Controllability and Observability of Partial Differential Equations: Some results and open problems

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Abstract

In these notes we present some of the recent progresses done on the problem of controllability of partial differential equations (PDE). Control problems for PDE arise in many different contexts and ways. A prototypical problem is that of controllability. Roughly speaking it consists in analyzing whether the solution of the PDE can be driven to a given final target by means of a control applied on the boundary or on a subdomain of the domain in which the equation evolves. In an appropriate functional setting this problem is equivalent to that of observability which concerns the possibility of recovering full estimates on the solutions of the uncontrolled adjoint system in terms of partial measurements done on the control region. Observability/controllability properties depend in a very sensitive way on the class of PDE under consideration. In particular, heat and wave equations behave in a significantly different way, because of their different behavior with respect to time reversal. In this paper we first recall the known basic controllability properties of the wave and heat equations emphasizing how their different nature affects their main controllability properties. We also recall the main tools to analyze these problems: the so-called Hilbert uniqueness method (HUM), multipliers, microlocal analysis and Carleman inequalities. We then discuss some more recent developments concerning the behavior of controls under homogenization processes, equations with low regularity coefficients, etc. We also analyze the way control and observability properties depend on the norm and regularity of these coefficients, a problem which is also relevant when addressing nonlinear models. We then present some recent results on coupled models of wave-heat equations arising in fluid-structure interaction. We also present some open problems and future directions of research.

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1 Introduction

In these notes we address some topics related to the controllability of partial differential equations (PDE) which, in the context of Control Theory, are also often refered to as Distributed Parameter Systems (DPS).

The controllability problem may be formulated roughly as follows. Consider an evolution system (either described in terms of Partial or Ordinary Differential Equations) on which we are allowed to act by means of a suitable choice of the control (the right hand side of the system, the boundary conditions, etc.). Given a time interval $0 < t < T$, and initial and final states, the goal is to determine whether there exists a control driving the given initial data to the given final ones in time T.

This is a classical problem in Control Theory and there is a large literature on it. We refer for instance to the book of Lee and Markus [120] for an introduction to the topic in the context of finite-dimensional systems described in terms of Ordinary Differential Equations (ODE). We also refer to the survey paper by Russell [174] and to the SIAM Review article and book by J.-L. Lions [122] and [123] for an introduction to the case of systems modelled by means of PDE.

There has been a very intensive research in this area in the last three decades and it would be impossible in these notes to report on the main progresses that have been made. For this reason we have chosen a number of specific topics to present some recent results. Our goal is to exhibit the variety and depth of the problems arising in this field and some of the mathematical tools that have been used and developed to deal with them. Of course, the list of topics we have chosen is limited and it is not intended to represent the whole field. We hope however that, through these notes, the reader will become familiar with some of the main research topics in this area. We have also included a long (but still incomplete) list of references for those readers interested is pursuing the study in this field and also a list of open problems for future research. As we shall see, many of them are closely related to other subtle questions of the theory of PDE, as unique continuation, asymptotic behavior of coupled system, spectral properties, etc.

Even in the specific context of PDE, in order to address controllability problems in a successful way, one has still to make further distinctions between linear and non-linear systems, time-reversible and time-irreversible ones, etc. In these notes we mainly focus on linear problems and discuss both the wave and the heat equations, as the two main prototypes of reversible and irreversible models.

The techniques we present for the wave equation apply, essentially, to other models like, for instance, Schrödinger and plate equations. Combining them with fixed point arguments, these results may be extended to some semilinear models too. But, other relevant issues like for instance the bilinear control of Schrödinger equations need important further developments and different techniques that we shall not develop in this article. At this respect we refer to the recent work by K. Beauchard [8] (see also [9] for a global version of the same result and the references therein) where this problem is solved by a combination of several tools including Coron's return method ([34]) and Nash-Moser's iteration.

On the other hand, the techniques we shall present on the use of Carleman inequalities and variational methods for the control of the heat equation, strongly inspired in the works by A. Fursikov and O. Imanuvilov [74], can be extended to a wider class of parabolic problems. In particular, with further important technical developments, this allows proving the local null controllability of Navier-Stokes equations. We refer to [65] for the latest results on this problem and to [67] for a survey on that topic. The Euler equations are also well known to be controllable (see [32], and [81]). However, because of the hyperbolic nature of the problem, this time Carleman inequalities may not be applied but rather the return method needs to be used.

Taking into account that the existing theory is able to cover quite successfully both hyperbolic and parabolic models or, in other words, structures and fluids, it is natural to address the important issue of fluid-structure interaction. Recently important progresses have been made also in this context too. First, existence results are available for a number of models in which the structure is condered to be a rigid body ([175]) or a flexible one ([14], [15], [38]). Part of this article will be devoted to report on these results. But we shall mainly focus on a simplified linearized model in which the wave and heat equation are coupled through an interface. We shall mainly discuss the problem of the asymptotic behavior of solutions. The techniques developed for the controllability of the wave equation will play a key role when doing that. As we shall see, some of the dynamical properties of the system we shall describe could seem unexpected. For instance, the damping effect that the heat equation introduces on the wave solutions is too strong and overdamping occurs and the decay rate fails to be exponentially uniform. The problem of controllability is by now only well understood in one space dimension. There is still to be done in this field to address controllability in several space dimensions and then for covering the nonlinear free boundary problems. One of the very few existing results on the subject is that in [16] that guarantees the local controllability of the Navier-Stokes equations, coupled with moving rigid bodies.

In these notes we do not address the issue of numerical approximation of controls. This is of course a very important topic for the implementation of the control theoretical results in practical applications. We refer to [213] for a recent survey article in this issue and [212] for a discussion in connection with optimal control problems.

As we said above, the choice of the topics in this article is necessarily limited. The interested reader may complement these notes with the survey articles [205], [211] for the controllability of PDE, [27] for the controllability and homogenization. We also refer to the notes [144] for an introduction to some of the most elementary tools in the controllability of PDE. The notes [144] are in fact published in a collective book which contains interesting survey and introductory paper in Control Theory. The article [199] contains a discussion of the state of the art on the controllability of semilinear wave equations, published in a collective work on unsolved problems in Control Theory that might be of interest for researchers in this area. However, our bibliography is not complete. There are for instance other books related to this and other closely related topics as, for instance, [58] and [111].

The content of this paper is as follows. In Section 2 we make a brief introduction to the topic in the context of linear finite-dimensional systems. Sections 3 and 4 are devoted to describe the main issues related to the controllability of the linear wave and heat equations, respectively, and the basic known results. In Section 5 we discuss the optimality of the known observability results for heat equations with potentials. We show that in the context of multi-dimensional parabolic systems the existing observability estimates are indeed sharp in what concerns the dependence on the L^∞-norm of the potential. In Section 6 we present some simple but new results on the observability of the heat equation with a low regularity variable coefficients ion the principal part, a topic which is full of interesting and difficult open problems. In Section 7 we discuss some models coupling heat and wave equations along a fixed interface, which may be viewed as a simplified and linearized version of more realistic models of fluid-structure interaction. We end up with a section devoted to present some open problems and future directions of research.

2 Preliminaries on finite-dimensional systems

2.1 Problem formulation

PDE can be viewed as infinite-dimensional versions of linear systems of ordinary differential equations (ODE). ODE generate finite-dimensional dynamical systems, while PDE correspond to infinite-dimensional ones. The fact that PDE are an infinite-dimensional version of finite dimensional ODE can be justified and is relevant in various different contexts. First, that is the case in Mechanics. While PDE are the common models for Continuum Mechanics, ODE arise in classical Mechanics, where the continuous aspect of the media under consideration is not taken into account. The same can be said in the context of Numerical Analysis. Numerical approximation schemes for PDE and, more precisely, those that are semi-discrete (discrete in space and continuous in time) yield finite-dimensional systems of ODE. This is particularly relevant in the context of control where, when passing to the limit from finite to infinite dimensions, unwanted and unexpected pathologies may arise (see [213]).

It is therefore convenient to first have a quick look to the problems under consideration in the finite-dimensional context. This will be useful when dealing with singular limits from finite to infinite dimensional systems and, in particular, when addressing numerical approximation issues. But it will also be usefull to better understand the problems and techniques we shall use in the context of PDE, where things are necessarily technically more involved and complex due to the much richer structure associated to continuous character of the media under consideration and the needed Functional Analytical tools.

There is by now an extensive literature on the control of finite-dimensional systems and the problem is completely understood for linear ones ([120], [179]). Here we shall only present briefly the problems and techniques we shall later employ in the context of PDE.

Consider the finite-dimensional system of dimension N:

$$x' + Ax = Bv, \quad 0 \leq t \leq T; \quad x(0) = x^0, \tag{2.1}$$

where $x = x(t)$ is the N-dimensional *state* and $v = v(t)$ is the M-dimensional *control*, with $M \leq N$. By $'$ we denote differentiation with respect to time t.

Here A is an $N \times N$ matrix with constant real coefficients and B is an $N \times M$ matrix. The matrix A determines the dynamics of the system and the matrix B models the way M controls act on it.

In practice, it is desirable to control the N components of the system with a low number of controls and the best would be to do it by a single one, in which case $M = 1$. As we shall see, this is possible provided B, the control operator, is chosen appropriately with respect to the matrix A governing the dynamics of the system. [1]

System (2.1) is said to be *controllable* in time T when every initial datum $x^0 \in \mathbf{R}^N$ can be driven to any final datum x^1 in $\mathbf{R}^N$ in time T by a suitable control $v \in (L^2(0,T))^M$, i. e. the

[1] This being possible for appropriate choices of the control operator B allows us to be optimistic when addressing PDE models, in which case the state variable is infinite-dimensional.

following final condition is satisfied

$$x(T) = x^1. \tag{2.2}$$

In other words the system is said to be controllable in time T when the set of reachable states

$$R\left(T; x^0\right) = \left\{x(T) : v \in (L^2(0,T))^M\right\},$$

covers the whole $\mathbf{R}^N$ and this for all $x^0 \in \mathbf{R}^N$. When this property holds, the system is said to be *exactly controllable*. Here "exactly" refers to the fact that the target (2.2) is achieved completely. This final condition can be relaxed in different ways leading to various weaker notions of controllability. However, as we shall see, since we are in finite dimensions, these apprarently weaker notions often coincide with the exact controllability one. For instance, the system is said to be *approximately controllable* when the set of reachable states is dense in $\mathbf{R}^N$. But, in $\mathbf{R}^N$, the only close affine dense subspace is the whole space itself. Thus, approximate and exact controllability are equivalent notions.

But let us analyze the problem of exact controllability.

There is a necessary and sufficient condition for (exact) controllability which is of purely algebraic nature. It is the so called *Kalman condition*: *System (2.1) is controllable in some time $T > 0$ iff*

$$\text{rank}[B, AB, ..., A^{N-1}B] = N. \tag{2.3}$$

Moreover, when this holds, the system is controllable for all time $T > 0$.

There is a direct proof of this result which uses the representation of solutions of (2.1) by means of the variation of constants formula. However, for addressing PDE models it is more convenient to use an alternative method which consists in transforming the control problem into a problem of observability for the adjoint system, since the later one can be solved by a combination of the existing methods to obtain a priori estimates on solutions of ODE and PDE.

2.2 Controllability $\equiv$ Observability

Let us introduce the problem of observability.

Consider the *adjoint system*

$$-\varphi' + A^*\varphi = 0, \quad 0 \le t \le T; \quad \varphi(T) = \varphi^0. \tag{2.4}$$

The following fundamental result establishes the equivalence between the controllability of system (2.1) and the observability property of the adjoint system (2.4).

Theorem 2.1 *System (2.1) is controllable in time T if and only if the adjoint system (2.4) is observable in time T, i. e. if there exists a constant $C = C(T) > 0$ such that, for all solution φ of (2.4),*

$$|\varphi^0|^2 \le C \int_0^T |B^*\varphi|^2 dt. \tag{2.5}$$

Both properties hold in all time T if and only if the Kalman rank condition (2.3) is satisfied.

Sketch of the proof: We first prove that the observability inequality (2.5) for the adjoint system (2.4) implies the controllability of the state equation (2.1). Our proof provides a constructive method to build controls.

We proceed in several steps:

Step 1: *Construction of controls as minimizers of a quadratic functional.*

Assume (2.5) holds and consider the quadratic functional $J : \mathbf{R}^N \to \mathbf{R}$:

$$J(\varphi^0) = \frac{1}{2}\int_0^T |B^*\varphi(t)|^2 dt - < x^1, \varphi^0 > + < x^0, \varphi(0) > . \tag{2.6}$$

If $\hat{\varphi}^0$ is a minimizer for J, $DJ(\hat{\varphi}^0) = 0$, and, the control

$$v = B^*\hat{\varphi}, \tag{2.7}$$

where $\hat{\varphi}$ is the solution of (2.4) with that datum $\hat{\varphi}^0$ at time $t = T$, is such that the solution x of (2.1) satisfies the control requirement $x(T) = x^1$.

Indeed, for all $\psi^0, \varphi^0 \in \mathbf{R}^N$,

$$< DJ(\psi^0), \varphi^0 >= \int_0^T B^*\psi(t) \cdot B^*\varphi(t) dt - < x^1, \varphi^0 > + < x^0, \varphi(0) > .$$

Thus, $DJ(\hat{\varphi}^0) = 0$ if and only if

$$\int_0^T B^*\hat{\varphi}(t) \cdot B^*\varphi(t) dt - < x^1, \varphi^0 > + < x^0, \varphi(0) >= 0,$$

for all $\varphi^0 \in \mathbf{R}^N$. In other words,

$$\int_0^T BB^*\hat{\varphi}(t) \cdot \varphi(t) dt - < x^1, \varphi^0 > + < x^0, \varphi(0) >= 0,$$

or,

$$\int_0^T Bv \cdot \varphi(t) dt - < x^1, \varphi^0 > + < x^0, \varphi(0) >= 0, \tag{2.8}$$

if v is chosen according to (2.7).

Here and in the sequel we denote by $\cdot$ or $< \cdot, \cdot >$ the scalar product in the Euclidean space (both in $\mathbf{R}^N$ and $\mathbf{R}^M$).

We claim that (2.8) is equivalent to the fact that the control v as above drives the solution x of (2.1) from x^0 to x^1. Indeed, multiplying the state equation (2.1) by any solution φ of the adjoint system (2.4), we get:

$$\int_0^T (x' + Ax) \cdot \varphi dt = \int_0^T Bv \cdot \varphi dt. \tag{2.9}$$

On the other hand,

$$\int_0^T (x' + Ax) \cdot \varphi dt = \int_0^T x \cdot (-\varphi' + A^*\varphi) dt + < x, \varphi > |_0^T =< x(T), \varphi^0 > - < x^0, \varphi(0) > . \tag{2.10}$$

Combining (2.8) and (2.10) we deduce that

$$< x(T) - x^1, \varphi^0 >= 0,$$

for all $\varphi^0 \in \mathbf{R}^N$. This is equivalent to the final condition (2.2) imposed to the control problem.

Thus, to solve the control problem it is sufficient to prove that the functional J in (2.6) achieves a minimizer. To do that, we apply the Direct Method of the Calculus of Variations (DMCV). The functional J being continuous, quadratic and convex, and defined in the finite dimensional Euclidean space, it is sufficient to prove its coercivity, i. e.

$$\underline{\lim}_{||\varphi^0||\to\infty} J(\varphi^0) = \infty. \tag{2.11}$$

This property holds if and only if the observability inequality is satisfied. Indeed, when (2.5) holds the following variant holds as well, with possibly a different constant $C > 0$:

$$|\varphi^0|^2 + |\varphi(0)|^2 \leq C \int_0^T |B^*\varphi|^2 dt. \tag{2.12}$$

In fact, both inequalities (2.5) and (2.12) are equivalent. This is so since $\varphi(t) = e^{A^*(t-T)}\varphi_0$ and the operator $e^{A^*(t-T)}$ is bounded and invertible.

In view of (2.12) the coercivity of J follows. This implies the existence of the minimizer for J and therefore that of the control we are looking for.

Step 2: *Equivalence between the observability inequality (2.12) and the Kalman condition.*

In the previous step we have shown that the observability inequality (2.12) implies the existence of the control. In this second step we show that the observability inequality is equivalent to the Kalman condition.

Since we are in finite-dimension and all norms are equivalent, (2.12) is equivalent to the following uniqueness property:

(2.13) Does the fact that $B^*\varphi$ vanish for all $0 \leq t \leq T$ imply that $\varphi \equiv 0$?

Taking into account that solutions φ of the adjoint system are analytic in time, $B^*\varphi$ vanishes if and only if all the derivatives of $B^*\varphi$ of any order vanish at time $t = T$. Since $\varphi = e^{A^*(t-T)}\varphi^0$ this is equivalent to the fact that $B^*[A^*]^k\varphi^0 \equiv 0$ for all $k \geq 0$. But, according to the Cayley-Hamilton's Theorem, this holds if and only if it is satisfied for all $k = 0, ..., N-1$. Therefore $B\varphi \equiv 0$ is equivalent to $[B^*, B^*A^*, ..., B^*[A^*]^{N-1}]\varphi^0 = 0$. But, the latter, when

$$\text{rank}[B^*, B^*A^*, ..., B^*[A^*]^{N-1}] = N,$$

is equivalent to the fact that $\varphi^0 = 0$ or $\varphi \equiv 0$. Obviously, this rank condition is equivalent to the Kalman one (2.3).

This concludes the proof of the fact that observability implies controllability. Let us now prove the reverse assertion, i.e. that controllability implies observability.

Let us assume that the state equation is controllable. We choose $x^1 = 0$. Then, for all $x^0 \in \mathbf{R}^N$ there exists a control $v \in (L^2(0,T))^M$ such that the solution of (2.1) satisfies $x(T) = 0$. The control is not unique thus it is convenient to choose the one of minimal norm. By the closed graph Theorem we deduce that there exists a constant $C > 0$ (that, in particular, depends on the control time T) such that

$$||v||_{(L^2(0,T))^M} \leq C|x^0|. \tag{2.14}$$

Then, multiplying the state equation (2.1) by any solution of the adjoint equation φ and taking into account that $x(T) = 0$ for the control v we have chosen, we deduce that

$$- < x^0, \varphi(0) >= \int_0^T v \cdot B^* \varphi dt.$$

Combining this identity with (2.14) we deduce that

$$| < x^0, \varphi(0) > | \leq C|x^0|||B^*\varphi||_{(L^2(0,T))^M},$$

for all $x^0 \in \mathbf{R}^N$, which is equivalent to

$$|\varphi(0)| \leq C||B^*\varphi||_{(L^2(0,T))^M}. \tag{2.15}$$

This estimate (2.15) is equivalent to the observability inequalities (2.5) and/or (2.12). This is so, once more, because of the continuity of the mapping $\varphi(0) \to \varphi^0$. $\square$

Remark 2.1 *The property of observability of the adjoint system (2.4) is equivalent to the inequality (2.5) because of the linear character of the system. In general, the problem of observability can be formulated as that of determining uniquely the adjoint state everywhere in terms of partial measurements.*

We emphasize that, in the finite-dimensional context under consideration, the observability inequality (2.5) is completely equivalent to (2.12) and/or (2.15). In other words, it is totally equivalent to formulate the problem of estimating the initial or final data of the adjoint system. This is so because the mapping $\varphi^0 \to \varphi(0)$ is continuous, and has continuous inverse. This is no longer necessarily true for infinite-dimensional systems. This fails, in particular, for time-irreversible equations as the heat equation.

There is another major difference with infinite dimensional systems written in terms of PDEs. Namely, the uniqueness property (2.13) may hold but this does not necessarily imply an observability inequality (2.5) to be true in the desired energy space. This is due to the fact that, in infinite dimensional Banach spaces, all norms are not necessariy equivalent. In other words, in infinite dimension a strict subspace may be dense, which never occurs in finite dimension.

Remark 2.2 *This proof of controllability provides a constructive method to build the control: minimizing the functional J. But it also yields explicit bounds on the controls. Indeed, since the functional $J \leq 0$ at the minimizer, and in view of the observability inequality (2.12), it follows that*

$$||v|| \leq 2\sqrt{C}[|x^0|^2 + |x^1|^2]^{1/2}, \tag{2.16}$$

C being the same constant as in (2.12). Therefore, we see that the observability constant is, up to a multiplicative factor, the norm of the control map associating to the initial and final data of the state equation (x^0, x^1) the control of minimal norm v. Actually, a more careful analysis indicates that the norm of the control can be bounded above in terms of the norm of the initial datum of

$e^{AT}x^0 - x^1$ *which measures the distance between the target* x^1 *and the final state* $e^{AT}x^0$ *that the uncontrolled dynamics would reach without implementing any control.*

Our proof above shows that the reverse is also true. In other words, the norm of the control map that associates the control v *to each pair of initial/final data* (x^0, x^1)*, also provides an explicit observability constant.*

Remark 2.3 *Furthermore, the approach above has also the interesting property of providing systematically the control of minimal* $L^2(0,T)$*-norm within the class of admissible ones. Indeed, given* T*, an initial datum and a final one, if the system is controllable, there are infinitely many controls driving the trajectory from the initial datum to the final target. To see this it is sufficient to argue as follows. In the first half of the time interval* $[0, T/2]$ *we can choose any function as controller. This drives the system to a new state, say, at time* $T/2$*. The system being controllable, it is controllable in the second half of the time interval* $[T/2, T]$*. This allows applying the variational approach above to obtain the control driving the sytem from its value at time* $t = T/2$ *to the final state* x^1 *at time* T *in that second interval. The superposition of these two controls provides an admissible control which has an arbitrary shape in the first interval* $[0, T/2]$*. This suffices to see that the set of admissible controls contains an infinite number of elements.*

As we said above, the variational approach we have described provides the control of minimal $L^2(0,T)$*-norm. Indeed, assume, to simplify the presentation, that* $x^1 = 0$*. Let* u *be an arbitrary control and* v *the control we have constructed by the variational approach. Multiplying by* φ *in the state equation and integrating by parts with respect to time, we deduce that both controls satisfy*

$$\int_0^T < u, B^*\varphi > dt = \int_0^T < v, B^*\varphi > dt = - < x^0, \varphi(0) >,$$

for any solution φ *of the adjoint system. In particular, by taking* $\hat{\varphi}$*, the solution of the adjoint system corresponding to the minimizer of* J *and that determines the control* v *(i. e.* $v = B^*\hat{\varphi}$*) it follows that*

$$\int_0^T < u, B^*\hat{\varphi} > dt = \int_0^T < v, B^*\hat{\varphi} > dt = \int_0^T |v|^2 dt = - < x^0, \hat{\varphi}(0) > .$$

Thus,

$$||v||^2_{L^2(0,T)} \leq \Big| \int_0^T < u, B^*\hat{\varphi} > dt \Big| \leq ||u||_{L^2(0,T)} ||B^*\hat{\varphi}||_{L^2(0,T)} = ||u||_{L^2(0,T)} ||v||_{L^2(0,T)},$$

which implies that $||v||_{L^2(0,T)} \leq ||u||_{L^2(0,T)}$*. This completes the proof of the minimality of the control we have built by the variational approach.*

Remark 2.4 *It is important to note that, in this finite-dimensional context, the time* T *of controllability/observability plays no role. Of course this is true, in particular, because the system under consideration is autonomous. In particular, whether a system is controllable (or its adjoint observable) is independent of the time* T *of control since these properties only depend on the algebraic Kalman condition. Note that the situation may be totally different for PDE. In particular, as we shall see, in the context of the wave equation, due to the finite velocity of propagation, the time needed to control/observe waves from the boundary needs to be large enough, of the order of the size of the ratio between size of the domain and velocity of propagation.*

Remark 2.5 *The set of controllable pairs* (A, B) *is open and dense. Indeed,*

- *If* (A, B) *is controllable there exists* $\varepsilon > 0$ *sufficiently small such that any* (A^0, B^0) *with* $\mid A^0 - A \mid < \varepsilon$, $\mid B^0 - B \mid < \varepsilon$ *is also controllable. This is a consequence of the Kalman rank condition and of the fact that the determinant of a matrix depends continuously on its entries.*

  This shows the robustness of the controllability property under (small) perturbations of the system.

- *On the other hand, if* (A, B) *is not controllable, for any* $\varepsilon > 0$, *there exists* (A^0, B^0) *with* $\mid A - A^0 \mid < \varepsilon$ *and* $\mid B - B^0 \mid < \varepsilon$ *such that* (A^0, B^0) *is controllable. This is a consequence of the fact that the determinant of a* $N \times N$ *matrix depends analytically on its entries and cannot vanish in a ball of* $\mathbb{R}^n$.

2.3 Bang-bang controls

In the previous section we have proved the equivalence of the controllability property of the state equation and the observability for the adjoint. This has been done in the $L^2(0, T)$-setting and we have developed a variational method allowing to obtain the control of minimal $L^2(0, T)$-norm, which turns out to be C^∞ smooth and even analytic in time, in view of its structure (2.7).

Smooth controllers are however difficult to implement in practice because of its continuous and subtle change in shape and intensity. In the opposite extreme we may think on bang-bang controls which are piecewise constant and discountinuous but easier to implement since they consist simply in switching from a constant state to another. Once the size of the bang-bang control is determined, it is completely identified by the location of the switching times.

The goal of this section is to show that, with the ideas we have developed above and some minor changes, one can show that, whenever the system is controllable, bang-bang controls exist, and to give a variational procedure to compute them.

To simplify the presentation, without loss of generality, we suppose that $x^1 \equiv 0$.

In order to build bang-bang controls, it is convenient to consider the quadratic functional

$$J_{bb}(\varphi^0) = \frac{1}{2}\left[\int_0^T \mid B^*\varphi \mid dt\right]^2 + \langle x^0, \varphi(0)\rangle \tag{2.17}$$

where φ is the solution of the adjoint system (2.4) with initial data φ^0 at time $t = T$.

It is interesting to note that J_{bb} differs from J in the quadratic term. Indeed, in J we took the $L^2(0, T)$-norm of $B^*\varphi$ while here we consider its $L^1(0, T)$-norm.

The functional J_{bb} is continuous, convex and also coercive because the unique continuation property (2.13) holds. It follows that J_{bb} attains a minimum in some point $\widehat{\varphi}^0 \in \mathbb{R}^n$. This can be easily seen using the Direct Method of the Calculus of Variations and taking into account that, in $\mathbf{R}^N$, all bounded sequences are relatively compact.

Note that for the coercivity of J_{bb} to hold one needs the following L^1-version of the observability inequality (2.5):

$$|\varphi^0|^2 \leq C\Big[\int_0^T |B^*\varphi| dt\Big]^2. \tag{2.18}$$

This inequality holds immediately as a consequence of the unique continuation property (2.13) because we are in the finite-dimensional setting. However, in the infinite dimensional setting things might be much more complex in the sense that the unique continuation property does not imply any specific observability inequality automatically. This will be particularly relevant when analyzing wave-like equations.

On the other hand, it is easy to see that

$$\lim_{h\to 0}\frac{1}{h}\left[\left(\int_0^T \mid f+hg\mid dt\right)^2-\left(\int_0^T \mid f\mid dt\right)^2\right]=2\int_0^T \mid f\mid dt\int_0^T \operatorname{sgn}(f(t))g(t)dt \tag{2.19}$$

if the Lebesgue measure of the set $\{t\in(0,T): f(t)=0\}$ vanishes.

