Null controllability of linear and semilinear heat equations in thin domains

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Abstract

We consider the linear heat equation with potential in a n-dimensional thin cilinder $\Omega_\varepsilon = \Omega \times (0, \varepsilon)$ where Ω is a bounded open smooth set of $\mathbb{R}^{n-1}$ with $n \geq 2$ and ε is a small parameter. We study the null controllability problem when the control acts in a cylindrical region $\omega_\varepsilon = \omega \times (0, \varepsilon)$, where $\omega \subset \Omega$ is an open and non-empty subset of Ω. We prove that, under appropriate boundary conditions, for a suitable class of potentials the heat equation is uniformly null controllable as $\varepsilon \to 0$. We also prove the convergence of the controls to a null control for the $n-1$-dimensional heat equation in Ω. Similar results are proved for the semilinear heat equation with globally Lipschitz nonlinearities.
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1 Introduction and main results

Let Ω be a bounded open smooth set of $\mathbb{R}^{n-1}$ with $n \geq 2$. Given $\varepsilon > 0$, we consider the n-dimensional cylinder $\Omega_\varepsilon = \Omega \times (0, \varepsilon)$.

We also consider a non-empty open subset ω of Ω and the corresponding cylindrical subset $\omega_\varepsilon = \omega \times (0, \varepsilon)$ of Ω_ε.

Given a control time $T > 0$ and a potential $a(x', x_n, t) \in L^\infty(Q_\varepsilon)$, $Q_\varepsilon = \Omega_\varepsilon \times (0, T)$, we consider the following heat equation in Ω_ε:

$$(1) \qquad \begin{cases} y_t - \Delta y + a(x', x_n, t) y = h 1_{\omega_\varepsilon} & \text{in } Q_\varepsilon \\ y = 0 & \text{on } [\partial\Omega \times (0,\varepsilon)] \times (0,T) \\ \frac{\partial y}{\partial n} = 0 & \text{on } \Gamma_\varepsilon^\pm \times (0,T) \\ y(0) = y^0 & \text{in } \Omega_\varepsilon. \end{cases}$$

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In (1) $y = y(x,t)$ is the state, $h = h(x,t)$ is the control function, and 1_{ω_ε} denotes the characteristic function of the subset ω_ε. Thus the control entering in the system is supported in $\omega_\varepsilon \times (0,T)$. The top and the bottom parts of the boundary of Ω_ε are denoted by

$$\Gamma_\varepsilon^+ = \{(x,\varepsilon)\colon x \in \Omega\}; \Gamma_\varepsilon^- = \{(x,0)\colon x \in \Omega\}.$$

On $\Gamma_\varepsilon^\pm$ we impose Neumann boundary conditions, while Dirichlet boundary conditions are considered on the lateral boundary $\partial\Omega \times (0,\varepsilon)$.

We shall use the notation $x = (x', x_n)$ with $x' \in \Omega$ and $x_n \in (0,\varepsilon)$. We shall denote by $\nabla', \Delta', \dots$ the gradient, Laplacian, ... with respect to the $(n-1)$-dimensional space variable x', while operators in the variable x will be denoted by $\nabla, \Delta, \dots$

Of course the solution y of (1) as well as the control h and the inital data y^0, in practice, will depend on the parameter ε. However we shall make this dependence explicit in the notation only when this will be convenient for the sake of clarity.

Given $y^0 \in L^2(\Omega_\varepsilon)$ and $h \in L^2(\omega_\varepsilon \times (0,T))$ system (1) admits a unique solution

$$y \in C([0,T]; L^2(\Omega_\varepsilon)) \cap L^2(0,T; V_\varepsilon)$$

where

$$V_\varepsilon = \{\phi \in H^1(\Omega_\varepsilon)\colon \phi = 0 \text{ on } \partial\Omega \times (0,\varepsilon)\}.$$

Moreover system (1) is known to be *null controllable* in any time $T > 0$. Let us recall that system (1) is said to be null controllable in time T if for any $y^0 \in L^2(\Omega_\varepsilon)$ there exists a control $h \in L^2(\omega_\varepsilon \times (0,T))$ such that the solution of (1) satisfies

$$y(T) = 0 \text{ in } \Omega_\varepsilon. \tag{2}$$

On the other hand, the formal limit of system (1) is the heat equation in $Q = \Omega \times (0,T)$, i. e.

$$\begin{cases} y_t - \Delta' y + a(x',t)y = h1_\omega & \text{in } Q \\ y = 0 & \text{on } \partial\Omega \times (0,T) \\ y(0) = y^0 & \text{in } \Omega \end{cases} \tag{3}$$

with a suitable potential a. For instance, when a in (1) is continuous, in (3) we get the potential $a(x',0,t)$.

In system (3) the spatial domain is reduced to the $(n-1)$-dimensional domain Ω. Dirichlet boundary conditions are kept on the lateral boundary $\partial\Omega \times (0,T)$ and the control is supported in the open non-empty subset ω.

System (3) is null-controllable as well in any time $T > 0$.

The main goal of this paper is to prove that the null controllability property for system (3) may be recovered as limit when $\varepsilon \to 0$ of the null controllability properties of system (1).

But before explaining in detail the content of this paper let us recall some basic references on the null controllability of heat equations. D. L. Russell in [11] proved that the constant coefficient heat equation is null controllable in any time T provided the wave equation is exactly controllable for some time T. Taking into account that the subdomain ω has to satisfy suitable geometric properties ([1]) in order to guarantee the exact controllability of the wave equation, this result imposes geometric conditions on ω for the null controllability of the heat equation, too. Later on G. Lebeau and L. Robbiano [10] proved that the null controllability of the heat equation holds without any geometric restriction on the open subset ω. Their proof used Fourier series decompositions and therefore did not apply to heat equations with potentials depending on space and time. The most general result was proved by A. Fursikov and O. Imanuvilov [8] using Carleman inequalities and duality arguments. The results in [8] apply to a large class of heat equations with variable coefficients and under various boundary conditions, both in the context of the internal control we are addressing here or in the case where the control acts through the boundary conditions (see also [9] for the null controllability of the heat equation involving convective terms). More recently, in [4], [5] and [6], the methods in [8] have been combined with the variational approach to controllability in [3] to prove null controllability results for heat equations with nonlinearities that grow at infinity in a superlinear way.

In this paper we adopt the approach in [5], [6] that combines the Carleman inequalities in [8] and the variational methods in [3].

The cylindrical structure of the domain Ω_ε and the boundary conditions we have considered (in particular the fact that we consider Neumann boundary conditions on the top and bottom boundaries) allow us to prove uniform Carleman inequalities (with constants that are independent of ε). This allows us to prove the uniform null controllability of system (1), i. e. the fact that the controls remain bounded as ε tends to zero when the initial data are bounded.

Here and all along the paper by "uniform" we refer to a property that is independent of the parameter ε that is devoted to tend to zero.

Thus, the first result of the paper is as follows:

Theorem 1. *Under the assumptions above, for any $y^0 \in L^2(\Omega_\varepsilon)$ and $0 < \varepsilon < 1$ there exists a control $h \in L^2(\omega_\varepsilon \times (0,T))$ such that the solution of (1) satisfies (2).*

Moreover, for any $R > 0$ there exists a constant $C = C(R) > 0$ (independent of ε) such that

$$\|h\|_{L^2(\omega_\varepsilon \times (0,T))} \leq C \|y^0\|_{L^2(\Omega_\varepsilon)} \tag{4}$$

for any $y^0 \in L^2(\Omega_\varepsilon)$ and for all potential $a \in L^\infty(\Omega_\varepsilon \times (0,T))$ such that $||a||_\infty \leq R$.

Remark 1. The constant C in (4) depends on the time T and on the potential a through its L^∞-norm but it is independent of $0 < \varepsilon < 1$. The proof of Theorem 1 we shall give below provides rather explicit estimates of how the constant C depends on $||a||_\infty$. But we shall not discuss this issue in detail since here we focus on the uniformity with respect to ε and on passing to the limit as this parameter tends to zero. ■

Remark 2. The parabolic nature of the problem under consideration is essential in the proof of Theorem 1. In the context of wave equations such a result fails since, even for $\varepsilon > 0$ fixed, in this geometric setting, the wave equation is not exactly controllable since the geometric control condition in [1] is not satisfied. This is due to the existence of rays of geometric optics that are reflected perpendicularly on the top and bottom boundaries of the domain without never intersecting the region in which the control acts. In [7] the same problem was considered for thin plates. In order to guarantee the exact controllability of the system a control was added on the top and bottom boundaries. It was then shown that, in the limit as $\varepsilon \to 0$, in addition to a boundary control on the lateral boundary $\partial\Omega \times (0,T)$ one also recovers a control distributed all along the domain, which is due to the contribution of the controls located on the top and bottom boundaries. ■

In order to carefully analyze the behavior as $\varepsilon \to 0$ it is convenient to rescale the variable x_n to work always on a reference domain $\Omega_1 = \Omega \times (0,1)$, independent of ε. Thus we make the following change of variables:

$$(5) \qquad \begin{cases} x_n = \varepsilon z_n,\ u_\varepsilon(x', z_n, t) = y(x', \varepsilon z_n, t),\ b_\varepsilon(x', z_n, t) = a(x', \varepsilon z_n, t), \\ g_\varepsilon(x', z_n, t) = h(x', \varepsilon z_n, t),\ u_\varepsilon^0(x', z_n) = y^0(x', \varepsilon z_n). \end{cases}$$

We remind that $Q_1 = \Omega_1 \times (0,T)$, with $\Omega_1 = \Omega \times (0,1)$ and $Q = \Omega \times (0,T)$.

