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GENERIC SIMPLICITY OF THE SPECTRUM AND STABILIZATION FOR A PLATE EQUATION*

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Abstract. In this work we prove the generic simplicity of the spectrum of the clamped plate equation in a bounded regular domain of $\mathbb{R}^d$. That is, given $\Omega \subset \mathbb{R}^d$, we show that there exists an arbitrarily small deformation of the domain u, such that all the eigenvalues of the plate system in the deformed domain $\Omega + u$ are simple. To prove this result we first prove a nonstandard unique continuation property for this system that also holds generically with respect to the perturbations of the domain. Both the proof of this generic uniqueness result and the generic simplicity of the spectrum use Baire's lemma and shape differentiation. Finally, we show an application of this unique continuation property to a result of generic stabilization for a plate system with one dissipative boundary condition.

Key words. spectral theory, plate equation, unique continuation property, stabilization

AMS subject classifications. 35P05, 35J40, 93D15

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1. Introduction and main results.

In this work we are interested in the study of the spectral properties for the plate system

$$
\begin{cases} \triangle^2 y = \lambda y & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with boundary of class C^4.

Problem (1.1) admits a sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \longrightarrow \infty,$$

which have finite multiplicity. The eigenfunctions $\{y_n\}_n \subset H_0^2(\Omega)$ of (1.1) can be chosen to form an orthonormal basis of $H_0^2(\Omega)$.

On the other hand, it is well known that the eigenvalues of (1.1) are not always simple. For instance, in [8], it is shown that the first eigenvalue is not simple in a suitable annular domain.

The problem of the simplicity of the spectrum arises in many contexts. This is, for instance, the case when analyzing stabilizability and controllability issues for evolution systems. When the spectrum is simple one can often reduce these problems to the analysis of suitable properties of eigenfunctions, which is an easier problem to deal with because of the lack of dependence with respect to time. We refer to J. L.

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Lions and E. Zuazua [21] for an example of a control problem where this strategy is successfully applied.

Concerning the simplicity of the spectrum, the problem we address is as follows. *Are there arbitrarily small deformations of the domain* $u \in W^{5,\infty}(\Omega, \mathbb{R}^d)$*, such that the spectrum of* (1.1) *in the deformed domain* $\Omega + u$ *is simple?*

In this paper we give a positive answer to this question. We show that the set of deformations of the domain $u \in W^{5,\infty}(\Omega, \mathbb{R}^d)$, such that the spectrum of the plate system is simple, is a dense subset of $W^{5,\infty}(\Omega, \mathbb{R}^d)$.

The generic simplicity of the spectrum of second order elliptic operators is by now well known. We refer to J. H. Albert [1] for perturbations of the coefficients of the operator and to A. M. Micheletti [22] and K. Uhlenbeck [29] for perturbations of the domain.

The result we prove in this paper and the methods we employ are inspired in [25] by the authors, where a similar result was proved for the $2-D$ Stokes system. The proof combines Baire's lemma and shape differentiation. These two tools reduce the problem to the obtainment of a suitable unique continuation property for the eigenfunctions. In the context of second order problems this uniqueness problem can be dealt with by means of Holmgrem's theorem. However, this is not the case when working with the plate equation. In this case the uniqueness problem cannot be analyzed in the context of the classical Cauchy problems since only three boundary conditions are known to vanish. We then proceed as in [25], showing that the uniqueness property holds generically with respect to the perturbations of the domain. But this turns out to be sufficient to complete the proof of the generic simplicity of the spectrum.

Our generic unique continuation property refers to the following uniqueness problem.

If y *solves* (1.1) *for some* $\lambda > 0$ *and*

$$\frac{\partial^2 y}{\partial n^2} = 0 \quad \text{on } \Gamma_0,$$

then, necessarily, $y \equiv 0$.

This property will be referred to as *spectral uniqueness.*

THEOREM 1.1. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain with boundary of class* C^4*. Let* Γ_0 *be an open nonempty subset of* $\partial\Omega$.

Then, the set of deformations of the domain $u \in W^{5,\infty}(\Omega, \mathbb{R}^d)$*, such that* $u = 0$ *on* $\partial\Omega \setminus \Gamma_0$ *and for which spectral uniqueness holds when* Ω *and* Γ_0 *are replaced by* $\Omega + u$ *and* $\Gamma_0 + u$*, is residual in*

$$W_0 = \{u \in W^{5,\infty}(\Omega, \mathbb{R}^d) : u = 0 \text{ on } \partial\Omega \backslash \Gamma_0\}.$$

In other words, it is a countable intersection of dense open sets of W_0*. In particular, it is dense in* W_0. □

Remark 1.1. (1) As far as we know, there is no example in the literature of bounded domain Ω and open subset Γ_0 of $\partial\Omega$ for which there exists a nontrivial eigenfunction of (1.1) such that the further condition

$$\frac{\partial^2 y}{\partial n^2} = 0 \quad \text{on } \Gamma_0 \tag{1.2}$$

is satisfied.

However, our result is of a generic nature, and, therefore, it does not apply to any domain Ω and open subset Γ_0 of $\partial\Omega$.

Therefore, whether this unique continuation property holds for any Ω and $\Gamma_0 \subset \partial\Omega$ remains an open problem.

(2) Using multiplier techniques (see Appendix I of [18], [19], and [13]), one can show that this unique continuation holds for any domain Ω provided Γ_0 is a subset of the boundary of the form

$$\Gamma_0 = \Gamma(x_0) = \{x \in \partial\Omega \,:\, (x - x_0) \cdot n(x) > 0\}\,.$$

In fact, multiplier methods allow us to show that uniqueness holds for the dynamic plate model, but always for subsets of the boundary of this particular form.

Note, however, that these subsets of the boundary are always large. In other words, there are many small subsets of the boundary Γ_0 that cannot be written in this form.

Consequently, our result is, as far we know, the first one that applies to arbitrarily small subsets of boundary, but it is of generic nature.

(3) Note also that the deformations we apply do deform the subset Γ_0 itself. Whether this unique continuation result applies when u preserves Γ_0 or not is an open problem. □

With the aid of this result we prove the main result on the generic simplicity of the spectrum.

THEOREM 1.2. *Let Ω be a bounded domain of $\mathbb{R}^d$ of class C^4. Let Γ_0 be an open nonempty subset of $\partial\Omega$.*

Then the set

$$A = \left\{u \in W^{5,\infty}(\Omega, \mathbb{R}^d) : u = 0 \text{ on } \partial\Omega \setminus \Gamma_0 \text{ and the spectrum of } (1.1) \text{ is simple}\right\}$$

is residual in

$$W_0 = \left\{u \in W^{5,\infty}(\Omega, \mathbb{R}^d) : u = 0 \text{ on } \partial\Omega \setminus \Gamma_0\right\}. \qquad \square$$

We also show an application of Theorem 1.1 to the study of the stabilization of the plate system with dissipative boundary conditions.

Let us consider the system

$$(1.3) \qquad \begin{cases} y_{tt} + \triangle^2 y = 0 & \text{in } \Omega \times (0,\infty)\,, \\ y = 0 & \text{on } \partial\Omega \times (0,\infty)\,, \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma_0 \times (0,\infty)\,, \\ \triangle y = -\dfrac{\partial y_t}{\partial n} & \text{on } \Gamma_0 \times (0,\infty)\,, \\ y\,(x,0) = y_0 & \text{in } \Omega, \\ y_t\,(x,0) = y_1 & \text{in } \Omega. \end{cases}$$

It is easy to see that for any $(y_0, y_1) \in X = X_1 \times X_2$ system (1.3) admits a unique solution $y \in C\left([0,\infty); X_1\right) \cap C^1\left([0,\infty); X_2\right)$. Here and in what follows $X_1 = \{\varphi \in H^2(\Omega) \cap H_0^1(\Omega) : \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma_0\}$ and $X_2 = L^2(\Omega)$.