Here and in the sequel the sign function "sgn" is defined as a multi-valued function in the following way

$$\operatorname{sgn}(s)=\begin{cases}1 & \text{when } s>0\\ \Big[-1,1\Big] & \text{when } s=0\\ -1 & \text{when } s<0.\end{cases}$$

Remark that in the previous limit there is no ambiguity in the definition of $\operatorname{sgn}(f(t))$ since the set of points $t\in[0,T]$ where $f=0$ is assumed to be of zero Lebesgue measure and does not affect the value of the integral.

Identity (2.19) may be applied to the quadratic term of the functional J_{bb} since, taking into account that φ is the solution of the adjoint system (2.4), it is an analytic function and therefore, $B^*\varphi$ changes sign finitely many times in the interval $[0,T]$ except when $\widehat{\varphi}^0=0$. In view of this, the Euler-Lagrange equation associated with the critical points of the functional J_{bb} is as follows:

$$\int_0^T \mid B^*\widehat{\varphi}\mid dt\int_0^T \operatorname{sgn}(B^*\widehat{\varphi})B^*\psi(t)dt+\langle x^0,\psi(0)\rangle=0$$

for all $\psi^0\in\mathbb{R}^N$, where ψ is the solution of the adjoint system (2.4) with initial data ψ^0. When applied to a vector, $\operatorname{sgn}(\cdot)$ is defined componentwise as before.

Consequently, the control we are looking for is $v=\int_0^T \mid B^*\widehat{\varphi}\mid dt\,\operatorname{sgn}(B^*\widehat{\varphi})$ where $\widehat{\varphi}$ is the solution of (2.4) with initial data $\widehat{\varphi}^0$, the minimizer of J_{bb}.

Note that when $M=1$, i. e. when the control u is a scalar function, it is of bang-bang form. Indeed, v takes only two values $\pm\int_0^T \mid B^*\widehat{\varphi}\mid dt$. The control switches from one to the other one when the function $B^*\widehat{\varphi}$ changes sign. This happens finitely many times. When $M>1$, the control v is a vector valued bang-bang function in the sense that each component is of bang-bang form. Note however that each component of v may change sign in different times, depending on the changes of sign of the corresponding component of $B^*\widehat{\varphi}$.

Remark 2.6 *Other types of controls can be obtained by considering functionals of the form*

$$J_p(\varphi^0)=\frac{1}{2}\left(\int_0^T \mid B^*\varphi\mid^p dt\right)^{2/p}+\langle x^0,\varphi^0\rangle$$

with $1<p<\infty$. *The corresponding controls are*

$$v=\left(\int_0^T \mid B^*\widehat{\varphi}\mid^p dt\right)^{(2-p)/p}|B^*\widehat{\varphi}|^{p-2}B^*\widehat{\varphi}$$

where $\widehat{\varphi}$ is the solution of (2.4) with initial datum $\widehat{\varphi}^0$, the minimizer of J_p.

It can be shown that, as expected, the controls obtained by minimizing these functionals give, in the limit when $p \to 1$, a bang-bang control.

In the previous section we have seen that the control obtained by minimizing the functional J is of minimal $L^2(0,T)$. We claim that the control obtained by minimizing the functional J_{bb} is of minimal $L^\infty(0,T)$-norm. Indeed, let u be any control in $L^\infty(0,T)$ and v be the one obtained by minimizing J_{bb}. Once more we have,

$$\int_0^T u \cdot B^* \hat{\varphi} dt = \int_0^T v \cdot B^* \hat{\varphi} dt = - < x^0, \widehat{\varphi}(0) > .$$

In view of the definition of v, it follows that

$$||v||^2_{L^\infty(0,T)} = \left(\int_0^T |B^* \widehat{\varphi}| dt \right)^2 = \int_0^T u \cdot B^* \widehat{\varphi} dt = \int_0^T v \cdot B^* \widehat{\varphi} dt.$$

Hence,

$$||v||^2_{L^\infty(0,T)} = \int_0^T u \cdot B^* \widehat{\varphi} dt \leq ||u||_{L^\infty(0,T)} \int_0^T |B^* \widehat{\varphi}| dt = ||u||_{L^\infty(0,T)} ||v||_{L^\infty(0,T)}$$

and the proof finishes.

3 Controllability of the linear wave equation.

3.1 Statement of the problem.

Let Ω be a bounded domain of $\mathbf{R}^n, n \geq 1$, with boundary Γ of class C^2. Let ω be an open and non-empty subset of Ω and $T > 0$.

Consider the linear controlled wave equation in the cylinder $Q = \Omega \times (0,T)$:

$$\text{(3.1)} \qquad \begin{cases} u_{tt} - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x) & \text{in} \quad \Omega. \end{cases}$$

In (3.1) Σ represents the lateral boundary of the cylinder Q, i.e. $\Sigma = \Gamma \times (0,T), 1_\omega$ is the characteristic function of the set $\omega, u = u(x,t)$ is the state and $f = f(x,t)$ is the control variable. Since f is multiplied by 1_ω the action of the control is localized in ω.

When $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f \in L^2(Q)$ system (3.1) has an unique finite energy solution $u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$.

The problem of *controllability* consists roughly in *describing the set of reachable final states*

$$R\left(T; (u^0, u^1)\right) = \left\{ (u(T), u_t(T)) : f \in L^2(Q) \right\}.$$

It is the affine subspace of the final states that the solutions reach at time $t = T$, starting from the initial datum (u^0, u^1), when the control f varies all over $L^2(Q)$. Note however that the action of the control is localized in ω. Thus, the controls may also be viewed to belong to $L^2(\omega \times (0,T))$.

One may distinguish the following notions of controllability:

(a) *Approximate controllability*: System (3.1) is said to be approximately controllable in time T if the set of reachable states is dense in $H_0^1(\Omega) \times L^2(\Omega)$ for every $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

(b) *Exact controllability*: System (3.1) is said to be exactly controllable at time T if

$$R\left(T; (u^0, u^1)\right) = H_0^1(\Omega) \times L^2(\Omega)$$

for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

(c) *Null controllability*: System (3.1) is said to be null controllable at time T if

$$(0,0) \in R\left(T; (u^0, u^1)\right)$$

for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Remark 3.1 *(a) Since we are dealing with solutions of the wave equation, due to the finite speed of propagation, for any of these properties to hold the control time T has to be sufficiently large, the trivial case in which the control subdomain ω coincides with the whole domain Ω being excepted.*

(b) Since system (3.1) is linear and reversible in time null and exact controllability are equivalent notions, as in the finite-dimensional case of the previous section. As we shall see, the situation is completely different in the case of the heat equation.

(c) Clearly, every exactly controllable system is approximately controllable too. However, system (3.1) may be approximately but not exactly controllable. In those cases it is natural to study the cost of approximate controllability, or, in other words, the size of the control needed to reach an ε-neighborhood of a final state which is not exactly reachable. This problem was analyzed by Lebeau in [112] in the context of wave equations with analytic coefficients. Roughly speaking, when exact controllability fails, the cost of reaching a target which does not belong to the subspace of reachable data, increases exponentially as the distance ε to the target tends to zero. Later on a slightly weaker version of this result was given by L. Robbiano [171] in the context of wave equations with lower order potentials by means of Carleman inequalities.

Obviously, this does not happen in the context of finite-dimensional systems since exact and approximate controllability are equivalent notions. This is so because, in $\mathbf{R}^N$, the only affine dense subspace of $\mathbf{R}^N$ is $\mathbf{R}^N$ itself.

(e) The controllability problem above may also be formulated in other function spaces in which the wave equation is well posed. For instance one can take initial and final data in $L^2(\Omega) \times H^{-1}(\Omega)$ and then the control in $L^2(0,T;H^{-1}(\omega))$ or, by the contrary, the initial data in $[H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)]$ and the control in $L^2(0,T;H_0^1(\omega))$. Similar results hold in all these cases. In these notes we have chosen to work in the classical context of finite-energy solutions of the wave equation to avoid unnecessary technicalities.

(f) Null controllability is a physically particularly interesting notion since the state $(0,0)$ is an equilibrium for system (3.1). Once the system reaches the equilibrium at time $t = T$, we can stop controlling (by taking $f \equiv 0$ for $t \geq T$) and the system naturally stays in the equilibrium configuration for all $t \geq T$.

(g) *Most of the literature on the controllability of the wave equation has been written on the framework of the boundary control problem. The control problems formulated above for system (3.1) are usually refered to as internal controllability problems since the control acts on the subset ω of Ω. Although the results are essentially the same in both cases, the boundary control problem is normally more complex from a technical point of view, because of the intrinsic difficulty of dealing with boundary traces and non-homogeneous boundary value problems. The closer analogies arise when considering boundary control problems on one side and, on the other one, internal controls localized in ω, a neighborhood of the boundary of the domain Ω or part of it (see [122]).*

In the context of boundary control the state equation reads

$$\begin{cases} u_{tt} - \Delta u = 0 & in \quad Q \\ u = v 1_{\Sigma_0} & on \quad \Sigma \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x) & in \quad \Omega, \end{cases} \tag{3.2}$$

where Σ_0 is the subset of the lateral boundary $\Sigma = \Gamma \times (0,T)$ where the control is applied. In most cases the subset of the boundary Σ_0 is taken to be cylindrical, i. e. $\Sigma_0 = \Gamma_0 \times (0,T)$ for a subset Γ_0 of $\partial\Omega$. But Σ_0 can be any non empty relative open subset of the lateral boundary Σ. The most natural functional setting is then that in which $v \in L^2(\Sigma_0)$ and $u \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$. In this setting the formulation of approximate, exact and null control problems is basically the same, except for the fact that, from a technical point of view, addressing them is more complex in this context of boundary control since one has to deal with fine trace results. In fact, proving that system (3.2) is well-posed in $C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$ with boundary data in $L^2(\Sigma_0)$ requires a quite subtle use of the method of transposition (see [122]). But the methods we shall develop below, based on the observability of the adjoint system, apply in this context too. The techniques we shall describe apply also to ther boundary conditions. However, the analysis of well-posedness for the corresponding non-homogeneous boundary value problems may present new difficulties (see [122]).

■

3.2 Exact controllability.

In the previous section we have explained the equivalence between the controllability of the state equation and a suitable observability property for the adjoint system in the context of finite-dimensional systems. The same is true for PDE. But, as we mentioned above, the problem is much more complex in the context of PDE since we are dealing with infinite-dimensional dynamical systems and not all norms are equivalent in this setting.

In the context of PDE, the unique continuation property by itself, i. e. the PDE analogue of (2.13), does not suffice and one has to directly address the problem of observability paying special attention to the norms involved on the observability inequality.

In this case the adjoint system is as follows:

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x), \varphi_t(x,T) = \varphi^1(x) & \text{in} \quad \Omega. \end{cases} \tag{3.3}$$

As it was shown by Lions [123], using the so called HUM (Hilbert Uniqueness Method), exact controllability is equivalent to the following inequality:

$$\left| (\varphi(0), \varphi_t(0)) \right|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt \tag{3.4}$$

for all solutions of the adjoint system (3.3). Note that this equivalence property is the analogue of the one (2.5) we have stated and proved in the previous section in the context of finite-dimensional systems.

This estimate is often also refered to as *continuous observability* since it provides a quantitative estimate of the norm of the initial data in terms of the observed quantity, by means of the observability constant C.

As we mentioned above, in contrast with the situation in finite-dimensional systems, for the observability inequality (3.4) to be true, it is not sufficient that the unique continuation property below holds:

$$If \quad \varphi \equiv 0 \quad in \quad \omega \times (0,T) \quad then \quad \varphi \equiv 0. \tag{3.5}$$

Indeed, as we shall see, it may happen that the unique continuation property (3.5) holds, but the corresponding observed norm $\left[\int_0^T \int_\omega \varphi^2 dxdt\right]^{1/2}$ to be strictly weaker that the energy in $L^2(\Omega) \times H^{-1}(\Omega)$.

Inequality (3.4), when it holds, allows to estimate the total energy of the solution of (3.24) at time $t = 0$ by means of a measurement in the control region $\omega \times (0,T)$. But, in fact, the $L^2(\Omega) \times H^{-1}(\Omega)$-energy is conserved in time, i.e.

$$\left| (\varphi(t), \varphi_t(t)) \right|^2_{L^2(\Omega)\times H^{-1}(\Omega)} = \left| (\varphi^0, \varphi^1) \right|^2_{L^2(\Omega)\times H^{-1}(\Omega)}, \; \forall t \in [0,T].$$

Thus, (3.4) is equivalent to

$$\left| (\varphi^0, \varphi^1) \right|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \tag{3.6}$$

or to

$$\int_0^T \left| (\varphi(t), \varphi_t(t)) \right|^2_{L^2(\Omega)\times H^{-1}(\Omega)} dt \leq C \int_0^T \int_\omega \varphi^2 dxdt. \tag{3.7}$$

When the observability inequality (3.4) holds the functional

$$J\left(\varphi^0, \varphi^1\right) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \left\langle (\varphi(0), \varphi_t(0)), \, (u^1, -u^0) \right\rangle - \left\langle (\varphi^0, \varphi^1), \, (v^1, -v^0) \right\rangle \tag{3.8}$$

has an unique minimizer $\left(\widehat{\varphi}^0, \widehat{\varphi}^1\right)$ *in* $L^2(\Omega) \times H^{-1}(\Omega)$ *for all* $\left(u^0, u^1\right), \left(v^0, v^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$. *The control* $f = \widehat{\varphi}$ *with* $\widehat{\varphi}$ *solution of (3.3) corresponding to the minimizer* $\left(\widehat{\varphi}^0, \widehat{\varphi}^1\right)$ *is such that the solution of (3.1) satisfies*

$$u(T) = v^0, \quad u_t(T) = v^1. \tag{3.9}$$

The proof of this result is similar to the one of the finite-dimensional case we developed in the previous section. Thus we omit it. On the other hand, as in the finite-dimensional context, the controls we have built by minimizing J are those of minimal $L^2(\omega \times (0,T))$-norm within the class of admissible controls.

But this observability inequality is far from being obvious and requires suitable geometric conditions on the control set ω, the time T and important technical developments.

Let us now discuss what is known about the observability inequality (3.6):

(a) **The method of multipliers.** Using multiplier techniques in the spirit of C. Morawetz [152], Ho in [92] proved that if one considers subsets of Γ of the form

$$\Gamma(x^0) = \left\{x \in \Gamma : (x - x^0) \cdot n(x) > 0\right\} \tag{3.10}$$

for some $x^0 \in \mathbf{R}^n$ (by $n(x)$ we denote the outward unit normal to Ω in $x \in \Gamma$ and by $\cdot$ the scalar product in $\mathbf{R}^n$) and if $T > 0$ is large enough, the following boundary observability inequality holds:

$$|| \left(\varphi(0), \varphi_t(0)\right) ||^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq C \int_0^T \int_{\Gamma(x^0)} \left|\frac{\partial \varphi}{\partial n}\right|^2 d\Gamma dt \tag{3.11}$$

for all $\left(\varphi^0, \varphi^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$.

This is the observability inequality that is required to solve the boundary controllability problem mentioned above in Remark 3.1, g).

Later on inequality (3.11) was proved in [123] for any $T > T(x^0) = 2 \parallel x - x^0 \parallel_{L^\infty(\Omega)}$. This is the optimal observability time that one may derive by means of multipliers. We refer to [103] for a simpler derivation of the minimal time $T(x^0)$ with an explicit observability constant.

Let us recall that the method of multipliers relies on using $(x - x^0) \cdot \nabla \varphi$, φ and φ_t as multipliers in the adjoint system. Integrating by parts and combining the identities obtained in this way one gets (3.11).

Proceeding as in [123], vol. 1, Chapter VII, Section 2.3, one can easily prove that (3.11) implies (3.4) when ω is a neighborhood of $\Gamma(x^0)$ in Ω, i.e. $\omega = \Omega \cap \Theta$ where Θ is a neighborhood of $\Gamma(x^0)$ in $\mathbf{R}^n$, with $T > 2 \parallel x - x^0 \parallel_{L^\infty(\Omega \setminus \omega)}$. To do that it is sufficient to observe that the energy concentrated on the boundary that, in the context of Dirichlet boundary conditions, is fully determined by the L^2-norm of the normal derivative $\partial\varphi/\partial\nu$, can be bounded above in terms of the energy on a neighborhood of that subset of the boundary. Thus, if the boundary observability inequality (3.11) is true, it should also hold when measurements are made on a neighborhood of the boundary. This provides an H^1-version of (3.4). Inequality (3.4) itself can be obtained by a lifting argument based on taking time integrals of solutions.

Later on Osses in [160] introduced a new multiplier, which is basically a rotation of the previous one, obtaining a larger class of subsets of the boundary for which the observability inequality (3.11) holds.

It is important to underline that the situation in which the boundary observability inequality is obtained by the method of multipliers is limited by at least two reasons:

- a) The time T needs to be large enough. This is in agreement with the property of the finite speed of propagation underlying the wave model under consideration. But the method of multipliers rarely provides the optimal control and minimal control time, some very particular geometries being excepted (for instance, the case of the ball, in which x^0 is taken to be its center).
- b) The geometry of the sets ω for which the inequality is proved using multipliers is very restrictive. One mainly recovers neighborhoods of subsets of the boundary of the form $\Gamma(x^0)$ as in (3.10) and these have a very special structure. In particular, when Ω is the square or a rectangle, the sets $\Gamma(x^0)$ are necessarily either two or three adjacent sides, or the whole boundary, depending on the location of x^0. When Ω is a circle, $\Gamma(x^0)$ is always larger than a half-circumference. On the other hand, the multiplier method in itself does not give a qualitative justification for the need of such strict geometric restrictions. The microlocal approach we shall describe below provides a good insight into this issue and also shows that, in fact, controllability holds for a much larger class of subdomains ω.

There is an extensive literature on the use of multiplier techniques for the control and stabilization of wave-like equations. Here we have chosen to quote only some of the basic ones.

(b) **Microlocal Analysis.** C. Bardos, G. Lebeau and J. Rauch [7] proved that, roughly, in the class of C^∞ domains, the observability inequality (3.4) holds if and only if (ω, T) satisfy the following *geometric control condition (GCC) in* Ω*: Every ray of Geometric Optics that propagates in* Ω *and is reflected on its boundary* Γ *enters* ω *in time less than* T*.*

This complete characterization of the sets ω and times T for which observability holds provides also a good insight to the underlying reasons of the strict geometric conditions we encountered when applying multiplier methods. Roughly speaking, around each ray of Geometric Optics, on an arbitrarily small tubular neighborhood of it, one can concentrate solutions of the wave equation, the so-called Gaussian beams, that decay exponentially away from the ray. This suffices to show that, in case one of the rays does not enter the control region ω in a time smaller than T, the observability inequality may not hold. The construction of the Gaussian beams was developed by J. Ralston in [163] and [164]. In those articles the necessity of the GCC for observability was also pointed out. The main contribution in [7] was to prove that GCC is also sufficient for observability. As we mentioned above, the proof in [7] uses Microlocal Analysis and reduces the problem to show that the complete energy of solutions can be estimated uniformly provided all rays reach the observation subset in the given time interval. In fact, in [7], the more difficult problem of boundary controllability was addressed. This was done using the theory of propagation of singularities and a lifting lemma that allows getting estimates along the ray in a neighborhood of the boundary from the boundary estimate for the normal derivative.

This result was proved by means of Microlocal Analysis techniques. Recently the microlocal approach has been greatly simplified by N. Burq [17] by using the microlocal defect measures introduced by P. Gérard [78] in the context of the homogenization and the kinetic equations. In [17] the GCC was shown to be sufficient for exact controllability for domains Ω of class C^3 and equations with C^2 coefficients.

For the sake of completeness let us give the precise definition of *bicharacteristic ray*. Consider the wave equation with a scalar, positive and smooth variable coefficient $a = a(x)$:

$$\varphi_{tt} - \mathrm{div}(a(x)\nabla\varphi) = 0. \tag{3.12}$$

Bicharacteristic rays solve the Hamiltonian system

$$\begin{cases} x'(s) = -a(x)\xi;\ t'(s) = \ \tau \\ \xi'(s) = \nabla a(x)|\xi|^2;\ \tau'(s) = 0. \end{cases} \tag{3.13}$$

Rays describe the microlocal propagation of energy. The projections of the bicharacteristic rays in the (x, t) variables are the rays of Geometric Optics that enter in the GCC. As time evolves the rays move in the physical space according to the solutions of (3.13). Moreover, the direction in the Fourier space (ξ, τ) in which the energy of solutions is concentrated as they propagate is given precisely by the projection of the bicharacteristic ray in the (ξ, τ) variables. When the coefficient $a = a(x)$ is constant all rays are straight lines and carry the energy outward, which is always concentrated in the same direction in the Fourier space, as expected.

But for variable coefficients the dynamics is more complex and can lead to some unexpected behaviour ([140]). GCC is still a sufficient and almost necessary condition for observability to hold. But one has to keep in mind that, in contrast with the situation of the constant coefficient wave equation, for variable coefficients, some rays may never reach the exterior boundary. There are for instance wave equations with smooth coefficients for which there are periodic rays that never meet the exterior boundary. Thus, the case in which ω is a neighborhood of the boundary of the domain Ω, for which observability holds for the constant coefficient wave equation, does not necessarily fulfill the GCC for variable coefficients. In those cases boundary observability fails. Our intuition is often strongly inspired on the constant coefficient wave equation for which all rays are straight lines tending to infinity at a constant velocity, which, in particular, implies that the rays will necessarily reach the exterior boundary of any bounded domain. But for variable coefficients rays are not straight lines any more and the situation may change drastically. We refer to [140] for a discussion of this issue. We also refer to the article by Miller [148] where this problem is analyzed from the point of view of "escape functions", a sort of Lyapunov functional allowing to test whether all rays tend to infinity or not.

On the other hand, this Hamiltonian system (3.13) describes the dynamics of rays in the interior of the domain where the equation is satisfied. But when rays reach the boundary they are reflected according to the laws of Geometric Optics.[2]

[2] Note however that tangent rays may be diffractive or even enter the boundary. We refer to [7] for a deeper discussion of these issues.

So far the microlocal approach is the one leading to the sharpest observability results, in what concerns the geometric requirements on the subset ω where the control is applied and on the control time, but it requires more regularity of coefficients and boundaries than multipliers do. The drawback of the multiplier method, which is much simpler to apply, is that it only works for restricted classes of wave equations and that it does not give sharp results as those that the Geometric Optics interpretation of the observability inequality predicts.

The multiplier approach was adapted to the case of non-smooth domains by P. Grisvard in [83]. But, as far as we know, this has not been done for the microlocal approach, although the existing works (see, for instance, [79]) on the diffusion of waves on corners, together with the techniques developed in [7] and [17] might well be sufficient to do it. In any case, this is an issue that needs further analysis for a complete clarification.

(c) **Carleman inequalities.** The third most common and powerful technique to derive observability inequalities is based on the so called Carleman inequalities. It can be viewed as a more developed version of the classical multiplier technique. It applies to a wide class of equations with variable coefficients, under less regularity conditions that the microlocal approach requires. The Carleman approach needs, roughly, that the coefficients of the principal part to be Lipschitz continuous. Thus, with respect to the method of multipliers, the Carleman approach has the advantage of being more flexible and allowing to address variable coefficients, and, with respect to the microlocal one, that it requires less regularity on coefficients and domain.

But one of the major advantages of this approach is that it allows considering, for instance, lower order perturbations and getting explicit bounds on the observability constant in terms of the potentials entering in it. We refer for instance to [95] and [194]. This is particularly important when dealing with nonlinear problems by means of linearization and fixed point arguments (see [199]). More generally, the Carleman inequality approach provides explicit bounds of the observability constant for systems depending on an extra parameter. For instance, in [138] the Carleman inequalities play a key role when deriving the continuity of controls for the following singular perturbation problem connecting dissipative wave equations with the heat equation

$$\varepsilon u_{tt} - \Delta u + u_t = f 1_\omega,$$

and in [36] where the same issue is addressed for the convective equation with vanishing viscosity and control in the interior or on the boundary:

$$u_t + u_x - \varepsilon u_{xx} = 0.$$

Note however that, as in the microlocal approach, the Carleman inequalities do not yield observability inequalities for all variable coefficients and that, in fact, as we said above, they fail to be true unless assumptions on coefficients are made. The way this is seen at the level of Carleman inequalities is not as explicit as in the microlocal approach where the methodology is based on the ray analysis. But for Carleman inequalities to yield observability estimates, suitable weight functions are needed, and this requires of suitable assumptions on the coefficients and its first order derivatives ([191]). This assumptions imply, in particular, that rays reach the observation region ω.

(d) **Spectral estimates.** More recently an interesting characterization of the observability inequality in terms of the spectrum of the underlying operator has been derived in [134], [20], [149] and [167]. The result reads essentially as follows. Let $\{\psi_k\}_{k\geq 1}$ be an orthonormal basis of $L^2(\Omega)$ constituted by the eigenfunctions of the Dirichlet Laplacian and let $\{\lambda_k\}_{k\geq 1}$ be the corresponding eigenvalues:

$$\begin{cases} -\Delta\psi_j = \lambda_j\psi_j & \text{in} \quad \Omega \\ \psi_j = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.14}$$

Then the observability inequality (3.4) holds for the wave equation if and only if the eigenfunctions satisfy the following property: *there exist some $\sigma > 0$ and $C > 0$ such that*

$$\sum_{I_k} |a_j|^2 \leq C \int_\omega |\sum_{I_k} a_j\psi_j(x)|^2 dx, \quad \forall k \geq 1, \, \forall a_j \in \mathbf{R}, \tag{3.15}$$

where the sums run over the sets of indexes I_k as follows:

$$I_k = \{j : |\sqrt{\lambda_j} - \sqrt{\lambda_k}| \leq \sigma\}. \tag{3.16}$$

Under this spectral condition (3.15), the time T needed for the observability estimate (3.4) to hold is $2\pi/\sigma$. This time is sharp as the $1-d$ case indicates, according to the classical Ingham inequality that we shall discuss below. It is however important to note that a spectral gap condition is needed to apply Ingham's inequality, a condition that is not fulfilled for the wave equation in several space dimensions. When the gap condition holds, the set of indices I_k is reduced to $\{k\}$ and then the eigenfunction estimate (3.15) is reduced to check it for isolated eigenfunctions.

The characterization above reduces the problem of observability of the wave equation to the obtention of the estimate (3.15) for linear combinations of eigenfunctions. But, of course, the proof of the latter is in general a non easy task. It requires, once more, of suitable geometric assumptions on the subdomain ω where the control is concentrated and can be developed by methods similar to those we described above for addressing directly the observability inequality for the wave equation. In particular, multipliers and Carleman inequalities may be used. But in the context of the condition (3.15) we can play with the advantage of using, for instance, multipliers that depend on the leading frequency of the wave packet under consideration. The two approaches, the dynamic one that consists in addressing directly the evolution equation (3.3) and the spectral one, end up giving similar results. However, because of its very nature, in the context of the wave equation, in order to obtain the control under the sharp GCC, in a way or another, the microlocal analysis needs to be used, even if it is in the obtention of (3.15).

In any case this spectral characterization of the observability inequality is of interest since, to some extent, it provides a natural extension of the methods based on non-harmonic Fourier series techniques and Ingham inequalities that so successfully apply in $1-d$ problems and that we describe now.

