theta^0\right). \tag{150}$$

By $(H^2\cap H_0^1(\Omega))'$ we denote the dual of $H^2\cap H_0^1(\Omega)$.

Of course there are other functional settings that make sense for this null-controllability problem. We have chosen this one since, first, problem (148) is well-posed in those spaces, i.e. under the assumptions above on the initial data and the control there exists an unique solution $(u, u_t, \theta) \in C\left([0,T];\ (L^2(\Omega))^n \times (H^{-1}(\Omega))^n \times (H^2\cap H_0^1(\Omega))'\right)$ and second, this controllability problem can be reduced easily to the decay properties for the system of thermoelasticity considered above.

We refer to [LZ1] for the null-controllability problem for the linear system of elasticity when periodic boundary conditions are considered (or in the more general case in which the system is considered on a manifold without boundary). Obviously, in this case, the null controllability does not hold since $w = \operatorname{curl} u$ satisfies the uncontrolled wave equation $w_{tt} - \mu\Delta w = 0$ with periodic boundary conditions.

However, in the present situation in which Dirichlet boundary conditions are imposed one can not exclude automatically the null controllability since the control acts on $\operatorname{curl} u$ too through the boundary due to the interaction between the longitudinal and transversal components of u.

The null controllability problem can be reduced to an observability estimate for the adjoint system:

$$\begin{cases} \psi_{tt} - \mu\Delta\psi - (\lambda+\mu)\nabla \operatorname{div}\psi + \beta\nabla\eta_t = 0 & \text{in} \quad \Omega\times(0,T) \\ -\eta_t - \Delta\eta - \alpha \operatorname{div}\psi = 0 & \text{in} \quad \Omega\times(0,T) \\ \psi = 0,\ \eta = 0 & \text{on} \quad \partial\Omega\times(0,T) \\ \psi(x,T) = \psi^0(x), \psi_t(x,T) = \psi^1(x), \eta(x,T) = \eta^0(x) & \text{in} \quad \Omega. \end{cases} \tag{151}$$

System (8.10) is well posed in $(\psi, \psi_t, \eta) \in (H_0^1(\Omega))^3 \times (L^2(\Omega))^3 \times (H^2\cap H_0^1(\Omega))$. On the other hand, multiplying in (8.7) by (ψ, η) and integrating by parts it follows that

$$\int_\Omega (\psi_t u - \psi u_t + \beta\nabla\eta u - \eta\theta) dx|_0^T - \int_0^T \int_\Omega f\eta_t dx dt = 0.$$

Note that when deriving this identity we have used the fact that f has compact support in time.

More precisely it can be proved that the null-controllability property above in time T implies the existence of a positive constant $C > 0$ such that

$$\| \psi(0) \|^2_{\left(H_0^1(\Omega)\right)^3} + \| \psi_t(0) \|^2_{(L^2(\Omega))^3} + \| \eta(0) \|^2_{H^2(\Omega)} \leq C \int_0^T \int_\Omega | \nabla \eta_t |^2 \, dxdt \tag{152}$$

holds for every solution of (151). On the other hand, if (152) holds for the adjoint system (151), then system (148) is null controllable for any $T' > T$ in the sense of (149)-(150).

In order to prove the last statement, given $(u^0, u^1, \theta^0) \in \left(L^2(\Omega)\right)^n \times \left(H^{-1}(\Omega)\right)^n \times \left(H^2 \cap H_0^1(\Omega)\right)'$ and ρ a non-negative smooth function such that $\rho \equiv 1$ in an interval of length T and $\rho(0) = \rho(T') = 0$ we consider the functional

$$\begin{aligned} J\left(\psi^0, \psi^1, \eta^0\right) &= \frac{1}{2} \int_0^{T'} \int_\Omega \rho(t) \left|\nabla \eta_t\right|^2 dxdt \\ &+ \; < \eta(0), \theta^0 + \beta \operatorname{div} u^0 > + < \psi(0), u^1 > - \int_\Omega \psi_t(0) \cdot u^0 dx, \end{aligned} \tag{153}$$

where $< \cdot, \cdot >$ denotes both the duality pairing between $H^2 \cap H_0^1(\Omega)$ and its dual and between $\left(H_0^1(\Omega)\right)^n$ and $\left(H^{-1}(\Omega)\right)^n$ and (ψ, η) solve (151) in the time interval $t \in [0, T']$.