We define the energy of the system as

$$(1.4) \qquad E\,(t) = \frac{1}{2}\int_\Omega \left[|y_t|^2 + |\triangle y|^2\right].$$

From the dissipative boundary conditions of (1.3) we have that

$$\frac{d}{dt}E(t) = -\int_{\Gamma_0} |\triangle y(x,t)|^2 \, d\Gamma \leq 0. \tag{1.5}$$

Thus, we deduce that the energy decreases for any solution y of (1.3) as $t \to \infty$. Therefore, the following question arises naturally.

Does the energy of any solution of (1.3) *tend to zero as* $t \to \infty$*? In other words, does*

$$E(t) \to 0 \quad \text{as } t \to \infty \tag{1.6}$$

for every solution of (1.3)*?*

We will see that the stabilization property (1.6) holds generically with respect to the domain Ω, as in the following theorem.

THEOREM 1.3. *Let* $\Omega \subset \mathbb{R}^d$ *be an open bounded set of class* C^4*, and let* Γ_0 *be an open nonempty subset of the boundary, such that the spectral uniqueness property holds.*

Then, every solution of (1.3) *verifies that*

$$E(t) \to 0 \quad \text{as } t \to +\infty. \quad \square \tag{1.7}$$

To prove this result we use La Salle's invariance principle, which allows us to reduce our stabilization problem to a unique continuation property. This unique continuation problem turns out to be the one we have solved generically in Theorem 1.1, i.e., the spectral uniqueness problem.

We could also consider system (1.3) with two dissipative boundary conditions instead of one. Consider, for instance,

$$\begin{cases} y = \dfrac{\partial y}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma_0 \times (0,\infty), \\ \triangle y = -\dfrac{\partial y_t}{\partial n} & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial \triangle y}{\partial n} = y_t & \text{on } \Gamma_0 \times (0,\infty). \end{cases} \tag{1.8}$$

Then the stabilization result above holds for every domain Ω and for every nonempty open subset $\Gamma_0 \subset \partial\Omega$. In this case, by means of La Salle's invariance principle, we can reduce our problem to a unique continuation one that may be solved by applying the classical uniqueness theorem by Holmgrem. Indeed, in this case, we add the extra boundary conditions $\triangle y = \frac{\partial y}{\partial n} = 0$ on Γ_0 to the solutions of (1.1), in which case we are in the context of the classical Cauchy problem.

Remark 1.2. The methods developed in this paper may be applied to other plate systems, such as when the plate equation is replaced by the one taking into account the rotational inertia term, i.e.,

$$y_{tt} - \gamma \triangle y_{tt} + \triangle^2 y = 0,$$

with $\gamma > 0$. Other boundary conditions may also be considered. For instance, the condition $\frac{\partial y}{\partial n} = 0$ on $\partial\Omega \setminus \Gamma_0$ may be replaced by $\triangle y = 0$ on $\partial\Omega \setminus \Gamma_0$.

The same result applies as well to plate equations with nonlinear monotone boundary damping. $\square$

Remark 1.3. When Γ_0 is a subset of the boundary of the form $\Gamma(x_0)$ we indicated in Remark 1.1, uniform stability properties may be proved for the systems under consideration, i.e.,

$$E(t) \le Ce^{-\alpha t}E(0)$$

for suitable $C, \alpha > 0$ (see, for instance, [15]). We do not address this problem here. In any case, one does not expect, in general, exponential decay to hold when the dissipative term acts on a small subset of the boundary. □

The rest of this work is organized as follows. In section 2 we present some preliminary results on the variational formulation of the plate equation and shape differentiation. In section 3 we prove a result of existence and regular dependence of the branches of eigenvalues and eigenfunctions of the bilaplacian with respect to the perturbation of the domain. In section 4 we compute the local variations of the eigenvalues and eigenfunctions of the bilaplacian. In section 5 we prove the unique continuation property of Theorem 1.1. In section 6 we prove the simplicity of the spectrum of Theorem 1.2. Finally, in section 7 we prove the stabilization result of Theorem 1.3.

2. Preliminaries.

2.1. Baire's lemma. First we remember the Baire's lemma, which will be a useful tool.

LEMMA 2.1 (Baire's lemma). *Let X be a complete metric space, and let A_n be an open dense subset of X for all $n \in \mathbb{N}$.*

Then $\cap_{n\in\mathbb{N}} A_n$ is dense in X. □

A direct consequence of Baire's lemma is the following result.

LEMMA 2.2. *Let X be a complete metric space, and let $\{A_n\}_{n\ge 0}$ be a sequence of open subsets of X such that*

1. *$A_0 = X$, and*
2. *A_{n+1} is a dense subset of A_n for all $n \ge 0$.*

Then $\cap_{n=1}^{\infty} A_n$ is dense in X. □

2.2. Variational formulation of the plate system. The variational formulation of the eigenvalue problem (1.1) is as follows: to find $y \in H_0^2(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$\int_\Omega \triangle y \triangle \varphi = \lambda \int_\Omega y\varphi \qquad \forall \varphi \in H_0^2(\Omega). \tag{2.1}$$

This variational eigenvalue problem can be handled in a standard way.

It is well known that there exists a positive constant $c > 0$ such that

$$\|f\|^2_{H^2(\Omega)} \le c \int_\Omega |\triangle f|^2 \quad \forall f \in H_0^2(\Omega). \tag{2.2}$$

Then

$$|f|_2 = \left(\int_\Omega |\triangle f|^2\right)^{\frac{1}{2}} \tag{2.3}$$

defines a norm in $H_0^2(\Omega)$, equivalent to the one induced by the norm of $H^2(\Omega)$.

Thus

$$b : H_0^2(\Omega) \times H_0^2(\Omega) \to \mathbb{R}, \quad b(\phi, \varphi) = \int_\Omega \triangle\phi \, \triangle\varphi \tag{2.4}$$

is a coercive and continuous bilinear form. Then, for each $f \in H^{-2}(\Omega)$, there exists a unique solution $y \in H_0^2(\Omega)$ of the problem

$$\int_\Omega \triangle y \triangle \varphi = \langle f, \varphi \rangle_{H^{-2} \times H_0^2} \qquad \forall \varphi \in H_0^2(\Omega).$$

This shows the existence and uniqueness of weak solutions of the elliptic problem

$$\begin{cases} \triangle^2 y = f & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Using the compactness of the imbedding $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$, one can show that the map $f \in L^2(\Omega) \to y \in L^2(\Omega)$ is compact. It is also easy to see that it is self-adjoint. Applying the classical spectral theory for compact, self-adjoint operators, we deduce that the eigenvalue problem (2.1) admits a sequence of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to +\infty.$$

Moreover, each eigenvalue has finite multiplicity, and the eigenfunctions $y_i \in H^4(\Omega) \cap H_0^2(\Omega)$ can be chosen to form an orthonormal basis of $L^2(\Omega)$.

2.3. Shape differentiation. An important tool for the study of the generic properties is the *shape differentiation.* For more details about this technique, we refer to [4], [27], [28], and the bibliographies therein.

Given a domain Ω and a function $u : \Omega \to \mathbb{R}^d$, we define the new domain $\Omega + u$ by

$$\Omega + u = \{y \in \mathbb{R}^d : y = x + u(x), \ x \in \Omega\}. \tag{2.6}$$

Let us consider perturbations u in the space $W^{k,\infty}(\Omega, \mathbb{R}^d)$ with norm

$$\|u\|_{k,\infty} = \sup_{0 \le |\alpha| \le k, \, x \in \Omega} \text{ess} \, |D^\alpha u(x)|.$$

The following results are well known.