In terms of the new vertical variable z_n, equation (1) reads as follows

$$(6) \qquad \begin{cases} u_{\varepsilon,t} - \Delta' u_\varepsilon - \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial z_n^2} u_\varepsilon + b_\varepsilon(x', z_n, t) u_\varepsilon = g_\varepsilon 1_{\omega_1} & \text{in } Q_1 \\ u_\varepsilon = 0 & \text{on } [\partial\Omega \times (0,1)] \times (0,T) \\ \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_1^\pm \times (0,T) \\ u_\varepsilon(0) = u_\varepsilon^0 & \text{in } \Omega_1. \end{cases}$$

Thus, we see that, in the new variables, the family of equations (1) becomes a family of singularly perturbed heat equations in the reference domain Ω_1 with controls supported in ω_1.

Concerning the passage to the limit as ε tends to zero, our main result is as follows:

Theorem 2. *Suppose that the potentials $b_\varepsilon(x', z_n, t) \in L^\infty(Q_1)$ are bounded as $\varepsilon \to 0$ in $L^\infty(Q_1)$ and that there exists a limit potential $a(x', t) \in L^\infty(Q)$ such that*

$$(7) \qquad b_\varepsilon(x', z_n, t) \to a(x', t) \ \text{as} \ \varepsilon \to 0 \quad \text{in } L^\infty(\Omega_1 \times (0,T)).$$

Let $u_\varepsilon^0 \in L^2(\Omega_1)$ be a sequence of initial data for system (6) such that

$$(8) \qquad u_\varepsilon^0(x', z_n) \to y^0(x') \ \text{strongly in } L^2(\Omega_1).$$

Then, there exists a sequence of null controls g_ε for system (6) corresponding to the initial data u_ε^0 such that

$$(9) \qquad \int_0^1 g_\varepsilon(x', z_n, t) dz_n \to h(x', t) \ \text{strongly in } L^2(\omega \times (0,T)) \ \text{as } \varepsilon \to 0$$

where $h = h(x', t)$ is a null control for the $(n-1)$-dimensional heat equation with data $y^0 = y^0(x')$.

More precisely, $h \in L^2(\omega \times (0,T))$ is such that the solution y of (3) satisfies

$$(10) \qquad y(T) = 0 \quad \text{in } \Omega.$$

Moreover

$$(11) \qquad \int_0^1 u_\varepsilon dz_n \to y, \qquad \text{in } L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).$$

Remark 3. Let us analyze the assumption (7) when the potential $a(x', x_n, t)$ in (1) is independent of ε, for instance. Let us recall that, according to the change of variables (5), we then have: $b_\varepsilon(x', z_n, t) = a(x', \varepsilon z_n, t)$. It is then easy to see that (7) holds when a is continuous in $\overline{Q_1}$. The assumption (7) may be weakened in various ways. However, here we have prefered to work under this assumption to make the proof simpler so that its main ingredients may be identified easily. ■

Remark 4. When $a_\varepsilon(x, x_n, t) \equiv 0$, i. e. in the absence of potentials, the second author in [15] proved the same result. However, the methods in [15], based on the Fourier decomposition of solutions, do not apply for equations with potentials depending both in space and time. In [15] it was proved that the limit control is basically due to the controls of the first Fourier components of the solutions of (1) (the projection of the solutions over the subspace of functions that are independent of z_n, i.e. its average with respect to z_n) while the other components of the control tend exponentially to zero. In the context of the present paper we may not use Fourier decomposition and therefore, the results we prove (see for instance (9) and (11)) are similar but less precise. ■

Remark 5. It is interesting to reinterpret the convergence result in Theorem 2 in the original variables (x', x_n, t). Note that the controls h_ε for system (1) are of the form: $h_\varepsilon(x', x_n, t) = g_\varepsilon(x', x_n/\varepsilon, t)$. Then, as a consequence of (9) the following holds:

$$\frac{1}{\varepsilon}\int_0^\varepsilon h_\varepsilon(x', x_n, t)dx_n \to h(x', t) \text{ strongly in } L^2(\omega \times (0,T)). \tag{12}$$

In a similar way, as a consequence of (11) it follows that

$$\frac{1}{\varepsilon}\int_0^\varepsilon y_\varepsilon(x', x_n, t)dx_n \to y(x', t) \text{ strongly in } L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)). \tag{13}$$

Consequently, the averages in the vertical variable x_n of the controls and solutions of system (1) converge respectively to the control and solution of the limit $(n-1)$-dimensional system. ■

As we said above, in order to prove the null controllability of system (1) we adopt the variational approach in [3] as in [15]. This approach reduces the proof of the null controllability to obtaining a suitable observability inequality for the adjoint system. In order to show the uniform null controllability property stated in Theorem 1, obviously, we need these observability inequalities to be uniform as well. This is the first main technical difficulty one encounters when addressing this problem.

To better explain this fact let us consider the adjoint system

$$\begin{cases} -\psi_t - \Delta\psi + a(x', x_n, t)\psi = 0 & \text{in } Q_\varepsilon \\ \frac{\partial\psi}{\partial n} = 0 & \text{on } \Gamma_\varepsilon^\pm \times (0,T) \\ \psi = 0 & \text{on } [\partial\Omega \times (0,\varepsilon)] \times (0,T) \\ \psi(T) = \psi^0 & \text{in } \Omega_\varepsilon. \end{cases} \tag{14}$$

Proceeding as in [8] and [4] the following may be proved:

Proposition 1. *For all $T > 0$ there exists $C = C(T)$ such that*

$$\| \psi(0) \|^2_{L^2(\Omega_\varepsilon)} \leq Ce^{C(1+\|a\|_{L^\infty(Q_\varepsilon)})} \int_0^T \int_{\omega\times(0,\varepsilon)} |\psi|^2 dxdt \tag{15}$$

for every solution of (14), for all $0 < \varepsilon < 1$ and for all potential $a \in L^\infty(\Omega_\varepsilon \times (0,T))$.

Remark 6. (a) Inequality (15) is usually referred to as observability inequality. It provides a bound on the solution everywhere in the domain Ω_ε in terms of the L^2-energy of the solution concentrated in ω_ε. Note however that this estimate is obtained at time $t = 0$, which is the final time for the adjoint system under consideration. Due to the irreversibility and the smoothing effect of the heat equation this inequality fails at the initial time $t = T$.

(b) Obviously, inequality (15) not only provides estimates that are uniform for classes of potentials $||a||_\infty \leq R$ but also provides an estimate on how the observability constant grows as R tends to ∞. Much more can be said about how the observability constant depends on the potential a and time T (see [5]).

(c) The observability inequality (15) holds in any time interval. Therefore it also holds in the time interval (t, T) for any $0 < t < T$. In fact, for every $0 < t < T$ there exists $C(t)$ such that

$$\| \psi(t) \|^2_{L^2(\Omega_\varepsilon)} \leq C(t) e^{C(t)(1+\|a\|_{L^\infty(Q_\varepsilon)})} \int_0^T \int_{\omega\times(0,\varepsilon)} |\psi|^2 dxdt \tag{16}$$

for all $0 < \varepsilon < 1$ and every solution of (14) and

$$\int_0^{T'} \int_\Omega \psi^2 dxdt \leq C_1(T') e^{C_2(T')(1+\|a\|_{L^\infty(Q_\varepsilon)})} \int_0^T \int_{\omega\times(0,\varepsilon)} \psi^2 dxdt. \tag{17}$$

Obviously, $C(t)$ in (16) and $C(T')$ in (17) tend to infinity as t and T' tend to T, since the observability inequality fails for $t = T$ as indicated in point (a) above. ■

As we shall see in section 2 below, the uniform null controllability property of Theorem 1 is an immediate consequence of Proposition 1. The proof of Proposition 1, that relies on Carleman inequalities, will be given in section 4. The passage to the limit, i. e. the proof of Theorem 2, will be given in section 3. Finally, in section 5, we give an extension of Theorem 2 to semilinear heat equations with globally Lipschitz nonlinearities.

2 Proof of the uniform null controllability.

All along this section we assume that Proposition 1 holds and we prove Theorem 1.

For any $0 < \varepsilon < 1$, given $\psi^0 \in L^2(\Omega_\varepsilon)$ we introduce the following norm

$$\| \psi^0 \|_{H_\varepsilon} = \left[\int_0^T \int_{\omega\times(0,\varepsilon)} |\psi_\varepsilon|^2 dxdt \right]^{\frac{1}{2}} \tag{18}$$

where ψ_ε is the solution of (14) corresponding to the initial data ψ^0.

It is clear that $\| \cdot \|_{H_\varepsilon}$ is a seminorm. To prove that it is a norm it is sufficient to use the unique continuation property of solutions of (14) (i.e. the fact that, if ψ solution of (14) vanishes in the control set $\omega_\varepsilon \times (0, T)$, then it vanishes everywhere [see [12]]).

We define the Hilbert space $H_\varepsilon = H_\varepsilon(a)$ as the completion of $L^2(\Omega_\varepsilon)$ with respect to the norm $\| \cdot \|_{H_\varepsilon}$.