3.3 Ingham inequalities and Fourier series techniques

We have described here the HUM, which reduces the control problem to an observability one for the adjoint wave equation, and some tools to prove observability inequalities. There are other techniques that are also useful to address the problem of observability. In particular the theory of nonharmonic Fourier series and the so-called Ingham inequality allows obtaining sharp observability inequalities for a large class of one-dimensional wave-like equations. As we mentioned above, the Ingham inequality ([99], [192]) plays, in $1-d$, a similar role to the spectral characterization (3.15) above. It reads as follows:

Theorem 3.1 Ingham's Theorem, [99]. *Let $\{\mu_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers such that*

$$\mu_{k+1}-\mu_k \geq \gamma > 0, \text{ for all } k\in\mathbb{Z}. \tag{3.17}$$

Then, for any $T>2\pi/\gamma$ there exists a positive constant $C(T,\gamma)>0$ such that

$$\frac{1}{C(T,\gamma)}\sum_{k\in\mathbb{Z}} \mid a_k \mid^2 \leq \int_0^T \left|\sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t}\right|^2 dt \leq C(T,\gamma)\sum_{k\in\mathbb{Z}} \mid a_k \mid^2 \tag{3.18}$$

for all sequences of complex numbers $\{a_k\}\in\ell^2$.

Remark 3.2 *1. Although the most common use of Ingham's theorem is precisely the inequality (3.18), in the original article by Ingham [100], it was also proved that, under the same gap condition, there exists a constant $C(T,\gamma)>0$ such that the following L^1-version is also true:*

$$\mid a_n \mid \leq C(T,\gamma)\int_0^T \left|\sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t}\right| dt, \tag{3.19}$$

for all $n\in\boldsymbol{Z}$. In fact, as proved by Ingham, the constant $C(T,\gamma)$ can be taken to be the same in (3.19) and in its L^2-analogue in (3.18).

2. The original Ingham inequality was proved under the gap condition (3.17). However it is by now well known that this gap condition can be weakened, extending the range of possible applications. It is for instance well known that for the inequalities (3.18) to be true it suffices that all eigenvalues are distinct and that the gap condition is fulfilled asymptotically for high frequencies. Under this asymptotic gap condition the time T for the first inequality in (3.18) is the same, γ being the asymptotic gap. We refer for instance to [143] where an explicit estimate on the constants in (3.18) is given. These constants depend on the global gap, on the asymptotic gap and also on how rapidly this asymptotic gap is achieved. We underline however that the time T only depends on the asymptotic gap. Further generalizations have also be given. We refer for instance to the book [104] which contains an extension of that inequality covering families of eigenfrequencies which are, roughly, a finite union of sequences, satisfying each of them separately a gap condition. This kind of generalization is very useful in particular when dealing with networks of vibrations (see [40]).

Let us develop the details of the application of the Ingham inequality for the $1-d$ wave equation to better explain the connection with observability.

Consider the $1-d$ domain $\Omega = (0,\pi)$ and the adjoint wave equation with Dirichlet boundary conditions:

$$
(3.20)\qquad \begin{cases} \varphi_{tt} - \varphi_{xx} = 0 & \text{in} \quad (0,\pi)\times(0,T) \\ \varphi(x,t) = 0 & \text{for} \quad x = 0,\pi,\ t\in(0,T), \\ \varphi(x,T) = \varphi^0(x), \varphi_t(x,T) = \varphi^1(x) & \text{in} \quad (0,\pi). \end{cases}
$$

Consider any non-empty subinterval ω of Ω as observation region.

The solutions of this wave equation can be written in Fourier series in the form

$$
\varphi(x,t) = \sum_{k\in\mathbb{Z}} a_k e^{ikt} \sin kx.
$$

When applying Ingham's inequality for this series the relevant eigenfrequencies are $\mu_k = k$. The gap condition (3.17) is then clearly satisfied in this case with $\gamma = 1$. Thanks to (3.18), obtaining observability estimates for the solutions of the wave equation, when $T > 2\pi$, can be reduced to the obtention of similar estimates for the eigenfunctions. More precisely, if $T > 2\pi$, (3.4) holds for the solutions of the $1-d$ wave equation because the eigenfunctions $\{\sin(kx)\}_{k\geq 1}$ satisfy

$$
(3.21)\qquad \int_\omega \sin^2(kx)dx \geq c_\omega, \quad \forall k \geq 1.
$$

The last condition is easy to obtain since $\sin^2(kx)$ converges weakly to $1/2$ in $L^2(0,\pi)$ as $k\to\infty$.

Indeed, applying Fubini's Lemma and Ingham's inequality for all $x\in\omega$ to the series $\varphi(x,t) = \sum_{k\in\mathbb{Z}} a_k e^{ikt}\sin kx$, viewing $a_k \sin kx$ as coeffficients and applying (3.21) we deduce that

$$
(3.22)\qquad \int_0^T \int_\omega \varphi^2 dxdt = \int_0^T \int_\omega \Big| \sum_{k\in\mathbb{Z}} a_k e^{ikt}\sin kx \Big|^2 dxdt
$$

$$
= \int_\omega \int_0^T \Big| \sum_{k\in\mathbb{Z}} a_k e^{ikt}\sin kx\Big|^2 dtdx \geq C\int_\omega \sum_{k\in\mathbb{Z}} |a_k|^2 \sin^2 kx\, dx \geq C\sum_{k\in\mathbb{Z}} |a_k|^2.
$$

This proves (3.4) because $\sum_{k\in\mathbb{Z}} |a_k|^2$ is equivalent to the $L^2\times H^{-1}$-norm of the initial data of (3.3). In other words, (3.4) holds.

Note that in the application to the $1-d$ wave equation on $(0,1)$, the Ingham inequality does not yield the observability inequality for $T = 2$ but rather only for $T > 2$. In fact it is well known that the Ingham inequality does not hold in general for the optimal time $T = 2\pi/\gamma$ but only for $T > 2\pi/\gamma$ (see [192]). But, in the present case, because of the orthogonality of trigonometric polynomials in $L^2(0,2)$ the estimate holds for $T = 2$ too. In fact, in the case (3.20), if the control subdomain is the subinterval $\omega = (a,b)$ of $(0,\pi)$, the minimal control time is $2\max(a, \pi - b)$.

Let us now comment on the relation between the Ingham inequality approach described here for $1-d$ problems and the spectral characetrization (3.15) developed in the previous section. Note that, because of the gap condition, whenever $\sigma < \gamma = 1$, the sets I_k in (3.15) are reduced to the single eigenvalue λ_k. Consequently, the inequality in (3.15) reduces to (3.21) that, as we have seen, trivially holds.

Consequently the spectral characterization (3.15) plays a similar role as the Ingham inequality, but in any space dimension. However, as we mentioned above, the use of the spectral condition (3.15)-(3.16) in several space dimensions is much more subtle since it forces us to deal with wave packets, while in $1-d$, one has only to check the uniform observability of individual eigenfunctions as in (3.21). Note that Ingham's inequality cannot be applied for the wave equation in multi-dimensional problems since they grow asymptotically as $\lambda_j \sim c(\Omega)j^{2/n}$ and the gap vanishes. Despite this fact the observability inequality (3.4) may hold under suitable geometric conditions on ω and for sufficiently large values of time T. This is precisely, as the spectral condition (3.15) indicates, because of the uniform observability of the spectral wave packets.

The Ingham inequality approach can be applied to a variety of $1-d$ problems in which the Fourier representation of solutions can be used (mainly when the coefficients are time-independent) provided the gap condition holds. In this way one can address wave equations with variable coefficients, Airy equations, beam and Schrödinger equations, etc. Ingham inequality is also useful to address problems in which the control is localized on an isolated point or other singular ways, situations that can not be handeled by multipliers, for instance (see [90] and [187]). We refer to [144] for a brief introduction to this subject and to the monographs by Avdonin and Ivanov [6] and Komornik [103] and that by Komornik and Loreti [104] for a more complete presentation and discussion of this approach, intimately related also to the moment problem formulation of the control problem. We refer to [174] for a discussion of the moment problem approach. We also refer to the book [40] for an application of non-harmonic Fourier series methods to the control of waves on networks.

It is also important to underline that both, the Ingham approach and the spectral characterization (3.15) apply only for equations allowing a spectral decomposition of solutions. Thus, for instance, it does not apply to wave equations with coefficients depending both on x and t, or with lower order potentials of the form

$$\varphi_{tt} - \Delta\varphi + a(x,t)\varphi = 0.$$

This is certainly the main drawback of the Fourier approach to observability of solutions of wave equations.

3.4 Approximate controllability.

So far we have analyzed the problem of exact controllability. Let us now briefly discuss the *approximate controllability problem.*

According to the definition above (see subection 3.1), the problem of approximate controllability is equivalent to being able to find controls f for all pairs of initial and final data in the energy space $H_0^1(\Omega) \times L^2(\Omega)$, such that the following holds:

$$\left| \left(u(T) - v^0, u_t(T) - v^1\right) \right|_{H_0^1(\Omega)\times L^2(\Omega)} \leq \varepsilon. \tag{3.23}$$

Note that the property of approximate controllability guarantees that one can drive the state of the system arbitrarily close to the final target (v^0, v^1). However, it does not ensure in itself that we can reach the final target exactly, i. e. that we can take $\varepsilon = 0$ in (3.23).

In fact, the value $\varepsilon = 0$ is reached when exact controllability holds, in which case there exists a control f such that the solution satisfies exactly (3.9). In this case the approximate controls f_ε are uniformly bounded (with respect to ε) and, as ε tends to zero, converge to an exact control.

In finite-dimension, approximate controllability and exact controllability are equivalent notions. But this is no longer the case in the context of PDE because of the intrinsic infinite-dimensional nature of the state space. Indeed, in infinite-dimensional spaces there are strict dense subspaces, while in finite-dimension they do not exist.

This is particularly important in the context of the wave equation. As we have seen, for the exact controllability property to hold, one needs to impose rather strict geometric conditions on the control set. However, as we shall see, these restrictions are not needed for approximate controllability. On the other hand, approximate controllability is relevant from the point of view of applications in which the notion of exact controllability might seem to introduce a too strong constraint on the final state. However, as we mentioned above, when exact controllability fails, the size of this control diverges typically exponentially as ε tends to zero (see [112], [171]). This is an important warning about the effective use of the property of approximate controllability when exact controllability fails.

The approximate controllability property is equivalent to an *unique continuation* one for the adjoint system:

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x), \varphi_t(x,T) = \varphi^1(x) & \text{in} \quad \Omega. \end{cases} \tag{3.24}$$

More precisely, system (3.1) is approximately controllable if and only if the following holds:

$$\varphi \equiv 0 \text{ in } \omega \times (0,T) \Rightarrow \left(\varphi^0, \varphi^1\right) \equiv (0,0). \tag{3.25}$$

This unique continuation property is the analogue of that in (2.13) arising in the finite-dimensional theory.

By using Holmgren's Uniqueness Theorem it can be easily seen that (3.25) holds if T is large enough, under the sole condition that ω is a non-empty open subset of Ω. We refer to [123], chapter 1 for a discussion of this problem. At this respect it is important to underline that, even if the unique continuation property holds for all subdomains ω, for the observability inequality to be true ω is required to satisfy the GCC.

Let us now, assuming that the uniqueness property (3.25) holds (we shall return to this issue later on), analyze how approximate controllability can be obtained out of it. There are at least two ways of checking that (3.25) implies the approximate controllability property:

(a) The application of Hahn-Banach Theorem.

(b) The variational approach developed in [124].

We refer to [144] for a presentation of these methods.

In fact, when approximate controllability holds, then the following (apparently stronger) statement also holds:

Theorem 3.2 ([203]) *Let E be a finite-dimensional subspace of $H_0^1(\Omega) \times L^2(\Omega)$ and let us denote by π_E the corresponding orthogonal projection. Then, if approximate controllability holds, or, equivalently, if the unique continuation property (3.25) is satisfied, for any $\left(u^0, u^1\right)$, $\left(v^0, v^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\varepsilon > 0$ there exists $f_\varepsilon \in L^2(Q)$ such that the solution u_ε of (3.1) satisfies*

$$\left| \left(u_\varepsilon(T) - v^0, u_{\varepsilon,t}(T) - v^1\right) \right|_{H_0^1(\Omega)\times L^2(\Omega)} \leq \varepsilon; \; \pi_E\left(u_\varepsilon(T), u_{\varepsilon,t}(T)\right) = \pi_E\left(v^0, v^1\right). \tag{3.26}$$

This result, that will be refered to as the *finite-approximate controllability property*, may be proved in several ways. But, in particular, it can be obtained easily by a suitable modification of the variational approach introduced in [124] that we shall describe at the end of this subsection. This variational approach, in all cases, provides the control of minimal L^2-norm within the class of admissible controls. This makes the method particularly interesting and robust. [3]

The functional to be minimized to get approximate controllability is as follows:

$$J_\varepsilon\left(\varphi^0,\varphi^1\right)=\frac{1}{2}\int_0^T\int_\omega \varphi^2 dxdt+\varepsilon\Big|\left(\varphi^0,\varphi^1\right)\Big|_{L^2(\Omega)\times H^{-1}(\Omega)} + \left\langle\left(\varphi(0),\varphi_t(0)\right),\,\left(u^1,-u^0\right)\right\rangle - \left\langle\left(\varphi^0,\varphi^1\right),\,\left(v^1,-v^0\right)\right\rangle. \tag{3.27}$$

When adding the term $\varepsilon\Big|\left(\varphi^0,\varphi^1\right)\Big|_{L^2(\Omega)\times H^{-1}(\Omega)}$ in the functional to be minimized, the corresponding Euler-Lagrange equations for the minimizers turn out to be (3.23) instead of (3.9), which corresponds to the property of exact controllability and to the minimization of the functional J in (3.8). Consequently, this added term acts as a regularization of the functional J and, consequently, relaxes the controllability property obtained as a direct consequence of the optimality condition satisfied by the minimizers. It is also interesting to observe that, when adding this term to the functional, its coercivity is much easier to derive since it holds as a direct consequence of the unique continuation property (3.5) of the adjoint system, without requiring the observability inequality (3.4) to hold. Note however that this argument does not give any information on the size of the control needed to reach the target up an ε-distance.

Similarly, the finite-approximately controllability property can be achieved as a consequence of the unique continuation property by minimizing the functional

$$J_{\varepsilon,E}\left(\varphi^0,\varphi^1\right)=\frac{1}{2}\int_0^T\int_\omega \varphi^2 dxdt+\varepsilon\Big|\left(I-\pi_E^*\right)\left(\varphi^0,\varphi^1\right)\Big|_{L^2(\Omega)\times H^{-1}(\Omega)} + \left\langle\left(\varphi(0),\varphi_t(0)\right),\,\left(u^1,-u^0\right)\right\rangle - \left\langle\left(\varphi^0,\varphi^1\right),\,\left(v^1,-v^0\right)\right\rangle. \tag{3.28}$$

Note that the main difference between the functionals J_ε and $J_{\varepsilon,E}$ is that in the latter the relaxation term is weaker since we only add the norm of the projection $I-\pi_E^*$ of the data of the adjoint system and not the full norm. It is important to observe that, in the proof of the coercivity of the functional $J_{\varepsilon,E}$, the fact that the operator π_E^* is compact plays a key role. For this reason the space E is assumed to be of finite-dimension. Observe that π_E^* is defined by duality as

$$<\pi_E^*(\varphi^0,\varphi^1),(u^1,-u^0)>=<(\varphi^0,\varphi^1),\pi_E(u^1,-u^0)>,$$

where $<\cdot,\cdot>$ denotes the duality pairing between $L^2(\Omega)\times H^{-1}(\Omega)$ and its dual $L^2(\Omega)\times H_0^1(\Omega)$.

[3] Robustness is one of the key requirements in control theoretical applications. Indeed, although the control strategy is built on the basis of some specific modelling (or *plant* in the engineering terminology), in practice, due to uncertainty or to the intrinsic inaccuracies of the model considered, one needs to be sure that the control will also work properly under those unavoidable perturbations such as measurement noise and external disturbances. Robustness is then a fundamental requirement to be fulfilled by the control mechanism. The advantage of building and using controls that come out of a variational principle, by minimizing a suitable quadratic, convex and coercive functional, is that, slight changes of the functional will produce a smooth behavior on the control, as can be proved by the classical techniques in Γ-convergence theory ([41]). This, of course, has to be carefully checked in each particular case, but it is a methodology that works satisfactorily well in most cases.

The functional $J_{\varepsilon,E}$ can be obtained by a duality argument in order to get the control of minimal norm among the admissible ones. We refer for instance to [122] that addresses this issue in the context of the exact controllability of the wave equation. To be more precise, the control of minimal norm f is characterized by the minimality condition (see [122]):

$$f \in \mathcal{U}_{ad} : ||f||_{L^2(\omega\times(0,T))} = \min_{g\in\mathcal{U}_{ad}} ||g||_{L^2(\omega\times(0,T))}, \tag{3.29}$$

where $\mathcal{U}_{ad}$ is the set of admissible controls. More precisely,

$$\mathcal{U}_{ad} = \{f \in L^2(\omega \times (0,T)) : \text{ the solution } u \text{ of (3.1) satisfies (3.26)}\}. \tag{3.30}$$

The dual, in the sense of Fenchel-Rockafellar, of this minimization problem turns out to be precisely that of the minimization of the functional $J_{\varepsilon,E}$ with respect to $\left(\varphi^0, \varphi^1\right)$ in $L^2(\Omega) \times H^{-1}(\Omega)$.

In all these cases the control f we are looking for is the restriction to ω of the solution of the adjoint system (3.24) with the initial data that minimize the corresponding functional.

The results above on uniqueness for the adjoint wave equation and approximate controllability of the state equation hold for wave equations with analytic coefficients too. Indeed, the approximate control problem can be reduced to the unique continuation one and the latter may be solved by means of Holmgren's Uniqueness Theorem when the coefficients of the equation are analytic. However, the problem is still not completely solved in the frame of the wave equation with lower order potentials $a \in L^\infty(Q)$ of the form

$$u_{tt} - \Delta u + a(x,t)u = f1_\omega \text{ in } Q. \tag{3.31}$$

Once again the problem of approximate controllability of this system is equivalent to the unique continuation property of its adjoint. We refer to Alinhac [1], Tataru [182] and Robbiano-Zuilly [172] for deep results in this direction. But, roughly, we can say that in the class of bounded coefficients $a = a(x,t)$ we still do not have local sharp results on unique continuation allowing to handle equations of the form (3.31) in full generality. The existing ones require either some analyticity properties of the coefficients [182] and [172] or some geometric constraints in ω to apply the Carleman inequalities techniques [194]. On the other hand, it is well known that the unique continuation property may fail in general ([1]). A complete picture is still to be found.

3.5 Quasi bang-bang controls

In the finite-dimensional setting we have shown that, by slightly changing the functional to be minimized to get the controls, one can build bang-bang controls.

There is a very natural way of adapting this idea in the context of the wave equation. Indeed, essentially, it consists in replacing the functional (3.27) by its L^1 -version

$$\begin{aligned} J_{bb,\varepsilon}\left(\varphi^0, \varphi^1\right) = \frac{1}{2}\Big[\int_0^T\int_\omega |\varphi| dxdt\Big]^2 + \varepsilon\Big|\left(\varphi^0,\varphi^1\right)\Big|_{L^2(\Omega)\times H^{-1}(\Omega)} \\ + \left\langle (\varphi(0), \varphi_t(0)),\, \left(u^1, -u^0\right)\right\rangle - \left\langle \left(\varphi^0,\varphi^1\right),\, \left(v^1, -v^0\right)\right\rangle. \end{aligned} \tag{3.32}$$

This functional, as we shall see, is motivated by the search of controls of minimal L^∞-norm too.

It is convex and continuous in the space $L^2(\Omega) \times H^{-1}(\Omega)$. It is also coercive as a consequence of the unique continuation property (3.5). Therefore, a minimizer exists. Let us denote it by

$(\hat{\varphi}^0, \hat{\varphi}^1)$. One can then see that there exists a quasi bang-bang control $f \in \int_0^T \int_\omega |\varphi| dxdt \, \mathrm{sgn}\,(\hat{\varphi})$, $\hat{\varphi}$ being the solution of the adjoint system corresponding to the minimizer, such that the approximate controllability condition (3.23) holds. Note however that the bang-bang structure of the control is not guaranteed. Indeed, the fact that $f \in \int_0^T \int_\omega |\varphi| dxdt \, \mathrm{sgn}\,(\hat{\varphi})$ means that $f = \pm \int_0^T \int_\omega |\varphi| dxdt$ in the set in which $\hat{\varphi} \neq 0$, but simply that $f \in \Big[- \int_0^T \int_\omega |\varphi| dxdt, \int_0^T \int_\omega |\varphi| dxdt \Big]$, in the set where $\hat{\varphi} = 0$. Obviously, one cannot exclude the null set $\hat{\varphi} \equiv 0$ to be large. As we have seen, in the context of finite-dimensional systems, because of the analyticity of solutions, this null set is reduced to be a finite number of switching times in which the bang-bang control changes sign. But for the wave equation one can not exclude it to be even a non-empty and open subset of $\omega \times (0, T)$. In fact the explicit computations in [87] show that, for the one-dimensional wave equation, the set of reachable states by means of bang-bang controls is rather restricted. This is natural to be expected and it is particularly easy to understand in the case of boundary control. Indeed, according to D'Alembert's formula, the effect of the boundary control in the state, solution of the $1-d$ wave equation, is roughly that of reproducing at time $t = T$ the structure of the controller (assuming we start from the null initial datum). Thus, if the control is of bang-bang form and, in particular, piecewise constant, that necessarily imposes a very simple geometry of the reachable functions.

A complete analysis of whether the quasi-bang controls we have obtained above are bang-bang or not and a characterization of the set of initial data for which the bang-bang controls exist in the multi-dimensional case is still to be done.

It is also worth noting that the same problem of the existence of quasi-bang-bang controls was investigated in [85] in the context of approximate controllability of the $1-d$ wave equation but by replacing the energy space $H_0^1(\Omega) \times L^2(\Omega)$ by its L^∞-version $W_0^{1,\infty}(\Omega) \times L^\infty(\Omega)$. In this case it was proved that relaxation occurs and that the controls that are obtained are not longer of quasi-bang-bang form. This is due to the fact that, when addressing this problem by the variational tools we have developed, one needs to modify the functional $J_{bb,\varepsilon}$ above by replacing the added term $\varepsilon \Big| (\varphi^0, \varphi^1) \Big|_{L^2(\Omega) \times H^{-1}(\Omega)}$ by its L^1-version. This makes the problem of minimization not to be formulated in a reflexive Banach space. Relaxation phenomena may not be excluded a priori and, in fact, as the explicit examples in [85] show, they occur making the minimizers to develop singularities and, eventually, making the controls obtained in this way not to be of quasi-bang-bang form.

The same problem can be considered in the case of exact controllability in which the functional $J_{bb,\varepsilon}$ has to be replaced by J_{bb}:

$$J_{bb}\left(\varphi^0, \varphi^1\right) = \frac{1}{2} \Big[\int_0^T \int_\omega |\varphi| dxdt \Big]^2 + \left\langle (\varphi(0), \varphi_t(0)) \, , \, \left(u^1, -u^0\right) \right\rangle - \left\langle \left(\varphi^0, \varphi^1\right) , \, \left(v^1, -v^0\right) \right\rangle . \tag{3.33}$$

The functional setting in which this functional has to be minimized is much less clear. In principle, one should work in the class of solutions of the adjoint system for which $\varphi \in L^1(\omega \times (0, T))$. Then, following the HUM, it would be natural to consider the functional J_{bb} as being defined in the Banach space defined as completion of test functions $(\varphi^0, \varphi^1) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ with respect to the norm

$$\int_0^T \int_\omega |\varphi| dxdt.$$

The characterization of the space X is a difficult open problem. Under the assumption that ω satisfies the GCC, if T is large enough, using the known observability estimates in energy spaces and Sobolev embeddings, one could show that X is continuously embedded in an energy space of the form $H^s(\omega)\times H^{s-1}(\Omega)$ for some negative $s<0$. But a complete characterization of the space X is certainly extremely hard to get, except, may be, in $1-d$, in which the D'Alembert formula holds. But even in $1-d$ this remains to be done. Note also that, in $1-d$ the L^1-version of the Ingham inequality (3.18) could also help. In particular, that would imply that the space X is continuously embedded in the space of solutions of the wave equation with Fourier coefficients in ℓ^∞. But, from this point of view, a complete characterization of X is also unknown.

But, regardless of what the exact characterization of X is, the space X turns out to be non-reflexive. Thus, the minimization problem is not guaranteed to have a solution. Relaxation phenomena could occur, and one could be obliged to work in the space of solutions of the wave equation that are bounded measures when restricted to $\omega\times(0,T)$. In [85] this relaxation process has been shown to arise in the context of the boundary control of the wave equation. These explicit results in [85] for the $1-d$ wave equation show that, in general, the controls are not of bang-bang form. The existence of the minimizer of J_{bb} in X and the regularity of the minimizers constitute then interesting open problems.

A complete analysis of this minimization problem and its connections with L^∞-minimal norm controls and its bang-bang structure is to be developed for multi-dimensional problems.

The same questions arise in the context of boundary control.

Remark 3.3 *Obviously, there is a one-parameter family of L^p variational problems making the link between the L^2-optimal controls considered in the previous section and the quasi bang-bang controls that we have analyzed above. Indeed, for instance, in the context of exact controllability (the same could be said about approximate controllability and finite-approximate controllability), one can consider the L^p-version of the functional to be minimized*

$$(3.34)\quad J_p\left(\varphi^0,\varphi^1\right)=\frac{1}{2}\Big[\int_0^T\int_\omega|\varphi|^p dxdt\Big]^{2/p}+\left\langle\left(\varphi(0),\varphi_t(0)\right),\,\left(u^1,-u^0\right)\right\rangle-\left\langle\left(\varphi^0,\varphi^1\right),\,\left(v^1,-v^0\right)\right\rangle.$$

This functional has to be analyzed in the corresponding Banach space X_p of solutions of the adjoint heat equation whose restrictions to $\omega\times(0,T)$ belongs to L^p. In this case the space is reflexive and the minimizers exist. The corresponding controls take the form

$$f=|\hat{\varphi}|^{p-2}\hat{\varphi},$$

where $\hat{\varphi}$ is the solution of the adjoint system corresponding to the minimizer. When $p=2$ this corresponds to the L^2-control, which is known to exist under the GCC. When $p=1$ it corresponds to the quasi bang-bang controls we have discussed. This justifies the use of the notation J_{bb} for J_1. Analyzing the behavior of the space X_p and of the minimizers as p goes from 2 to 1 is an interesting open problem.

The functional J_p can also be obtained by the Fenchel-Rockafellar duality principle when searching controls of minimal $L^{p'}$-norm.

3.6 Stabilization

In this article we have not addressed the problem of stabilization. Roughly, in the context of linear equations, it can be formulated as the problem of producing the exponential decay of solutions through the use of suitable dissipative feedback mechanisms.

The main differences with respect to controllability problems are that:

- In controllability problems the time varies in a finite interval $0 < t < T < \infty$, while in stabilization problems, the time t tends to infinity;
- In control problems, the control can enter in the system freely in an open-loop manner, while in stabilization the control is of feedback or closed-loop form.