LEMMA 2.3 (see [28]). *Let $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$, and let $k \ge 1$ be such that $\|u\|_{k,\infty} \le \frac{1}{2}$. Then the map $(I + u) : \Omega \to \Omega + u$ is invertible. Furthermore, there exists $w \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ such that $(I + u)^{-1} = I + w$ and $\|w\|_{k,\infty} \le C_k \|u\|_{k,\infty}$, where C_k is a constant independent on u.* □

Remark 2.1. According to this result, if Ω is of class C^j, we can choose $k = j + 1$ (and therefore the perturbation space $W^{k,\infty}(\Omega, \mathbb{R}^d)$) such that our new domain $\Omega + u$ is also of class C^j. In particular, if Ω is of class C^4, then $\Omega + u$ is also of class C^4, and the solutions of the eigenvalue problem for the bilaplacian in the new domain $\Omega + u$ satisfy $y(u) \in H^4(\Omega + u) \cap H_0^2(\Omega + u)$ for every $u \in W^{5,\infty}(\Omega, \mathbb{R}^d)$ small enough. This is the functional framework we shall work in. □

LEMMA 2.4 (see [28]). *Let $k \geq 1$, and consider the function*

$$\begin{array}{rrcl}\gamma: & W^{k,\infty}(\Omega,\mathbb{R}^d) & \to & W^{k-1,\infty}(\Omega,\mathbb{R}),\\ & u & \to & \gamma(u) = Jac\,(I+u) = \left|\det\left[\partial_j\,(I+u)_i\right]\right|.\end{array}$$

This function γ is differentiable at $u=0$. Furthermore, the directional derivative in the direction w at the point $u=0$ is $\operatorname{div} w$; *that is,*

$$\langle D\gamma(0), w\rangle = \operatorname{div}\, w \quad \forall\, w \in W^{k,\infty}(\Omega,\mathbb{R}^d). \qquad \square$$

LEMMA 2.5 (see [28]). *Let $k \geq 1$. The map*

$$\begin{array}{rrcl}\beta: & \mathcal{W} \subset W^{k,\infty}(\Omega,\mathbb{R}^d) & \to & \mathcal{M}_{d\times d}\left(W^{k-1,\infty}(\Omega,\mathbb{R})\right),\\ & u & \to & {}^t\left[\partial_j\,(I+u)_i\right]^{-1},\end{array}$$

where $\mathcal{W}$ is a neighborhood of $u=0$ on $W^{k,\infty}(\Omega,\mathbb{R}^d)$, is differentiable on $u=0$. Its directional derivative on $u=0$ in the direction w is given by the matrix $-^t\left[\partial_j w_i\right]$, where ${}^t\left[\partial_j w_i\right]$ denotes the adjoint of $\left[\partial_j w_i\right]$. In other words,

$${}^t\left[\partial_j\,(I+u)_i\right]^{-1} = [I] -^t \left[\partial_j u_i\right] + \theta(u),$$

where the matrix $\theta(u)$ satisfies

$$\frac{\|\theta(u)\|_{k-1,\infty}}{\|u\|_{k,\infty}} \to 0 \quad \text{as} \quad \|u\|_{k,\infty} \to 0. \qquad \square$$

Now, we consider a function

$$\begin{array}{rrcl}v: & W^{k,\infty}(\Omega,\mathbb{R}^d) & \to & W^{m,r}(\Omega+u),\\ & u & \to & v(u),\end{array}$$

where $1 \leq r < \infty$ and $m \leq k$ are integer numbers. In practice, $v(u)$ will be the solution of a suitable problem, which depends on the perturbation function u (for instance, a solution of our eigenvalue problem (1.1)).

We are interested in the study of the regularity of the function $v(u)$ with respect to the perturbation parameter u.

DEFINITION 2.1 (see [28]). *Let $k \geq m \geq 1$, $1 \leq r < \infty$. We say that the function $v(u)$ has a first order local variation at $u=0$ on $W^{m-1,r}_{loc}(\Omega)$ if $v(u) \in W^{m,r}(\Omega+u)$ for all $u \in W^{k,\infty}(\Omega,\mathbb{R}^d)$ and there exists a linear map $v'(\Omega;u)$ defined from $u \in W^{k,\infty}(\Omega,\mathbb{R}^d)$ to $W^{m-1,r}_{loc}(\Omega)$ such that, for each open set $\omega \subset\subset \Omega$,*

$$v(u) = v(0) + v'(\Omega;u) + \widehat{\theta}(u) \quad \text{in } \omega,$$

when $\|u\|_{k,\infty}$ is small enough and

$$\frac{\widehat{\theta}(u)}{\|u\|_{k,\infty}} \to 0 \quad \text{in } W^{m-1,r}(\omega) \quad \text{as } \|u\|_{k,\infty} \to 0. \qquad \square$$

Remark 2.2. From Definition 2.1 it follows that the *first local variation* can be defined as

$$v'(\Omega;u) = \lim_{t\to 0} \frac{v(tu)|_\omega - v(0)|_\omega}{t} \quad \text{in } \omega, \tag{2.7}$$

where $\omega \subset\subset \Omega$ and $v(tu)|_\omega$, $v(0)|_\omega$ are the restrictions of the functions $v(tu)$ and $v(0)$ to ω. □

In what follows, to simplify the notation, we will write $v'(u) = v'(\Omega; u)$.

The following theorem provides sufficient conditions for the existence of the first local variation for functions which depend on the deformation u. Furthermore, it provides an expression for the local variation on the boundary in terms of the normal derivative of $v(0)$.

THEOREM 2.6 (see [28]). *Let Ω be a $C^{0,1}$ domain. Consider a map $u \to v(u) \in W^{m,r}(\Omega + u)$ defined on a neighborhood of $u = 0$ in $W^{k,\infty}(\Omega, \mathbb{R}^d)$, with $k \geq m \geq 1$ and $1 \leq r < \infty$.*

Let us assume that there exists a linear continuous map $u \to \dot{v}(u)$ defined on $W^{k,\infty}(\Omega, \mathbb{R}^d)$ with values in $W^{m,r}(\Omega)$, such that

$$v(u) \circ (I + u) = v(0) + \dot{v}(u) + \theta(u) \ \ \text{in } W^{m,r}(\Omega)$$

for all $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ small enough, where

$$\frac{\theta(u)}{\|u\|_{k,\infty}} \to 0 \quad \text{on } W^{m,r}(\Omega) \quad \text{as } \|u\|_{k,\infty} \to 0.$$

Furthermore, assume that for each $u \in W^{k,\infty}(\Omega, \mathbb{R}^d)$ small enough,

$$v(u) = 0 \quad \text{on} \quad \partial(\Omega + u).$$

Then, for each $\omega \subset\subset \Omega$, the function $u \to v_\omega(u) = v(u)|_\omega$, defined on a neighborhood of $u = 0$ in $W^{k,\infty}(\Omega, \mathbb{R}^d)$ with values in $W^{m-1,r}(\omega)$, is differentiable at $u = 0$.

Moreover, the map $u \to v(u)$ has a local derivative at $u = 0$ (see Definition 2.1) and the local derivative at $u = 0$, in the direction u, denoted by $v'(u)$, verifies $v'(u) \in W^{m-1,r}(\Omega)$ and

$$v'(u) = -(u \cdot n)\frac{\partial v(0)}{\partial n} \quad \text{on} \quad \partial\Omega,$$

where n is the unit outward normal vector to Ω. □

In what follows we will use the notation

$$\mathcal{W} = \{u \in W^{k,\infty}(\Omega, \mathbb{R}^d) : \|u\|_{k,\infty} < c_\Omega\},$$

where $k \geq 1$ and $c_\Omega < 1/2$ is small enough such that all the previous results hold.

LEMMA 2.7 (see [4, Lemma 9]). *Let $u \in \mathcal{W}$. If $f \in H_0^1(\Omega + u)$, there exists a unique $g \in H_0^1(\Omega)$ such that $f \circ (I + u) = g$. Moreover,*

$$\left(\frac{\partial f}{\partial z_i}\right) \circ (I + u) = \sum_j M_{ij}(u)\frac{\partial g}{\partial x_j} = D_i(u)\, g, \tag{2.8}$$

where the matrix $M(u)$ is defined as

$$M(u) = [M_{ij}(u)] = {}^t\left[\frac{\partial}{\partial x_j}(I + u)_i\right]^{-1},$$

and

$$z_i = x_i + u_i(x) \quad \forall\, x \in \Omega. \qquad \square$$

3. Regularity of the eigenvalues and eigenfunctions.

3.1. Some results of spectral theory. To prove the existence and regularity of the eigenvalues and eigenfunctions of the plate system with respect to the perturbation parameter u, we will use the Lyapunov–Schmidt method (see [30], [7, p. 30]).