Observe that for $\psi^0 \in H_\varepsilon$ there exists a sequence $\psi^0_n \in L^2(\Omega_\varepsilon)$ such that $\psi^0_n \to \psi^0$ strongly in H_ε. By definition of the norm $||\cdot||_{H_\varepsilon}$, the corresponding solutions ψ_n of (14) constitute a Cauchy sequence in $L^2(\omega_\varepsilon \times (0,T))$. Moreover, in view of Proposition 1 and Remark 6 (c) above, this sequence is bounded in $L^2(\Omega_\varepsilon \times (0,T'))$ for all $T' < T$. Therefore, there exists a solution ψ of (14) taking ψ^0 as datum for $t = T$ such that

$$\psi_n \to \psi \text{ strongly in } L^2(\omega_\varepsilon \times (0,T)).$$

Furthermore, $\psi \in L^2(\Omega_\varepsilon \times (0,T'))$ for all $T' < T$. On the other hand, the observability inequalities (16) and (17) remain true for this solution and $\psi(t) \in L^2(\Omega_\varepsilon)$ for all $0 \le t < T$. But, of course, we can not guarantee that ψ^0 belongs to $L^2(\Omega_\varepsilon)$.

Given $y^0 \in L^2(\Omega_\varepsilon)$ we introduce the following quadratic functional J_ε defined on H_ε:

$$J_\varepsilon(\psi^0) = \frac{1}{2} \parallel \psi^0 \parallel_{H_\varepsilon}^2 + \int_{\Omega_\varepsilon} y^0 \psi_\varepsilon(0) dx$$

with ψ_ε the solution of (14) with data $\psi^0 \in H_\varepsilon$.

We have the following result:

Proposition 2. *J_ε is continuous, convex and coercive in H_ε. More precisely,*

$$\lim_{\|\psi^0\|_{H\varepsilon} \to \infty} \frac{J_\varepsilon(\psi_\varepsilon^0)}{\parallel \psi_\varepsilon^0 \parallel_{H_\varepsilon}} = \infty. \tag{19}$$

Therefore J_ε reaches its minimum at a unique $\hat{\psi}_\varepsilon^0 \in H_\varepsilon$. Moreover, $\hat{\psi}_\varepsilon^0 \neq 0$ whenever $y_0 \neq 0$. At the minimum, the following optimality condition is satisfied

$$0 = < \hat{\psi}_\varepsilon^0, \psi_\varepsilon^0 >_{H_\varepsilon} + \int_{\Omega_\varepsilon} y^0 \psi_\varepsilon(0) \tag{20}$$

for any $\psi_\varepsilon^0 \in H_\varepsilon$, where $< \cdot, \cdot >_{H_\varepsilon}$ denotes the internal product in H_ε and ψ_ε is the solution of (14) with datum ψ_ε^0.

Moreover, under the assumptions of Theorem 1, there exists $C > 0$ independent of $0 < \varepsilon < 1$ such that

$$\parallel \hat{\psi}_\varepsilon^0 \parallel_{H_\varepsilon} \le C e^{C(1+\|a\|_\infty)} \parallel y^0 \parallel_{L^2(\Omega_\varepsilon)}, \tag{21}$$

for all $y^0 \in L^2(\Omega_\varepsilon)$.

Remark 7. The spaces H_ε and the functionals J_ε depend on the potential a involved in the adjoint system (14). When needed, in order to make explicit their dependence on the potential we shall use the notations $H_\varepsilon(a)$, $J_\varepsilon(\cdot; a_\varepsilon)$, etc. Note also that the functional J_ε also depends on the initial datum y_0 of (1) to be controlled. When needed we shall also use the notation $J_\varepsilon(\cdot; a_\varepsilon, y_0)$ to make explicit the dependence on y_0. ■

Proof: Taking into account that, by definition of the space H_ε and Proposition 1, there exists a constant C_ε such that

$$||\psi(0)||_{L^2(\Omega_\varepsilon)} \le C_\varepsilon ||\psi^0||_{H_\varepsilon},$$

for all $\psi^0 \in H_\varepsilon$, the continuity of J_ε is straightforward. The convexity of J_ε is also clear. On the other hand, by the definition of the norm $||\cdot||_{H_\varepsilon}$ and the observability inequality (15), we also have

$$\frac{J_\varepsilon(\psi_n^0)}{\parallel \psi_n^0 \parallel} \geq \frac{1}{2} \parallel \psi_n^0 \parallel_{H_\varepsilon} - C\, e^{C(1+\|a\|_\infty)} \parallel y^0 \parallel_{L^2(\Omega_\varepsilon)} . \tag{22}$$

From (22) (19) holds immediately. The proof of the optimality condition (20) is also standard.

On the other hand, the uniform inequality (21) is a consequence of (15). In fact,

$$0 \geq J_\varepsilon(\hat{\psi}_\varepsilon^0) \geq \frac{1}{2} \parallel \hat{\psi}_\varepsilon^0 \parallel_{H_\varepsilon}^2 - C\, e^{C(1+\|a\|_\infty)} \parallel \hat{\psi}_\varepsilon^0 \parallel_{H_\varepsilon} \parallel y^0 \parallel_{L^2(\Omega_\varepsilon)}$$

and (21) holds.

■

Proposition 3. *Let $0 < \varepsilon < 1$, $a \in L^\infty(Q_\varepsilon)$ and $y_0 \in L^2(\Omega_\varepsilon)$. Let $\hat{\psi}^0 \in H_\varepsilon$ be the minimizer of $J_\varepsilon(\cdot; a)$ and $\hat{\psi}$ the corresponding solution of (14). Then the solution y_ε of (1) with*

$$h_\varepsilon = \hat{\psi}_\varepsilon \tag{23}$$

satisfies

$$y_\varepsilon(T) = 0.$$

Proof: Let us consider system (1) with the control $h_\varepsilon = \hat{\psi}_\varepsilon$. Multiplying by any solution ψ_ε of (14) and integrating by parts we obtain for all $\psi^0 \in H_\varepsilon$

$$\int_0^T \int_{\omega_\varepsilon} \hat{\psi}_\varepsilon \psi_\varepsilon dxdt = < y_\varepsilon(T), \psi^0 > - \int_{\Omega_\varepsilon} y_\varepsilon(0)\psi_\varepsilon(0)dx, \tag{24}$$

where $< \cdot, \cdot >$ denotes the duality product $H'_\varepsilon - H_\varepsilon$.

Combining the optimality condition (20) and (24) we deduce that

$$< y_\varepsilon(T), \psi^0 > = 0,$$

for all ψ^0 in H_ε. This implies that $y_\varepsilon(T) \equiv 0$ as we wanted to prove. ■

As a consequence of Proposition 3 the proof of Theorem 2 is complete. Indeed, it shows that $h_\varepsilon = \hat{\psi}_\varepsilon$ is the control we were looking for. Estimate (21) clearly implies (4). ■

3 Proof of the convergence result

This section is devoted to the proof of Theorem 2. One of the main points in the proof is Proposition 1. For the moment we will suppose that it holds true. The proof of Proposition 1 will be given in the next section.

First of all it is useful to rewrite the uniform controllability results of section 2 in the rescaled variables (x', z_n, t). This is done in the following subsection. We also recall some basic facts about the controllability of the limit system (3).

3.1 Preliminaries

As indicated in the introduction, the null controllability of systems (1) and (6) are equivalent properties as it can be seen through the change of variables (5).

The null control g_ε for system (6) one obtains when doing the change of variables (5) on the control of system (1) can also be characterized by a variational principle.

Consider the rescaled adjoint system:

$$
(25) \qquad \begin{cases} -\partial_t \eta_\varepsilon - \Delta' \eta_\varepsilon - \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial z_n}\eta_\varepsilon + b_\varepsilon(x', z_n, t)\eta_\varepsilon = 0 & \text{in } Q_1 \\ \frac{\partial \eta_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_1^\pm \times (0,T) \\ \eta_\varepsilon = 0 & \text{on } [\partial\Omega \times (0,1)] \times (0,T) \\ \eta_\varepsilon(T) = \eta^0 & \text{in } \Omega_1. \end{cases}
$$

Introduce the Hilbert space $\tilde{H}_\varepsilon$ of functions defined in Ω_1, obtained from H_ε by the change of variables (5). This space coincides with the completion of $L^2(\Omega_1)$ with respect to the norm

$$
||\eta^0||_{\tilde{H}_\varepsilon} = (\int_0^T \int_{\omega_1} \eta_\varepsilon^2 dx' dz_n dt)^{1/2},
$$

where η_ε denotes the solution of (25) with initial datum η^0.

Consider the following functional, defined in $\tilde{H}_\varepsilon$:

$$
\tilde{J}_\varepsilon(\psi^0) = \frac{1}{2} \parallel \eta^0 \parallel^2_{\tilde{H}_\varepsilon} + \int_{\Omega_1} u_\varepsilon^0 \eta_\varepsilon(0) dx' dz_n
$$

with η_ε the solution of (25) with data $\eta^0 \in \tilde{H}_\varepsilon$. The functional $\tilde{J}_\varepsilon$ is exactly the same as J_ε, but written under the change of variables (5).

It is easy to see that $\tilde{J}_\varepsilon$ has a unique minimizer $\hat{\eta}_\varepsilon^0$ in $\tilde{H}_\varepsilon$. The control $g_\varepsilon = \hat{\eta}_\varepsilon$, where $\hat{\eta}_\varepsilon$ is the solution of (25) with this minimizer $\hat{\eta}_\varepsilon^0$ as initial datum is such that the solution of (6) vanishes at $t = T$. Obviously, this control coincides with the one one gets by means of the change of variables (5) applied to the control we have obtained in section 2 minimizing the functional J_ε in H_ε.