As in the context of controllability there are several degrees of stability or stabilizability of a system that are of interest:

- One can simply analyze the decay of solutions. This is typically done using LaSalle's invariance principle (see [89]). At the level of controllability this would correspond to a situation in which approximate controllability holds. Indeed, both problems are normally reduced to proving an unique continuation property. Still, the one corresponding to stabilization is normally easier to deal with since the time t varies in the whole $(0, \infty)$ and, because of this, often, the unique continuation problem is reduced to analyzing it at the spectral level;
- The more robust and strong stability property one can look for is that in which the energy of solutions (the norm of the solution in the state-space) tends to zero exponentially uniformly as $t \to \infty$. This normally requires of very efficient feedback mechanisms and, at the level of controllability, corresponds to the property of exact controllability;
- An interesting intermediate situation is that in which the uniform exponential decay fails but one is able to prove the polynomial decay of the solutions in the domain of the generator of the underlying semigroup. This normally corresponds, at the control level, to situations in which the control mechanism is unable to yield exact controllability properties, but guarantees the controllability of all data with slightly stronger regularity properties. That is for instance the case in the context of multi-structures as those considered in [88] where a model for the vibrations of strings coupled with point masses is considered and in [40] where wave equations on graphs are addressed.

But not only these two properties (controllability/stabilizability) are closely related but, in fact, one can prove various rigorous implications. We mention some of them in the context of the linear wave equation under consideration, although the same holds in a much larger class of systems including also plate and Schrödinger equations

- Whenever exact controllability holds, solving an infinite horizon Linear Quadratic Regulator (LQR) problem one can prove the existence of feedback operators, obtained as solutions of suitable Riccati operator equations, for which the corresponding semigroup has the property of exponential decay (see, for instance, [123]);
- Whenever the uniform exponential decay property holds, the system is exactly controllable as well, with controls supported precisely in the set where the feedback damping mechanism is active. This result is known as *Russells stabilizability* $\Longrightarrow$ *controllability principle* (see [174]);
- When exact controllability holds in the class of bounded control operators, stabilization holds as well (see [200]).

There is an extensive literature on the topic. Although in some cases, as we have mentioned, the stabilizability can be obtained as a consequence of controllability, this is not always the case. Consequently, the problem of stabilization needs often to be addressed directly and independently. The main tools for doing it are essentially the same: multipliers, microlocal analysis and Carleman inequalities.

Let us briefly mention some of the techniques and the type of results one may expect.

- The obtention of decay rates for solutions of dampoed wave equations has been the object of intensive research. One of the most useful tools for doing that is building new functionals, which are equivalent to the energy one, and for which differential inequalities can be obtained leading to the uniform exponential decay. These new functionals are built by perturbing the original energy one by adding terms that make explicit the effect of the mechanism on the various components of the system. At this respect it is important to note that the state of the wave equation involves in fact two components, the solution itself and its velocity. Thus, the way typical velocity feedback mechanisms affect the whole solution needs some analysis. On the other hand, in practice, the feedback is localized in part of the domain or its boundary, as controls do in controllability problems. Thus, how they affect the state everywhere else in the domain needs also some analysis. We refer to [91] where this method was introduced in the context of damped wave equations with damping everywhere in the interior of the domain. In this case the main multiplier that needs to be used is the solution of the equation itself since it allows obtaining the so-calle "equipartition of energy" estimate that makes explicit the effect of the velocity feedback on the solution itself. In [105] the method was adapted to the boundary stabilization of the wave equation. In that case one needs to use the same multipliers as for the boundary observability of the wave equation with Dirichlet boundary conditions. We also refer to [206] where the method has been applied to deal with nonlinear feedback terms for which the decay is polynomial.

- In the context of nonlinear systems the obtention of uniform exponential decay rates is more delicate. Indeed, due to the presence of the nonlinearity, the exponential rate of decay may depend on the size of the solutions under consideration. In fact the nonlinearity has to satisfy some "good sign" properties at infinity to guarantee that the exponential decay rate is independent of the initial data. We refer to [200] where the uniform exponential decay has been proved for the semilinear wave equation with linear damping concentrated on a neighborhood of the boundary. In this case the exponential decay has been directly proved by proving observability estimates without building Lyapunov functionals. We refer also to [43] where these results have been extended, by means of Strichartz estimates, to supercritical nonlinearities that energy methods do not allow to handle.

- The obtention of uniform exponential decay rates not only needs of appropriate feedback mechanisms but they also need to be supported in regions that guarantee the Geometric Control Condition (GCC) to hold. In fact the microlocal techniques apply in stabilization problems as well (see [7]). When the GCC fails, due to the existence of gaussiam beam solutions that are exponentially concentrated away from the support of the damper, the property of uniform exponential decay fails. In that case one may only prove logarithimc decay rates for solutions with data in the domain of the operator. We refer to [198] where this type of result has been obtained in the context of a coupled wave-heat system.

- In some cases, even if the damping mechanism is supported in a subdomain of the domain itself or of its boundary that satisfies the GCC, the uniform exponential decay may fail if the damping does not damp the energy of the system itself but rather a weaker one. This happens typically if, instead of the wave equation with velocity damping supported everywhere in the domain

$$u_{tt} - \Delta u + u_t = 0, \tag{3.35}$$

one considers

$$u_{tt} - \Delta u + K u_t = 0, \tag{3.36}$$

where $K : L^2(\Omega) \to L^2(\Omega)$ is a compact positive operator (for isntance $K = (-\Delta)^{-s}$ for some $s > 0$). In the first case the energy

$$E(t) = \frac{1}{2} \int \Big[|u_t|^2 + |\nabla u|^2 \Big] dx,$$

satisfies the energy dissipation law

$$\frac{d}{dt} E(t) = - \int_\Omega |u_t|^2 dx, \tag{3.37}$$

while, in the other one, it follows that

$$\frac{d}{dt} E(t) = - \int_\Omega K u_t u_t dx. \tag{3.38}$$

While in the first case the energy dissipation rate is proportional to the kinetic energy, in the second one the dissipation is weaker, because of the compactness of the operator K. In the latter one can not expect the uniform exponential decay rate to hold. In fact, in the simplest case in which $K = (-\Delta)^{-s}$ the spectrum of the system can be computed explicitly and one sees that the spectral abscissa vanishes.

However, even if the damping is too weak, one can get a polynomial decay rate within the class of solutions in the domain of the operator. It is important to observe that dissipative semigroups are necessarily such that either the norm of the semigroup decays exponentially or the semigroup is of unit norm for all $t \geq 0$. Therefore, one may not expect any other uniform decay rate when the uniform exponential decay property fails. For that reason one needs to restrict the class of solutions under consider. A natural way of doing that is considering initial data in the domain of the generator of the semigroup.

There are several mechanical systems in which these phenomena arise. One of them is the system of thermoelasticity in which the damping introduced through the heat equation dissipates at most a lower order energy with a loss of one derivative (see [119]). The same occurs often in the context of multi-structures (see for instance [88]) where a system coupling two vibrating strings with a point mass is considered). We also refer to [168] where a $2-d$ plate with dynamical boundary conditions is considered. The polynomial decay property is proved by using a multiplier of the form $(x - x^0) \cdot \nabla u E(t)$, the novelty being that the multiplier is not linear on the solutions but rather of cubic homogeneity.

The problem of stabilization of wave equations is also intimately related to other issues in the theory of infinite-dimensional dissipative dynamical systems. We refer to [89] for an introduction to this topic. We also refer to [61] where the issue of attractors for semilinear wave equations with locally distributed damping is addressed.

4 The heat equation.

4.1 Problem formulation

With the same notations as above we consider the linear controlled heat equation:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{4.1}$$

We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (4.1) admits an unique solution

$$u \in C\left([0,T]\,; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right).$$

We introduce the reachable set $R(T;u^0) = \left\{u(T) : f \in L^2(Q)\right\}$. In this case the different notions of controllability can be formulated as follows:

(a) System (4.1) is said to be *approximately controllable* if $R(T;u^0)$ is dense in $L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.

(b) System (4.1) is *exactly controllable* if $R(T;u^0) = L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.

(c) System (4.1) is *null controllable* if $0 \in R(T;u^0)$ for all $u^0 \in L^2(\Omega)$.

Summarizing, the following can be said about these notions:

(a) It is easy to see that exact controllability may not hold, the trivial case in which $\omega = \Omega$ being excepted.[4] Indeed, due to the regularizing effect of the heat equation, solutions of (4.1) at time $t = T$ are smooth in $\Omega \backslash \overline{\omega}$. Therefore $R(T;u^0)$ is strictly contained in $L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.

(b) Approximate controllability holds for every open non-empty subset ω of Ω and for every $T > 0$. As we shall see, as in the case of the wave equation, the problem can be reduced to an uniqueness one that can be solved applying Holmgren's uniqueness Theorem. The controls of minimal norm can be characterized as the minima of suitable quadratic functionals. As in the context of the wave equation, as a consequence of approximate controllability, we can ensure immediately that finite-approximate controllability also holds.

(c) The system being linear, null controllability implies that all the range of the semigroup generated by the heat equation is reachable too. In other words, if $0 \in R(T;u^0)$ then, $S(T)[L^2(\Omega)] \subset R(T;u^0)$, where $S = S(t)$ is the semigroup generated by the uncontrolled heat

[4] In the later controllability holds in the space H^1 instead of L^2.

equation. This result might seem surprising in a first approach. Indeed, the sole fact that the trivial state $u^1 \equiv 0$ is reachable, implies that all the range of the semigroup is it.

But, in fact, a more careful analysis shows that the reachable set is slightly larger. We shall return to this matter.

(d) Null controllability in time T implies approximate controllability in time T. Proving it requires the use of the density of $S(T)[L^2(\Omega)]$ in $L^2(\Omega)$.

In the case of the linear heat equation this can be seen easily developing solutions in Fourier series. In the absence of control ($f \equiv 0$), the solution can be written in the form

$$u(x,t) = \sum_{j\geq 1} a_j e^{-\lambda_j t}\psi_j(x). \tag{4.2}$$

The initial datum u^0 being in $L^2(\Omega)$ is equivalent to the condition that its Fourier coefficients $\{a_j\}_{j\geq 1}$ satisfy that $\{a_j\}_{j\geq 1} \in \ell^2$.

Then, the range $S(T)[L^2(\Omega)]$ of the semigroup can be characterized as the space of functions of the form

$$\sum_{j\geq 1} a_j e^{-\lambda_j T}\psi_j(x), \tag{4.3}$$

with $\{a_j\}_{j\geq 1} \in \ell^2$. This space is small, in particular, it is smaller than any finite order Sobolev space in Ω. But it is obviously dense in $L^2(\Omega)$ since it contains all finite linear combinations of the eigenfunctions.

If the equation contains time dependent coefficients the density of the range of the semigroup still holds, but can not be proved by Fourier expansions. One has rather to use a duality argument that reduces the problem to that of the backward uniqueness. This property is by now well known for the Dirichlet problem in bounded domains for the heat equation with lower order terms (see Lions and Malgrange [126] and Ghidaglia [80]). It reads as follows: If $y \in C([0,T]; H_0^1(\Omega))$ solves

$$\begin{cases} y_t - \Delta y + a(x,t)y = 0 & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma, \end{cases} \tag{4.4}$$

and

$$y(x,T) \equiv 0 \quad \text{in } \Omega,$$

then, necessarily, $y \equiv 0$.

In fact, the proof of this backward uniqueness result can be made quantitave, yielding an energy estimate which, roughly, depends exponentially on the ratio $R(0) = ||\nabla y(0)||^2_{L^2(\Omega)}/||y(0)||^2_{L^2(\Omega)}$ and the L^∞-norm of the potential $a = a(x,t)$. Note that the initial datum $y(0)$ on that problem is assumed to be unkown but, in view of the regularity condition imposed on the solution the ratio $R(0)$ it is known to be finite. This estimate allows getting upper bounds on the L^2-norm of solutions at time t_1 in terms of the L^2-norm in time t_2 with $t_1 < t_2$. In particular, when $y(T) \equiv 0$ this estimate implies that $y(t) \equiv 0$ for all $0 \leq t \leq T$. In fact, one can obtain rather explicit estimates on the exponential growth of the norm of solutions backwards in time. This has been used systematically in [68] to get explicit estimates on the cost of approximate controllability.

Note also that, in fact, the density of the range of the semigroup is also true for heat equations with globally Lipschitz nonlinearities [57].

■

Let us now develop some of these results in more detail.

4.2 Approximate controllability.

We first discuss the *approximate and the finite-approximate controllability problems.*

As we said above, system (4.1) is approximately controllable for any open, non-empty subset ω of Ω and $T > 0$. To see this one can apply Hahn-Banach's Theorem or the variational approach developed in [124] and that we have presented in the previous section in the context of the wave equation. In both cases the approximate controllability is reduced to the unique continuation property for the adjoint system

$$(4.5) \qquad \begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases}$$

More precisely, approximate controllability holds if and only if the following uniqueness property is true:

$$(4.6) \qquad \text{If } \varphi \text{ solves (4.5)and } \varphi = 0 \text{ in } \omega \times (0,T) \text{ then, necessarily}, \varphi \equiv 0, \text{ i.e. } \varphi^0 \equiv 0.$$

This uniqueness property holds for every open non-empty subset ω of Ω and $T > 0$ by Holmgren's Uniqueness Theorem.

As we have seen above in the context of the wave equation, when this unique continuation property holds, not only the system is approximately controllable but it is also finite-approximately controllable too.

Following the variational approach of [124] described in the previous subsection in the context of the wave equation, the control can be constructed as follows. First of all we observe that, the system being linear and well-posed in $L^2(\Omega)$, it is sufficient to consider the particular case $u^0 \equiv 0$. Then, for any u^1 on $L^2(\Omega), \varepsilon > 0$ and E finite-dimensional subspace of $L^2(\Omega)$ we introduce the functional

$$(4.7) \qquad J_\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \varepsilon||(I-\pi_E)\varphi^0||_{L^2(\Omega)} - \int_\Omega \varphi^0 u^1 dx$$

where π_E denotes the orthogonal projection from $L^2(\Omega)$ over E. Strictly speaking, this functional corresponds to the case $u^0 \equiv 0$. Note however that, without loss of generality, the problem can be reduced to that particular case because of the linearity of the system under consideration. In general the functional to be considered would be:

$$(4.8) \qquad J_\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \varepsilon \left|(I-\pi_E)\varphi^0\right|_{L^2(\Omega)} - \int_\Omega \varphi^0 u^1 dx + \int_\Omega \varphi(0) u^0 dx.$$

The functional J_ε is continuous and convex in $L^2(\Omega)$. On the other hand, in view of the unique continuation property above, one can prove that

$$\lim_{\|\varphi^0\|_{L^2(\Omega)}\to\infty} \frac{J_\varepsilon(\varphi^0)}{\|\varphi^0\|_{L^2(\Omega)}} \geq \varepsilon \tag{4.9}$$

Let us, for the sake of completeness, give the proof of this coercivity property.

In order to prove (4.9) let $\left(\varphi_j^0\right) \subset L^2(\Omega)$ be a sequence of initial data for the adjoint system with $\|\varphi_j^0\|_{L^2(\Omega)} \to \infty$. We normalize them by

$$\widetilde{\varphi}_j^0 = \varphi_j^0 / \|\varphi_j^0\|_{L^2(\Omega)},$$

so that $\|\widetilde{\varphi}_j^0\|_{L^2(\Omega)} = 1$.

On the other hand, let $\widetilde{\varphi}_j$ be the solution of (4.5) with initial data $\widetilde{\varphi}_j^0$. Then

$$J_\varepsilon(\varphi_j^0)/\|\varphi_j^0\|_{L^2(\Omega)} = \frac{1}{2}\|\varphi_j^0\|_{L^2(\Omega)} \int_0^T\int_\omega |\widetilde{\varphi}_j|^2\,dxdt + \varepsilon\|(I-\pi_E)\widetilde{\varphi}_j^0\|_{L^2(\Omega)} - \int_\Omega u^1\widetilde{\varphi}_j^0 dx.$$

The following two cases may occur:

1) $\underline{\lim}_{j\to\infty}\int_0^T\int_\omega |\widetilde{\varphi}_j|^2 > 0$. In this case we obtain immediately that

$$J_\varepsilon(\varphi_j^0)/\|\varphi_j^0\|_{L^2(\Omega)} \to \infty.$$

2) $\underline{\lim}_{j\to\infty}\int_0^T\int_\omega |\widetilde{\varphi}_j|^2 = 0$. In this case since $\widetilde{\varphi}_j^0$ is bounded in $L^2(\Omega)$, by extracting a subsequence we can guarantee that $\widetilde{\varphi}_j^0 \rightharpoonup \psi^0$ weakly in $L^2(\Omega)$ and $\widetilde{\varphi}_j \rightharpoonup \psi$ weakly in

$$L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)),$$

where ψ is the solution of (4.5) with initial data ψ^0 at $t=T$. Moreover, by lower semicontinuity,

$$\int_0^T\int_\omega \psi^2 dxdt \leq \underline{\lim}_{j\to\infty}\int_0^T\int_\omega |\widetilde{\varphi}_j|^2\,dxdt = 0$$

and therefore $\psi = 0$ in $\omega\times(0,T)$.

Holmgren's Uniqueness Theorem implies that $\psi\equiv 0$ in $\Omega\times(0,T)$ and consequently $\psi^0 = 0$.

Therefore, $\widetilde{\varphi}_j^0 \rightharpoonup 0$ weakly in $L^2(\Omega)$ and consequently $\int_\Omega u^1\widetilde{\varphi}_j^0 dx$ tends to 0 as well. Furthermore, E being finite-dimensional, π_E is compact and then $\pi_E\widetilde{\varphi}_j^0 \to 0$ strongly in $L^2(\Omega)$. Consequently

$$\|(I-\pi_E)\widetilde{\varphi}_j^0\|_{L^2(\Omega)} \to 1, \quad \text{as } j\to\infty.$$

Hence

$$\underline{\lim}_{j\to\infty}\frac{J_\varepsilon(\varphi_j^0)}{\|\varphi_j^0\|} \geq \underline{\lim}_{j\to\infty}\left[\varepsilon - \int_\Omega u^1\widetilde{\varphi}_j^0 dx\right] = \varepsilon,$$

and (4.9) follows.

Then, J_ε admits an unique minimizer $\widehat{\varphi}^0$ in $L^2(\Omega)$. The control $f=\widehat{\varphi}$ where $\widehat{\varphi}$ solves (4.5) with $\widehat{\varphi}^0$ as data is such that the solution u of (4.1) with $u^0=0$ satisfies

$$\| u(T)-u^1 \|_{L^2(\Omega)} \leq \varepsilon,\ \pi_E\left(u(T)\right)=\pi_E(u^1). \tag{4.10}$$

Indeed, suppose that J_ε attains its minimum value at $\widehat{\varphi}^0\in L^2(\Omega)$. Then for any $\psi^0\in L^2(\Omega)$ and $h\in\mathbb{R}$ we have $J_\varepsilon(\widehat{\varphi}^0)\leq J_\varepsilon\left(\widehat{\varphi}^0+h\psi^0\right)$. On the other hand,

$$\begin{aligned} J_\varepsilon\left(\widehat{\varphi}^0+h\psi^0\right) &= \frac{1}{2}\int_0^T\int_\omega |\,\widehat{\varphi}+h\psi\,|^2\,dxdt+\varepsilon \,\|\,(I-\pi_E)(\widehat{\varphi}^0+h\psi^0)\,\|_{L^2(\Omega)}-\int_\Omega u^1(\widehat{\varphi}^0+h\psi^0)dx \\ &= \frac{1}{2}\int_0^T\int_\omega |\,\widehat{\varphi}\,|^2\,dxdt+\frac{h^2}{2}\int_0^T\int_\omega |\,\psi\,|^2\,dxdt+h\int_0^T\int_\omega \widehat{\varphi}\psi dxdt \\ &\qquad +\varepsilon\,\|\,(I-\pi_E)(\widehat{\varphi}^0+h\psi^0)\,\|_{L^2(\Omega)}-\int_\Omega u^1(\widehat{\varphi}^0+h\psi^0)dx. \end{aligned}$$

Thus

$$\begin{aligned} 0\leq\quad & \varepsilon\left[\|\,(I-\pi_E)(\widehat{\varphi}^0+h\psi^0)\,\|_{L^2(\Omega)}-\|\,(I-\pi_E)\widehat{\varphi}^0\,\|_{L^2(\Omega)}\right]+\frac{h^2}{2}\int_{(0,T)\times\omega}\psi^2dxdt \\ & +h\left[\int_0^T\int_\omega\widehat{\varphi}\psi dxdt-\int_\Omega u^1\psi^0dx\right]. \end{aligned}$$

Since

$$\|\,(I-\pi_E)(\widehat{\varphi}^0+h\psi^0)\,\|_{L^2(\Omega)}-\|\,(I-\pi_E)\widehat{\varphi}^0\,\|_{L^2(\Omega)}\leq |h|\,\|\,(I-\pi_E)\psi^0\,\|_{L^2(\Omega)}$$

we obtain

$$0\leq\varepsilon\,|h|\,\|\,(I-\pi_E)\psi^0\,\|_{L^2(\Omega)}+\frac{h^2}{2}\int_0^T\int_\omega\psi^2dxdt+h\int_0^T\int_\omega\widehat{\varphi}\psi dxdt-h\int_\Omega u^1\psi^0dx$$

for all $h\in\mathbb{R}$ and $\psi^0\in L^2(\Omega)$.

Dividing by $h>0$ and by passing to the limit $h\to 0$ we obtain

$$0\leq\varepsilon\,\|\,(I-\pi_E)\psi^0\,\|_{L^2(\Omega)}+\int_0^T\int_\omega\widehat{\varphi}\psi dxdt-\int_\Omega u^1\psi^0dx. \tag{4.11}$$

The same calculations with $h<0$ give that

$$\left|\int_0^T\int_\omega\widehat{\varphi}\psi dxdt-\int_\Omega u^1\psi^0dx\right|\leq\varepsilon\,\|\,(I-\pi_E)\psi^0\,\|\quad\forall\psi^0\in L^2(\Omega). \tag{4.12}$$

On the other hand, if we take the control $f=\widehat{\varphi}$ in (4.1), by multiplying in (4.1) by ψ solution of (4.5) and by integrating by parts we get that

$$\int_0^T\int_\omega\widehat{\varphi}\psi dxdt=\int_\Omega u(T)\psi^0dx. \tag{4.13}$$

From the last two relations it follows that

$$\left|\int_\Omega(u(T)-u^1)\psi^0dx\right|\leq\varepsilon||\psi^0||_{L^2(\Omega)},\quad\forall\psi^0\in L^2(\Omega) \tag{4.14}$$

which is equivalent to

$$||u(T) - u^1||_{L^2(\Omega)} \leq \varepsilon. \tag{4.15}$$

We have shown that the variational approach allows to prove the property of finite-approximate controllability, as soon as the unique continuation property for the adjoint system holds. The controls we obtain this way are those of minimal $L^2(\omega \times (0,T))$-norm.

This method can be extended to the L^p-setting and, in particular, be used to build bang-bang controls.

4.3 Null controllability.

Let us now analyze the *null controllability* problem.

This problem is also a classical one. In recent years important progresses have been done combining the variational techniques we have described and the Carleman inequalities yielding the necessary observability estimates. We shall describe some of the key ingredients of this approach in this section. However, the first results in this context were obtained in one space dimension, using the moment problem formulation and explicit estimates on the family of biorthogonal functions. We refer to [174] for a survey of the first results obtained by those techniques (see also [59], [60] and [173]).

The null controllability problem for system (4.1) is equivalent to the following observability inequality for the adjoint system (4.5):

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{4.16}$$

Due to the time-irreversibility of the system (4.5), (4.16) is not easy to prove. For instance, multiplier methods, that are so efficient for wave-like equations, do not apply. Nevertheless, inequality (4.16) is rather weak. Indeed, in contrast with the situation we encountered when analyzing the wave equation, in the present case, getting estimates on the solution of the adjoint system at $t = 0$ is much weaker than getting estimates of φ^0 at time $t = T$. Indeed, due to the very strong irreversibility of the adjoint system (4.5), which is well-posed in the backward sense of time, one cannot get estimates of the initial datum φ^0 out of the estimate (4.16). We shall come back to this issue later.

There is an extensive literature on the null control of the heat equation. In [174] the boundary null controllability of the heat equation was proved in one space dimension using moment problems and classical results on the linear independence in $L^2(0,T)$ of families of real exponentials. On the other hand, in [173] it was shown that *if the wave equation is exactly controllable for some $T > 0$ with controls supported in ω, then the heat equation* (4.1) *is null controllable for all $T > 0$ with controls supported in ω.* We refer to [6] for a systematic and more recent presentation of this method. As a consequence of this result and in view of the controllability results of the previous section for the wave equation, it follows that the heat equation (4.1) is null controllable for all $T > 0$ provided ω satisfies the GCC and the observability inequality (4.16) holds.

The fact that the control time $T > 0$ is arbitrary for the heat equation is in agreement with the intrinsic infinite velocity of propagation of the heat model. However, the GCC does not seem to be a natural sharp condition in the context of the heat equation. Indeed, in view of the diffusion and regularizing process that the heat equation induces one could expect the heat equation to be null-controllable from any open non-empty subset ω. This result was proved by Lebeau and Robbiano [117]. Simultaneously, the same was proved independently by Imanuvilov in [94] (see also [74]) for a much larger class of heat equations with lower order potentials by using parabolic Carleman inequalities. We shall return to this issue below.

Let us first discuss the method in [117] which is based on the Fourier decomposition of solutions. A simplified presentation was given in [118] where the linear system of thermoelasticity was also addressed. The main ingredient of it is the following observability estimate for the eigenfunctions of the Laplace operator:

Theorem 4.1 ([117], [118]) *Let Ω be a bounded domain of class C^∞. For any non-empty open subset ω of Ω there exist positive constants $C_1, C_2 > 0$ such that*

$$C_1 e^{-C_2\sqrt{\mu}} \sum_{\lambda_j \leq \mu} | a_j |^2 \leq \int_\omega \left| \sum_{\lambda_j \leq \mu} a_j \psi_j(x) \right|^2 dx \tag{4.17}$$

for all $\{a_j\} \in \ell^2$ and for all $\mu > 0$.

The proof of (4.17) is based on Carleman inequalities (see [117] and [118]).

Although the constant in (4.17) degenerates exponentially as $\mu \to \infty$, it is important that it does it exponentially on $\sqrt{\mu}$. As we shall see, the strong dissipativity of the heat equation allows compensating this fact. Indeed, estimate (4.17) provides a measure of the degree of linear independence of the traces of linear finite combinations of eigenfunctions over ω. By inspection of the gaussian heat kernel it can be shown that this estimate, i. e. the degeneracy of the constant in (4.17) as $\exp(-C_2\sqrt{\mu})$ for some $C_2 > 0$, is sharp even in $1-d$.