LEMMA 3.1 (see [7, Lemma 4.1, p. 31]). *Suppose that X and Z are Hilbert spaces and $A : X \longrightarrow Z$ is a continuous linear operator. Let $U : X \longrightarrow N(A)$, $E : Z \longrightarrow R(A)$ be the orthogonal projections from X and Z on the kernel and range of A, respectively.*

Then, there exists a bounded linear operator $Q : R(A) \longrightarrow N(A)^{\perp}$, called the right inverse of A, such that

$$AQ = I : R(A) \longrightarrow R(A), \qquad QA = I - U : Z \longrightarrow N(A)^{\perp}.$$

Let Λ be a closed subset of a Banach space, such that $Int\Lambda \neq \emptyset$. If $N : \Lambda \times X \longrightarrow Z$ is a continuous operator, then the problem

$$Ax - N(x, \lambda) = 0 \tag{3.1}$$

is equivalent to the equations

$$z - QEN(y + z, \lambda) = 0, \tag{3.2}$$

$$(I - E)N(y + z, \lambda) = 0, \tag{3.3}$$

where $x = y + z$, $y \in N(A)$, and $z \in N(A)^{\perp}$. □

Assume that the operator N verifies that

$$N(0, 0) = 0, \qquad \frac{\partial N}{\partial x}(0, 0) = 0,$$

and consider (3.2) for (x, λ) in a neighborhood of $(0, 0)$ in $X \times \Lambda$. Applying the implicit function theorem to (3.2), we deduce the existence of a neighborhood $V \subset N(A) \times \Lambda$ of $(0, 0)$ and a function $z^* : V \longrightarrow N(A)^{\perp}$ with the same regularity of N providing the solution of (3.1). Therefore, if $\{y_1, \dots, y_h\}$ is an orthonormal basis of $N(A)$, the solution $x(\lambda)$ of (3.1) satisfies

$$x(\lambda) = \sum_{i=1}^{h} c_i(\lambda) y_i + z^* \left(\sum_{i=1}^{h} c_i(\lambda) y_i, \lambda \right) = 0 \tag{3.4}$$

for suitable coefficients $c_1, \dots, c_h$. Then, $(x, \lambda) \in V$ satisfy (3.1) iff

$$(I - E)\, N \left(\sum_{i=1}^{h} c_i(\lambda) y_i + z^* \left(\sum_{i=1}^{h} c_i(\lambda) y_i, \lambda \right), \lambda \right) = 0, \tag{3.5}$$

which is a finite dimensional system of equations on the constants $c_1, \dots, c_h$.

Now we have the following result, which is a slight variation of a theorem due to J. H. Albert [2].

THEOREM 3.2. *Let E be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let Λ be a Banach space. Let $P : D(P) \subset E \to E$ be a self-adjoint operator densely defined* *in E. Assume that λ is an eigenvalue of multiplicity h of P, and let $\phi_1, \dots, \phi_h$ be the orthonormal eigenfunctions associated to λ. Moreover, assume that there exists a bounded linear operator $Q : E \to E$, such that $Q\Pi_N = 0$ and $Q(P+\lambda) = I - \Pi_N$, Π_N being the orthogonal projection in $N = \operatorname{Ker}(P+\lambda)$.*

Let $R(u)$ be an analytic self-adjoint map in $B(E,F)$ for every u in a neighborhood of $u=0$ in Λ, such that $R(0)=0$ and $P(u) = P + R(u)$.

Then there exist h analytic functions defined in a neighborhood of $u=0$ in Λ with values in $\mathbb{R}$, $u \to \lambda_i(u)$, and h analytic functions $u \to \phi_i(u)$, with values in E, $i = 1, \dots, h$, defined in a neighborhood of $u=0$ in Λ, such that the following hold.

1. $\lambda_j(0) = \lambda, \qquad j = 1, \dots, h.$
2. *For all u small enough, $(\lambda_j(u), \phi_j(u))$ is a solution of the eigenvalue problem $P(u)\phi_j(u) = \lambda_j(u)\phi_j(u)$.*
3. *For all u small enough the set $\{\phi_1(u), \dots, \phi_h(u)\}$ is orthonormal in E.*
4. *For each interval $I \subset \mathbb{R}$ such that $\overline{I}$ contains only the eigenvalue λ of (P), there exists a neighborhood U of $u=0$ such that there are exactly h eigenvalues (counting the multiplicity) $\lambda_1(u), \dots, \lambda_h(u)$ of (P_u) contained on I.* ◻

Remark 3.1. To prove Theorem 3.2, we prove first that we can find functions $u \to \lambda(u) \in \mathbb{R}$ and $u \longrightarrow \phi(u)$ such that $\phi(u)$ is an eigenfunction of (P_u) associated to the eigenvalue $\lambda(u)$. To find the other $h-1$ branches of eigenvalues and eigenfunctions, we apply in an iterative form the method described in the following proposition. ◻

Proposition 3.3. *Under the hypotheses of Theorem* 3.2*, if λ is an eigenvalue of multiplicity h of P and $\phi_1, \dots, \phi_h$ are orthonormal eigenfunctions associated to λ, there exists at least a function $u \longrightarrow (\lambda(u), \phi(u)) \in \mathbb{R} \times E$ which is analytic in a neighborhood of $u=0$ in Λ such that*

1. $\lambda(0) = \lambda$, *and*
2. *$\phi(u)$ is an eigenfunction of $P(u)$, associated to the eigenvalue $\lambda(u)$.* ◻

Proof of Proposition 3.3. Let λ be an eigenvalue of multiplicity h of P, and let $\phi_1, \dots, \phi_h$ be the orthonormal eigenfunctions associated to λ.

Suppose that the maps $u \to \lambda(u)$, $u \to \phi(u)$, such that

$$(P(u) + \lambda(u))\,\phi(u) = 0, \tag{3.6}$$

do exist. Then

$$\begin{aligned}(P+\lambda)\,\phi(u) &= (P + \lambda - P(u) - \lambda(u) + P(u) + \lambda(u))\,\phi(u)\\ &= (-R(u) + \lambda - \lambda(u))\,\phi(u)\\ &= -(R(u) + \lambda(u) - \lambda)\,\phi(u).\end{aligned} \tag{3.7}$$

Since $Q(P+\lambda) = I - \Pi_N$, we obtain that

$$\phi(u) = -\left[Q\left(R(u) + \lambda(u) - \lambda\right)\right]\phi(u) + \psi(u), \tag{3.8}$$

where $\psi(u) \in N = \operatorname{Ker}(P+\lambda)$.

Thus

$$\psi(u) = \left[I + Q\left(R(u) + \lambda(u) - \lambda\right)\right]\phi(u), \tag{3.9}$$

and therefore

$$\phi(u) = \left[I + Q\left(R(u) + \lambda(u) - \lambda\right)\right]^{-1}\psi(u). \tag{3.10}$$

Moreover, the map $[I+Q\,(R\,(u)+\lambda\,(u)-\lambda)]$ has an inverse in a neighborhood of $u=0$ in Λ.

Thus, if we know the functions $u\rightarrow\lambda\,(u)$ and $u\rightarrow\psi\,(u)$, we can obtain the map $u\rightarrow\phi\,(u)$.

Let $\phi_1,\ldots,\phi_h$ be an orthonormal basis of $N=\mathrm{Ker}\,(P+\lambda)$. We must find constants $c_j(u)$ such that

$$\psi\,(u)=\sum_{j=1}^{h}c_j\,(u)\,\phi_j. \tag{3.11}$$

We can see that

$$[R\,(u)+\lambda\,(u)-\lambda]\,\phi\,(u)\in N^{\perp}, \tag{3.12}$$

because, according to (3.6), we have that $[R\,(u)+\lambda\,(u)-\lambda]\,\phi\,(u)\in R\,(P+\lambda)$.