The functional $J : \left(H_0^1(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n \times H^2 \cap H_0^1(\Omega) \to I\!R$ is continuous and strictly convex. On the other hand, from (152) and the properties of ρ, J is also coercive. Therefore, J has a unique minimizer $\left(\widehat{\psi^0}, \widehat{\psi^1}, \widehat{\eta^0}\right)$ in $\left(H_0^1(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n \times H^2 \cap H_0^1(\Omega)$. It is easy to check that the control $g = \frac{\partial}{\partial t}\left(\rho(t) \Delta \widehat{\eta}_t\right)$ where $\left(\widehat{\psi}, \widehat{\eta}\right)$ solves (151) with this minimizer as data fulfills the control requirements (149) and (150) at time $t = T'$.

The main result on the null-controllability of system (148) is as follows:

Theorem 8.1 *Under the assumptions of Theorems 5.1 and 5.2 there is no T such that* (152) *holds for solutions of* (151). *Therefore, system* (148) *is no null-controllable.*

Proof. It is sufficient to observe that (ψ, η) solve (151) if an only if

$$\varphi(x,t) = \psi(x, T-t); \xi(x,t) = \eta_t(x, T-t)$$

satisfy

$$\begin{cases} \varphi_{tt} - \mu \Delta \varphi - (\lambda + \mu) \nabla \operatorname{div} \varphi + \beta \nabla \xi = 0 & \text{in} \quad \Omega \times (0, \infty) \\ \xi_t - \Delta \xi + \alpha \operatorname{div} \varphi_t = 0 & \text{in} \quad \Omega \times (0, \infty) \\ \varphi = 0, \xi = 0 & \text{on} \quad \partial\Omega \times (0, \infty) \\ \varphi(x,0) = \psi^0(x), \varphi_t(x,0) = \psi^1(x), \xi(x,0) = -\Delta \eta^0(x) - \alpha \operatorname{div} \psi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{154}$$

On the other hand, (152) is equivalent to

$$\begin{aligned} \| \varphi(T) \|^2_{\left(H_0^1(\Omega)\right)^3} + \| \varphi_t(T) \|^2_{(L^2(\Omega))^3} &+ \; \| \xi(T) + \alpha \operatorname{div} \varphi(T) \|^2_{L^2(\Omega)} \\ &\leq \; C \int_0^T \int_\Omega | \nabla \xi |^2 \, dxdt \end{aligned}$$

or, equivalently, to

$$E(T) \leq C \int_0^T \int_\Omega | \nabla \xi |^2 \, dxdt \tag{155}$$

where E denotes the energy

$$E(t) = \frac{1}{2}\int_{\Omega}\left[\mid \varphi_t \mid^2 + \mu \mid \nabla\varphi \mid^2 + (\lambda+\mu) \mid \operatorname{div}\varphi \mid^2 + \frac{\beta}{\alpha} \mid \xi \mid^2\right] dx. \tag{156}$$

Taking into account that

$$\frac{dE}{dt} = -\frac{\beta}{\alpha}\int_{\Omega} \mid \nabla\xi \mid^2 dx$$

we deduce that (155) is equivalent to

$$E(0) \leq C\int_0^T\int_{\Omega} \mid \nabla\xi \mid^2 dxdt. \tag{157}$$

This is precisely the inequality we have shown does not hold under the assumptions of Theorems 5.1 and 5.2.