Under the assumptions of Theorem 2, the sequence of initial data u_ε^0 is bounded in $L^2(\Omega_1)$ and therefore the sequence of controls g_ε is bounded in $L^2(\omega_1 \times (0,T))$. In particular the vertical averages $\int_0^1 g_\varepsilon(x', z_n, t) dz_n$ are uniformly bounded in $L^2(\omega \times (0,T))$. As we shall see, as stated in Theorem 2, the limit of these vertical averages will provide the control we are looking for the limit system (3).

Let us now analyze the limit system (3). Consider the corresponding adjoint system

$$
(26) \qquad \begin{cases} -\partial_t \eta - \Delta' \eta + a(x', t)\eta = 0 & \text{in } Q \\ \eta = 0 & \text{on } \partial\Omega \times (0,T) \\ \eta(T) = \eta^0 & \text{in } \Omega. \end{cases}
$$

Introduce the Hilbert space H_0, completion of $L^2(\Omega)$ with respect to the norm

$$
||\eta^0||_{H_0} = (\int_0^T \int_\omega \eta^2 dx' dt)^{1/2},
$$

where η denotes the solution of (26) with inital datum η^0. Let us also consider the functional

$$J_0(\eta^0) = \frac{1}{2}||\eta^0||^2_{H_0} + \int_\Omega y^0(x')\eta(x',0)dx'.$$

Let $\hat{\eta}^0$ be the unique minimizer of J_0 in H_0. Then, the control

$$h(x',t) = \hat{\eta}$$

with $\hat{\eta}$ the solution of (26) with the minimizer $\hat{\eta}^0$ of J_0 in H_0 as datum is such that the solution of (3) satisfies (10).

Summarizing, we see that the null control both for the ε and the limit systems are characterized as minima of suitable quadratic functionals. This will be essential when proving, in the following subsection, the convergence of the controls stated in Theorem 2.

3.2. Proof of Theorem 2.

As indicated above, the controls g_ε are bounded in $L^2(\omega_1 \times (0,T))$. We also recall that g_ε are in fact the restrictions to $\omega_1 \times (0,T)$ of the solutions $\hat{\eta}_\varepsilon$ of the rescaled adjoint system (25) with the minimizers of $\tilde{J}_\varepsilon$ as initial data. Consequently, there exists a subsequence (still denoted by the index ε to simplify the notation) such that

$$\int_0^1 \hat{\eta}_\varepsilon(x', z_n, t)dz_n \rightharpoonup h(x',t) \quad \text{weakly in} \quad L^2(\omega \times (0,T)), \tag{27}$$

as ε tends to zero.

We have to show that the weak limit $h(x',t)$ coincides in $\omega \times (0,T)$ with the minimizer of J_0 in H_0. To do this, we use the ideas of Γ-convergence (see [2]). The following technical Lemma is needed.

Lemma 1. *Under the assumptions of Theorem 2, the limit $h(x',t)$ in (27) coincides in $\omega \times (0,T)$ with a solution $\eta = \eta(x',t)$ of the limit adjoint system (26). Moreover, for any $0 < T' < T$ the following convergences hold:*

$$\int_0^1 \hat{\eta}_\varepsilon(x', z_n, t)dz_n \rightharpoonup \eta(x',t) \quad \textit{weakly in} \quad L^2(\Omega \times (0,T')), \tag{28}$$

$$\int_0^1 \hat{\eta}_\varepsilon(x', z_n, T')dz_n \to \eta(x',T') \quad \textit{strongly in} \quad L^2(\Omega), \tag{29}$$

as ε tends to zero.

Furthermore, since

$$\int_0^T \int_\omega \eta^2 dx'dt < \infty, \tag{30}$$

η is a solution of (26) with initial datum η^0 in H_0.

Remark 8. In this Lemma we do not identify the initial datum η^0 corresponding to the limit solution $\eta(x', t)$ as the minimizer of J_0 in H_0. This will be done later on. ■

Proof: According to the uniform observability inequality and that $g_\varepsilon = \hat{\eta}_\varepsilon$ is bounded in $L^2(\omega_1 \times (0, T))$, we deduce that the sequence $\hat{\eta}_\varepsilon$ of solutions of the rescaled adjoint system (25) is bounded in $L^2(\Omega_1 \times (0, T'))$, for any $T' < T$. Consequently, the averages in z_n of the controls g_ε are bounded in $L^2(\Omega \times (0, T'))$ for all $T' < T$ and therefore, convergence (28) holds as well.

We now show that the limit $\eta(x', t)$, which belongs to $L^2(\Omega \times (0, T'))$ for all $T' < T$ and to $L^2(\omega \times (0, T))$, is a solution of the limit adjoint system (26). To do this, we integrate system (25) with respect to z_n in $(0, 1)$. We obtain that

$$\varphi_\varepsilon(x', t) = \int_0^1 \hat{\eta}_\varepsilon(x', z_n, t) dz_n$$

solves

$$\begin{cases} -\partial_t \varphi_\varepsilon - \Delta' \varphi_\varepsilon + \int_0^1 b_\varepsilon(x', z_n, t) \hat{\eta}_\varepsilon(x', z_n, t) dz_n = 0 & \text{in } Q \\ \varphi_\varepsilon = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \tag{31}$$

Thus, it is sufficient to show that $\int_0^1 b_\varepsilon(x', z_n, t) \hat{\eta}_\varepsilon(x', z_n, t) dz_n$ converges in the sense of distributions in $\Omega \times (0, T)$ to $a(x', t)\eta(x', t)$. We introduce the following decomposition:

$$\int_0^1 b_\varepsilon(x', z_n, t) \hat{\eta}_\varepsilon(x', z_n, t) dz_n - a(x', t)\eta(x', t) = \int_0^1 [b_\varepsilon(x', z_n, t) - a(x', t)] \hat{\eta}_\varepsilon(x', z_n, t) dz_n$$

$$+ a(x', t)[\int_0^1 \hat{\eta}_\varepsilon(x', z_n, t) dz_n - \eta(x', t)] = I_{1,\varepsilon} + I_{2,\varepsilon}.$$

In view of the convergence (28) and the fact that $a(x', t) \in L^\infty(\Omega \times (0, T))$, it is immediate to see that $I_{2,\varepsilon}$ converges weakly to zero in $L^2(\Omega \times (0, T'))$ for all $T' < T$. Therefore, it is sufficient to analyze the term $I_{1,\varepsilon}$. We have

$$\|I_{1,\varepsilon}\|_{L^1(\Omega \times (0,T'))} \leq \| \|b_\varepsilon(x', z_n, t) - a(x', t)\|_{L^2_{z_n}((0,1))} \|\hat{\eta}_\varepsilon(x', z_n, t)\|_{L^2_{z_n}((0,1))} \|_{L^1(\Omega \times (0,T'))}$$

$$\leq \|b_\varepsilon(x', z_n, t) - a(x', t)\|_{L^2(\Omega_1 \times (0,T'))} \|\hat{\eta}_\varepsilon(x', z_n, t)\|_{L^2(\Omega_1 \times (0,T'))}.$$

In view of the assumption (7) on the potentials and the fact that $\hat{\eta}_\varepsilon(x', z_n, t)$ is uniformly bounded in $L^2([\Omega \times (0, 1)] \times (0, T'))$ for all $T' < T$, we deduce that $I_{1,\varepsilon}$ converges to zero in $L^1(\Omega \times (0, T'))$ for all $T' < T$. This completes the proof of the fact that the limit η is a solution of the limit adjoint system.

The fact that (29) holds can also be proved in a classical way. Using the fact that $\varphi_\varepsilon(x', t) = \int_0^1 \hat{\eta}_\varepsilon(x', z_n, t) dz_n$ satisfies (31), the regularizing effect of the heat equation in Ω with Dirichlet boundary conditions, and the fact that $\hat{\eta}_\varepsilon(x', z_n, t)$ is uniformly bounded in $L^2(\Omega_1 \times (0, T'))$ for all $T' < T$, we deduce that $\varphi_\varepsilon(x', t)$ is uniformly bounded in $L^2(0, T'; H^2(\Omega)) \cap H^1(0, T'; L^2(\Omega))$ for all $T' < T$. Classical compactness results ([13]) allow us to prove that the convergence of φ_ε towards η holds actually in $C([0, T']; H^s(\Omega))$ for all $s < 2$ and $0 < T' < T$. In particular, (29) holds. This completes the proof of the Lemma. ■

To continue with the proof of Theorem 2 we have to show that the initial data $\eta^0 \in H_0$ of the solution η of (26) obtained as limit in Lemma 1 of the averages in z_n of the ε-controls is actually the minimizer of J_0 in H_0.