Thus

$$\begin{aligned}0&=\langle[R\,(u)+\lambda\,(u)-\lambda]\,\phi\,(u)\,,\phi_j\rangle\\&=\langle[R\,(u)+\lambda\,(u)-\lambda]\,\{I+Q\,[R\,(u)+\lambda\,(u)-\lambda]^{-1}\}\psi\,(u)\,,\phi_j\rangle\\&=\sum_{i=1}^{h}c_j\,(u)\,\langle[R\,(u)+\lambda\,(u)-\lambda]\,\{I+Q\,[R\,(u)+\lambda\,(u)-\lambda]^{-1}\}\phi_i,\phi_j\rangle,\end{aligned} \tag{3.13}$$

which is a linear system of equations on the unknowns $c_j(u)$.

This system has a nontrivial solution iff

$$\det(\langle[R\,(u)+\lambda\,(u)-\lambda]\,\{I+Q\,[R\,(u)+\lambda\,(u)-\lambda]^{-1}\}\phi_i,\phi_j\rangle)=0. \tag{3.14}$$

Now we show the existence of the constants $c_1\,(u)\,,\ldots,c_h\,(u)$, not all of them being zero simultaneously.

We replace $\lambda(u)-\lambda$ by α, and we define

$$f_{ij}\,(\alpha,u)=\langle[R\,(u)+\alpha]\{I+Q[R\,(u)+\alpha]^{-1}\}\phi_i,\phi_j\rangle \tag{3.15}$$

and

$$F\,(\alpha,u)=\det\left(f_{ij}\,(\alpha,u)\right). \tag{3.16}$$

For u small enough, the map $u\longrightarrow[I+Q\,[R(u)+\alpha])]^{-1}$ is well defined. Indeed, for $\alpha=0$ and $u=0$ we have that $[I+QR(0)]=I$, and the map is analytic in a neighborhood of $u=0$ in Λ. On the other hand, as we mentioned above, if $F(\alpha,u)=0$, system (3.13) has a nontrivial solution $c_1(u),\ldots,c_h(u)$, and then

$$\lambda(u)=\lambda+\alpha \tag{3.17}$$

is an eigenvalue of $P(u)$.

Moreover, from (3.8) and (3.11) we deduce that

$$\phi(u)=\sum_{j=1}^{h}c_j(u)\,[I+Q\,(R(u)+\lambda(u)-\lambda)]^{-1}\,(v_j,p_j) \tag{3.18}$$

is an eigenfunction of $P(u)$ associated to the eigenvalue $\lambda(u)$.

According to our previous discussion, for these values of $\alpha(u)$ and setting $\lambda(u) = \lambda + \alpha(u)$, system (3.13) admits a solution $c_1(u), \dots, c_h(u)$, not all the components being zero simultaneously. We have that

$$
\begin{aligned}
f_{ij}(\alpha, 0) &= \langle [R(0) + \alpha] \{I + Q[R(0) + \alpha]\}^{-1} \phi_i, \phi_j \rangle \\
&= \langle \alpha \{I + \alpha Q\}^{-1} \phi_i, \phi_j \rangle \\
&= \alpha \langle \{I + \alpha Q\}^{-1} \phi_i, \phi_j \rangle \\
&= \alpha \delta_{ij},
\end{aligned} \tag{3.19}
$$

because $[I + \alpha Q] \phi_i = \phi_i$.

Therefore, we have that $F(\alpha, 0) = \det(\alpha I) = \alpha^h$.

Applying the Weierstrass preparation theorem, we deduce that

$$
F(\alpha, u) = \left(\alpha^h + a_1(u)\alpha^{h-1} + \dots + a_h(u)\right) E(\alpha, u)
$$

with $E(\alpha, u) \neq 0$ in a neighborhood of $(0, 0)$. Then for (α, u) small enough we have that $E(\alpha, u) \neq 0$, and functions $a_j(u)$ are analytic in a neighborhood of $u = 0$.

Then, $F(\alpha, u) = 0$ iff

$$
\alpha^h + a_1(u)\alpha^{h-1} + \dots + a_h(u) = 0. \tag{3.20}
$$

Let $\alpha_j(u), j = 1, \dots, h$ be the complex roots of (3.20). Then there exist constants $c_1(u), \dots, c_h(u)$, not all vanishing simultaneously, which are the solution of system (3.13).

Thus, from (3.18) we obtain that

$$
\phi(u) = \sum_{j=1}^{h} c_j(u) \left[I + Q\left(R(u) + (\lambda(u) - \lambda)\right)\right]^{-1} \phi_j
$$

and $\lambda(u) = \lambda + \alpha_1(u)$ constitute an eigenpair.

Notice that if $c_j(u)$ is complex, it is enough to consider the real part $\mathcal{R}c_j(u)$ to get a real eigenfunction. Since the operator $P(u)$ is self-adjoint, we have that $\alpha_j(u)$ is real, which completes the proof of Proposition 3.3. □

Remark 3.2. Proposition 3.3 provides the existence of one branch of eigenpairs associated to the root $\alpha(u)$ of (3.20). We do not use the eigenpairs associated to the other roots α_j by now since, so far, we do not know whether they coincide or not with the eigenpair associated to $\alpha_1(u)$. □

Now we prove Theorem 3.2.

Proof of Theorem 3.2. Using induction on h, we prove the existence of the h analytic functions $u \to (\lambda_i(u), \phi_i(u))$, such that

$$
(P(u) + \lambda_i(u)) \phi_i(u) = 0. \tag{3.21}
$$

From Proposition 3.3, there exists an analytic function $u \to (\lambda_1(u), \phi_1(u))$ defined in a neighborhood of $u = 0$ in Λ with values in $\mathbb{R} \times E$, which verifies (3.21).

Therefore, Theorem 3.2 holds for $h = 1$. We must prove it for $h \geq 2$.

Let $\Pi_1(u) : E \longrightarrow E$ be the orthogonal projection on the eigenspace generated by $\phi_1(u)$. Then we define the map

$$
B(u) = P(u) - \Pi_1(u). \tag{3.22}
$$

Then

$$B(0)\phi_j = (P(0) - \Pi_1(0))\,\phi_j = \lambda\phi_j - \delta_{1j}\phi_j; \tag{3.23}$$

that is,

$$B(0)\phi_j = \lambda\phi_j, \quad j = 2, \dots, h,$$

and

$$B(0)\phi_1 = (\lambda - 1)\phi_1.$$

Then, λ is an eigenvalue of multiplicity $h-1$ of the operator $B = B(0)$, with eigenfunctions $v_2, \dots, v_h$.

Note that other linearly independent eigenfunctions of B associated to λ do not exist. Indeed, if ϕ is another eigenfunction of B associated to the eigenvalue λ such that $\langle \phi, \phi_j \rangle = 0$, $j = 2, \dots, h$, then $\langle \phi, \phi_1 \rangle = 0$ (since ϕ_1 is an eigenfunction associated to the eigenvalue $\lambda - 1$) and $B\phi = \lambda\phi$. Then

$$P\phi = B\phi + \Pi_1\phi = B\phi + \langle \phi, \phi_1 \rangle v_1 = \lambda\phi;$$

that is, ϕ is an eigenfunction of P associated to λ, and thus λ is an eigenvalue of multiplicity $h+1$, which is impossible because the multiplicity of λ is h.

We can see that $B(u)$ satisfies the hypotheses of Proposition 3.3. This allows us to apply Proposition 3.3 in an iterative form and to obtain $h-1$ analytic functions in a neighborhood of $u = 0$ in Λ, $u \longrightarrow \lambda_i(u)$, and $u \longrightarrow \phi_i(u)$, with $i = 1, \dots, h$ such that

$$B(u)\phi_i(u) = \lambda_i(u)\phi_i(u).$$

Moreover, the functions $\phi_2(u), \dots, \phi_h(u)$ form an orthonormal set in E.