As a consequence of (4.17) one can prove that the observability inequality (4.16) holds for solutions of (4.5) with initial data in $E_\mu = \text{span}\{\psi_j\}_{\lambda_j \leq \mu}$, the constant being of the order of $\exp\left(C\sqrt{\mu}\right)$. This shows that the projection of solutions of (4.1) over E_μ can be controlled to zero with a control of size $\exp\left(C\sqrt{\mu}\right)$.[5] Thus, when controlling the frequencies $\lambda_j \leq \mu$ one increases the $L^2(\Omega)$-norm of the high frequencies $\lambda_j > \mu$ by a multiplicative factor of the order of $\exp\left(C\sqrt{\mu}\right)$. However, solutions of the heat equation (4.1) without control ($f = 0$) and such that the projection of the initial data over E_μ vanishes, decay in $L^2(\Omega)$ at a rate of the order of $\exp(-\mu t)$. This can be easily seen by means of the Fourier series decomposition of the solution. Thus, if we divide the time interval $[0, T]$ in two parts $[0, T/2]$ and $[T/2, T]$, we control to zero the frequencies $\lambda_j \leq \mu$ in the interval $[0, T/2]$ and then allow the equation to evolve without control in the interval $[T/2, T]$, it follows that, at time $t = T$, the projection of the solution u over E_μ vanishes and the norm of the high frequencies does not exceed the norm of the initial data u^0.

This argument allows to control to zero the projection over E_μ for any $\mu > 0$ but not the whole solution. To do that an iterative argument is needed in which the interval $[0, T]$ has to be

[5] In fact the same is true for any evolution equation allowing a Fourier expansion on the basis of these eigenfunctions (Schrödinger, plate, wave equations, etc.) The novelty of the argument in [117] when applied to the heat equation is that its dissipative effect is able to compensate the growth of the control as μ tends to infinity, a fact that does not hold for conservative systems. This is sharp and natural, to some extent, since we know that the wave equation does not have the property of being controllable from any open set.

decomposed in a suitably chosen sequence of subintervals $[T_k, T_{k+1})$ and the argument above is applied in each subinterval to control an increasing range of frequencies $\lambda_j \leq \mu_k$ with $\mu_k \to \infty$ at a suitable rate. We refer to [117] and [118] for the proof.

Once (4.16) is known to hold for the solutions of the adjoint heat equation (4.5) one can obtain the control with minimal $L^2(\omega \times (0,T))$-norm among the admissible ones. To do that it is sufficient to minimize the functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0) u^0 dx \tag{4.18}$$

over the Hilbert space

$$H = \{\varphi^0 : \text{ the solution } \varphi \text{ of (4.5) satisfies } \int_0^T \int_\omega \varphi^2 dxdt < \infty\},$$

endowed with its canonical norm.

To be more precise, H should be defined as the completion of $\mathcal{D}(\Omega)$ with respect to the norm

$$||\varphi^0||_H = \Big(\int_0^T \int_\omega \varphi^2 dxdt\Big)^{1/2}.$$

The space H is very large. In fact, due to the regularizing effect of the heat equation, any initial (at time $t = T$) datum φ^0 of the adjoint heat equation in H^{-s}, whatever $s > 0$ is, belongs to H because the corresponding solutions satisfy $\varphi \in L^2(\omega \times (0,T))$. We shall return in the following section to the discussion of the nature and structure of this space.

Observe that J is convex and continuous in H. On the other hand (4.16) guarantees the coercivity of J and the existence of its minimizer. The minimizer of J provides the control we are looking for, which is of minimal $L^2(\omega \times (0,T))$-norm.

There is an easy way to build null controls and avoiding working in the space H. Indeed, we can build, for all $\varepsilon > 0$ an approximate control f_ε such that the solution u_ε of (4.1) satisfies the condition

$$\Big| u_\varepsilon(T) \Big|_{L^2(\Omega)} \leq \varepsilon. \tag{4.19}$$

Recall that, for this to be true, the unique continuation property (4.6) of the adjoint system suffices. But, the fact that the observability inequality (4.16) holds adds an important information to this: the sequence of controls $\{f_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded. Assuming for the moment that this holds let us conclude the null controllability of (4.1) out of these results. In view of the fact that controls $\{f_\varepsilon\}_{\varepsilon>0}$ are uniformly bounded in $L^2(\omega \times (0,T))$, by extracting subsequences, we have $f_\varepsilon \to f$ weakly in $L^2(\omega \times (0,T))$. Using the continuous dependence of the solutions of the heat equation (4.1) on the right hand side term, we can show that $u_\varepsilon(T)$ converges to $u(T)$ weakly in $L^2(\Omega)$. In view of (4.19) this implies that $u(T) \equiv 0$. The limit control f then fulfills the null-controllability requirement.

In order to see that the controls f_ε are bounded, we have to use its structure. Note that $f_\varepsilon = \hat{\varphi}_\varepsilon$ where $\hat{\varphi}_\varepsilon$ solves (4.5) with initial data at time $t = T$ obtained by minimizing the functional (4.8) when $E = \{0\}$ and $u^1 \equiv 0$. At the minimizer $\hat{\varphi}^0_\varepsilon$ we have $J_\varepsilon(\varphi^0_\varepsilon) \leq J_\varepsilon(0) = 0$. This implies that

$$\int_{\omega\times(0,T)} |\hat{\varphi}_\varepsilon|^2 dxdt \leq ||u^0||_{L^2(\Omega)} ||\varphi_\varepsilon(0)||_{L^2(\Omega)}.$$

This, together with the observability inequality (4.16), implies that

$$\Big[\int_{\omega\times(0,T)}|f_\varepsilon|^2dxdt\Big]^{1/2}=\Big[\int_{\omega\times(0,T)}|\hat{\varphi}_\varepsilon|^2dxdt\Big]^{1/2}\leq C||u^0||_{L^2(\Omega)},$$

which yields the desired bound on the approximate controls.

Note that, the approximate controllability in itself (or, in other words, the unique continuation property of the adjoint system) does not yield this bound. We have rather used the fact that observability inequality (4.16) holds. The argument above simply avoids minimizing the functional in H, a space whose nature will be investigated below.

As a consequence of the internal null controllability property of the heat equation one can deduce easily the null boundary controllability with controls in an arbitrarily small open subset of the boundary. To see this it is sufficient to extend the domain Ω by a little open subset attached to the subset of the boundary where the control needs to be supported. The arguments above allow to control the system in the large domain by means of a control supported in this small added domain. The restriction of the solution to the original domain satisfies all the requirements and its restriction or trace to the subset of the boundary where the control had to be supported, provides the control we were looking for.

Note however that the boundary control problem may be addressed directly. As a consequence of Holmgren's Uniqueness Theorem, the corresponding unique continuation result holds and, as a consequence we obtain approximate and finite-approximate controllability. On the other hand, Carleman inequalities yield the necessary observability inequalities to derive null controllability as well (see for instance [94], [74], [63]). As a consequence of this, as in the context of the internal control problem, null controllability holds in an arbitrarily small time and with boundary controls supported in an arbitrarily small open non-empty subset of the boundary.

The method of proof of the null controllability property we have described is based on the possibility of expanding solutions in Fourier series. Thus it can be applied in a more general class of heat equations with variable but time-independent coefficients. The same can be said about the methods of [173]. In the following section we shall present a direct Carleman inequality approach proposed and developed in [94] and [74] for the parabolic problem which allows circumventing this difficulty.

Recently L. Miller in [149] used for control purposes the so-called Kannai transform allowing to write the solutions of the heat equation in terms of those of the wave equation to derive null controllability results for the heat equation as a consequence of the exact controllability of the wave equation. This approach, the so-called transmutation method according to the terminology in [149], plays the role in the physical space of that used by Russell [173] which consists in performing a change of variable in the frequency domain. Both approaches give similar results. The advantage of the transmutation method is that it allows getting explicit estimates of the norms of the controls more easily, but the drawback, as in Russell's approach, is that it only applies to heat equations with coefficients which are independent of time. Previously similar arguments and transformations have been used by K. D. Phung [154] to analyze the cost of controllability for Schrödinger equations.

4.4 Parabolic equations of fractional order

The iterative argument developed in [117] and [118] based on the spectral estimate (4.17) suggests that the regularizing effect of the heat equation is far too much to guarantee the null controllability. Indeed, controlling low frequencies $\lambda_j \leq \mu$ costs $\exp(C\sqrt{\mu})$ while the dissipation rate of the controlled one is $\exp(-\mu)$. In view of this it would be natural to consider equations of the form

$$u_t + (-\Delta)^\alpha u = f 1_\omega, \tag{4.20}$$

where $(-\Delta)^\alpha$ denotes the α-th power of the Dirichlet laplacian. This problem was addressed in [146] where it was proved that:

- The system is null controllable for all $\alpha > 1/2$. This result is not hard to guess from the iterative construction above. The range $\alpha > 1/2$ is that in which the dissipative effect dominates and compensates the increasing cost of controlling the low frequencies, as its range increases.

- The system is not null controllable when $\alpha \leq 1/2$. In particular, null controllability fails in the critical case $\alpha = 1/2$. This is due to the following result on the lack of linear independence of the sums of real exponentials that was previously proved in [145]:

  Proposition 4.1 *Assume that $\alpha \leq 1/2$. Then there is no sequence $\{\rho_n\}_{n\geq 1}$ of positive weights, i. e. $\rho_n > 0$ for all $n \geq 1$, such that*

  $$\sum_{n\geq 1} \rho_n |a_n|^2 \leq \int_0^T \left| \sum_{n\geq 1} a_n e^{-n^{2\alpha} t} \right|^2 dt \tag{4.21}$$

  for all sequence of real numbers $\{a_n\}$.

The inequality (4.21) is the one that is required to obtain an observability inequality of the form (4.16) for the solutions of the adjoint fractional parabolic equation

$$-\varphi_t + (-\Delta)^\alpha \varphi = 0, \tag{4.22}$$

in one space dimension.

Note that inequalities of the form (4.21) are well-known to hold when $\alpha > 1/2$ (see [60]).

The fact that the inequality does not hold, whatever the weights $\{\rho_n\}_{n\geq 1}$ are, indicates that not only (4.16) does not hold but that any weakened version of it fails as well. The lack of controllability of the system for $\alpha \leq 1/2$ has then some catastrophic nature in the sense that it can not be compensated by restricting the class of initial data under consideration.

The fact that inequalities (4.21) fail to hold was proved in [145] in the context of the control of the heat equation in unbounded domains. There it was proved that, despite of the fact that the model has infinite speed of propagation, there is no compactly supported smooth initial data that can be controlled to zero by means of L^2-controls localized in a bounded set. This result was later interpreted (and extended in a significant way) in [53] as a backward unique continuation one, in the absence of boundary conditions in the complement of a bounded set. The proof in [145] was based on the fact that, when writing the heat equation in conical domains with similarity variables, the underlying elliptic operator turns out to have a discrete spectrum and the eigenvalues grow in a linear way. This corresponds precisely to the critical case $\alpha = 1/2$ in model (4.22) in one space dimension in which, according to Proposition 4.1, controllability fails to hold.

4.5 Carleman inequalities for heat equations with potentials

The null controllability of the heat equation with lower order time-dependent terms of the form

$$(4.23)\qquad \begin{cases} u_t - \Delta u + a(x,t)u = f1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega \end{cases}$$

has been proved for the first time in a series of works by Fursikov and Imanuvilov (see for instance [94], [28], [74], [76] and [98] and the references therein). Their approach, is based on a direct application of Carleman inequalities to the adjoint system

$$(4.24)\qquad \begin{cases} -\varphi_t - \Delta\varphi + a(x,t)\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases}$$

More precisely, observability inequalities of the form (4.16) are directly derived for system (4.24), for all $T > 0$ and any open non-empty subset ω of Ω. We shall describe this approach in the following section.

This method has been extremely successful when dealing with observability inequalities for parabolic problems as the reader may see from the articles in the list of references in the end of this paper devoted to that issue. The method being very flexible, the same ideas have been applied in a variety of problems, including Navier-Stokes equations, and also heat equations with variable coefficients in the principal part. Carleman inequalities require these coefficients to be, roughly, Lipschitz continuous (or a suitable L^p bound on its derivatives). But, as far as we know, there is no result in the literature showing the lack of null controllability of the heat equation with bounded measurable coefficients. This is an interesting and possibly difficult open problem that we shall discuss later on in more detail.

4.6 Bang-bang controls

A slight change on the functional J_ε introduced in (4.7) to prove finite-approximate controllability allows building *bang-bang* controls. Indeed, we set

$$(4.25)\qquad J_{bb,\varepsilon}(\varphi^0) = \frac{1}{2}\left(\int_0^T \int_\omega |\, \varphi \,|\, dxdt\right)^2 + \varepsilon \parallel (I - \pi_E)\varphi^0 \parallel_{L^2(\Omega)} - \int_\Omega u^1 \varphi^0 dx.$$

The functional $J_{bb,\varepsilon}$ is continuous and convex in $L^2(\Omega)$ and satisfies the coercivity property (4.9) too.

Let $\widehat{\varphi}^0$ be the minimizer of $J_{bb,\varepsilon}$ in $L^2(\Omega)$ and $\widehat{\varphi}$ the corresponding solution of (4.5). We set

$$(4.26)\qquad f = \int_0^T \int_\omega |\, \widehat{\varphi} \,|\, dxdt\, \operatorname{sgn}(\widehat{\varphi})$$

where sgn is the multivalued sign function: $\operatorname{sgn}(s) = 1$ if $s > 0$, $\operatorname{sgn}(0) = [-1, 1]$ and $\operatorname{sgn}(s) = -1$ when $s < 0$. The control f given in (4.26) is such that the solution u of (4.1) with null initial data satisfies (4.10).

Due to the regularizing effect of the heat equation, the solution $\hat{\varphi}$ is analytic and its zero set is of zero $(n+1)-$dimensional Lebesgue measure. Thus, the control f in (4.26) is of *bang-bang* form, i.e. $f = \pm\lambda$ a. e. in $\omega \times (0,T)$ where $\lambda = \int_0^T \int_\omega \mid \widehat{\varphi} \mid dxdt$.

We have proved the following result:

Theorem 4.2 ([203]) *Let ω be any open non-empty subset of Ω and $T > 0$ be any positive control time. Then, for any $u^0, u^1 \in L^2(\Omega)$, $\varepsilon > 0$ and finite-dimensional subspace E of $L^2(\Omega)$ there exists a bang-bang control $f \in L^\infty(Q)$ such that the solution u of* (4.1) *satisfies* (4.10). *The control that the variational approach provides is of minimal L^∞-norm among the class of admissible ones.*

The fact that the control obtained when minimizing $J_{bb,\varepsilon}$ is of minimal L^∞-norm was proved in [56] by using a classical duality principle (see [51]). In [56] we also considered linear equations with potentials and semilinear ones. In those cases the fact that the control obtained this way is bang-bang is less clear since one can not use the analyticity of solutions to directly obtain that the zero-set of solutions of the adjoint system is of null Lebesgue measure. In that case one rather needs to use more sophisticated results in that direction as those in [2]. In general, the method described above always yields quasi bang-bang controls, as for the wave equation, even if one does not have results about the null measure of the zero level set of solutions of the corresponding adjoint system. Whether controls are actually of bang-bang form is a widely open problem. Obviously, it can be viewed as a problem of unique continuation in the sense that it can be reformulated as follows: *Does the fact that the measure of the zero set $\{(x,t) : \varphi(x,t) = 0\}$ be positive, imply that $\varphi \equiv 0$?*

Bang-bang controls also exist in the context of null controllability. This is specific to the heat equation and, as we shall see, its very strong dissipative effect plays a key role on the proof of this result.

To address the problem of null controllability we have to take $\varepsilon = 0$ and consider the functional

$$J_{bb}(\varphi^0) = \frac{1}{2}\left(\int_0^T \int_\omega \mid \varphi \mid dxdt\right)^2 - \int_\Omega u^1 \varphi^0 dx. \tag{4.27}$$

The functional J_{bb} is well defined and continuous in the Banach space X constituted by the solutions of the adjoint heat equation (4.5) such that $\varphi \in L^1(\omega \times (0,T))$. The space X is endowed with its canonical norm, namely, $||\varphi||_X = \int_0^T \int_\omega \mid \varphi \mid dxdt$. Note that the coercivity of J_{bb} in X is not obvious at all. In fact, for that to be true one has to show an observability inequality of the form

$$||\varphi(0)||_{L^2(\Omega)} \leq C \int_0^T \int_\omega |\varphi| dxdt. \tag{4.28}$$

Note that we have not mentioned this inequality so far. It is an observability inequality in which the localized observation is made in the L^1-norm instead of the L^2-norm considered so far. This estimate was proved in Proposition 3.2 of [69] in order to get bounded controls. In fact it was proved not only for the heat equation (4.5) but also for equations with zero order potentials with explicit estimates on the observability constant in terms of the potential.

According to the observability estimate (4.28) the functional is also coercive.

However, because of the lack of reflexivity of the space X, the existence of the minimizer of J_{bb} in X may not be guaranteed. Indeed, the minimizer can only be guaranteed to exist in the closure of X with respect to the weak convergence in the sense of measures. We denote that space by $\tilde{X}$. More precisely, $\tilde{X}$ is the space of solutions of the adjoint heat equation whose restriction to $\omega \times (0,T)$ is a bounded measure. We denote by $\hat{\varphi}$ the minimizer of J_{bb} in $\tilde{X}$.

We claim that the minimizer is in fact smooth. This is due to the regularizing effect and to an improved observability estimate. Indeed, in view of the results in [69] we can improve (4.28) to obtain

$$\int_0^{T-\tau}\int_\Omega |\varphi|^2 dxdt \leq C_\tau \Big[\int_0^T\int_\omega |\varphi| dxdt\Big]^2. \tag{4.29}$$

This estimate can be extended by density to $\tilde{X}$. In that case the L^1-norm on $\omega \times (0,T)$ has to be replaced by the total measure of the solution on that set.

Estimate (4.29) shows, in particular, that the minimizer $\hat{\varphi}$ is such that

$$\hat{\varphi} \in L^2(\Omega \times (0, T-\tau)),\ \forall \tau > 0.$$

Thus, the minimizer is a smooth solution of the heat equation except possibly at $t = T$.

Accordingly, the Euler-Lagrange equations associated to the minimizer show that

$$f = \int_0^T \int_\omega |\hat{\varphi}| dxdt \,\mathrm{sgn}\, \hat{\varphi}$$

is a null control for the heat equation. This control is of quasi-bang-bang form. In fact it is strictly bang-bang since, because of the analyticity of solutions of the heat equation, its zero set is of null Lebesgue measure.

4.7 Discussion and comparison

In these sections we have presented the main controllability properties under consideration and some of the fundamental results both for the wave and the heat equation. There are some clear differences in what concerns the way each of these equations behaves:

- For the wave equation exact and null controllability are equivalent notions. However, for the heat equation, the exact controllability property may not hold and the null controllability property is the most natural one to address. Null controllability is in fact equivalent to an apparently stronger property of control to trajectories, which guarantees that every state which is the value at the final time of a solution of the uncontrolled equation is reachable from any initial datum by means of a suitable control.
- The exact controllability property for the wave equation holds provided the Geometric Control Condition is satisfied. This imposes severe restrictions on the subset where the control acts and requires the control time to be large enough. At the contrary, the null controllability property for the heat equation holds in an arbitrarily small time and with controls in any open non-empty subset of the domain.

In view of this, we can say that, although exact controllability may not hold for the heat equation, at the level of null controllability it behaves much better than the wave equation since no geometric requirements are needed for it to hold in an arbitrarily small time.

We have described above a number of methods allowing to prove that the heat equation is null controllable for all $T > 0$ whenever the corresponding wave equation is controllable for some time $T > 0$. This result is not optimal when applied to the constant coefficient heat equation (in particular since geometric restrictions are needed on the subset where the control applies) but it has the advantage of yielding results in situations in which the parabolic methods described are hard to apply directly. This procedure has been recently used in a number of situations:

- In [70] the problem of the null controllability of the $1 - d$ heat equation with variable coefficients in the principal part is addressed. Using this procedure, null controllability is proved in the class of BV coefficients. This is, as far as we know, the best result in this direction.

  However, the Global Carleman inequalities do not seem to yield this result, since globally Lipschitz coefficients are required. In [46] parabolic equations with piecewise constant coefficients were addressed by means of Global Carleman inequalities in the multi-dimensional case. There observability and null controllability was proved but only under suitable monotonicity conditions on the coefficients along interfaces.

  This is due to the fact that, when applying Carleman inequalities to heat equations with discontinuous coefficients, integration by parts generates some singular terms on the interfaces. These terms cannot be absorbed as lower order ones. Therefore, a sign condition has to be imposed. This type of monotonicity condition is natural in the context of wave equations where it is known that interfaces may produce trapped waves ([140]). But as far as we know, there is no evidence for the need of this kind of monotonicity condition for parabolic equations. This is an interesting and probably deep open problem. We shall return to this matter below.

  Of course, in $1 - d$ this monotonicity conditions are unnecessary, as the results in [70] for BV coefficients show. But the situation is unclear in the multidimensional case. In any case, even in $1 - d$, Carleman inequalities seem to need monotonicity conditions to work.

- Recently an unified approach for Carleman inequalities for parabolic and hyperbolic equations has been presented in [71]. There a pointwise weighted identity is derived for general second order operators. In this way the author is able to recover the existing Carleman inequalities for heat, wave, plate and Schrödinger equations. A viscous Schrödinger equation in between the heat and the Schrödinger equations is also addressed.

- In [40] the heat equation in a planar $1 - d$ network is addressed. A number of null controllability results are proved by means of this procedure, as a consequence of the previously proved ones for the corresponding wave equation in the same network. In the context of networks, the wave equation is easier to deal with since one may use propagation arguments, sidewise energy estimates, D'Alembert's formula, ... So far the null controllability of parabolic equations on networks has not been addressed directly by means of Carleman inequalities. The difficulty for doing that is the treatment of the nodes of the network where various segments are interconnected. There, as in the context of parabolic equations with discontinuous coefficients, it is hard to match the Carleman inequality along each segment and to deduce a global observability estimate. On the other hand, as the spectral analysis shows, depending of the structure of the network and the mutual lengths of the segments entering on it, there may exist concentrated eigenfunctions making observability impossible. Thus, the difficulty is not merely technical. The understanding of this issue by means of Carleman inequalities is an interesting open subject.

As we mentioned above, more recently similar results have been obtained using Kannai's tranform ([154] and [149]). Both approaches are limited to the case where the coefficients of the equations are independent of time. As far as we know, there is no systematic method to transfer control results for wave equations into control results for heat equations with potentials depending on x and t.

We have also observed important differences in what concerns bang-bang controls. Bang-bang controls exist for the heat equation both in the context of approximate and exact controllability. However, for the wave equation, only quasi-bang-bang controls can be found and this in the framework of approximate controllability. The same analysis fails for the exact controllability of the wave equation.

5 Sharp observability estimates for the linear heat equation

5.1 Sharp estimates

In the previous section we have mentioned that the heat equation (4.23) with lower order potentials depending both on x and t is null controllable. This is equivalent to an observability inequality for the adjoint heat equation (4.24). The only existing method that allows dealing with equations of this form are the so called Global Carleman Inequalities. They were introduced in this context by Imanuvilov (see [94], and the books [74], [73]) and have allowed to solve a significant amount of complex control problems for parabolic equations, including the Navier-Stokes equations (see [35], [77], [65], for instance). In this section we present the inequality in the form it was derived in [68] following the method in [74]. The observability constant depends on the norm of the potential in an, apparently, unexpected manner. But, as we shall see in the following section, according to the recent results in [50], the estimate turns out to be sharp.

The following holds:

Proposition 5.1 ([74], [68]) *There exists a constant $C > 0$ that only depends on Ω and ω such that the following inequality holds*

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq \exp\left[C \left(1 + \frac{1}{T} + T \| a \|_\infty + \| a \|_\infty^{2/3} \right) \right] \int_0^T \int_\omega \varphi^2 dxdt \tag{5.1}$$

for any φ solution of (4.24), for any $T > 0$ and any potential $a \in L^\infty(Q)$.

Furthermore, the following global estimate holds

$$\int_0^T \int_\Omega e^{-\frac{A(1+T)}{T-t}} \varphi^2 dxdt \leq \exp\left[C\left(1+\frac{1}{T}+\parallel a\parallel_\infty^{2/3}\right)\right]\int_0^T \int_\omega \varphi^2 dxdt \tag{5.2}$$

with a constant A that only depends on the domains Ω and ω as well.

Remark 5.1 *Several remarks are in order:*

- *Note that (5.1) provides the observability inequality for the adjoint heat equation (4.24) with an explicit estimate of the observability constant, depending on the control time T and the potential a. The observability inequality (5.2) differs from that in (5.1), on the fact that it provides a global estimate on the solution in $\Omega \times (0,T)$, but with a weight function that degenerates exponentially at $t = T$. In fact, using Carleman inequalities one first derives (5.2) to later obtain the pointwise (in time $t = 0$) estimate out of it. When doing that one needs to apply Gronwall's inequality for the time evolution of the $L^2(\Omega)$ norm of the solution. This yields the extra term $e^{T||a||_\infty}$ in the observability constant.*
- *Inequality (5.1) plays an important role when dealing, for instance, with the null control of nonlinear problems. Using this explicit observability estimate, and, in particular, the fact that it depends exponentially on the power $2/3$ of the potential a, in [69] the null controllability was proved for a class of semilinear heat equations with nonlinearities growing at infinity slower than $s\log^{3/2}(s)$. This is a surprising result since, in this range of nonlinearities, in the absence of control, solutions may blow-up in finite time. The presence of the control avoids blow-up to occur and makes the solution reach the equilibrium at time $t = T$.*

The estimates in Proposition 5.1 are a direct consequence of the Carleman estimates that we briefly describe now.

We introduce a function $\eta^0 = \eta^0(x)$ such that:

$$\begin{cases} \eta^0 \in C^2(\bar{\Omega}) & \\ \eta^0 > 0 & \text{in} \quad \Omega,\ \eta^0 = 0 \quad \text{in} \quad \partial\Omega \\ \nabla\eta^0 \neq 0 & \text{in} \quad \overline{\Omega\backslash\omega}. \end{cases} \tag{5.3}$$

The existence of this function was proved in [74]. In some particular cases, for instance when Ω is star-shaped with respect to a point in ω, it can be built explicitly without difficulty. But the existence of this function is less obvious in general, when the domain has holes or its boundary oscillates, for instance.

Let $k > 0$ such that

$$k \geq 5 \max_{\bar{\Omega}} \eta^0 - 6 \min_{\bar{\Omega}} \eta^0$$

and let

$$\beta^0 = \eta^0 + k, \bar{\beta} = \frac{5}{4}\max \beta^0,\ \rho^1(x) = e^{\lambda\bar{\beta}} - e^{\lambda\beta^0}$$

with λ sufficiently large. Let be finally

$$\gamma = \rho^1(x)/(t(T-t)); \rho(x,t) = \exp(\gamma(x,t))$$

and the space of functions

$$Z = \left\{ q : Q \to \mathbf{R} : q \in C^2(\bar{Q}),\, q = 0 \text{ in } \Sigma \right\}.$$

The following Carleman inequality holds:

Proposition 5.2 ([74]) *There exist positive constants $C_*, s_1 > 0$ such that*

$$\begin{aligned} &\frac{1}{s}\int_Q \rho^{-2s} t(T-t)\left[|q_t|^2 + |\Delta q|^2\right] dxdt \\ &+ s\int_Q \rho^{-2s} t^{-1}(T-t)^{-1} |\nabla q|^2\, dxdt + s^3 \int_Q \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt \\ &\le C_* \left[\int_Q \rho^{-2s} |\partial_t q + \Delta q|^2\, dxdt + s^3 \int_0^T \int_\omega \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt \right] \end{aligned} \tag{5.4}$$

for all $q \in Z$ and $s \ge s_1$.