To do this we first observe that

$$\int_{\Omega_1} u_\varepsilon^0 \hat{\eta}_\varepsilon(0) dx' dz_n \to \int_\Omega y^0 \eta(0) dx \tag{32}$$

as ε tends to zero. Indeed, we use the decomposition

$$\int_{\Omega_1} u_\varepsilon^0 \hat{\eta}_\varepsilon(0) dx' dz_n - \int_\Omega y^0 \eta(0) dx' = \int_{\Omega_1} [u_\varepsilon^0(x', z_n) - y^0(x')] \hat{\eta}_\varepsilon(x', z_n, 0) dx' dz_n \tag{33}$$

$$+ \int_\Omega y^0(x') [\int_0^1 \hat{\eta}_\varepsilon(x', z_n, 0) dz_n - \eta(x', 0)] dx' = I_{1,\varepsilon} + I_{2,\varepsilon}.$$

The first term $I_{1,\varepsilon}$ tends to zero since $u_\varepsilon^0(x', z_n) - y^0(x')$ tends strongly to zero in $L^2(\Omega_1)$ by hypothesis and $\hat{\eta}_\varepsilon(x', z_n, 0)$ is uniformly bounded in $L^2(\Omega_1)$. In order to pass to the limit in the second one, it is sufficient to observe that $\int_0^1 \hat{\eta}_\varepsilon(x', z_n, 0) dz_n - \eta(x', 0)$ converges strongly to zero in $L^2(\Omega_1)$ as proved in the Lemma above.

On the other hand, by the weak convergence (28), we have

$$\int_0^T \int_\omega \eta^2 dx' dt \leq \liminf_{\varepsilon \to 0} \int_0^T \int_\omega [\int_0^1 \hat{\eta}_\varepsilon dz_n]^2 dx' dt \leq \liminf_{\varepsilon \to 0} \int_0^T \int_\omega \int_0^1 \hat{\eta}_\varepsilon^2 dz_n dx' dt. \tag{34}$$

Consequently,

$$J_0(\eta^0) \leq \liminf_{\varepsilon \to 0} \tilde{J}_\varepsilon(\hat{\eta}_\varepsilon^0). \tag{35}$$

In order to show that η^0 is the minimizer of J_0 in H_0 we have to prove that

$$J_0(\eta^0) \leq J_0(\rho^0), \quad \forall \rho^0 \in H_0. \tag{36}$$

By the density of $L^2(\Omega)$ in H_0 and the continuity of J_0 in H_0, it is in fact sufficient to see that this holds for all $\rho^0 \in L^2(\Omega)$.

In order to prove (36), taking into account that (35) holds, and that $\hat{\eta}_\varepsilon^0$ is the minimizer of $\tilde{J}_\varepsilon$ in $\tilde{H}_\varepsilon$, it is sufficient to show that

$$J_0(\rho^0) = \lim_{\varepsilon \to 0} \tilde{J}_\varepsilon(\rho^0), \tag{37}$$

for all ρ^0 in $L^2(\Omega)$.

Here, we use the fact that $\rho^0 = \rho^0(x')$, which is independent of z_n, can also be viewed as a function in $L^2(\Omega_1)$ and therefore in $\tilde{H}_\varepsilon$.

Let ρ_ε be the solution of (25) with $\rho^0 = \rho^0(x') \in H_0$ as initial data. We have to show that

$$\int_0^T \int_{\omega \times (0,1)} \rho_\varepsilon^2 dx' dz_n dt \to \int_0^T \int_\omega \rho^2 dx' dt \tag{38}$$

and

$$\int_{\Omega\times(0,1)} u_\varepsilon^0(x',z_n)\rho_\varepsilon(x',z_n,0)dx'dz_n \to \int_\Omega y^0(x')\rho(x',0)dx' \tag{39}$$

as ε tends to zero.

In order to prove these two facts we set $v_\varepsilon = \rho_\varepsilon - \rho$, where ρ_ε and ρ are respectively the solutions of (25) and (26) with initial datum ρ^0. Then, v_ε satisfies

$$\begin{cases} -\partial_t v_\varepsilon - \Delta' v_\varepsilon - \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial z_n} v_\varepsilon + b_\varepsilon(x',z_n,t)v_\varepsilon = (a-b_\varepsilon)\rho & \text{in } Q_1 \\ \frac{\partial v_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_1^\pm \times (0,T) \\ v_\varepsilon = 0 & \text{on } [\partial\Omega\times(0,1)]\times(0,T) \\ v_\varepsilon(T) = 0 & \text{in } \Omega_1. \end{cases} \tag{40}$$

Here we have used the fact that $\rho(x',t)$ is a solution of (26) independent of z_n.

Multiplying (40) by v_ε and integrating by parts in Ω_1 we get,

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}\int_{\Omega_1}|v_\varepsilon|^2dx'dz_n + \int_{\Omega_1}|\nabla' v_\varepsilon|^2dx'dz_n + \frac{1}{\varepsilon^2}\int_{\Omega_1}|\frac{\partial}{\partial z_n}v_\varepsilon|^2dx'dz_n + \\ &+\int_{\Omega_1} b_\varepsilon|v_\varepsilon|^2dx'dz_n = \int_{\Omega_1}(a-b_\varepsilon)\rho v_\varepsilon dx'dz_n. \end{aligned} \tag{41}$$

Observe that

$$|\int_{\Omega_1}(a-b_\varepsilon)\rho v_\varepsilon dx'dz_n| = |\int_\Omega \rho\left(\int_0^1 (a-b_\varepsilon)v_\varepsilon dz_n\right)dx'| \leq \|\rho\|_{L^2(\Omega)}\|\int_0^1(a-b_\varepsilon)v_\varepsilon dz_n\|_{L^2(\Omega)}$$

$$\leq \|\rho\|_{L^2(\Omega)}\|a-b_\varepsilon\|_{L^\infty(\Omega_1)}\|v_\varepsilon\|_{L^2(\Omega_1)}$$

and therefore

$$|\int_{\Omega_1}(a-b_\varepsilon)\rho v_\varepsilon dx'dz_n| \leq \frac{\|a-b_\varepsilon\|_{L^\infty(\Omega_1)}}{2}\|\rho\|^2_{L^2(\Omega)} + \frac{\|a-b_\varepsilon\|_{L^\infty(\Omega_1)}}{2}\|v_\varepsilon\|^2_{L^2(\Omega_1)}.$$

Then, taking into account that $\rho \in L^\infty(0,T;L^2(\Omega))$, $b_\varepsilon \to a$ in $L^\infty(\Omega_1)$ and applying Gronwall's Lemma, we deduce that v_ε tends to zero in $C([0,T];L^2(\Omega_1))$ as $\varepsilon \to 0$, or, in other words, ρ_ε tends to ρ in $C([0,T];L^2(\Omega_1))$. In particular,

$$\int_0^T\int_0^1\int_\omega|\rho_\varepsilon|^2dx'dz_ndt \to \int_0^T\int_0^1\int_\omega|\rho|^2dx'dz_ndt = \int_0^T\int_\omega|\rho|^2dx'dt \tag{42}$$

and

$$\int_0^1\int_\Omega u_\varepsilon^0\rho_\varepsilon(0)dx'dz_n \to \int_0^1\int_\Omega y^0\rho(0)dx'dz_n = \int_\Omega y^0\rho(0)dx'. \tag{43}$$

Observe that (42), (43) imply (37). This, together with (35) implies that η^0 is the minimizer of J_0 in H_0.

Let us now prove that the controls actually converge strongly. In view of the weak convergence, it is sufficient to show that

$$\int_0^T \int_\omega (\int_0^1 g_\varepsilon(x', z_n, t)dz_n)^2 dx'dt \to \int_0^T \int_\omega h^2(x', t)dx'dt. \tag{44}$$

We now know that $h = \hat{\eta}$ in $\omega \times (0, T)$, where $\hat{\eta}$ is the solution of (26) associated with the initial datum $\hat{\eta}^0$ that minimizes J_0 in H_0. We denote by I_ε (resp. I_0) the value of the minimum of $\tilde{J}_\varepsilon$ (resp. J_0) in $\tilde{H}_\varepsilon$ (resp. H_0). We claim that I_ε tends to I_0. Indeed, for any $\delta > 0$ there exists η^0 in $L^2(\Omega)$ such that

$$J_0(\eta^0) \le J_0(\hat{\eta}^0) + \delta = I_0 + \delta.$$

On the other hand, as we have proved above, $\tilde{J}_\varepsilon(\eta^0)$ tends to $J_0(\eta^0)$. Taking into account that $\hat{\eta}^0_\varepsilon$ minimizes $\tilde{J}_\varepsilon$, we deduce that

$$\limsup_{\varepsilon\to 0} I_\varepsilon \le J_0(\eta^0) \le I_0 + \delta$$

for all $\delta > 0$ and consequently

$$\limsup_{\varepsilon\to 0} I_\varepsilon \le I_0.$$

On the other hand, from the weak convergence results on the controls proved above it is also clear that

$$I_0 \le \liminf_{\varepsilon\to 0} I_\varepsilon.$$

We conclude that

$$I_0 = \lim_{\varepsilon\to 0} I_\varepsilon.$$

Since, on the other hand, $\int_{\Omega_1} u^0_\varepsilon \hat{\eta}_\varepsilon(0)dx'dz_n$ converges to $\int_\Omega y^0\hat{\eta}(0)dx'$, we deduce that (44) holds. This completes the proof of the strong convergence of the controls.