This shows us the existence of the h branches of eigenpairs.

Now we prove the last part of the theorem.

Since the eigenvalues $u \to \lambda_i(u)$ are analytic in a neighborhood of $u = 0$, there exist constants c_i such that

$$|\lambda_i(u) - \lambda_i(v)| \le c_i \|u - v\|.$$

Let $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n \dots$ be the eigenvalues of the P, and assume that

$$\dots \le \lambda_{n-1} < \lambda = \lambda_n = \dots = \lambda_{n+h-1} < \lambda_{n+h} \le \cdots.$$

Let $I \subset \mathbb{R}$ be an interval such that λ is the unique eigenvalue contained in I.

Then there exists $\delta > 0$ such that $I \subset (\lambda_{n-1} + \delta, \lambda_{n+h} - \delta)$. Let $u \in B(0, \delta/c)$, with $c = \max\{c_i : i = 1, \dots, n+h\}$. Then

$$|\lambda_{n-1}(u) - \lambda_{n-1}| \le c_{n-1}\|u\| < c_{n-1}\frac{\delta}{c} \le \delta,$$

and

$$|\lambda_{n+h}(u) - \lambda_{n+h}| \le c_{n+h}\|u\| < c_{n+h}\frac{\delta}{c} \le \delta.$$

Therefore, $\lambda_{n-1}(u) \notin \overline{I}$ and $\lambda_{n+h}(u) \notin \overline{I}$; that is, $P(u)$ has at most h eigenvalues contained in $\overline{I}$ counting multiplicity. This completes the proof of Theorem 3.2. ∎

3.2. Equivalent formulation for the plate system. Now, for each $u \in W^{5,\infty}(\Omega, \mathbb{R}^d)$, we consider the eigenvalue problem for the plate system

$$\begin{cases} \triangle^2 y &= \lambda y \qquad \text{in } \Omega + u, \\ y &= 0 \qquad \text{on } \partial(\Omega + u), \\ \dfrac{\partial y}{\partial n} &= 0 \qquad \text{on } \partial(\Omega + u). \end{cases} \tag{3.24}$$

Let $\{\lambda(u), y(u)\}$ be a solution of (3.24), and define the function $Y(u) = y(u) \circ (I + u)$.

Thus, our problem is to find $\lambda(u) \in \mathbb{R}$ and $Y(u)$ such that

$$\begin{cases} D_j^2(u)\left(Jac(I+u)\, D_i^2(u)\, Y(u)\right) &= \lambda(u)\, Y(u)\, Jac(I+u) \qquad \text{in } \Omega, \\ Y(u) &\in H_0^2(\Omega), \end{cases} \tag{3.25}$$

where

$$D_i(u)\, g = \sum_j M_{ij}(u)\, \partial_j f,$$

with $M_{ij}(u)$ defined as in (2.8) and $g = f \circ (I + u)$.

3.3. Regularity of the eigenvalues and eigenfunctions. Now, we will see the existence of the branches of eigenvalues $u \to \lambda(u)$ and eigenfunctions $u \to y(u)$ of the plate system.

LEMMA 3.4. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set of class C^4.*

Then the map

$$P : W^{5,\infty}(\Omega, \mathbb{R}^d) \longrightarrow \mathcal{L}\left(H_0^2(\Omega); H^{-2}(\Omega)\right),$$

such that

$$P(u)\phi = \frac{1}{Jac(I+u)} D_j^2(u)\left(Jac(I+u)\, D_i^2(u)\, \phi\right) \tag{3.26}$$

is analytic in a neighborhood of $u = 0$ in $W^{5,\infty}(\Omega, \mathbb{R}^d)$. □

Proof. We can see that $Jac(I + u)$ is a polynomial on the first partial derivatives of u. Then the map $u \longrightarrow Jac(I + u)$ is analytic in a neighborhood of $u = 0$ in $W^{5,\infty}(\Omega, \mathbb{R}^d)$. On the other hand,

$$M_{ij}(u) = \frac{1}{Jac(I+u)}(\delta_{ij} + a_{ij}),$$

where $a_{ij}(u)$ is the minor of the matrix $M^{-1}(u)$ associated to its ijth element, which is also a polynomial of the first partial derivatives of u. Moreover, for u small enough $Jac(I + u) > 0$. Therefore, $u \longrightarrow M(u)$ is analytic in a neighborhood of $u = 0$ in $W^{5,\infty}(\Omega, \mathbb{R}^d)$ as well.

Since $D(u)\varphi = M(u)\nabla\varphi$, from the analyticity of the functions $u \longrightarrow Jac(I + u)$ and $u \longrightarrow M(u)$ we obtain that the map $P(u)$ is analytic in a neighborhood of $u = 0$ on $W^{5,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ with values in $\mathcal{L}\left(H_0^2(\Omega); H^{-2}(\Omega)\right)$, and the proof is complete. □

Now, we can apply Theorem 3.2 to the operator $P(u)$ to obtain the existence of the h analytic branches $u \to (\lambda_i(u), y_i(u))$, with $i = 1, \dots, h$.

From Lemma 3.1 we have that the map $A = P - \lambda I$ has a right inverse operator Q which satisfies the hypotheses of Theorem 3.2. Furthermore, we have that if $u \in W^{5,\infty}(\Omega, \mathbb{R}^d)$, the new domain $\Omega + u$ has a boundary of class C^4, and then, the eigenfunctions satisfy that $y_i \in H^4(\Omega) \cap H_0^2(\Omega)$. Thus we have the following result.

THEOREM 3.5. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain of class C^4. Let λ be an eigenvalue of multiplicity h of the plate system* (1.1) *for $u = 0$ with associated eigenfunctions $y_1, \dots, y_h$.*

Then, there exist h analytic functions with values in $\mathbb{R}$, $u \to \lambda_i(u)$, and h analytic functions $u \to y_i(u)$, with values in $H^4(\Omega + u) \cap H_0^2(\Omega + u)$, $i = 1, \dots, h$, defined in a neighborhood of $u = 0$ in $W^{5,\infty}(\Omega, \mathbb{R}^d)$, such that the following hold.

1. $\lambda_j(0) = \lambda, \qquad j = 1, \dots, h.$
2. *For all u small enough, $(\lambda_j(u), y_j(u))$ is a solution of the plate system defined in the new domain $\Omega + u$.*
3. *For all u small enough, the set $\{y(u), \dots, y(u)\}$ is orthonormal in $L^2(\Omega+u)$.*
4. *For each interval $I \subset \mathbb{R}$ such that $\overline{I}$ contains only the eigenvalue λ of* (1.1), *there exists a neighborhood U of $u = 0$ such that there are exactly h eigenvalues (counting the multiplicity) $\lambda_1(u), \dots, \lambda_h(u)$ of (P_u) contained on I.* ∎

4. Local variations of the eigenvalues and the eigenfunctions. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with boundary of class C^4. Let λ be an eigenvalue of (1.1) of multiplicity h, and let y_i, $i = 1, \dots, h$ be the associated eigenfunctions, normalized in $L^2(\Omega)$.

Let $y_i(u) \in H^4(\Omega + u) \cap H_0^2(\Omega + u)$, $i = 1, \dots, h$, be the eigenfunctions of (3.24) associated to the eigenvalue $\lambda_i(u)$, where $\lambda = \lambda_i(0)$, $y_i(0) = y_i$, $i = 1, \dots, h$.