Moreover, C_* depends only on Ω and ω and s_1 is of the form

$$s_1 = s_0(\Omega, \omega)(T + T^2),$$

where $s_0(\Omega, \omega)$ only depends on Ω and ω.

We refer to the Appendix in [68] for a proof of this Carleman inequality.

Applying (5.4) with $q = \varphi$ we deduce easily (5.2) , taking into account that the first term on the right hand side of (5.4) coincides with $a\varphi$ when φ is a solution of (4.24). In order to absorb this term we make use of the third term on the left hand side of (5.4). This imposes the choice of the parameter s as being of the order of $||a||_\infty^{2/3}$ and yields that factor on the exponential observability constants in (5.1) and (5.2).

As observed in ([68]), when $a \equiv 0$ in the adjoint heat equation (4.24) or, more generally, when the potential is independent of t, these estimates can be written in terms of the Fourier coefficients $\{a_k\}$ of the datum of the solution of the adjoint system at $t = T$:

$$\varphi^0(x) = \sum_{k \ge 1} a_k \psi_k(x).$$

The following holds:

Theorem 5.1 ([68]) *Let $T > 0$ and ω be an open non-empty subset of Ω. Then, there exist $C, c > 0$ such that*

$$\sum_{k=1}^{\infty} |a_k|^2 e^{-c\sqrt{\lambda_k}} \le C \int_0^T \int_\omega \varphi^2 dxdt \tag{5.5}$$

for all solution of (4.5), where $\{\psi_k\}$ denotes the orthonormal basis of $L^2(\Omega)$ constituted by the eigenfunctions of the Dirichlet Laplacian, $\{\lambda_k\}$ the sequence of corresponding eigenvalues and $\{a_k\}$ the Fourier coefficients of φ^0 on this basis.

Remark 5.2 *Note that the left hand side of (5.5) defines a norm of φ^0 that corresponds to the one in the domain of the operator* $\exp\left(-c\sqrt{-\Delta}\right)$. *Characterizing the best constant c in (5.5) in terms of the geometric properties of the domains Ω and ω is an open problem. Obviously, the constant may also depend on the length of the time interval T. The problem may be made independent of T by considering the analogue in infinite time:*

$$\sum_{k=1}^{\infty} |a_k|^2 e^{-c\sqrt{\lambda_k}} \leq C \int_{-\infty}^{0} \int_{\omega} \varphi^2 dx dt, \tag{5.6}$$

φ being now the solution of the adjoint system for $t \leq 0$.

As far as we know the characterization of the best constant $c > 0$ in (5.6) is an open problem. This problem is intimately related to the characterization of the best constant $A > 0$ in (5.2) for $a \equiv 0$, which is also an open problem.

By inspection of the proof of the inequality (5.2), as a consequence of the Carleman inequality (5.4), one sees that A depends on the geometric properties of the weight η^0 in (5.3). But how this is translated into the properties of the domains Ω and ω is to be investigated. Some lower bounds on A have been obtained in terms of the Gaussian heat kernels in [68] and [150]. But further investigation is needed for a complete characterization of the sharp value of A as well.

Observe also that the observability inequality (5.2) is stronger than (5.1). Indeed, (5.2) provides a global estimate on φ away from $t = 0$ and yields, in particular, (5.6) with weights $e^{-c\sqrt{\lambda_k}}$. Inequality (5.1) is much weaker since it provides only estimates on $\varphi(0)$ and therefore involves weights of the form $e^{-\lambda_k T}$ in its Fourier representation.

Remark 5.3 *In [68] these estimates were used to obtain sharp estimates on the cost of approximate and finite-approximate controllability, i. e. on the size of the control f_ε needed to reach (4.10). As we mentioned above, roughly speaking, when the final datum is not reachable, for instance when u^1 is the characteristic function of some measurable subset of Ω, the cost of controlling to an ε distance grows exponentially as ε tends to zero.*

In the same article the connections between optimal control and approximate control were also explored and quantified. It is well known that the approximate controllability property can be achieved as a limit of optimal control problems with a penalization parameter k tending to ∞ that enhances the requirement of getting close to the target. More precisely, when looking for the optimal control $f \in L^2(\omega \times (0,T)$ that minimizes the functional

$$I_k(f) = \frac{1}{2} \int_0^T \int_\omega f^2 dx dt + \frac{k}{2} ||u(T) - u||^2_{L^2(\Omega)}, \tag{5.7}$$

the minimizer f_k is a control such that the corresponding solution u_k satisfies $u_k(T) \to u^1$ as k tends to ∞ in $L^2(\Omega)$. In [68] a logarithmic convergence rate was proved for this procedure.

Remark 5.4 *As we mentioned above, the heat equation, despite the infinite speed of propagation behaves quite differently in unbounded domains. In [145] it was proved that, even if approximate controllability holds, null controllability does not hold for the heat equation in the whole line when the control acts in a bounded subdomain. But, approximate controllability holds, and can be even extended to semilinear equations ([183], [185]). Null controllability may be achieved when the support of the control is such that its complement is a bounded set. In that case the situation is* *fairly similar to the case where the equation holds in a bounded domain [21]. Similar results hold also, roughly, when the uncontrolled domain is unbounded but the distance to the controlled region is uniformly bounded ([22], [151]).*

Remark 5.5 *In this section we have discussed heat equations with zero order bounded potentials. Similar estimates, with different exponents, can be obtained when the potential belongs to L^p, for p large enough (see [50]). But we could also consider equations with convective terms. For instance,*

$$
\left\{
\begin{array}{lll}
-\varphi_t - \Delta\varphi + \mathrm{div}(W(x,t)\varphi) + a(x,t)\varphi = 0 & in & Q \\
\varphi = 0 & on & \Sigma \\
\varphi(x,T) = \varphi^0(x) & in & \Omega,
\end{array}
\right.
\tag{5.8}
$$

where $W = W(x,t)$ is a bounded convective potential (see [98]). The observability inequalities are true in this more general case, the observability constant being affected by an added term of the form $exp(C||W||^2)$. This allows addressing control problems for semilinear heat equations depending on the gradient. In this case the growth of the nonlinearity on ∇u has to be asymptotically smaller than $s\log^{1/2}(s)$ (see [48], [3]).

5.2 Optimality

The observability constant in (5.1) includes three different terms. More precisely:

$$
\exp\left(C\left(1+\frac{1}{T}+T \parallel a \parallel_\infty + \parallel a \parallel_\infty^{2/3}\right)\right) = C_1^*(T,\, a)C_2^*(T,\, a)C_3^*(T,\, a),
\tag{5.9}
$$

where

$$
C_1^*(T,\, a) = \exp\left(C\left(1+\frac{1}{T}\right)\right), \quad C_2^*(T,\, a) = \exp(CT \parallel a \parallel_\infty), \quad C_3^*(T,\, a) = \exp\left(C \parallel a \parallel_\infty^{2/3}\right).
\tag{5.10}
$$

The role that each of these constants plays in the observability inequality is of different nature. It is roughly as follows:

- When $a \equiv 0$, i.e. in the absence of potential, the observability constant is simply $C_1^*(T,\, a)$. This constant blows-up exponentially as $T \downarrow 0$. This growth rate is easily seen to be optimal by inspection of the heat kernel and has been analyzed in more detail in [68] and [150], in terms of the geometry of Ω and ω. We refer also to [176] for a discussion of the optimal growth rate in one space dimension.

- The second constant $C_2^*(T,\, a)$ is very natural as it arises when applying Gronwall's inequality to analyze the time evolution of the L^2-norm of solutions. More precisely, it arises when getting (5.1) out of (5.2).

- The constant $C_3^*(T,\, a)$, which, actually, only depends on the potential a, is the most intriguing one. Indeed, the 2/3 exponent does not seem to arise naturally in the context of the heat equation since, taking into account that the heat operator is of order one and two in the time and space variables respectively, one could rather expect terms of the form $\exp(c \parallel a \parallel_\infty)$ and $\exp\left(c \parallel a \parallel_\infty^{1/2}\right)$, as a simple ODE argument would indicate.

In the recent paper [50] we show that, surprisingly, to some extent, the optimality of this last contribution $C_3^*(T,\,a)$ to the observability constant. This happens for systems of two heat equations, in even space dimension and in the range of values of time T:

$$\| a \|_\infty^{-2/3} \lesssim T \lesssim \| a \|_\infty^{-1/3} . \tag{5.11}$$

Here and in the remainder of this section we refer to systems of heat equations of the form:

$$\begin{cases} -\varphi_t - \Delta\varphi + A(x,t)\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega, \end{cases} \tag{5.12}$$

where $\varphi = \varphi(x,t)$ is a vector valued function with N components and the potential $A = A(x,t)$ is matrix valued, with bounded measurable coefficients. System (5.12) is weakly coupled since it is diagonal in the principal part and it is only coupled through the zero order term.

The Carleman inequality (5.4) yields for these systems observability inequalities of the form (5.1), following exactly the same method as for scalar equations. This is so because, as we said, the system is uncoupled in the principal part. In fact, as mentioned above, the Carleman inequality refers to the principal part of the operator, which in the present case is the heat operator componentwise, and the lower order term is simply treated as a perturbation.

There is still a lot to be understood for these problems: scalar equations, one space dimension, other time intervals, etc. But the interest of this first optimality result is to confirm the need of the unexcepted term $C_3^*(T,\,a)$ in the observability inequality (5.1). This is also relevant in view of applications to non-linear problems, since it is this constant that determines the maximal growth of the nonlinearity for which null-controllability is known to hold in an uniform time: $s\log^{3/2}(s)$, [69].

The following holds:

Theorem 5.2 *([50]) Assume that the space dimension $n \geq 2$ is even and that the number of equations of the parabolic system is $N \geq 2$. Let Ω be a bounded domain of $\mathbb{R}^n$ and ω a non-empty open subset of Ω. Then there exists $c > 0$, $\mu > 0$, a family $(A_R)_{R>0}$ of matrix-valued potentials such that*

$$\| A_R \| \underset{R\to+\infty}{\longrightarrow} +\infty,$$

and a family $(\varphi_R^0)_{R>0}$ of initial conditions in $(L^2(\Omega))^N$ so that the corresponding solutions φ_R of (5.12) with $A = A_R$ satisfies

$$\lim_{R\to\infty} \left\{ \inf_{T\in I_\mu} \frac{\| \varphi_R(0) \|^2_{(L^2(\Omega))^N}}{\exp(c \| A_R \|_\infty^{2/3}) \int_0^T \int_\omega |\varphi_R|^2 dxdt} \right\} = +\infty. \tag{5.13}$$

where $I_\mu \triangleq \left(0, \mu \| A_R \|^{-1/3}\right]$.

Let us briefly sketch its proof. We refer to [50] for more details and other results and open problems related with this issue. In particular, in [50] the wave equation with lower order terms is also considered and sharp observability inequalities are proved.

Theorem 5.2 is a consequence of the following known result:

Theorem 5.3 (Meshkov [141]). *Assume that the space dimension is $n = 2$. Then, there exists a nonzero complex-valued bounded potential $q = q(x)$ and a non-trivial complex valued solution $u = u(x)$ of*

$$\Delta u = q(x)u, \qquad \text{in } \mathbb{R}^2, \tag{5.14}$$

with the property that

$$| u(x) | \leq C \exp(- | x |^{4/3}), \qquad \forall\, x \in \mathbb{R}^2 \tag{5.15}$$

for some positive constant $C > 0$.

This construction by Meshkov provides a complex-valued bounded potential $q = q(x)$ in $\mathbb{R}^2$ and a non-trivial solution u of the elliptic equation (5.14) with the decay property $| u(x) | \leq \exp\left(-|x|^{4/3}\right)$. This decay estimate turns out to be sharp as proved by Meshkov by Carleman inequalities. In other words, if, given a bounded potential q, the solution of (5.14) decays faster than $\exp\left(-C|x|^{4/3}\right)$, for all $C > 0$, then, necessarily, this solution is the trivial one. Meshkov's construction may be generalized to any even dimension by separation of variables. We refer to [50] for a similar construction in odd dimension with a slightly weaker decay rate (essentially the same exponential decay up to a multiplicative logarithmic factor).

Theorem 5.2 holds from the construction by Meshkov by scaling and localization arguments. To simplify the presentation we focus in the case of two space dimensions $n = 2$ and of systems with two components $N = 2$ in which case Meshkov's result can be applied in a more straightforward way.

Its proof is divided into several steps.

Step 1: Construction on $\mathbb{R}^n$.

Consider the solution u and potential q given by Theorem 5.3. By setting

$$u_R(x) = u(Rx), \qquad A_R(x) = R^2 q(Rx), \tag{5.16}$$

we obtain a one-parameter family of potentials $\{A_R\}_{R>0}$ and solutions $\{u_R\}_{R>0}$ satisfying

$$\Delta u_R = A_R(x) u_R, \qquad \text{in } \mathbb{R}^n \tag{5.17}$$

and

$$| u_R(x) | \leq C \exp\left(-R^{4/3} \mid x \mid^{4/3}\right), \qquad \text{in } \mathbb{R}^n. \tag{5.18}$$

These functions may also be viewed as stationary solutions of the corresponding parabolic systems. Indeed, $\psi_R(t, x) = u_R(x), \quad x \in \mathbb{R}^n,\ t > 0$ satisfies

$$\psi_{R,t} - \Delta\psi_R + A_R \psi_R = 0, \qquad x \in \mathbb{R}^n,\ t > 0 \tag{5.19}$$

and

$$| \psi_R(x, t) | \leq C \exp(-R^{4/3} \mid x \mid^{4/3}), \qquad x \in \mathbb{R}^n,\ t > 0. \tag{5.20}$$

Step 2: Restriction to Ω.

Let us now consider the case of a bounded domain Ω and ω to be a non-empty open subset Ω such that $\omega \neq \Omega$. Without loss of generality (by translation and scaling) we can assume that $B \subset \Omega \backslash \bar{\omega}$.

We can then view the functions $\{\psi_R\}_{R>0}$ above as a family of solutions of the Dirichlet problem in Ω with non-homogeneous Dirichlet boundary conditions:

$$\begin{cases} \psi_{R,t} - \Delta \psi_R + A_R \psi_R = 0, & \text{in } Q, \\ \psi_R = \varepsilon_R, & \text{on } \Sigma, \end{cases} \tag{5.21}$$

where $\varepsilon_R = \psi_R\Big|_{\partial\Omega} = u_R\Big|_{\partial\Omega}$.

Taking into account that both ω and $\partial\Omega$ are contained in the complement of B, we deduce that, for a suitable C:

$$| \psi_R(t,x) | \leq C \exp\left(-R^{4/3}\right), \qquad x \in \omega,\ 0 < t < T, \tag{5.22}$$

$$| \varepsilon_R(t,x) | \leq C \exp\left(-R^{4/3}\right), \qquad x \in \partial\Omega,\ 0 < t < T; \tag{5.23}$$

$$\| \psi_R(T) \|^2_{L^2(\Omega)} \sim \| \psi_R(T) \|^2_{L^2(\mathbb{R}^n)} = \| u_R \|^2_{L^2(\mathbb{R}^n)} = \frac{1}{R^n} \| u \|^2_{L^2(\mathbb{R}^n)} = \frac{c}{R^n} \tag{5.24}$$

$$\| A_R \|_{L^\infty(\Omega)} \| A_R \|_{L^\infty(\mathbb{R}^n)} = C R^2. \tag{5.25}$$

We can then correct these solutions to fulfill the Dirichlet homogeneous boundary condition. For this purpose, we introduce the correcting terms

$$\begin{cases} \rho_{R,t} - \Delta \rho_R + a_R \rho_R = 0, & \text{in } Q, \\ \rho_R = \varepsilon_R, & \text{on } \Sigma, \\ \rho_R(0,x) = 0, & \text{in } \Omega, \end{cases} \tag{5.26}$$

and then set

$$\varphi_R = \psi_R - \rho_R. \tag{5.27}$$

Clearly $\{\varphi_R\}_{R>0}$ is a family of solutions of parabolic systems of the form (5.12) with potentials $A_R(x) = R^2 q(Rx)$.

The exponential smallness of the Dirichlet data ε_R shows that ρ_R is exponentially small too. This allows showing that φ_R satisfies essentially the same properties as ψ_R in (5.22)-(5.25). Thus, the family φ_R suffices to show that the statement in Theorem 5.2 holds.

6 Parabolic equations with low regularity coefficients

In this section we briefly discuss the problem of controllability for parabolic equations with low regularity coefficients in the principal part.

The same issue is relevant for wave equations too. In that case, according to the results in [26], we know that observability inequalities and exact controllability properties may fail for wave equations with Hölder continuous coefficients even in one space dimension. In $1-d$ we also know that exact controllability holds with BV-coefficients ([39], [70]). The picture is not complete in the multi-dimensional case, in which the various existing methods require different regularity properties.

The method of multipliers requires coefficients to be C^1 or Lipschitz continuous because, when integrating by parts, one is forced to take one derivative of the coefficients in the principal part of the operator. Roughly speaking, the same happens for the Carleman inequality approach (although the Lipschitz condition can be replaced by a suitable L^p estimate on the first order derivatives). Obviously, in both cases, other structural assumptions are needed on the coefficients (not only regularity) to guarantee that the observability inequality holds. The microlocal approach requires more regular coefficients. Indeed, $C^{1,1}$ coefficients are needed in order to prove existence, uniqueness and stability of bicharacteristic rays and, as far as we know, this is the only context in which the GCC is known to suffice. In fact, the extension of the GCC for less regular coefficients has not been formulated since, as we said, when coefficients fail to be $C^{1,1}$, the Hamiltonian system determining the bicharacteristic rays is not necessarily well-posed.

Much less is known for parabolic operators. The Carleman inequality approach works for Lipschitz continuous coefficients. But we do not know whether this assumption is needed or not. Indeed, there is no counterexample in the literature justifying the need of regularity assumptions on the coefficients other than being merely bounded and measurable. A first result for piecewise constant coefficients by means of Carleman inequalities has been established in [46] but imposing monotonicity conditions on the interfaces. Indeed Carleman inequalities, as multipliers for wave equations, generate spurious terms on interfaces and, so far, the only way of getting rid of them is precisely imposing these monotonicity conditions on the interfaces to guarantee they have the good sign. For wave equations these conditions are known to be natural since they avoid trapped waves ([140]). But in the context of heat equations there is no evidence of the need of such conditions. In [70] it has been proved that BV-regularity of coefficients suffices in $1-d$. This shows that the monotonicity conditions are not always required. But the method in [70] is based on transmutation arguments which allow showing the null controllability of the heat equation as a consequence of the exact controllability of the corresponding wave model and it only applies to coefficients depending only on x. Consequently, the problem of getting observability inequalities for heat equations with non-smooth coefficients in widely open. Even in $1-d$ the problem is open for Hölder continuous coefficients depending only on x.

In this section we pursue a classical argument in the theory of PDE that consists in considering small L^∞ perturbations of a constant coefficient heat operator. The basic ingredient for doing that is the Carleman inequality (5.4) that not only yields estimates on the solution of the heat equation but also on the leading order terms. Before considering heat equations, in order to illustrate the methods, we discuss elliptic equations by means of the sharp Carleman inequalities proved in [98], [96], [97] and, more precisely, the problem of unique continuation of eigenfunctions.

6.1 Ellliptic equations

We consider the elliptic problem

$$\begin{cases} -\Delta y &= f + \sum_{j=1}^{n} \partial_j f_j, \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Let ω be an open non-empty subset of Ω and consider the weight function η^0 in (5.3). Set $\tilde{\rho}(x) = \exp(\exp(\lambda\eta^0(x))$, where η^0 is as in (5.3).The following Carleman estimate was proved in [98] (see also [96], [97] for an extension to non-homogeneous boundary conditions).

Theorem 6.1 *There exist positive constants $C > 0$, s_0 and λ_0, which only depend on Ω and ω, such that for all $s \geq s_0$ and $\lambda \geq \lambda_0$ the following inequality holds for every solution of (6.1):*

$$
\begin{aligned}
(6.2) \qquad & \int_\Omega \Big[\tilde{\rho}^{2s}|\nabla y|^2 + s^2\lambda^2 \exp(2\lambda\eta^0)\tilde{\rho}^{2s}|y|^2\Big]dx \\
&\leq \hat{C}\Big[\frac{1}{s\lambda^2}\int_\Omega \frac{\tilde{\rho}^{2s}}{\exp(\lambda\eta^0)}f^2 dx \\
&+s\int_\Omega \exp(\lambda\eta^0)\tilde{\rho}^{2s}\sum_{j=1}^n |f_j|^2 dx + \int_\omega \tilde{\rho}^{2s}\Big(|\nabla y|^2 + s^2\lambda^2\exp(2\lambda\eta^0)|y|^2\Big)dx\Big].
\end{aligned}
$$

In [96] and extension of this result has been proved including elliptic problems with variable smooth coefficients in the principal part and non-homogeneous Dirichlet data in $H^{1/2}(\partial\Omega)$. Here, for the sake of simplicity we restrict our attention to the case of homogeneous boundary conditions. Strictly speaking, by viewing the solutions of (6.1) as time-independent solutions of the corresponding parabolic problem, in the present case, inequality (6.2) is a consequence of the parabolic inequalities in [98].

Note that the estimate is sharp in what concerns the order of the different terms entering in it. Indeed, an estimate of the right-hand side term in $H^{-1}(\Omega)$ allows getting estimates on the solutions in $H^1(\Omega)$ in appropriate weighted norms. In fact, the proof of Theorem 6.1 requires of Carleman estimates and duality arguments to deal with right-hand side terms in H^{-1}.

Let us now consider an elliptic operator with bounded coefficients. To simplify the presentation we consider the case

$$
(6.3) \qquad \begin{cases} -\mathrm{div}((1+\varepsilon(x))\nabla y) &= f + \sum_{j=1}^n \partial_j f_j, \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega, \end{cases}
$$

where ε is assumed to belong to $L^\infty(\Omega)$ and small so that $||\varepsilon||_{L^\infty(\Omega)} < 1$. This guarantees the ellipticity of the operator.

In order to extend the Carleman inequality to this class of operators, it is natural to view the leading term as follows:

$$
-\mathrm{div}((1+\varepsilon(x))\nabla y) = -\Delta y - \mathrm{div}(\varepsilon(x)\nabla y).
$$

We can then rewrite (6.3) in the form (6.1) but with f_j replaced by $\tilde{f}_j = f_j + \varepsilon(x)\partial_j y$. When doing that and applying (6.2) we deduce that

$$
\begin{aligned}
(6.4) \quad & \int_\Omega \Big[\tilde{\rho}^{2s}|\nabla y|^2 + s^2\lambda^2\exp(2\lambda\eta^0)\tilde{\rho}^{2s}|y|^2\Big]dx \\
&\leq C\Big[\frac{1}{s\lambda^2}\int_\Omega \frac{\tilde{\rho}^{2s}}{\exp(\lambda\eta^0)}f^2 dx \\
&+s\int_\Omega \exp(\lambda\eta^0)\tilde{\rho}^{2s}\sum_{j=1}^n\Big(|f_j|^2 + ||\varepsilon||^2_{L^\infty(\Omega)}|\partial_j y|^2\Big)dx + \int_\omega \tilde{\rho}^{2s}\Big(|\nabla y|^2 + s^2\lambda^2\exp(2\lambda\eta^0)|y|^2\Big)dx\Big].
\end{aligned}
$$

Thus, with respect to (6.2) this adds the extra term

$$
\hat{C}s\int_\Omega \exp(\lambda\eta^0)\tilde{\rho}^{2s}||\varepsilon||^2_{L^\infty(\Omega)}|\nabla y|^2 dx.
$$

But this term can be absorbed by the left-hand side term in (6.4) provided

$$\hat{C} s \exp(\lambda ||\eta^0||_{L^\infty(\Omega)}) ||\varepsilon||^2_{L^\infty(\Omega)} < 1. \tag{6.5}$$

To be more precise, assume that ε satisfies the smallness condition

$$||\varepsilon||^2_{L^\infty(\Omega)} < \delta \hat{C} s_0^{-1} \exp(-\lambda_0 \max_{\Omega}(\eta^0)), \tag{6.6}$$

for some $\delta < 1$. Then (6.2) holds also for the solutions of (6.3), with a larger constant $C = \hat{C}/\delta > 0$, where $\hat{C}$ is the one in (6.2).

Accordingly, the following holds:

Theorem 6.2 *Assume that ε satisfies the smallness condition (6.6) with $\delta < 1$ and where $\hat{C} > 0$, s_0 and λ_0 are as in (6.2). Then (6.2) holds for the solutions of (6.3) for a larger observability constant $\hat{C}/\delta > 0$.*

Whether the smallness condition (6.6) is needed for (6.2) to hold or not is an open problem.

Note that, in particular, these Carleman inequalities may be used to prove unique continuation properties. Let us consider for instance the spectral problem:

$$\begin{cases} -\mathrm{div}((1+\varepsilon(x))\nabla y) &= \gamma^2 y, \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{6.7}$$

When applying the inequality to the solution of (6.7) we get

$$\begin{aligned} &\int_\Omega \left[\tilde{\rho}^{2s} |\nabla y|^2 + s^2 \lambda^2 \exp(2\lambda\eta^0) \tilde{\rho}^{2s} |y|^2 \right] dx \\ &\leq C \Big[\frac{\gamma^4}{s\lambda^2} \int_\Omega \frac{\tilde{\rho}^{2s}}{\exp(\lambda\eta^0)} y^2 dx \\ &+ s \int_\Omega \exp(\lambda\eta^0) \tilde{\rho}^{2s} \sum_{j=1}^n \Big(||\varepsilon||^2_{L^\infty(\Omega)} |\partial_j y|^2 \Big) dx + \int_\omega \tilde{\rho}^{2s} \Big(|\nabla y|^2 + s^2 \lambda^2 \exp(2\lambda\eta^0) |y|^2 \Big) dx \Big]. \end{aligned} \tag{6.8}$$

Absorbing the two remainder terms

$$\hat{C} \frac{\gamma^4}{s\lambda^2} \int_\Omega \frac{\tilde{\rho}^{2s}}{\exp(\lambda\eta^0)} y^2 dx$$

and

$$s\hat{C} \int_\Omega \exp(\lambda\eta^0) \tilde{\rho}^{2s} \sum_{j=1}^n \Big(||\varepsilon||^2_{L^\infty(\Omega)} |\partial_j y|^2 \Big) dx$$

on the right-hand side requires the smallness condition (6.6) together with the bound

$$\hat{C} \gamma^4 < s^3 \lambda^4 \exp(3\lambda \min_{\Omega} \eta^0) = s^3 \lambda^4. \tag{6.9}$$

This means that in order to cover larger and larger eigenfrequencies γ one is required to assume that the perturbation ε is smaller and smaller.

As we shall see this limitation appears also when dealing with evolution problems. This is natural indeed, since, when considering evolution problems all the spectrum of the underlying elliptic operator is involved simultaneously.