The fact that $\int_0^1 u_\varepsilon dz_n$ converges to y in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ may be proved easily as when dealing with $v_\varepsilon = \rho_\varepsilon - \rho$ since we already know that the averages of the controls and the initial data with respect to z_n converge. Indeed, let $w_\varepsilon = \int_0^1 (u_\varepsilon - y)dz_n$. Then w_ε satisfies

$$\begin{cases} w_{\varepsilon,t} - \Delta' w_\varepsilon + \int_0^1 (b_\varepsilon(x', x_n, t)u_\varepsilon - a(x', t)y)dz_n = \int_0^1 (g_\varepsilon - h)dz_n 1_\omega & \text{in } Q \\ w_\varepsilon = 0 & \text{on } \partial\Omega \times (0, T) \\ w_\varepsilon(0) = \int_0^1 (u^0_\varepsilon - y^0)dz_n & \text{in } \Omega. \end{cases} \tag{45}$$

From the assumptions of Theorem 2 we already know that the initial data converge in $L^2(\Omega)$. We also have proved the convergence in $L^2(\Omega \times (0, T))$ of the right hand side terms. On the other hand,

$$|\int_0^1 (b_\varepsilon(x', x_n, t)u_\varepsilon - a(x', t)y)dz_n| \le$$

$$\le |\int_0^1 (b_\varepsilon(x', x_n, t) - a(x', t))u_\varepsilon dz_n| + |\int_0^1 a(x', t)(u_\varepsilon - y)dz_n| \le$$

$$\le \int_0^1 |b_\varepsilon(x', x_n, t) - a(x', t)||u_\varepsilon|dz_n + |a(x', t)\int_0^1 (u_\varepsilon - y)dz_n| = I_{1,\varepsilon} + |a(x', t)w_\varepsilon.|$$

The integral $I_{1,\varepsilon}$ converges to zero in $L^\infty(\Omega\times(0,T)$ since b_ε converges to a in $L^2(Q_1)$ and u_ε is uniformly bounded in $L^2(Q_1)$.

Taking all this into account and by classical energy estimates we deduce that w_ε tends to zero in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$.

This completes the proof of Theorem 2.

4 Proof of the uniform Carleman inequality

In order to prove Proposition 1 we introduce an auxiliary function whose existence is guaranteed by the following result due to Fursikov and Imanuvilov in [8].

Theorem 3. *Let $X\subset\mathbb{R}^m$ be an open and bounded set of class C^2. Let $\mathcal{O}\subset X$ be an open and non empty subset. Then, there exists $\eta\in C^\infty(X)$ satisfying $\eta(x)>1$ in X, $\frac{\partial\eta}{\partial\nu}<0$ on ∂X , $|\nabla\eta|>0$ in $\Omega\backslash\mathcal{O}$. Furthermore, given $x_0\in\mathcal{O}$ and $\delta>0$ such that $B_\delta(x_0)\subset\mathcal{O}$, where $B_\delta(x_0)$ denotes a ball of radius δ centered at x_0, η can be chosen satisfying $|\nabla\eta|>0$ in $X\backslash B_\delta(x_0)$.*

In this section $\|\cdot\|$ and $((\cdot,\cdot))$ will stand for the norm and the scalar product in $L^2(Q_\varepsilon)$, respectively. In order to prove our result, we choose $\eta=\eta(x')$ given by Theorem 3 for the sets $X=\Omega\subset\mathbb{R}^{n-1}$, and $\mathcal{O}=\omega\subset\Omega$. Observe that, in particular, η is independent of the vertical variable z_n and therefore it satisfies $\frac{\partial\eta}{\partial\nu}|_{\Gamma_\varepsilon^\pm}=0$. We define

$$\varphi(x,t)=\frac{e^{\lambda\eta}}{t(T-t)},\qquad \tilde\varphi(x,t)=\frac{e^{\lambda\eta}-e^{2\lambda\|\eta\|_\infty}}{t(T-t)}.$$

Let $c(x,t)\in L^\infty(Q_\varepsilon)$ and let w be a solution of

$$\begin{cases} w_t-\Delta w+c(x,t)w=f & \text{in } Q_\varepsilon\\ \dfrac{\partial w}{\partial\nu}=0 & \text{on } \Gamma_\varepsilon^\pm\times(0,T)\\ w=0 & \text{on } \partial\Omega\times(0,\varepsilon)\times(0,T)\end{cases}\tag{46}$$

such that $w\in W_\varepsilon=L^2_{loc}(0,T;H^2(\Omega_\varepsilon)\cap V_\varepsilon)\cap H^1_{loc}(0,T;L^2(\Omega_\varepsilon))$ and f such that

$$\int_0^T\int_{\Omega_\varepsilon}f^2e^{2s\tilde\varphi}dxdt<\infty.$$

We have the following result, similar to one proved by Fursikov and Imanuvilov [8] for the heat equation with Dirichlet homegeneous boundary conditions.

Proposition 4. *Let $r>\delta$ be such that $B_\delta\subset B_r\subset\omega$. Then, there exist constants $\hat\lambda>1$, $\hat s(\Omega,T,\|c\|_\infty)$ and $C(\Omega,T)$ such that the following estimate holds for every $w\in W_\varepsilon$ solution of (46), $0<\varepsilon<1$ and for every $\lambda>\hat\lambda$, $s>\hat s$:*

$$\begin{aligned}&\int_0^T\int_{\Omega_\varepsilon}\frac{1}{s\varphi}e^{2s\tilde\varphi}(|\Delta w|^2+|w_t|^2)+(s\lambda^2\varphi|\nabla w|^2+s^3\lambda^4\varphi^3|w|^2)e^{2s\tilde\varphi}dxdt\\ &\le C\left(\int_0^T\int_{\Omega_\varepsilon}e^{2s\tilde\varphi}f^2dxdt+\int_0^T\int_0^\varepsilon\int_{B_r}s^3\lambda^4\varphi^3|w|^2e^{2s\tilde\varphi}dxdt\right).\end{aligned}\tag{47}$$

Furthermore, $\hat{s}$ can be chosen of the form $\sigma_1 + \sigma_2\|c\|_\infty^{2/3}$ where σ_1 and σ_2 are positive constants that depend only on Ω and T.

Remark 9. Note that all the constants appearing in this inequality are uniformly bounded with respect to $0 < \varepsilon < 1$. ■

Proof: We proceed as in [8]. Set $v = e^{s\tilde{\varphi}}w$. Define $f_s = e^{s\tilde{\varphi}}f + s\lambda\varphi v\Delta\eta - c(x,t)v - s\lambda^2\varphi v|\nabla\eta|^2$. Observe that from (46), we get that v solves

$$f_s = L_1 v + L_2 v \tag{48}$$

with

$$L_1 v = -\Delta v - s^2\lambda^2\varphi^2|\nabla\eta|^2 v - s\varphi_t v \tag{49}$$

$$L_2 v = v_t + 2s\lambda\varphi\nabla\eta\cdot\nabla v. \tag{50}$$

We deduce the following identity

$$\|f_s\|^2 = \|L_1 v\|^2 + 2((L_1 v, L_2 v)) + \|L_2 v\|^2. \tag{51}$$

Let us analyse the scalar product in (51):

$$\begin{aligned}((L_1 v, L_2 v)) = & -\int_{Q_\varepsilon}\Delta v v_t dxdt - \int_{Q_\varepsilon} 2s\lambda\varphi\Delta v\nabla\eta\cdot\nabla v dxdt \\ & -\int_{Q_\varepsilon} s^2\lambda^2\varphi^2|\nabla\eta|^2 v v_t dxdt - \int_{Q_\varepsilon} 2s^3\lambda^3\varphi^3|\nabla\eta|^2\nabla\eta\cdot\nabla v v dxdt \\ & -\int_{Q_\varepsilon} s\varphi_t v v_t dxdt - \int_{Q_\varepsilon} 2s^2\lambda\varphi\varphi_t\nabla\eta\cdot\nabla v v dxdt.\end{aligned}$$

Let us compute $\int_{Q_\varepsilon}\Delta v v_t dxdt$. Integrating by parts we get

$$\begin{cases}\displaystyle\int_{Q_\varepsilon}\Delta v v_t dxdt & \displaystyle= -\int_0^T\int_{\Omega_\varepsilon}\nabla v\cdot\nabla v_t + \int_0^T\int_{\partial\Omega_\varepsilon}\frac{\partial v}{\partial\nu}v_t \\ & \displaystyle= -\frac{1}{2}\int_0^T\int_{\Omega_\varepsilon}\frac{d}{dt}|\nabla v|^2 + \int_0^T\int_{\partial\Omega_\varepsilon}\frac{\partial v}{\partial\nu}v_t.\end{cases}$$

Observe that since $\nabla v(t) = 0$ at $t = T$ and $t = T$, $\int_0^T\int_{\Omega_\varepsilon}\frac{d}{dt}|\nabla v|^2 = 0$. On the other hand,

$$\int_{\partial\Omega_\varepsilon}\frac{\partial v}{\partial\nu}v_t = \int_0^T\int_{\partial\Omega\times(0,\varepsilon)}\frac{\partial v}{\partial\nu}v_t + \int_0^T\int_{\Gamma_\varepsilon^\pm}\frac{\partial v}{\partial\nu}v_t.$$

The first term vanishes since $v = 0$ on $\partial\Omega\times(0,\varepsilon)$. For the second term, we observe that

$$\int_0^T\int_{\Gamma_\varepsilon^\pm}\frac{\partial v}{\partial\nu}v_t = \int_0^T\int_{\Gamma_\varepsilon^\pm}e^{s\tilde{\varphi}}\frac{\partial w}{\partial\nu}v_t + \int_0^T\int_{\Gamma_\varepsilon^\pm}s\lambda\varphi\frac{\partial\eta}{\partial\nu}v v_t = 0.$$

Note that, here, in addition to the boundary conditions that φ satisfies, we have used in an essential way the fact that $\partial\eta/\partial\nu \equiv 0$.