According to the results of the previous section, the branches of the eigenvalues $u \longrightarrow \lambda_i(u) \in \mathbb{R}$ and the eigenfunctions $u \longrightarrow y_i(u) \in H^4(\Omega + u) \cap H_0^2(\Omega + u)$ are analytic with respect to the perturbation parameter u in a neighborhood of $u = 0$ in $W^{5,\infty}(\Omega, \mathbb{R}^d)$. The first local variation of the branches solves the system

$$\text{(4.1)} \qquad \left\{ \begin{array}{lll} \triangle^2 y_i'(u) & = \lambda_i'(u)\, y_i + \lambda y_i'(u) & \text{in } \Omega, \\ y_i'(u) & = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial y_i'(u)}{\partial n} & = -(u \cdot n)\, \dfrac{\partial^2 y}{\partial n^2} & \text{on } \partial\Omega. \end{array} \right.$$

The following result provides an identity for the local derivative of the eigenvalues.

LEMMA 4.1. *Under the above conditions, the first local derivatives of the eigenvalues verify*

$$\text{(4.2)} \qquad \delta_{ij} \lambda_i'(u) = -\int_{\partial\Omega} (u \cdot n)\, \frac{\partial y_i}{\partial n} \cdot \frac{\partial^2 y_j}{\partial n^2} \quad \forall i, j = 1, \dots, h. \qquad \square$$

Note that here and in what follows, $\lambda_i'(u)$ denotes the derivative of λ_i at $u = 0$ in the direction u.

Proof. Multiplying (4.1) by $w \in H_0^2(\Omega)$, we obtain that

$$\text{(4.3)} \qquad \int_\Omega \triangle y_i'(u)\, \triangle w = \lambda_i'(u) \int_\Omega y_i w + \lambda \int_\Omega y_i'(u)\, w.$$

Taking $w = y_j$ in (4.3), we have that

$$\int_\Omega \triangle y_i'(u)\, \triangle y_j = \lambda_i'(u) \int_\Omega y_i y_j + \lambda \int_\Omega y_i'(u)\, y_j.$$

Since $y_i \in H^4(\Omega)$, $i = 1, \dots, h$ (see [10, Theorem 7.1.2]), integrating by parts, we deduce that

$$\int_\Omega \triangle^2 y_j \, y_i'(u) + \int_{\partial\Omega} \frac{\partial y_i'(u)}{\partial n} \triangle y_j(u) = \lambda_i'(u)\,\delta_{ij} + \lambda \int_\Omega y_i'(u)\, y_j.$$

Therefore,

$$\begin{aligned} \delta_{ij}\lambda_i'(u) &= \int_{\partial\Omega} \frac{\partial y_i'(u)}{\partial n} \triangle y_j \\ &= -\int_{\partial\Omega} (u \cdot n) \frac{\partial^2 y_i}{\partial n^2} \triangle y_j = -\int_{\partial\Omega} (u \cdot n) \frac{\partial^2 y_i}{\partial n^2} \frac{\partial^2 y_j}{\partial n^2}. \end{aligned} \tag{4.4}$$

The proof is complete. ☐

5. Proof of Theorem 1.1. In this section, we prove the generic unique continuation property stated in Theorem 1.1.

First, we state a unique continuation result for the evolution plate system, which is a consequence of the classical Holmgrem uniqueness theorem and which is needed in the proof. Being more precise, in the proof of Theorem 1.1 we use this result only for the eigenfunctions of the plate system or, more precisely, for the corresponding separated variables solutions of (5.1). However, we state the result for general solutions of the evolution plate system for the sake of completeness.

LEMMA 5.1 (see [18, Lemma 3.6, p. 276]). *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with boundary of class C^4. Let $\Gamma_0 \subset \partial\Omega$ be a nonempty open set, and let $T > 0$.*

Then, if y solves

$$\begin{cases} y'' + \triangle^2 y = 0 & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ \dfrac{\partial^2 y}{\partial n^2} = 0 & \text{on } \Gamma_0 \times (0,T), \\ \dfrac{\partial \triangle y}{\partial n} = 0 & \text{on } \Gamma_0 \times (0,T), \end{cases} \tag{5.1}$$

then, necessarily, $y \equiv 0$. ☐

Proof of Theorem 1.1. Let $\Gamma_0 \subseteq \partial\Omega$ be an open nonempty set.

Assume that λ is an eigenvalue of (1.1) of multiplicity h, and let y_i, $i = 1, \dots, h$, be the associated eigenfunctions.

We define the set

$$A_0 = W_0 = \left\{ u \in W^{5,\infty}(\Omega, \mathbb{R}^d) \,:\, u = 0 \text{ on } \partial\Omega \setminus \Gamma_0 \right\},$$

and for each $n \in \mathbb{N}$ we consider the set

$$A_n = \{ u \in W^{5,\infty}(\Omega, \mathbb{R}^d) \,:\, u = 0 \text{ on } \partial\Omega \setminus \Gamma_0 \text{ and the unique continuation property of Theorem 1.1 holds for the first } n \text{ branches of eigenvalues} \}.$$

Note that $A_{n+1} \subset A_n$ for all $n \geq 0$. We will prove that A_n is an open subset of W_0 and A_{n+1} is a dense subset of A_n for all $n \geq 0$. Thus, by using Baire's lemma (see Lemmas 2.1 and 2.2), we will prove that $\bigcap_{n\in\mathbb{N}} A_n$ is residual in W_0.

Obviously, this completes the proof since $\bigcap_{n\in\mathbb{N}} A_n$ coincides with the set of admissible perturbations of the domain Ω such that the unique continuation property of Theorem 1.1 holds simultaneously for all the branches of eigenvalues.

To apply Baire's lemma, the following properties of the sets A_n are needed.

(i) A_n is open in W_0.

It is clear that A_0 is an open set in W_0. To see that each set A_n is open in $W^{5,\infty}(\Omega, \mathbb{R}^d) \cap W_0$ for $n \geq 1$, we argue by contradiction. Suppose that A_n^c is not closed. Then, there exists a sequence of deformations $\{u_k\}_k \subset A_n^c$ such that u_k converges to $\widetilde{u} \in A_n$ as $k \to \infty$.

Since $\{u_k\}_k \subset A_n^c$, there exist $\{\lambda(u_k), y(u_k)\}_k$ such that

$$\begin{cases} \triangle^2 y(u_k) &= \lambda(u_k)\, y(u_k) & \text{in } \Omega + u_k, \\ y(u_k) &= 0 & \text{on } \partial\Omega + u_k, \\ \dfrac{\partial y(u_k)}{\partial n(\Omega + u_k)} &= 0 & \text{on } \partial\Omega + u_k, \\ \displaystyle\int_{\Omega+u_k} |y(u_k)|^2 &= 1, \end{cases}$$

and

$$\frac{\partial^2 y(u_k)}{\partial n^2(\Omega + u_k)} = 0 \quad \text{on } \Gamma_0 + u_k,$$

$\lambda(u_k)$ belonging to one of the branches $\lambda_1(u), \ldots, \lambda_n(u)$ and $y(u_k)$ being the corresponding eigenfunction.

From Theorem 3.5 we have that the branches $u \to \lambda(u)$, $u \to y(u)$ are analytic functions in a neighborhood of $u = 0$ in $W^{5,\infty}(\Omega, \mathbb{R}^d)$ with values in $\mathbb{R}$ and $H^4(\Omega + u) \cap H_0^2(\Omega + u)$, respectively. Thus the eigenpair $(\lambda(\widetilde{u}), y(\widetilde{u}))$ is a solution of the problem

$$\begin{cases} \triangle^2 y(\widetilde{u}) &= \lambda(\widetilde{u})\, y(\widetilde{u}) & \text{in } \Omega + \widetilde{u}, \\ y(\widetilde{u}) &= 0 & \text{on } \partial\Omega + \widetilde{u}, \\ \dfrac{\partial y(\widetilde{u})}{\partial n(\Omega + \widetilde{u})} &= 0 & \text{on } \partial\Omega + \widetilde{u}, \\ \dfrac{\partial^2 y(\widetilde{u})}{\partial n^2(\Omega + \widetilde{u})} &= 0 & \text{on } \Gamma_0 + \widetilde{u}, \\ \displaystyle\int_{\Omega+\widetilde{u}} |y(\widetilde{u})|^2 &= 1 \end{cases}$$

for some $\lambda(\widetilde{u})$ belonging to one of the first n branches $\lambda_1(\widetilde{u}), \ldots, \lambda_h(\widetilde{u})$. But this is impossible, since $\widetilde{u} \in A_n$. This shows that A_n is an open set.