Once the smallness conditions (6.6) and (6.9) are imposed one guarantees the following observability inequality to hold:

$$\int_{\Omega}\tilde{\rho}^{2s}\Big[|\nabla y|^2+s^2\lambda^2\exp(2\lambda\eta^0)\tilde{\rho}^{2s}|y|^2\Big]dx\leq C\int_{\omega}\tilde{\rho}^{2s}\Big(|\nabla y|^2+s^2\lambda^2\exp(2\lambda\eta^0)|y|^2\Big)dx, \tag{6.10}$$

for some $C>0$. This implies, in particular, the property of unique continuation: *If* $y\equiv 0$ *in* ω*, then, necessarily,* $y\equiv 0$ *everywhere.*

At this respect it is important to note that, in what concerns unique continuation, in two space dimensions, this property is guaranteed for bounded measurable coefficients without further regularity assumptions ([11] and [12]). But the techniques of proof are specific to $2-d$. However these results do not provide quantitative estimates as those we obtained above.

The situation is totally different in higher dimensions. Indeed, for $n\geq 3$ it is well known that unique continuation fails, in general, for elliptic equations with measurable (and even Hölder continuous) coefficients (see [147] and, for elliptic equations in non-divergence form, [162]). Therefore it is natural that the methods we develop here, based on global Carleman inequalities, that do not distinguish the various space dimensions, require restrictions on the size of the bounded measurable perturbations of the coefficients allowed.

The situation is different for equations in which the density is perturbed. In this case the corresponding eigenvalue problem reads

$$\left\{\begin{array}{lcl}-\Delta y&=&\gamma^2(1+\varepsilon(x))y,\quad \text{in }\Omega,\\ y&=&0\quad\text{on }\partial\Omega.\end{array}\right. \tag{6.11}$$

In this case one gets (6.10) under a suitable smallness assumption on ε, but, contrarily to the elliptic problem (6.7), for all the spectrum simultaneously.

In the following subsection we apply the same ideas to a parabolic equation with bounded small perturbations in the leading density coefficient.

6.2 Parabolic equations

Let us now consider the following parabolic equation

$$\left\{\begin{array}{lll}-(1+\varepsilon(x,t))\varphi_t-\Delta\varphi=0&\text{in}&Q\\ \varphi=0&\text{on}&\Sigma\\ \varphi(x,T)=\varphi^0(x)&\text{in}&\Omega.\end{array}\right. \tag{6.12}$$

This is an adjoint heat equation with a variable density $\rho(x,t)=1+\varepsilon(x,t)$. We assume that $\varepsilon\in L^\infty(\Omega\times(0,T))$ satisfies the smallness condition $||\varepsilon||_{L^\infty(\Omega\times(0,T))}<1$ so that the system is parabolic and well-posed.

As in the elliptic case it is natural to decompose the parabolic operator into the heat one plus a small perturbation. The heat equation in (6.12) can then be written in the form

$$\Big[-\varphi_t-\Delta\varphi\Big]=\varepsilon(x,t)\varphi_t.$$

Applying the Carleman inequality (5.4) to the solution of (6.12) we get

$$\begin{aligned}(6.13)\qquad &\frac{1}{s}\int_Q \rho^{-2s} t(T-t)\left[|\varphi_t|^2+|\Delta\varphi|^2\right]dxdt\\ &+s\int_Q \rho^{-2s}t^{-1}(T-t)^{-1}|\nabla\varphi|^2 dxdt + s^3\int_Q \rho^{-2s}t^{-3}(T-t)^{-3}\varphi^2 dxdt\\ &\le C_*\left[\int_Q \rho^{-2s}|\varepsilon(x,t)\varphi_t|^2 dxdt + s^3\int_0^T\int_\omega \rho^{-2s}t^{-3}(T-t)^{-3}\varphi^2 dxdt\right]\end{aligned}$$

The term $\int_Q \rho^{-2s}|\varepsilon(x,t)\varphi_t|^2 dxdt$ on the right-hand side can be viewed as a remainder. In order to get rid of it and to get an observability estimate for the solutions of the perturbed system (6.12) we need to assume that

$$(6.14)\qquad \frac{\varepsilon^2(x,t)}{t(T-t)} \le \frac{s}{C_*}.$$

This is clearly a smallness condition on the perturbation ε of the constant coefficient. But, it also imposes that ε vanishes at $t=0$ and $t=T$. Indeed, in order to see this it is better to write

$$(6.15)\qquad \varepsilon(x,t)=\sqrt{t}\sqrt{T-t}\,\delta(x,t).$$

Then, the smallness condition reads

$$(6.16)\qquad ||\delta||^2_{L^\infty(Q)} \le \frac{s}{C_*}.$$

According to this analysis, the following holds:

Theorem 6.3 *Let Ω be a bounded smooth domain and ω a non-empty open subset. Let $T>0$. Let also s be large enough so that (5.4) holds with constant C_*. Consider the variable density heat equation (6.12) with ε as in (6.15) and satisfying the smallness condition (6.16). Then, the observability inequalities (5.1) and (5.2) hold for the solutions of (6.12).*

Strictly speaking, the arguments above provide a Carleman ienquality of the form (5.4) for the solutions of (6.12). Estimates of the form (5.1) and (5.2) can then be obtained following classical arguments (see [68]). One first derives (5.2) as an immediate consequence of the Carleman inequality to later obtain (5.1) as a consequence of the well-posedness of (6.12) . Indeed, multiplying in (6.12) by φ_t and integrating by parts we deduce the energy identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\varphi|^2 dx = \int_\Omega (1+\varepsilon(x,t))|\varphi_t|^2 dx \ge 0.$$

Thus,

$$\int_\Omega |\nabla\varphi|^2(x,t)dx \le \int_\Omega |\nabla\varphi|^2(x,T)dx$$

for all $0\le t\le T$. In view of this and, as a consequence of the Carleman inequality (5.4) we deduce that

$$(6.17)\qquad \int_0^T\int_\Omega e^{-\frac{A}{T-t}}|\nabla\varphi|^2 dxdt \le C\int_0^T\int_\omega \varphi^2 dxdt$$

and

$$\int_\Omega |\nabla\varphi(x,0)|^2 dxdt \le C\int_0^T\int_\omega \varphi^2 dxdt \tag{6.18}$$

for suitable constants $C, A > 0$. By Poincaré inequality this yields:

$$\int_0^T\int_\Omega e^{-\frac{A}{T-t}}\varphi^2 dxdt \le C\int_0^T\int_\omega \varphi^2 dxdt \tag{6.19}$$

and

$$\int_\Omega \varphi^2(x,0) dxdt \le C\int_0^T\int_\omega \varphi^2 dxdt. \tag{6.20}$$

Whether the smallness conditions (6.15) and (6.16) on ε are needed or not is an open problem. It is however convenient to distinguish between $t = 0$ and $t = T$.

The restriction at $t = 0$ can be relaxed. Indeed, using the regularizing effect of the heat equation, a variant of (5.4) can be obtained so that the weight function involved in it does not degenerate at $t = 0$ (see [74]). In this way, a similar result would hold under the restriction

$$\frac{\varepsilon^2(x,t)}{(T-t)} \le \frac{s}{C_*}, \tag{6.21}$$

instead of (6.14).

At the contrary getting rid of the condition that $\varepsilon^2(x,t)/(T-t)$ is small at $t = T$ is probably very difficult. This smallness condition could even be necessary. This is related to the nature of the observability inequalities for the adjoint heat equation that are unable to provide estimates on the solutions at $t = T$ because of the very strong smoothing effect. In fact, as indicated in the context of the constant-coefficient heat equation, even when estimating the L^2-norm of the solutions, an exponentially vanishing weight is needed at $t = T$ (see (5.2)).

This difficulty is probably also related to the one we encountered in the previous subsection when dealing with the spectrum of the system. There, we could only deal with an eigenvalue range whose width depended on the smallness condition on the perturbation of the coefficient. As indicated there, the perturbation needed to be smaller and smaller to be able to cover eventually the whole range of frequencies. Obviously, when dealing with the evolution problem the whole range of frequencies is involved. It is therefore natural to require an smallness assumption that vanishes as $t \to T$.

The observability result for the adjoint heat equation (6.12) we have proved yields immediately results on the null-controllability of the corresponding state equation

$$\begin{cases} \partial_t((1+\varepsilon(x,t))u) - \Delta u = f1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{6.22}$$

The following holds:

Theorem 6.4 *Under the assumptions of Theorem 6.3 system (6.22) is null-controllable.*

The control for (6.22) can be obtained from the variational methods we have developed in the context of the constant-coefficient heat equation. It may be built being of minimal L^2-norm or of minimal L^∞-norm, in which case it will be of quasi-bang-bang form.

As in Theorem 6.3, Theorem 6.4 requires smallness conditions of the form (6.15)-(6.16) on the coefficient ε. As we said above, getting rid of this smallness condition at $t = T$ is probably a very difficult problem. This can also be interpreted in the context of the control of the state equation (6.22). Indeed, since we are trying to drive the state u to rest at time $t = T$, the oscillations of the density coefficients ε could be a major obstacle for doing that. Note however that, as mentioned in the introduction of this section, there is no example of heat equation with bounded coefficients for which the null controllability property fails. The situation is different in what concerns the degeneracy condition of the coefficient at $t = 0$. As we said above, this condition is probably unnecessary for observability and for null controllability too, but this remains to be investigated.

A similar analysis could be developed for heat equations with bounded small perturbations on the coefficients entering in the second order operator, i. e., for equations of the form

$$\text{(6.23)} \qquad \begin{cases} -(1+\varepsilon(x,t))\varphi_t - \operatorname{div}((1+\sigma(x,t))\nabla\varphi) = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega, \end{cases}$$

and

$$\text{(6.24)} \qquad \begin{cases} \partial_t((1+\varepsilon(x,t))u) - \operatorname{div}((1+\sigma(x,t))\nabla u) = f1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases}$$

For that purpose we need a parabolic version of the elliptic inequalities in the previous section. This was developed in [98] (see also [63]). Let us recall it briefly. Consider the adjoint heat equation

$$\text{(6.25)} \qquad \begin{cases} -\varphi_t - \Delta\varphi = f + \sum_{j=1}^n \partial_j f_j & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases}$$

The following holds ([98], [63]):

$$\text{(6.26)} \qquad \begin{aligned} &\int_Q \rho^{-2s}\Big[s\lambda^2 \frac{\exp(\lambda\eta^0)}{(t(T-t))^3}|\nabla\varphi|^2 + s^3\lambda^4 \frac{\exp(3\lambda\eta^0)}{(t(T-t))^3}|\varphi|^2\Big]dxdt \\ \le\ & C\Big[\int_Q \rho^{-2s} f^2 dxdt + s^2\lambda^2 \int_Q \frac{\exp(2\lambda\eta^0)}{(t(T-t))^2}\rho^{-2s}\sum_{j=1}^n |f_j|^2 dxdt \\ +\ & s^3\lambda^4 \int_{\omega\times(0,T)} \rho^{-2s}\frac{\exp(3\lambda\eta^0)}{(t(T-t))^3}|\varphi|^2 dxdt\Big]. \end{aligned}$$

As in the elliptic case, this inequality allows absorbing the effect of the term that the bounded perturbation on the principal part of the operator in (6.25) adds. More precisely, the term $\operatorname{div}(\sigma(x,t)\nabla\varphi)$ can be absorbed by a suitable smallness condition on σ:

$$\text{(6.27)} \qquad Cs \max_Q \exp(\lambda\eta^0)\frac{|\sigma(x,t)|^2}{t(T-t)} < 1,$$

which is similar to (6.14). Once more the smallness condition may be relaxed so that the vanishing weight is not needed at $t = 0$.

In fact these parabolic estimates contain those we have obtained in the previous subsection on elliptic equations. Indeed, by viewing solutions of the elliptic equations as time independent solutions of the parabolic ones it is easy to see that the elliptic estimates are contained in the parabolic ones under similar smallness conditions on the perturbations of the coefficients.

In order to deal with the general parabolic equation (6.25) and to address variable densities ε and coefficients σ in the second order elliptic operator we have to combine (5.4) and (6.26). In this way we get the Carleman inequality under similar smallness conditions both on ε and σ.

As an immediate consequence of these results we deduce the null controllability of the state equation (6.24).

Remark 6.1 *In [66] similar developments have been done in the context of the adjoint equation*

$$-\partial_t((1+\varepsilon(x,t))\varphi) - \Delta\varphi = 0.$$

Note that, in this case, the perturbation has to be viewed as an element of $H^{-1}(0,T;L^2(\Omega))$ *what addas further technical difficulties. Some applications to the controllability of quasilinear parabolic problems have also been given. These two issues are closely related, as in the semilinear case, because of the use of the fixed point method which reduces the control of the nonlinear problem to a sharp estimate of the cost of controllability of the linearized one.*

Control of quasilinear heat equations is also a widely open subject of research. Very likely, the approach based on linearization and a sharp analysis of the cost of controlling linear systems is insufficient to cover the new phenomena that quasilinear equations present.

Remark 6.2 *It is important to observe that the approach we have developed in this section requires smallness conditions on the perturbations of the coefficients. In particular, we could consider coefficients that, for all* t *are piecewise constant and possibly discontinuous. The results in this section do not require the monotonicity conditions in [46]. But, at the contrary, they need the jumps to be small, and to vanish as* t *tend to* T *at order* $\sqrt{T-t}$.

It is also important to observe that the results of this section apply to coefficients which depend both on space and time, a framework that might be much richer than that of parabolic equations with low regularity coefficients depending only on x. *For instance, as indicated above, in [70] we prove that the* $1-d$ *heat equation with* BV *coefficients depending only on* x *is null controllable. But it is unkown whether the same result is true for* BV *coefficients depending both on* x *and* t.

7 Fluid-structure interaction models

7.1 Problem formulation

So far we have only discussed two model systems: the heat and the wave equation. But most of the techniques we have developed are also useful to address more sophisticated and realistic models. That is the case, for instance, in the context of the Navier-Stokes equations in which most of the developments we have presented based on Carleman and observability inequalities, duality and variational principles allow obtaining a number of controllability results. It is by now well known, for instance, that the Navier-Stokes equations are locally null controllable (see, for instance, [33], [35], [65] , [72], [76], and the references therein). Despite of the fact that the techniques described above apply, important further developments are needed to deal with the pressure term, the incompressibility condition, the lack of regularity of the convective potentials when linearizing the system around weak solutions, etc. We refer to the survey article [67] for un updated discussion of these issues. Controllability also holds for the Euler equations in an appropriate functional and geometric setting. However, because of the lack of viscosity, Carleman inequalities do not apply and different techniques have to be applied. We refer for instance to [32], and to [81] where the problem is solved in $2D$ and $3D$, respectively, by the so called "return method" due to J.-M. Coron [34].

Much less is known in the context of fluid-structure interaction models. These models are indeed very hard to deal with because of their mixed hyperbolic-parabolic nature. Roughly speaking they can be viewed as the coupling of a Navier-Stokes system for the fluid, with a system of elasticity for the structure, coupled along a moving interface determined by the boundary of the deformed elastic body. In fact, even the problem of the well-posedness of these problems is badly understood. We refer to [14], [15] and [38] for some results in that direction. The inviscid case has been also treated (see [158], [159]). To the best of our knowledge, nothing is known on the controllability of this system.

This model may be simplified by assuming that the structure is a rigid body. In that case the modelling consists in coupling between the Navier-Stokes equations and the ordinary differential equations for the motion of the rigid body. In $1-d$ in which a fluid modelled by the Burgers equation is coupled with a finite number of mass points, existence and uniqueness of global solutions is known. In particular it is known that two solid particles may not collide in finite time, a problem that is still open in several space dimensions ([190]). For this model, in the presence of one single particle and with controls on both sides, null controllability has been proved in [47]. The difficulty one encounters when dealing with this apparently simple system from a control theoretical point of view is similar to that we found when considering heat equations with piecewise constant and discontinuous coefficients: it is hard to derive observability estimates by means of Carleman inequalities because of the interface terms. In particular, the problem of control of this $1-d$ fluid-mass model is open when the control acts on one side of the mass only. Very recently a very interesting result has been obtained in $2-d$ in [16]. It guarantees the local null controllability for the Navier-Stokes equations coupled with the motion of a finite number of rigid bodies, the control being applied on an arbitrary open subset of the fluid and in an arbitrarily small time. In this sense the result is better than in $1-d$ where the control is assumed to be on both sides of the mass. There is however an intuitive explanation of this fact. In $2-d$ the fluid envelopes the rigid bodies so that the effect of local controls on the fluid may propagates to all of it and then acts on the masses too, while in $1-d$ the control, when applied only on one side, needs to cross the point-mass. The problem of controlling large initial data is still open. It is related to the fact that, in $2-d$ it is not known whether two rigid bodies may collide or whether they may collide with the external boundary. In [175] it has been proved that, even if they collide, one can extend the solution for all time $t \geq 0$ by a weak solution. But the estimates that Carleman inequalities yield becomes singular when collision occurs. Analyzing whether the control may avoid collision to occur (in case it occurs in the absence of control) and controlling to zero large initial data is certainly a very interesting and difficult problem.

These models are free boundary ones. When linearizing them around the equilibrium configuration they become evolution equations on two adjacent domains separated by a fixed interface. In this section we summarize some recent results on the asymptotic behavior of a linearized model arising in fluid-structure interaction, where a wave and a heat equation evolve in two adjacent bounded domains, with natural transmission conditions at the interface. The content of this section is based on joint work with J. Rauch and X. Zhang [170], [195], [196], [197], [198].

The system under consideration may be viewed as an approximate and simplified model for the motion of an elastic body immersed in a fluid, which, as we mentioned above, in its most rigorous modelling should be a nonlinear free boundary problem, with a moving interface between the fluid and the elastic body.

In the model we consider here the heat unknown is coupled with the velocity of the wave solution along the interface. A slightly simpler case in which the states of the heat and wave equations are directly coupled was addressed in [170]. Note however that the coupling conditions we consider here are more natural from the point of view of fluid-structure interaction.

In this section we mainly discuss the problem of the decay of solutions as t tends to zero. A similar study has been undertaken previously for the system of thermoelasticity (see [118]), another natural situation in which wave and heat equations are coupled. Note however that, in thermoelasticity, both the heat and the wave equation evolve in the same domain, while in the fluid-structure interaction model under consideration they evolve on two different domains, separated through an interface.

The model we consider here can be viewed as the coupling of the purely conservative dynamics generated by the wave equation and the strongly dissipative one that the heat equation produces. The total energy of solutions, addition of the thermal and elastic one, is dissipated through the heat domain. Therefore, studying the rate of decay of solutions of the whole system, is a way of addressing the issue of how strongly the two dynamics are coupled. Indeed, one could expect that, in case the two components of the system are coupled strongly enough along the interface, then solutions should decay with an exponential rate. This corresponds to the situation in which the semigroup $S(\cdot)$ generated by the system, which is dissipative, is such that $||S(T)|| < 1$ for some $T > 0$ in the norm of the energy space. At the contrary, the lack of uniform exponential decay could be considered as a proof of the lack of strong coupling.

But the issue is more complex. Indeed, it is well known that, when the damping introduced on a wave-like equation is too strong, overdamping phenomena may occur and the exponential decay may be lost. This is for instance the case for the damped wave equation

$$u_{tt} - \Delta u + k u_t = 0,$$

with homogeneous Dirichlet boundary conditions. In this case the energy of solutions is given by

$$E(t) = \frac{1}{2} \int \Big[|u_t|^2 + |\nabla u|^2 \Big] dx,$$

and the energy dissipation law

$$\frac{dE(t)}{dt} = -k \int |u_t|^2 dx.$$

In view of this energy dissipation law one could expect a faster decay rate for larger values of the dissipation parameter $k > 0$. But the exponential decay rate is not monotonic on $k > 0$.

Indeed, despite the exponential decay rate increases as $k > 0$ is increasing and small, the decay rate diminishes when $k \to \infty$.

The damping that the heat equation introduces is an unbounded perturbation of the wave dynamics. This predicts that the exponential decay may be lost. This is indeed the case and it is independent of the geometry of the subdomain in which the heat and wave equations hold. In the case where the domain Ω is a polygon and the interface is a hyperplane, the lack of exponential decay can be proved by means of a plane wave analysis that allows exhibiting a class of solutions whose energy is mainly concentrated in the wave domain and therefore, very weakly dissipated through the heat mechanism. In general domains and for curved interfaces this construction needs of a more careful development based on the use of gaussian beams.

But, on the other hand, due to the presence of the wave motion, and in view of our experience on the control and stabilization of the wave equation, one expects that the system will be more stable when the heat domain satisfies the GCC and, more precisely, when all rays in the wave domain enter the heat one in an uniform time. This is indeed the case. When the heat subdomain satisfies this GCC the decay rate of smooth solutions is polynomial but, in general, one can only guarantee a polynomial decay rate.

The main conclusions of the series of works we have mentioned above are roughly as follows:

- Whenever the heat subdomain is non-empty, the energy of solutions tends to zero as time tends to infinity;
- The decay rate is never exponentially uniform, regardless of the geometric properties of the heat sudomain. In other words, the dissipative semigroup generated by the system is of unit norm for all $t > 0$;
- When the heat domain satisfies the GCC, then the energy of smooth solutions decays polynomially.
- When the heat domain does not satisfy the GCC smooth solutions decay logarithmically for the simplified interface conditions in which continuity of the states is imposed. The problem is open for the more natural boundary conditions we shall consider here.

Some other issues are by now also well-understood. In $1-d$ the problem of controllability has been solved in [210] and [195] and [196]. When the control acts on the exterior boundary of the wave domain null controllability can be easily proved using sidewise energy estimates for the wave equation and Carleman inequalities for the heat one. However, when the control acts on the extreme of the heat domain the space of controllable data is very small. Roughly, the controllable initial data have exponentially small Fourier coefficients on the basis of the eigenfunctions of the generator of the semigroup. In fact in [195] and [196] a complete asymptotic analysis of the spectrum of the system has been developed. According to it, the spectrum can be decomposed in two branches: the parabolic one and the hyperbolic one. The parabolic eigenvalues are asymptotically real and tend to $-\infty$ and the energy of the corresponding eigenfunctions is more and more concentrated on the heat domain. The hyperbolic one has vanishing asymptotic real part and their energy is concentrated on the wave domain. As a consequence of this fact the high frequency hyperbolic eigenfunctions are very badly controlled from the heat domain. Thus, for controlling a given initial datum, its Fourier components on the hyperbolic eigenfunctions need to vanish exponentially at high frequencies. These results, which have been obtained by means of $1-d$ methods are completely open in several space dimensions. In fact, according to them one also expects important differences for the multi-dimensional problem. In this section we only address the problem of the rate of decay. We do it in the multi-dimensional case. Thus, we do not employ spectral methods. Rather we use plane wave analysis and gaussian beams to show that, whatever the geometry is, the decay rate is never exponential, and the existing hyperbolic observability estimates to prove the polynomial decay under the GCC. Although the results we present here are far from answering to the problem of controllability, by combining our understanding of the problem that the analysis of the stabilization problem yields, and the behavior of the control problem in $1-d$, one may at least guess what kind of results should be expected for the control problem in multi$-d$. It is natural for instance to expect that, if the control enters in a subset of the wave domain satisfying the GCC, then one should expect null controllability on the energy space. However, when controlling on the heat subdomain the space of controllable data should be very small, even if the heat domain envelops the wave one and satisfies the GCC. This is a widely open subject of research. In one space dimension other closely related models have all also been investigatigated. In particular, a model coupling the wave equation with an equation of viscoelasticity

7.2 The model

Let $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}^*$) be a bounded domain with C^2 boundary $\Gamma = \partial\Omega$. Let Ω_1 be a sub-domain of Ω and set $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. We denote by γ the interface, by $\Gamma_j = \partial\Omega_j \setminus \overline{\gamma}$ ($j = 1, 2$) the exterior boundaries, and by ν_j the unit outward normal vector of Ω_j ($j = 1, 2$). We assume $\gamma \neq \emptyset$ and γ is of class C^1.

Consider the following hyperbolic-parabolic coupled system:

$$(7.1) \qquad \begin{cases} y_t - \Delta y = 0 & \text{in } (0,\infty) \times \Omega_1, \\ z_{tt} - \Delta z = 0 & \text{in } (0,\infty) \times \Omega_2, \\ y = 0 & \text{on } (0,\infty) \times \Gamma_1, \\ z = 0 & \text{on } (0,\infty) \times \Gamma_2, \\ y = z_t, \quad \frac{\partial y}{\partial \nu_1} = -\frac{\partial z}{\partial \nu_2} & \text{on } (0,\infty) \times \gamma, \\ y(0) = y_0 & \text{in } \Omega_1, \\ z(0) = z_0, \quad z_t(0) = z_1 & \text{in } \Omega_2. \end{cases}$$

As we said above, this is a simplified and linearized model for fluid-structure interaction. In system (7.1), y may be viewed as the velocity of the fluid; while z and z_t represent respectively the displacement and velocity of the structure. More realistic models should involve the Stokes (*resp.* the elasticity) equations instead of the heat (*resp.* the wave) ones.

In [170] and [197], the same system was considered but for the transmission condition $y = z$ on the interface instead of $y = z_t$. But, from the point of view of fluid-structure interaction, the transmission condition $y = z_t$ in (7.1) is more natural. Note also that, as indicated above, the interface in this model is fixed. This corresponds to the fact that the system is a linearization around the trivial solution of a free boundary problem.

Set $H^1_{\Gamma_1}(\Omega_1) \triangleq \left\{ h|_{\Omega_1} \,\middle|\, h \in H^1_0(\Omega) \right\}$ and $H^1_{\Gamma_2}(\Omega_2) \triangleq \left\{ h|_{\Omega_2} \,\middle|\, h \in H^1_0(\Omega) \right\}$. System (7.1) is well-posed in the Hilbert space

$$H \stackrel{\triangle}{=} L^2(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2).$$

When Γ_2 is a non-empty open subset of the boundary (or, more generally, of positive capacity), in H the following norm is equivalent to the canonical one:

$$|f|_H = \left[|f_1|^2_{L^2(\Omega_1)} + |\nabla f_2|^2_{(L^2(\Omega_2))^n} + |f_3|^2_{L^2(\Omega_2)} \right]^{1/2}, \qquad \forall\, f = (f_1, f_2, f_3) \in H.$$

In this case the only stationary solution is the trivial one. This is due to the fact that Poincaré inequality holds.

When Γ_2, vanishes, $|\cdot|_H$ is no longer a norm on H. In this case, there are non-trivial stationary solutions of the system. Thus, the asymptotic behavior is more complex and one should rather expect the convergence of each individual trajectory to a specific stationary solution. To simplify the presentation in this section we assume that the capacity of Γ_2 is positive.

The energy of system (7.1) is given by

$$E(t) \stackrel{\triangle}{=} E(y, z, z_t)(t) = \frac{1}{2} |(y(t), z(t), z_t(t))|^2_H$$

and satisfies the dissipation law

$$\frac{d}{dt} E(t) = -\int_{\Omega_1} |\nabla y|^2 dx. \tag{7.2}$$

Therefore, the energy of (7.1) decreases as $t \to \infty$.

In fact $E(t) \to 0$ as $t \to \infty$, without any geometric conditions on the domains Ω_1 and Ω_2 (other than the capacity of Γ_2 being positive). However, due to the lack of compactness of the domain of the generator of the underlying semigroup of system (7.1) for $n \geq 2$, one can not use directly the LaSalle's invariance principle to prove this result. Instead, using the "relaxed invariance principle" ([178]), we conclude that y and z_t tend to zero strongly in $L^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively; while z tends to zero weakly in $H^1_{\Gamma_2}(\Omega_1)$ as $t \to \infty$. Then, we use the special structure of (7.1) and the key energy dissipation law (7.2) to obtain the strong convergence of z in $H^1_{\Gamma_2}(\Omega_1)$ ([198]).