The results of sections 3.2 and 7 allow to deduce that null controllbility holds for most two-dimensional domains provided the control g is taken in $H^{-2}(0,T;H^{-1}(\Omega))$ instead of taking it in $H^{-1}(0,T;H^{-1}(\Omega))$. This is due to the fact that the following weaker version of (8.11) holds:

$$\| \psi(0) \|^2_{\left(H_0^1(\Omega)\right)^3} + \| \psi_t(0) \|^2_{(L^2(\Omega))^3} + \| \eta(0) \|^2_{H^2(\Omega)} \leq C\int_0^T\int_{\Omega} \mid \nabla\eta_{tt} \mid^2 dxdt. \tag{158}$$

8.3 Divergence-free eigenfunctions of the Lamé system

In this section we discuss the existence of divergence-free eigenfunctions for the Lamé system, i. e. the existence of vector-valued functions φ and real numbers γ such that

$$-\Delta\varphi = \gamma^2\varphi \text{ in } \Omega; \quad \operatorname{div}\varphi = 0 \text{ in } \Omega; \quad \varphi = 0 \text{ on } \partial\Omega. \tag{159}$$

When Ω si a ball or a spherical annulus, system (159) admits infinitely many linearly independent solutions in any space dimension. The results of section 7.1 show that for most two-dimensional domains the subspace of solution of (159) is of finite dimension. Indeed, as we have seen in section 7.1 this holds as soon as Ω is a two-dimensional bounded smooth domain without contacts of infinite order with its tangent and such that it is not a ball or a spherical annulus of the form $\Omega = \mathcal{O}\backslash\lambda\mathcal{O}$, $\mathcal{O}$ being a ball and $0 < \lambda < 1$. Thus, roughly speaking, a necessary and sufficient condition for the subspace of solutions of (159) to be of infinite dimension is Ω to be a ball or a spherical annulus. This is essentially a result due to C. A. Berenstein [B]. Indeed, in [B] it was proved that if Ω is a two-dimensional simply connected $C^{2,\alpha}$ domain in which (8.18) has an infinite number of linearly independent solutions, then Ω is a ball.

The characterization of the domains such that the set of non trivial eigenfunctions satisfying (159) is non-empty is an open problem.

■

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Appendix A: *A decomposition Lemma*

The object of this Appendix is to give a detailed proof of a decomposition result for vector fields into solenoidal plus potential ones that will be used in Appendix B below.

Let I be an interval in $I\!R$. Let us denote by A the elliptic operator associated with the Lamé system $A = -\mu\Delta - (\lambda+\mu)\nabla \operatorname{div}$. Let $u \in \left(H^1(\Omega \times I)\right)^n$ be a solution of

$$u_{tt} + Au = 0 \text{ in } \left(\mathcal{D}'(\Omega \times I)\right)^n \tag{A.1}$$

with $n = 2$ or 3.

When $n = 2$ we set $n' = 1$. Then, given a scalar function $w = w(x_1, x_2)$ we denote by $\operatorname{curl} w = (\partial_2 w, -\partial_1 w)$ the curl of w. When $n = 3$ we set $n' = 3$.

Let J be a compact interval strictly contained in I.

The following holds:

Lemma A.1 *The vector field u can be decomposed as*

$$u = \nabla v + \operatorname{curl} w \text{ in } \left(\mathcal{D}'(\Omega \times J)\right)^n \tag{A.2}$$

with

$$v \in L^2(J \times \Omega), v_{tt} - c_L^2 \Delta v = 0 \text{ in } \mathcal{D}'(\Omega \times J); \tag{A.3}$$

$$w \in \left(L^2(\Omega \times J)\right)^{n'}; w_{tt} - c_T^2 \Delta w = 0 \text{ in } \left(\mathcal{D}'(\Omega \times J)\right)^{n'} \tag{A.4}$$

with

$$\operatorname{div} w = 0 \text{ when } n = 3. \tag{A.5}$$

Moreover, (v, w) can be chosen such that $(v, w) = R(w)$ where R is a linear continuous operator from $\left(H^1(\Omega \times I)\right)^n$ into $\left(L^2(\Omega \times J)\right)^{1+n'}$.