Proceeding as before, after integration by parts and some manipulation we get:

$$\begin{aligned}((L_1 v, L_2 v)) &= Y_1 + X_1 + 3\int_{Q_\varepsilon} s^3\lambda^4\varphi^3|\nabla\eta|^4 v^2 dxdt \\ &+2\int_{Q_\varepsilon} s\lambda^2\varphi|\nabla\eta\cdot\nabla v|^2 dxdt - \int_{Q_\varepsilon} s\lambda^2\varphi|\nabla\eta|^2|\nabla v|^2 dxdt\end{aligned} \tag{52}$$

with

$$Y_1 = -\int_0^T \int_{\partial\Omega\times(0,\varepsilon)} s\lambda\varphi|\frac{\partial v}{\partial\nu}|^2\frac{\partial\eta}{\partial\nu} \tag{53}$$

$$|X_1| \le C\left[\int_{Q_\varepsilon} s\lambda\varphi|\nabla v|^2 dxdt + \int_{Q_\varepsilon} s^2\lambda^3\varphi^3 v^2 dxdt\right]. \tag{54}$$

Note that, by construction of ψ,

$$Y_1 \ge 0. \tag{55}$$

In other words, we concentrate in X_1 the lower powers of λ. Now, multiply (48) by $\lambda^2 s\varphi v|\nabla\eta|^2$. We get

$$\begin{aligned}\int_{Q_\varepsilon} f_s\lambda^2 s\varphi v|\nabla\eta|^2 dxdt = &\int_{Q_\varepsilon} \lambda^2 s\varphi|\nabla\eta|^2 L_2 v^2 dxdt - \int_{Q_\varepsilon} \lambda^2 s\varphi|\nabla\eta|^2 v\Delta v dxdt \\ &- \textstyle\int_{Q_\varepsilon} \lambda^4 s^3\varphi^3|\nabla\eta|^4 v^2 dxdt - \displaystyle\int_{Q_\varepsilon} \lambda^2 s^2\varphi\varphi_t v^2|\nabla\eta|^2 dxdt.\end{aligned} \tag{56}$$

After integration by parts and some computations we get

$$\int_{Q_\varepsilon} \lambda^4 s^3\varphi^3|\nabla\eta|^4 v^2 dxdt = \int_{Q_\varepsilon} \lambda^2 s\varphi|\nabla\eta|^2|\nabla v|^2 dxdt + X_2 \tag{57}$$

with

$$|X_2| \le \frac{1}{16}\int_{Q_\varepsilon} |L_2 v|^2 dxdt + C_1\int_{Q_\varepsilon} s^2\lambda^4\varphi^3 v^2 + \frac{1}{2}\|f_s\|^2. \tag{58}$$

In other words, we concentrate in X_2 the lower powers of s. Sustituting

$$2\int_{Q_\varepsilon} \lambda^4 s^3\varphi^3|\nabla\eta|^4 v^2 dxdt$$

in (52) by the expression (57) we get:

$$\begin{aligned}((L_1 v, L_2 v)) &= Y_1 + X_1 + 2X_2 + \int_{Q_\varepsilon} s^3\lambda^4\varphi^3|\nabla\psi|^4 v^2 dxdt \\ &+2\int_{Q_\varepsilon} s\lambda^2\varphi|\nabla\eta\cdot\nabla v|^2 dxdt + \int_{Q_\varepsilon} s\lambda^2\varphi|\nabla\eta|^2|\nabla v|^2 dxdt.\end{aligned} \tag{59}$$

From the estimates (54), (58) and having in mind that η may vanish in B_δ, it is clear that there exists $\hat{\lambda}$ and $\hat{s}$ large enough such that

$$
(60)\qquad \begin{aligned} \|f_s\|^2 &\geq \frac{1}{4}\|L_2 v\|^2 + \frac{1}{3}\|L_1 v\|^2 \\ &+(\frac{2}{3}-\hat{\nu})\int_0^T\int_0^\varepsilon\int_{\Omega\backslash B_\delta} s^3\lambda^4\varphi^3 dxdt + (\frac{2}{3}-\hat{\nu})\int_0^T\int_0^\varepsilon\int_{\Omega\backslash B_\delta} s\lambda^2\varphi|\nabla v|^2 \end{aligned}
$$

for every $\lambda > \hat{\lambda}$, $s > \hat{s}$. The constant $\hat{\nu}$ can be chosen arbitrarily small but the election of $\hat{\lambda}$ and $\hat{s}$ depend on $\hat{\nu}$.

We observe that $L_1 v - s^2\lambda^2\varphi^2|\nabla\eta|^2 v - s\varphi_t v = \Delta v$. Therefore

$$
(61)\qquad C\left(\frac{1}{s\varphi}|L_1 v|^2 + s^3\lambda^4\varphi^3 v^2\right) \geq \frac{|\Delta v|^2}{s\varphi}.
$$

Proceeding as in (61) we obtain:

$$
(62)\qquad C\left(\frac{1}{s\varphi}|L_2 v|^2 + s\lambda^2\varphi|\nabla v|^2\right) \geq \frac{|v_t|^2}{s\varphi}.
$$

From (60), (61) and (62) we get

$$
(63)\qquad \begin{aligned} &\frac{1}{8}\int_{Q_\varepsilon}\frac{1}{s\varphi}|\Delta v|^2 dxdt + \frac{1}{6}\int_{Q_\varepsilon}\frac{1}{s\varphi}|v_t|^2 dxdt + (\frac{5}{6}-\hat{\nu})\int_{Q_\varepsilon} s^3\lambda^4\varphi^3 v^2 dxdt \\ &+(\frac{1}{6}-\hat{\nu})\int_{Q_\varepsilon} s\lambda^2\varphi|\nabla v|^2 dxdt \\ &\leq \int_0^T\int_0^\varepsilon\int_{B_\delta} s\lambda^2\varphi|\nabla v|^2 dxdt + \int_0^T\int_0^\varepsilon\int_{B_\delta} s^3\lambda^4\varphi^3|v|^2 dxdt + \int_{Q_\varepsilon} f^2 e^{2s\tilde{\varphi}} dxdt. \end{aligned}
$$

Observe that in the right hand side of (63) we have a term involving the gradient of v on the ball B_δ that we want to eliminate. In this aim let $\rho(x) \in C_0^\infty(\omega)$ with $\rho(x) = 1$ in B_δ and $\rho(x) = 0$ in $\Omega\backslash B_r$. We multiply (48) by $\rho(x)s\lambda^2\varphi v$ in $L^2(Q)$. After integrating by parts and some manipulations we get

$$
(64)\qquad \int_0^T\int_0^\varepsilon\int_{B_\delta} s\lambda^2\varphi|\nabla v|^2 dxdt \leq C\left[\int_0^T\int_0^\varepsilon\int_{B_r} s^3\lambda^4\varphi^3|v|^2 dxdt + \int_{Q_\varepsilon} f^2 e^{2s\tilde{\varphi}} dxdt\right].
$$

The last step is to put together (63), (64), recall $v = e^{s\tilde{\varphi}}w$ and apply Hölder and then Young inequality with appropriate constants. ■

Proof of Proposition 1. Let ψ be a solution of (14). Observe that $w(t) = \psi(T-t)$ solves (46) with $f \equiv 0$, $c(x,t) = a(x, T-t)$. Applying (47) to ψ, we get

$$
\int_0^T\int_{\Omega_\varepsilon} s^3\lambda^4\varphi^3|\psi|^2 e^{2s\tilde{\varphi}} dxdt \leq Ce^{c(1+\|a\|_\infty)}\int_0^T\int_0^\varepsilon\int_\omega s^3\lambda^4\varphi^3|\psi|^2 e^{2s\tilde{\varphi}} dxdt.
$$

Observe that $s^3\lambda^4\varphi^3 e^{2s\tilde{\varphi}} \leq C$ (once s, λ are fixed). Then

$$\int_0^T \int_{\Omega_\varepsilon} s^3\lambda^4\varphi^3|\psi|^2 e^{2s\tilde{\varphi}} dxdt \leq Ce^{c(1+\|a\|_\infty)} \int_0^T \int_0^\varepsilon \int_\omega |\psi|^2 dxdt. \tag{65}$$

On the other hand, ψ is a solution of (14) with $\psi^0 \in L^2(\Omega_\varepsilon)$, then $\psi \in C([0,T]; L^2(\Omega_\varepsilon))$. Clearly, the trace of ψ at $T-\delta$ is defined in $L^2(\Omega_\varepsilon)$ for every $\delta > 0$. Then, $\psi(T-\delta) \in L^2(\Omega_\varepsilon)$. By classical estimates for the heat eqution, for every $0 \leq t < T-\delta$

$$\int_{\Omega_\varepsilon} \psi^2(T-\delta)dx \leq \frac{1}{\delta} \int_{T-2\delta}^{T-\delta} \int_{\Omega_\varepsilon} \psi^2(s)dxds. \tag{66}$$

And, by construction of φ there exists $C = C(\delta)$ such that

$$\int_{T-2\delta}^{T-\delta} \int_{\Omega_\varepsilon} \psi^2(s)dxds \leq C(\delta) \int_0^T \int_{\Omega_\varepsilon} s^3\lambda^4\varphi^3|\psi|^2 e^{2s\tilde{\varphi}} dxdt. \tag{67}$$

Combining (65),(66) and (67) we conclude the proof. ■

5 The semilinear heat equation with globally Lipschitz nonlinearities

It is well known that the semilinear heat equation is null controllable provided the nonlinearity is globally Lipschitz. In fact, by now we also know that null controllability holds for nonlinearities that grow at infinity in some (slight) superlinear way ([4], [6]).