Once the energy of each individual trajectory has been shown to tend to zero as t goes to ∞, we analyze the rate of decay. In particular, it is natural to analyze whether there is an uniform exponential decay rate, i. e. whether there exist two positive constants C and α such that

$$E(t) \leq C E(0) e^{-\alpha t}, \qquad \forall\, t \geq 0 \tag{7.3}$$

for every solution of (7.1).

According to the energy dissipation law (7.2), the uniform decay problem (7.3) is equivalent to showing that: there exist $T > 0$ and $C > 0$ such that every solution of (7.1) satisfies

$$|(y_0, z_0, z_1)|^2_H \leq C \int_0^T \int_{\Omega_1} |\nabla y|^2 dx dt, \qquad \forall\, (y_0, z_0, z_1) \in H. \tag{7.4}$$

Inequality (7.4) can be viewed as an observability estimate for equation (7.1) with observation on the heat subdomain. In principle, whether it holds or not depends very strongly on how the two components y and z of the solution are coupled along the interface. Indeed, the right-hand side term of (7.4) provides full information on y in Ω_1 and, consequently, also on the interface. Because of the continuity conditions on the interface this also yields information on z and its normal derivative on the interface. But how much of the energy of z we are able to obtain from this interface information has to be analyzed in detail. It depends on two facts. First it may depend in a very significant way on whether the interface γ controls geometrically the wave domain Ω_2 or not. Second, on the Sobolev norm of the interface trace information we recover z and its normal derivative.

Remark 7.1 *This argument also shows the close connections of the problems of control and that of the exponential decay of solutions of damped systems. Both end up being reducible to an observability inequality. This is particularly clear for the wave equation with localized damping:*

$$u_{tt} - \Delta u + 1_\omega u_t = 0,$$

In this case the energy is given by

$$E(t) = \frac{1}{2}\int \Big[|u_t|^2 + |\nabla u|^2\Big] dx,$$

and the energy dissipation law reads

$$\frac{dE(t)}{dt} = -\int_\omega |u_t|^2 dx.$$

The energy has an uniform exponential decay rate if and only if there exists some time T and constant $C > 0$ such that

$$E(0) \leq C \int_0^T \int_\omega |u_t|^2 dx dt.$$

Moreover, this observability estimate holds for the dissipative equation satisfied by u if and only if it holds for the conservative wave equation

$$\varphi_{tt} - \Delta\varphi = 0.$$

Thus we see that exponential decay is equivalent to observability which, as we know from previous sections, is also equivalent to controllability. This establishes a clear connection between controllability and stabilization. Here the argument has been developed for the wave equation but similar developments could be done for plate and Schrödinger equations and, more generallly, for conservative evolution equations.

The fact that the exponential decay is equivalent to an observability inequality is also important for nonlinear problems. We refer to [200] and [43] for the analysis of the stabilization of nonlinear wave equations.

Returning to the coupled heat-wave system, as indicated in [196], there is no uniform decay for solutions of (7.1) even in one space dimension. The analysis in [196] exhibits the existence of a hyperbolic-like spectral branch such that the energy of the eigenvectors is concentrated in the wave domain and the eigenvalues have an asymptotically vanishing real part. This is obviously incompatible with the exponential decay rate. The approach in [196], based on spectral analysis, does not apply to multidimensional situations. But the $1-d$ result in [196] is a warning in the sense that one may not expect (7.4) to hold.

Exponential decay property also fails in several space dimensions, as the $1-d$ spectral analysis suggests. To prove this fact one has to build a familiy of solutions of the coupled system whose energy is mainly concentrated in the wave domain. This has been done in [198] following [170] using Gaussian Beams ([164] and [140]) to construct approximate solutions of (7.1) which are highly concentrated along the generalized rays of the D'Alembert operator in the wave domain Ω_2 and are almost completely reflected on the interface γ. As we mentioned above, in the particular case of polygonal domains with a flat interface, one can do a simpler construction using plane waves.

This result on the lack of uniform exponential decay, which is valid for all geometric configurations, suggests that one can only expect a polynomial stability property of smooth solutions of (7.1) even under the Geometric Control Condition, i.e., when the heat domain where the damping of the system is active is such that all rays of Geometric Optics propagating in the wave domain meet the interface in an uniform time. To prove this, we need to derive a weakened observability inequality. This can be done by viewing the whole system as a perturbation of the wave equation in the whole domain Ω, an argument that was introduced in [170] for the simpler interface conditions.

These results are summarized in the following section.

7.3 Decay properties

First of all, as mentioned above, solutions tend to zero as t goes to infinity but the decay rate is not exponential:

Theorem 7.1 *For any given $(y_0, z_0, z_1) \in H$, the solution (y, z, z_t) of (7.1) tends to 0 strongly in H as $t \to \infty$, without any geometric assumption on the heat and wave domains other than Γ_2 being of positive capacity.*

But, at the contrary, there is no uniform exponential decay. In other words, the norm of the semigroup generated by the system $S(t)$ is one, $||S(t)|| = 1$, as a linear and continuous operator form H to H, and this for all time $t > 0$.

To prove some decay rate it is convenient to view the whole coupled system as a perturbation of the wave equation in the union of the wave and heat domains. But for this method to work we need to assume that the heat domain satisfies the GCC. In this case the solutions of the wave equation in Ω:

$$\begin{cases} \zeta_{tt} - \Delta\zeta = 0 & \text{in} \quad \Omega \times (0,T), \\ \zeta = 0 & \text{on} \quad \Gamma \times (0,T), \\ \zeta(0) = \zeta_0, \quad \zeta_t(0) = \zeta_1 & \text{in} \quad \Omega \end{cases}$$

satisfy the following observability inequality (see [7])

$$|\zeta_0|^2_{H^1_0(\Omega)} + |\zeta_1|^2_{L^2(\Omega)} \leq C \int_0^T \int_{\Omega_1} |\zeta_t|^2 dxdt, \qquad \forall\, (\zeta_0, \zeta_1) \in H^1_0(\Omega) \times L^2(\Omega),$$

for T sufficiently large.

Under this condition the following holds:

Theorem 7.2 *Assume that Ω_1 satisfies the GCC in ω. Then there is a constant $C > 0$ such that for any $(y_0, z_0, z_1) \in D(\mathcal{A})$, the solution of (7.1) satisfies*

$$|(y(t), z(t), z_t(t))|_H \leq \frac{C}{t^{1/6}} |(y_0, z_0, z_1)|_{D(\mathcal{A})}, \quad \forall\, t > 0. \tag{7.5}$$

Remark 7.2 *The domain of the generator of the semigroup of the coupled system is given by*

$$D(\mathcal{A}) = \Big\{(Y_1, Y_2, Y_3) \in H \;\Big|\; \Delta Y_1 \in L^2(\Omega_1),\ \Delta Y_2 \in L^2(\Omega_2),\ Y_3 \in H^1(\Omega_2),$$
$$Y_1|_{\Gamma_1} = Y_3|_{\Gamma_2} = 0,\ Y_1|_\gamma = Y_3|_\gamma,\ \frac{\partial Y_1}{\partial \nu_1}\Big|_\gamma = -\frac{\partial Y_2}{\partial \nu_2}\Big|_\gamma \Big\}.$$

As we have mentioned above it is not compactly embedded in H, except for the dimension $n = 1$.

Remark 7.3 *Theorem 7.2 is not sharp for $n = 1$ since, as proved in [196] using spectral analysis, the decay rate is $1/t^2$. Whether estimate (7.5) is sharp in several space dimensions is an open problem. However, its proof is rather rough and therefore it is very likely that it might be improved by a more subtle analysis of the interaction of the wave and heat components on the interface.*

The proof of Theorem 7.2 is based on the following key weakened observability inequality for equation (7.1):

Theorem 7.3 *Assume that Ω_1 satisfies the GCC in Ω. Then there exist two constants T_0 and $C > 0$ such that for any $(y_0, z_0, z_1) \in D(\mathcal{A}^3)$, and any $T \geq T_0$, the solution of (7.1) satisfies*

$$|(y_0, z_0, z_1)|_H \leq C|\nabla y|_{H^3(0,T;(L^2(\Omega_1))^n)}. \tag{7.6}$$

Remark 7.4 *Estimate (7.6) is a weakened observability inequality. Comparing it to (7.4), which is needed for the uniform exponential decay, we see that on the right-hand side term we are using a much stronger norm involving time derivatives up to order three. In order to get a better polynomial decay rate one should improve this observability estimate, using less time derivatives on the right-hand side term.*

The main idea to prove Theorem 7.3 is as follows. Setting $w = y\chi_{\Omega_1} + z_t\chi_{\Omega_2}$, noting (7.1) and recalling that $\partial z_t/\partial\nu_2 = -\partial y_t/\partial\nu_1$ on $(0,T) \times \gamma$, and by $(y_0, z_0, z_1) \in D(\mathcal{A}^2)$, one sees that $w \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ satisfies

$$\begin{cases} w = (y_{tt} - y_t)\chi_{\Omega_1} + \left(\frac{\partial y}{\partial \nu_1} - \frac{\partial y_t}{\partial \nu_1}\right)\delta_\gamma & \text{in } (0,T)\times\Omega, \\ w = 0 & \text{on } (0,T)\times\Gamma, \\ w(0) = y_0\chi_{\Omega_1} + z_1\chi_{\Omega_2},\ w_t(0) = (\Delta y_0)\chi_{\Omega_1} + (\Delta z_0)\chi_{\Omega_2} & \text{in } \Omega. \end{cases} \tag{7.7}$$

Then, this weakened observability inequality holds from the GCC condition and energy estimates.

These results do not yield any decay rate in the case in which Ω_1 does not satisfy the GCC. This is for instance the case when Ω_1 is a convex subdomain in a convex domain Ω surrounded by the wave domain Ω_2. In this case one expects a logarithmic decay rate. This was proved in [198] for the simpler interface conditions ($y = z$ instead of $y = z_t$). But the problem is open in the present case. This is once more due to the lack of compactness of the domain of the generator of the semigroup.

Remark 7.5 *the decay rates obtianed in this section have been recently improved by Th. Duyckaerts in [49] for C^∞-domains, using sharp results.*

8 Some open problems

In this section we present some open problems related to the topics we have addressed in this paper.

- **Sharp observability estimates.** In the context of the constant coefficient heat equation in (8.3) we refered to the sharp observability inequality

$$\sum_{k=1}^{\infty} |a_k|^2 \, e^{-c\sqrt{\lambda_k}} \leq C \int_0^T \int_\omega \varphi^2 dx dt \tag{8.1}$$

for the solutions φ of the adjoint heat equation (4.5). Note that the left-hand side of this inequality defines a norm of φ^0 that corresponds to the one in the domain of the operator $\exp\left(-c\sqrt{-\Delta}\right)$. Characterizing the best constant c in this inequality in terms of the geometric properties of the domains Ω and ω is an open problem. Obviously, the constant may also depend on the length of the time interval T. The problem may be made independent of T by considering the analogue in the infinite time. Indeed, for

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad \Omega \times (-\infty, 0) \\ \varphi = 0 & \text{on} \quad \partial\Omega \times (-\infty, 0) \\ \varphi(x,0) = \varphi^0(x) & \text{in} \quad \Omega \end{cases} \tag{8.2}$$

the following inequality holds

$$\sum_{k=1}^{\infty} |a_k|^2 \, e^{-c\sqrt{\lambda_k}} \leq C \int_{-\infty}^0 \int_\omega \varphi^2 dx dt, \tag{8.3}$$

and the problem of determining the best constant $c > 0$ makes sense.

This problem is intimately related to the characterization of the best constant $A > 0$ in (5.2) for $a \equiv 0$, which is also an open problem. The problem of the best constant A in the inequality (5.2) can be also formulated independently of the length of the time interval T. For, it is sufficient to consider the adjoint heat equation (8.2) in the infinite time $(-\infty, 0)$. The problem then consists in identifying the best constant $A > 0$ such that

$$\int_{-\infty}^0 e^{-A/|t|} \varphi^2 dx dt \leq C \int_{-\infty}^0 \int_\omega \varphi^2 dx dt. \tag{8.4}$$

As we said above, by inspection of the proof of the inequality, one can get some rough estimates on A in terms of the weight function appearing in the Carleman inequality. Some lower bounds on A can also be obtained in terms of the Gaussian heat kernels in ([68]) and ([150]). The best one seems to be that

$$A \geq \delta^2(\omega, \Omega)/4$$

that can be achieved as a consequence of the upper gaussian bounds on the heat kernel in [42] (Th. 3.2.7, p. 89). Here $\delta(\omega, \Omega)$ stands for the largets geodesic distance between the set ω and any point in the domain Ω.

But further investigation is needed for a complete charcaterization of the sharp value of A.

The question of establishing a direct relation between the best constants c and A in (8.1) and (8.2) is also open.

The same can be said about the spectral estimate in (4.17). Characterizing the best constant C_2 on the exponential degeneracy of the observability constant of finite linear combinations of eigenfunctions is an open problem. How this constant C_2 is related to the best constants c and A in the inequalities above is an open problem.

Actually, as far as we know, there is no direct proof of the fact that the spectral observability inequality (4.17) implies the observability inequality for the heat equation. The existing proof is that due to Lebeau and Robbiano and passes through the property of null controllability and duality [117].

Observe also that the right hand side term of inequality (8.3) can be written in Fourier series. indeed, taking into account that $\varphi(x,t) = \sum_{j\geq 1} a_j \exp(\lambda_j t)\psi_j(x)$, we deduce that

$$\int_{-\infty}^{0}\int_{\omega}\varphi^2 dxdt = \sum_{j,k\geq 1} a_j a_k \frac{\int_{\omega}\psi_j(x)\psi_k(x)dx}{\lambda_j+\lambda_k}. \tag{8.5}$$

Combining (8.3) and (8.5) we deduce that

$$\sum_{k=1}^{\infty} |a_k|^2 e^{-c\sqrt{\lambda_k}} \leq C \sum_{j,k\geq 1} a_j a_k \frac{\int_{\omega}\psi_j(x)\psi_k(x)dx}{\lambda_j+\lambda_k}. \tag{8.6}$$

Whether inequality (4.17) can be derived from (8.6) is an open problem.

- **Bang-bang controls.** As described in Section 3.5, the problem of bang-bang controls for wave-like equations is still badly understood. In finite-dimensional dynamical systems and also for the heat equation, controls of minimal L^∞-norm are of bang-bang form.

  This is not the case for the wave equation as shown in [87] in $1-d$ using the D'Alembert formula to explicitly represent solutions. As explained in Section 3.5, in the context of the approximate controllability of the wave equation, quasi bang-bang controls can be built both in one and several space dimensions. But a complete analysis of the actual structure of these controls, which is intimately related with the structure of the nodal sets of the minimizers of the quadratic functionals for the adjoint system, is still to be done. The analysis in [85] shows that, even in $1-d$, these quasi-bang-bang controls very rarely have an actual bang-bang structure. In the context of exact controllability the situation is even worse since relaxation phenomena occur and the minimizers of the corresponding minimization problem for the adjoint system may develop singular measures. The results in [85] also establish a clear relation between the lack of existence of bang-bang controls and the non existence of minimizers for the corresponding variational principle. The problem is totally unexplored in multi-d although, in view of the results in [85], obviously, one expects even a more complex picture.

A complete characterization of the set of data that are controllable by means of bang-bang controls has been obtained by M. Gugat in [85] in $1-d$ using the explicit representation of solutions by means of D'Alembert's formula. But the problem is totally open in several space dimensions.

We have discussed the bang-bang principle in the following context. The time of control $T>0$ and the initial and data final are given and we have considered the problem of finding bang-bang controls and relating this fact to the property of being of minimal L^∞-norm within the class of admissible controls. Bang-bang controls arise also in the following alternate way. Assuming that controllability holds for some time T and considering initial and final data, and a given bound of the maximum size of the control allowed, one can look for the minimal time in which the system is controllable for those data under that bound. One expects the control to be of bang-bang form in that case too. This problem of minimal time control is well understood for finite-dimensional systems but much less is known in the context of PDE. We refer to the book [106] for the study of this problem for $1-d$ wave and heat processes.

- **Control of semilinear heat equations.** As we mentioned above, in [69] it was proved that semilinear heat equations are null controllable in an arbitrarily small time and from any open subset ω of the domain Ω, for nonlinearities that, at infinity, grow slower than $s\log^{3/2}(s)$. On the other hand, there are examples in which, because of blow-up phenomena, this result fails to be true when the nonlinearity grows as $s\log^p(s)$ with $p>2$. The optimality results of section 5.2 show that one can not expect the classical method (based on linearization, Carleman inequalities, and fixed points) to work in the range $3/2<p\leq 2$. Whether null controllability holds in an arbitrarily small time in that range of nonlinearities is an interesting open problem.

  For power-like nonlinearities, or even for nonlinearities growing at infinity as $s\log^p(s)$ with $p>2$, null controllability may fail because of blow-up ([69]). On the other hand, for nonlinearities with the good sign it is also well-known that for null controllability to hold the time of control has to be taken large enough, depending on the size of the initial datum to be controlled ([4]). Recently, it has also been proved that, despite these possible nonlinear effects, these equations are controllable in the sense that two stationary solutions in the same connected component (within the space of stationary solutions), can be connected by means of a suitable control (*see* [37]).

  Determining what the situation is for the Navier-Stokes equations constitutes an open problem. So far the existing null controllability results are local and need the time to be large when the initial data are large. Whether this is necessary or not is an open problem. In [64] it has been recently proved in the context of the $1-d$ Burgers equation that the time needs to be large when the initial data are large. But the multi$-d$ analogue for the Navier-Stokes equations is open.

- **Degenerate parabolic problems.** Heat equations with degenerate coefficients in the principal part and possible applications to nonlinear parabolic equations as the porous medium or the p-Laplacian one, for instance, is a widely open subject of research. We refer to [23] for the first results in this direction, concerning the $1-d$ linear heat equation with a space-dependent coefficient degenerating on a single point, in a polynomial way.

- **Equations with low regularity coefficients.** To generalize the eigenfunction estimate in Theorem 4.16 for elliptic operators and systems with low regularity coefficients and in non smooth domains is an open problem. That would allow adapting Lebeau and Robbiano [117] strategy for proving the null controllability of the underlying parabolic equation/system. So far, the technique in [117], based on Carleman inequalities, only applies for smooth coefficients and domains.

  As mentioned in Section 6, Carleman inequalities can be directly applied to obtain observability inequalities for heat equations with variable coefficients in the principal part which are a small L^∞ perturbation of constants coefficients. Furthermore, the smallness condition depends on time. Whether these smallness conditions are necessary or not is an open problem. The same can be said about the observability estimates for elliptic equations addressed in Section 6.

  As far as we know, there is no result in the literature showing the lack of null controllability of the heat equation with bounded measurable coefficients. This is an interesting a possibly difficult open problem. The results in [70] show that, in $1-d$, the same is true for heat equations with BV-coefficients. But whether the heat equation with bounded coefficients is null controllable is open even in $1-d$.

- **Spectral characterization of controllability of parabolic equations.** In the context of the wave equation we have given a spectral necessary and sufficient condition (3.15) for controllability in terms of the observability of wave packets, combination of eigenfunctions corresponding to nearby eigenfunctions. A similar characterization is unknown for the heat equation and, more generally, for time-irreversible systems.

  The iterative method in [117] uses the eigenfunction estimate (4.17) in Theorem 4.16 but does not suffice in itself to yield the observability estimate for the heat equation directly. The argument in [117] passes through the property of null controllability of the controlled system and, as a corollary, gives the observability property. It would be interesting to develop a direct iterative argument showing that the eigenfunction estimate (4.17) implies the observability inequality for the heat equation.

  The spectral estimate also implies the controllability and observability of fractional order equations of the form (4.22) with $\alpha > 1/2$. Thus, a complete characterization of (4.17) in terms of the associated evolution problems is still to be found. It is however important to note that the approach based on eigenfunctions of the form Theorem 4.16 is certainly more limited than that consisting in addressing directly the heat equation by Carleman inequalities, which allows, for instance, considering heat equations with coefficients depending both on space x and time t, as in section 6.

- **Systems of parabolic equations.** Most of the literature on the null control of parabolic equations refers to the scalar case. A lot remains to be done to better understand the null controllability of systems. In view of the results we have described for the scalar heat equation one could expect that parabolic systems share the same property of being null controllable in any time and from any open non-empty subset. But systems also have the added possibility of controlling the whole state with less components of controls than the state of the system has. In fact, a very natural condition for that to hold is that the underlying algebraic structure of the system satisfies the Kalman rank condition. Some preliminary results in that direction can be found in [177], [82] and [84] but a complete answer is still to be found.

  Another source of problems in which Carleman and observability inequalities for systems arise is that of the insensitizing control. This notion was introduced in the infinite-dimensional context by J.-L. Lions in [125], the goal being to reduce the sensitivity of partial measurements of solutions of a given equation with respect to perturbations (of initial data, right hand side terms, etc.). As pointed out by Lions, in an appropriate setting, this amounts to prove an observability estimate (or unique continuation property if the notion of insensitivity is relaxed to some approximate insensitivity one) for a cascade system in which the equations are coupled in a diagonal way, but with the very peculiarity that one is forward in time, the other one being a backwards equation. Interesting results in this direction have been proved in [44], [13], [62], in particular. Still, a complete theory is to be developed, in particular in the case where the control acts on a region with empty intersection with the subset of the domain to be insensitized (the first results in that direction for the multi-dimensional heat equation, using Fourier series techniques have been recently obtained by [45]).

- **Control and homogenization.** In this article we have not addressed the issue of homogenization in controllability, and more generally, that on the behavior of the controllability properties under singular perturbations of the system under consideration. These problems can be formulated for a wide class of equations. In particular, in the context of homogenization of PDE with rapidly oscillating coefficients it consists roughly in analyzing whether the controls converge to the control of the limit homogenized equation as the frequency of oscillation of the coefficients tends to infinity, or the microstructure gets finer and finer. The problem is relevant in applications. Indeed, when the convergence of controls holds, one can use the control of the homogeneous limit homogenized medium (which is much easier to compute because of the lack of heterogeneities) to control the heterogeneous equation. This property holds often in the context of approximate controllability, but rarely does for exact or null controllability problems.

We now summarize the existing results and some open problems in this field. We refer to [27] for further developments in this direction and for a complete discussion of this issue:

a) The controls have been proved to converge in the context of $1-d$ the heat equation with rapidly oscillating periodic coefficients [139]. The multi-dimensional analogue constitutes an interesting open problem.

b) In the frame of approximate controllability, the convergence of controls was proved for the heat equation with rapidly oscillating coefficients in [208]. The same proof applies to a wide class of PDE. It is sufficient to apply the variational techniques we have developed to construct approximate controls, combined with Γ-convergence arguments [41]. Despite this fact, addressing exact or null controllability problems is of much greater complexity since it requires of observability estimates which are independent of the vanishing parameter measuring the size of the microstructure.

c) In the context of the wave equation the property of exact controllability fails to be uniform as the microstructure gets finer and finer. This is due to resonance phenomena of high frequency waves for which the wave-length is of the order of the microstructure. However, the controls can be proved to be uniformly bounded, and one can pass to the limit to get the control of the homogenized equation, when one relaxes the controllability condition to that of controlling only the low frequencies. The controlled low frequencies are precisely those that avoid resonance with the microstructure. We refer to [27] for a survey article on this topic. This result was first proved by Castro and Zuazua in $1-d$ in [25] and later extended to the multi-dimensional case by Lebeau [114] using Bloch waves decompositions and microlocal analysis techniques.

In view of this and using the methods of transmuting control results for wave equations into control results for heat equations, one can show that the heat equation with rapidly oscillating coefficients is uniformly partially null controllable when the control subdomain satisfies the GCC for the underlying homogenized problem. Here partial null controllability refers to the possibility of controlling to zero the projection of solutions on the low frequency components avoiding resonances with the microstructure. Getting the uniform null controllability out of this is an open problem. The three-step method developed in [138] and [139] could be applied if we had a Carleman inequality for the multi$-d$ heat equation with rapidly oscillating coefficients with an observability constant of the form $\exp(C/\varepsilon^\alpha)$ with $\alpha < 2$. But such an estimate is unknown. The direct application of the existing Carleman inequalities provides an estimate of the form $\exp(\exp(C/\varepsilon))$ which is far from being sufficient.

Another drawback of this result is that for it to hold one needs to impose a GCC on the control subset, a fact whose necessity has not been justified in the context of parabolic equations.

d) Similar problems arise in the context of perforated domains. One may consider, for instance, the case of a periodically perforated domains with small holes in sense of Cioranescu and Murat [31]. In that case, for Dirichlet boundary conditions, the limit effective equation is the wave equation itself in the whole domain, with possibly a lower order perturbation. In that situation it is known that the exact controllability property passes to the limit provided the controls are applied everywhere in the boundary: the external one and that of the holes (see [30]). The question of whether by filtering the high frequencies one can achieve the uniform controllability from the exterior boundary, as for rapidly oscillating coefficients in [114], is open.

e) The same can be said about the null controllability of the heat equation.

- **Fluid-Structure interaction.** Important work also remains to be done in the context of the models for fluid-structure interaction we have discussed in Section 7. We include here a brief description of some of the most relevant ones:

• *Logarithmic decay without the GCC.* Inspired on [171], it seems natural to expect a logarithmic decay result for system (7.1) without the GCC. This has been done successfully in [170] when replacing the interface condition $y = z_t$ by $y = z$. However, there is a difficulty when addressing the interface condition $y = z_t$ directly which is related to the lack of compactness for (7.1) in multi-dimensions.

• *More complex and realistic models.* The model under consideration would be more realistic replacing the wave equation in system (7.1) by the system of elasticity and the heat equation by the Stokes system, and the fluid-solid interface by a free boundary. It would be interesting to extend the results in Section 7 to these situations but this remains to be done.

The problem would be even more realistic when considering nonlinear equations, as for instance the Navier-Stokes equations for the fluid. But, to the best of our knowledge, very little is known about the well-posedness and the long time behavior for the solutions to the corresponding equations (We refer to [14] and [15] for some existence results of weak solutions and and [38] for local smooth solutions in $3-d$).

• *Control problems.* In [196], we analyzed the null controllability problem for system (7.1) in one space dimension by means of spectral methods. It was found that the controllability results depend strongly on whether the control enters the system through the wave component or the heat one. When the control acts on the boundary of the wave interval one obtains null controllability in the energy space. However, when the control acts of the boundary of the heat interval, null controllability holds in a much smaller space. This is due to the existence of an infinite branch of eigenfunctions that are weakly dissipated and strongly concentrated on the wave interval. Therefore, the control affects these spectral components exponentially weakly at high frequencies. Because of this the initial data to be controlled need to have exponentially small Fourier coefficients on that spectral branch. This problem is completely open in several space dimensions. Two different situations need to be considered also in the multi-dimensional case depending on whether the control acts on a subset of the wave or heat domain. The answer to the problem may also depend on whether the set in which the control enters controls geometrically the whole domain or not. In $1-d$ this condition is automatically satisfied because the only possible rays are segments that cover the whole domain under propoagation.

- **Controllability of Stochastic PDE.** Extending the results be have presented in this article to the stochastic framework is a widely open subjet of research. There are however some interestingv results in this direction. We refer for instance to the article [180] where Carleman inequalities were proved for stochastic parabolic equations.

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