Proof. Let $k = k(x,t)$ be the solution of

$$\Delta k = u \text{ in } \Omega \times I; \; k = 0 \text{ on } \partial\Omega \times I. \qquad (A.6)$$

Taking into account that $u \in (H^1(\Omega \times I))^n$ solves $(A.1)$ it is easy to see that $\partial_t^j u \in L^2\left(I; (H^{1-j}(\Omega))^n\right)$ for $j = 0,1,2$. Therefore $\partial_t^j k \in L^2\left(I; (H^{3-j}(\Omega))^n\right)$ for $j = 0,1,2$. We set

$$f = \left(\partial_t^2 - c_L^2 \Delta\right) (\operatorname{div} k) \in L^2(\Omega \times I);$$
$$g = \left(\partial_t^2 - c_T^2 \Delta\right) (-\operatorname{curl} k) \in \left(L^2(\Omega \times I)\right)^{n'}.$$

Given $\varphi \in \mathcal{D}(I)$ such that $\varphi = 1$ in J, let α and β be the solutions of $\partial_t^2 \alpha = \varphi f, \partial_t^2 \beta = \varphi g$ such that $\alpha \equiv \beta \equiv 0$ for $t << 0$ small enough. We define

$$v = \operatorname{div} k - \alpha; \; w = -\operatorname{curl} k - \beta. \qquad (A.7)$$

Let us check that (u, w) as in $(A.7)$ satisfy the conditions of the Lemma.

Taking into account that $A = -c_L^2 \nabla \operatorname{div} + c_T^2 \operatorname{curl}\operatorname{curl}$ we deduce that

$$\nabla f + \operatorname{curl} g = \left(\partial_t^2 + A\right) (\nabla \operatorname{div} k - \operatorname{curl}\operatorname{curl} k) = \left(\partial_t^2 + A\right)(\Delta k) = 0.$$

This implies that $\nabla \alpha + \operatorname{curl} \beta = 0$. Therefore,

$$\nabla v + \operatorname{curl} w = \nabla \operatorname{div} k - \operatorname{curl}\operatorname{curl} k = \Delta k = u.$$

On the other hand

$$\begin{aligned} \left(\partial_t^2 - c_L^2 \Delta\right) v &= \left(\partial_t^2 - c_L^2 \Delta\right)(\operatorname{div} k) - \left(\partial_t^2 - c_L^2 \Delta\right) \alpha \\ &= f - \partial_t^2 \alpha + c_L^2 \Delta \alpha = 0 \text{ in } \mathcal{D}'(\Omega \times H), \end{aligned}$$

since $\partial_t^2 \alpha = f$ and $\nabla \alpha + \operatorname{curl} \beta = 0$ implies $\Delta \alpha = 0$.

Equation (A.4) can be checked in a similar way.

When $n = 3, \operatorname{div} w = -\operatorname{div} \beta$ satisfies $\partial_t^2(\operatorname{div} \beta) = \varphi \operatorname{div} g = 0$ and $\operatorname{div} \beta = 0$ for $t << 0$. This implies that $\operatorname{div} \beta \equiv 0$.

The fact that $(v, w) = R(u)$ with R linear and continuous from $(H^1(\Omega \times (0,T)))^n$ into $(L^2(\Omega \times (0,T)))^{1+n'}$ is clear from the construction above.

■

Appendix B: *H^1 propagation along transversal rays.*

In this Appendix Ω denotes an open, bounded, smooth set of $\mathbb{R}^n$ with $n = 2$ or 3 without contacts of infinite order with its tangents.

As in Appendix A, A denotes the elliptic operator associated with the Lamé system. We will also use the notions of characteristic manifold Char and transversal characteristic manifold Char$(\mathcal{T})$ introduced in previous Sections.

Let us also recall that, as we said in Section 4, for any $\rho \in$ Char$(\mathcal{T})$ there is a unique ray $s \mapsto \gamma(s) \in$ Char$(\mathcal{T})$ such that $\gamma(0) = \rho$.