Therefore, it is natural to ask whether the convergence results on thin domains we have proved above extend to semilinear equations.

Let us consider the semilinear heat equation:

$$\begin{cases} y_t - \Delta y + f(y) = h1_{\omega_\varepsilon} & \text{in } Q_\varepsilon \\ y = 0 & \text{on } [\partial\Omega \times (0,\varepsilon)] \times (0,T) \\ \frac{\partial y}{\partial n} = 0 & \text{on } \Gamma_\varepsilon^\pm \times (0,T) \\ y(0) = y^0 & \text{in } \Omega_\varepsilon. \end{cases} \tag{68}$$

Here $f : \mathbb{R} \to \mathbb{R}$ denotes a globally Lipschitz function.

We also assume that $f(0) = 0$ so that all the systems above admit the trivial zero solution as a stationary solution.

We also consider the rescaled version of system (68):

$$\begin{cases} \partial_t u_\varepsilon - \Delta' u_\varepsilon - \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial z_n^2} u_\varepsilon + f(u_\varepsilon) = g_\varepsilon 1_{\omega_1} & \text{in } Q_1 \\ u_\varepsilon = 0 & \text{on } [\partial\Omega \times (0,1)] \times (0,T) \\ \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_1^\pm \times (0,T) \\ u_\varepsilon(0) = u_\varepsilon^0 & \text{in } \Omega_1. \end{cases} \tag{69}$$

Consider also the limit $(n-1)$-dimensional system:

$$\begin{cases} y_t - \Delta' y + f(y) = h 1_\omega & \text{in } Q \\ y = 0 & \text{on } \partial\Omega \times (0,T) \\ y(0) = y^0 & \text{in } \Omega. \end{cases} \tag{70}$$

The following holds:

Theorem 4. *Under the assumptions of Theorem 2, and provided f is globally lipschitz and $f(0) = 0$, there exists a sequence g_ε of null controls for the semilinear ε-system (69) such that, after extraction of subsequences,*

$$\int_0^1 g_\varepsilon(x', z_n, t) dz_n \to h(x', t) \text{ strongly in } L^2(\omega \times (0,T)) \text{ as } \varepsilon \to 0 \tag{71}$$

where $h = h(x', t)$ is a null control for the $(n-1)$-dimensional semilinear heat equation (70) with data $y^0 = y^0(x')$.

Moreover, along this subsequence,

$$\int_0^1 u_\varepsilon dz_n \to y, \quad \text{in } L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)). \tag{72}$$

Remark 10. Note that this Theorem does not guarantee the convergence of the whole sequence. Analyzing the proof we give below it is easy to see that the whole sequence of controls g_ε we build is such that all its accumulation points provide a null control for the limit system. However, taking into account that the null control for system (68) is not unique, we have not been able to identify the limit control in a unique way and to guarantee the convergence of the whole sequence. ■

Sketch of the proof. First of all, we recall that the null controls for both system (68) and its rescaled version (69) and the limit system (70) may be build by a fixed point method (see for instance [4] and [6]). Let us recall that the key ingredient for this fixed point argument to work for globally Lipschitz nonlinearities is that all the estimates on the controls of previous sections are uniform when the potentials arising in the linear heat equation (with a lower order term in which the potential arises) are uniformly bounded in $L^\infty(Q_1)$. The fixed point argument provides controls that fulfill the same uniform boundedness properties. This is natural to expect since the nonlinear term in the equations above may be written as

$$f(y) = F(y)y \tag{73}$$

with

$$F(s) = f(s)/s, \quad \text{when } s \neq 0 \quad \text{and } F(0) = f'(0). \tag{74}$$

Taking into account that f is globally Lipschitz and that $f(0) = 0$ we see that F is uniformly bounded, i. e. $F \in L^\infty(\mathbb{R})$. Thus, the nonlinear term in systems above can be viewed as the product of a bounded potential with the state, the bound on the potential being independent of the state, actually $\|F\|_{L^\infty(\mathbf{R})}$.

Thus, the controls we get for system (69) are such that

$$\|g_\varepsilon\|_{L^2(\omega_1\times(0,T))} \le C, \tag{75}$$

for all $0<\varepsilon<1$.

It is then easy to see by an energy estimate that u_ε is bounded in $L^\infty(0,T;L^2(\Omega_1))$ and that

$$\int_0^T\int_{\Omega_1}[|\nabla' u_\varepsilon|^2 + \frac{1}{\varepsilon^2}|\frac{\partial^2 u_\varepsilon}{\partial z_n^2}|^2]dx'dz_ndt \le C. \tag{76}$$

In view of (76) and, more precisely, of the fact that

$$\frac{1}{\varepsilon^2}\int_0^T\int_{\Omega_1}|\frac{\partial^2 u_\varepsilon}{\partial z_n^2}|^2dx'dz_ndt \le C. \tag{77}$$

it is easy to see that

$$u_\varepsilon - \int_0^1 u_\varepsilon dz_n \to 0 \quad \text{as } \varepsilon\to 0 \text{ in } L^2(\Omega_1\times(0,T)). \tag{78}$$

Moreover, (78) together with the uniform boundedness of the sequence u_ε in $L^\infty(0,T;L^2(\Omega_1))$, implies that the convergence (78) holds in $L^p(0,T;L^2(\Omega_1))$ for all $1\le p<\infty$. Taking into account that f is globally lipschitz, we deduce that

$$f(u_\varepsilon) - f(\int_0^1 u_\varepsilon dz_n) \to 0 \quad \text{as } \varepsilon\to 0 \text{ in } L^p(\Omega_1\times(0,T)), \tag{79}$$

for all $1\le p<\infty$.

By extracting subsequences it is then easy to prove that

$$\int_0^1 g_\varepsilon(x',z_n,t)dz_n \rightharpoonup h(x',t) \text{ weakly in } L^2(\omega\times(0,T)) \text{ as } \varepsilon\to 0 \tag{80}$$

and

$$\int_0^1 u_\varepsilon dz_n \rightharpoonup y, \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega)). \tag{81}$$

On the other hand, the limit control h, as stated in the Theorem, is a null control for the limit system (70).

In order to complete the proof of the Theorem, we have to show that the convergence (80) is actually strong.

Note that, once this is done the fact that (81) is strong will be an immediate consequence of classical energy estimates.

In order to show the strong convergence in (80) we proceed as follows. The controls g_ε are, by construction, the controls of minimal $L^2(\omega_1 \times (0,T))$-norm for the "linearized" system

$$(82) \qquad \begin{cases} \partial_t w_\varepsilon - \Delta' w_\varepsilon - \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial z_n^2} w_\varepsilon + F(u_\varepsilon) w_\varepsilon = g_\varepsilon 1_{\omega_1} & \text{in } Q_1 \\ w_\varepsilon = 0 & \text{on } [\partial\Omega \times (0,1)] \times (0,T) \\ \frac{\partial w_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_1^\pm \times (0,T) \\ w_\varepsilon(0) = u_\varepsilon^0 & \text{in } \Omega_1. \end{cases}$$

In fact, they coincide with $\hat{\eta}_\varepsilon$, the solution of the adjoint system

$$(83) \qquad \begin{cases} -\partial_t \eta_\varepsilon - \Delta' \eta_\varepsilon - \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial z_n^2} \eta_\varepsilon + F(u_\varepsilon) \eta_\varepsilon = 0 & \text{in } Q_1 \\ \eta_\varepsilon = 0 & \text{on } [\partial\Omega \times (0,1)] \times (0,T) \\ \frac{\partial \eta_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_1^\pm \times (0,T) \\ \eta_\varepsilon(T) = \eta^0 & \text{in } \Omega_1, \end{cases}$$

with initial data $\hat{\eta}_\varepsilon^0$, the minimizer of the functional $\tilde{J}_\varepsilon$ in $\tilde{H}_\varepsilon$, the functional and space associated with the null control of (82), as in previous sections. It is convenient to identify the limit control h as the one minimizing the functional J_0 in the space H_0. Here J_0 and H_0 stand for the functional and space associated with the null control of the "linearized" $(n-1)$-dimensional systems

$$(84) \qquad \begin{cases} \partial_t w - \Delta' w + F(y) w = h 1_\omega & \text{in } Q \\ w = 0 & \text{on } \partial\Omega \times (0,T) \\ w(0) = y^0 & \text{in } \Omega \end{cases}$$

and

$$(85) \qquad \begin{cases} -\partial_t \eta - \Delta' \eta + F(y) \eta = 0 & \text{in } Q \\ \eta = 0 & \text{on } \partial\Omega \times (0,T) \\ \eta(T) = \eta^0 & \text{in } \Omega. \end{cases}$$

This can be done with the arguments used in the proof of Theorem 2. Only some slight changes are needed since, this time, the potentials $F(u_\varepsilon)$ converge to $F(y)$ in $L^p(\Omega_1 \times (0,T))$ for all $p < \infty$, but we do not know whether this convergence holds in $L^\infty(\Omega_1 \times (0,T))$. ■

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