The object of this Appendix is to prove the following result that guarantees that the H^1_ρ microlocal regularity of solutions of the Lamé system with homogeneous Dirichlet boundary conditions propagates along transversal rays:

Theorem B.1. *Let* $u \in (\mathcal{D}'(\Omega \times (0,T)))^n$ *be a solution of*

$$u_{tt} + Au = 0 \ \textit{in} \ \left(\mathcal{D}'(\Omega \times (0,T))\right)^n; u = 0 \ \textit{on} \ \partial\Omega \times (0,T) \tag{B.1}$$

with

$$\operatorname{div} u \in L^2(\Omega \times (0,T)). \tag{B.2}$$

Then, H^1-regularity propagates along transversal characteristic rays. In other words, if $s \mapsto \gamma(s) \in$ $\mathrm{Char}(\mathcal{T})$ *is a transversal characteristic ray*

$$u \in H^1_{\gamma(s_1)} \Leftrightarrow u \in H^1_{\gamma(s_2)}, \ \forall s_1, s_2. \tag{B.3}$$

Proof. Equation $(\partial_t^2 + A)\, u = 0$ guarantees that $WF_b(u) \subset \mathrm{Char}$. On the other hand, for $\rho = (x, \xi, t, s) \in \mathrm{Char}$ such that $x \in \Omega$ and $\mid \tau \mid = c_L \mid \xi \mid$, condition $\operatorname{div} u \in L^2(\Omega \times (0,T))$ implies that $u \in H^1_\rho$. In order to prove $(B.3)$ we use the decomposition $u = \nabla v + \operatorname{curl} w$ of Lemma A.1.

For $\rho \in \mathrm{Char}(\mathcal{T}), \Delta v = \operatorname{div} u \in L^2_\rho$ and therefore $v_{tt} \in L^2_\rho$ as well. Thus $v \in H^2_\rho$ since $\tau(\rho) \neq 0$. Consequently, $u \in H^1_\rho$ if and only if $\operatorname{curl} w \in H^1_\rho$ or, equivalently, if $w \in H^2_\rho$ because of the fact that, when $n = 3, \operatorname{div} w = 0$.

The statement $(B.3)$ is of local nature. Thus, it is sufficient to analyze the propagation near each ρ in one of the following four situations:

$$\begin{array}{ll} (i) & \rho = (x, t, \xi, s),\ x \in \Omega,\ \mid \tau \mid = c_T \mid \xi \mid; \\ (ii) & \rho = (y, t, \eta, s),\ y \in \partial\Omega,\ \mid \eta \mid c_L < \mid \tau \mid; \\ (iii) & \rho = (y, t, \eta, s),\ y \in \partial\Omega,\ \mid \eta \mid c_L = \mid \tau \mid; \\ (iv) & \rho = (y, t, \eta, s),\ y \in \partial\Omega,\ \mid \tau \mid < c_L \mid \eta \mid. \end{array}$$

Case (i): This case is a consequence of the classical result on the propagation of singularities in the interior: We have $u \in H^1_\rho$ if an only if $w \in H^2_\rho$ and since $\left(\partial_t^2 - c_T^2 \Delta\right) w = 0$, the microlocal H^2 regularity propagates along transversal rays.

Case (ii): In this case we are led to analyze the propagation near the boundary over points that are hyperbolic both for $\partial_t^2 - c_T^2\Delta$ and $\partial_t^2 - c_L^2\Delta$. Since $v \in H^2_\rho$, we also have $\nabla v|_{\partial\Omega\times(0,T)} \in H^1_\rho$ and since $u|_{\partial\Omega\times(0,T)} = 0$, $\operatorname{curl} w|_{\partial\Omega\times(0,T)} \in H^1_\rho$ as well. Let γ^+ and γ^- be the outcoming and incoming open half rays at ρ. Suppose for instance that $u \in H^1_\rho$ on γ^-. Then $w \in H^2_\rho$ on γ^-. Thus, if $w = w_+ + w_-$ is the decomposition of w solution of $\left(\partial_t^2 - c_T^2\Delta\right) w = 0$ near ρ in outcoming and incoming waves, we have $w_- \in H^2_\rho$. Then since ρ is hyperbolic with respect to $\left(\partial_t^2 - c_T^2\Delta\right)$ we have $\operatorname{curl}(w_+)|_{\partial\Omega\times(0,T)} \in H^1_\rho$ and $\operatorname{div} w_+|_{\partial\Omega\times(0,T)} \in H^1_\rho$. Taking into account that both $\operatorname{curl}(w_+)$ and $\operatorname{div} w_+$ satisfy an equation of the form $\left(\partial_t^2 - c_T^2\Delta\right) h \in C^\infty$ we deduce that $\operatorname{curl} w_+ \in H^1_\rho$ and $\operatorname{div} w_+ \in H^1_\rho$ over γ^+ as well.

Case (iii): It corresponds to those points that are hyperbolic for $\partial_t^2 - c_T^2\Delta$ and glancing for $\partial_t^2 - c_L^2\Delta$. This situation is described in the following figure:

Figure B1.

We have to check, with the notations of the previous case, that $w \in H^2_\rho$ when $w \in H^2$ over γ_-. We have again $w = w_+ + w_-$ with $w_- \in H^2_\rho$. Then

$$\nabla v|_{\partial\Omega\times(0,T)} + (\operatorname{curl} w_+)|_{\partial\Omega\times(0,T)} \in H^1_\rho; \tag{B.4}$$

$$(\operatorname{div} w_+)|_{\partial\Omega\times(0,T)} \in H^1_\rho; \tag{B.5}$$

$$\left.\frac{\partial w_+}{\partial n}\right|_{\partial\Omega\times(0,T)} = E_1(w_+|_{\partial\Omega\times(0,T)}) \tag{B.6}$$

where E_1 is a pseudo-differential operator of degree 1 with principal symbol $ie_1 = i\sqrt{\frac{\tau^2}{c_T^2} - \mid \eta \mid^2}$.

From $(B.4) - (B.6)$ we deduce that

$$v|_{\partial\Omega\times(0,T)} - E_{-1}\left(\left.\frac{\partial v}{\partial n}\right|_{\partial\Omega|\times(0,T)}\right) \in H^2_\rho$$

where E_{-1} is a pseudo-differential operator of degree -1 an principal symbol $ie_1/\mid \eta \mid^2$.

Therefore, the scalar function v satisfies

$$\begin{cases} \partial_t^2 v - c_L^2 \Delta v = 0; v \in H^2_\rho \\ v|_{\partial\Omega\times(0,T)} - E_{-1}\left(\left.\frac{\partial v}{\partial n}\right|_{\partial\Omega\times(0,T)}\right) \in H^2_\rho. \end{cases} \tag{B.7}$$

The boundary condition in $(B.7)$ satisfies the Lopatinski condition at the glancing point ρ. Therefore, as a consequence of $(B.7)$, we deduce that

$$v|_{\partial\Omega\times(0,T)} \in H^2_\rho; \ \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega\times(0,T)} \in H^1_\rho. \tag{B.8}$$

Using now $(B.4) - (B.5)$ we deduce, as in case (ii) above, that $w_+ \in H^2_\rho$.

Case (iv): It corresponds to elliptic points for $\partial_t^2 - c_L^2\Delta$. At these points the Theorem of propagation of the wave front set for solutions of

$$u_{tt} + Au = 0; \ u|_{\partial\Omega\times(0,T)} = 0$$

by K. Yamamoto [Y2] is reduced to the propagation along transversal characteristic rays. On the other hand, it is by now well known that the propagation result for the wave front set for systems that are well posed in L^2 implies the propagation of the H^s regularity for any s (see, for instance, Th. 3.3, p. 1045 of [BLR]).

■