(ii) A_{n+1} is dense in A_n.

Now we will see that A_{n+1} is dense in A_n for all $n \geq 0$; in particular, A_1 is dense in $A_0 = W_0$.

Suppose that A_{n+1} is not dense in A_n. Then, there exists $u \in A_n \setminus A_{n+1}$ and a neighborhood $\mathcal{V}$ of u such that $\mathcal{V} \subset A_n \setminus A_{n+1}$. Without lost of generality, we may assume that $u = 0$.

For any $u \in \mathcal{V} \subset A_n \setminus A_{n+1}$ there exists a nontrivial eigenfunction $y(u) \in H_0^2(\Omega + u)$ associated to the $(n+1)$th eigenvalue $\lambda(u) = \lambda_{n+1}(u)$ such that

$$\frac{\partial^2 y(u)}{\partial^2 n(\Omega + u)} = 0 \text{ on } \Gamma_0 + u. \tag{5.2}$$

On the other hand, from Theorem 2.6, the local variations satisfy

$$\left\{\begin{array}{rcll} \triangle^2 y'(u) & = & \lambda'(u)\, y + \lambda y'(u) & \text{in } \Omega, \\ y'(u) & = & 0 & \text{on } \partial\Omega, \\ \dfrac{\partial y'(u)}{\partial n} & = & 0 & \text{on } \partial\Omega, \\ \dfrac{\partial^2 y'(u)}{\partial n^2} & = & -(u \cdot n)\dfrac{\partial^3 y}{\partial n^3} & \text{on } \Gamma_0 \end{array}\right. \tag{5.3}$$

for all $u \in \mathcal{V}$, where

$$y(0) = y, \quad \lambda(0) = \lambda.$$

Since $u = 0$ on $\partial\Omega \setminus \Gamma_0$, we have that

$$\lambda'(u) = -\int_{\partial\Omega} (u \cdot n) \left|\frac{\partial^2 y}{\partial n^2}\right|^2 = 0,$$

and therefore

$$\left\{\begin{array}{rcll} \triangle^2 y'(u) & = & \lambda y'(u) & \text{in } \Omega, \\ y'(u) & = & 0 & \text{on } \partial\Omega, \\ \dfrac{\partial y'(u)}{\partial n} & = & 0 & \text{on } \partial\Omega, \\ \dfrac{\partial^2 y'(u)}{\partial^2 n} & = & -(u \cdot n)\dfrac{\partial^3 y}{\partial n^3} & \text{on } \Gamma_0. \end{array}\right. \tag{5.4}$$

That is, for each $u \in \mathcal{V}$, $y'(u)$ is an eigenfunction of the plate system associated to the eigenvalue λ.

We now distinguish two cases.

Case 1. λ is a simple eigenvalue. Since $y'(u)$ is an eigenfunction of the plate system associated to the eigenvalue λ, there exists a constant c_u such that $y'(u) = c_u y$, and therefore

$$\frac{\partial^2 y'(u)}{\partial n^2} = -(u \cdot n)\frac{\partial^3 y}{\partial n^3} = c_u \frac{\partial^2 y}{\partial n^2} = 0 \text{ on } \Gamma_0.$$

Thus, we obtain that $\frac{\partial^3 y}{\partial n^3} = 0$ on Γ_0 as well.

From Lemma 5.1 we conclude that $y \equiv 0$ in Ω, which is impossible since y is an eigenfunction of the plate system.

Case 2. λ is a multiple eigenvalue. First, we assume that the eigenvalue λ has multiplicity two. Let y_1, y_2 be the associated eigenfunctions.

We will show that there exists a perturbation u, as small as we want, such that the eigenvalue $\lambda(u)$ is simple or the unique continuation property holds for the branch $u \to y_1(u)$. Notice that the case where $\lambda(u)$ is simple has been addressed in Case 1 above.

We can proceed in an analogous form considering the branch $u \to y_2(u)$.

We argue by contradiction.

We suppose that the eigenvalue $\lambda(u)$ remains of multiplicity two and the corresponding eigenfunction $y_1(u)$ satisfies (5.2) in a neighborhood $\mathcal{V}$ of $u = 0$. If $\lambda(u)$ is simple, we can proceed as in Case 1. In other words,

$$\frac{\partial^2 y_1(u)}{\partial n^2(\Omega + u)} = 0 \quad \text{on } \Gamma_0 + u \ \forall u \in \mathcal{V}.$$

Then, as in the case of simple eigenvalues, the local variation $y_1'(u)$ is an eigenfunction associated to the eigenvalue λ. Then, there exist constants $c_1(u)$, $c_2(u)$ such that $y_1'(u) = c_1(u)\, y_1 + c_2(u)\, y_2$. Thus, from (5.4) we have that

$$\frac{\partial^2 y_1'(u)}{\partial n^2} = -(u \cdot n)\, \frac{\partial^3 y_1}{\partial n^3} \text{ on } \Gamma_0$$

and

$$\frac{\partial^2 y_1'(u)}{\partial n^2} = c_1(u)\, \frac{\partial^2 y_1}{\partial n^2} + c_2(u) \frac{\partial^2 y_2}{\partial n^2} = c_2(u) \frac{\partial^2 y_2}{\partial n^2} \quad \text{on } \Gamma_0.$$

Hence, for every pair of deformations u_1 and u_2 we have that

$$c_2(u_1)\, \frac{\partial^2 y_2}{\partial n^2} = -(u_1 \cdot n)\, \frac{\partial^3 y_1}{\partial n^3} \text{ on } \Gamma_0,$$
$$c_2(u_2)\, \frac{\partial^2 y_2}{\partial n^2} = -(u_2 \cdot n)\, \frac{\partial^3 y_1}{\partial n^3} \text{ on } \Gamma_0.$$

If $\frac{\partial^3 y_1}{\partial n^3} = 0$ on Γ_0, in view of Lemma 5.1 we immediately deduce that $y_1 \equiv 0$, which is a contradiction. Thus, we can assume that $\frac{\partial^3 y_1}{\partial n^3} \not\equiv 0$ on Γ_0 and consequently that $\frac{\partial^2 y_2}{\partial n^2} \not\equiv 0$ on Γ_0 as well.

Then

$$\frac{(u_1 \cdot n)}{(u_2 \cdot n)} = \frac{c_2(u_1) \dfrac{\partial^2 y_2}{\partial n^2}}{c_2(u_2) \dfrac{\partial^2 y_2}{\partial n^2}} = \frac{c_2(u_1)}{c_2(u_2)} = \text{constant}.$$

That is, necessarily $(u_1 \cdot n) = c\,(u_2 \cdot n)$ for a suitable constant c. This is impossible since the functions u_1 and u_2 may be chosen arbitrarily.

Thus we reach a contradiction.

Assume now that λ is an eigenvalue of multiplicity $h > 2$, and let $y_1, \dots, y_h$ be the associated eigenfunctions normalized in $L^2(\Omega)$.

We claim that there exists a deformation u, arbitrarily small, such that the eigenvalue $\lambda(u)$ has multiplicity at most $h - 1$ or the unique continuation property holds for the branch $y_1(u)$.

To prove this we argue by contradiction. Suppose that $\lambda(u)$ is an eigenvalue of multiplicity h for all $u \in \mathcal{V}$, with associated eigenfunctions $y_1(u), \dots, y_h(u)$. Moreover, assume that

$$\frac{\partial^2 y_1(u)}{\partial n(\Omega + u)^2} = 0 \quad \text{on } \Gamma_0 + u \tag{5.5}$$

for each $u \in \mathcal{V}$.

Then, from (5.3) we have that $y_1'(u)$ is an eigenfunction of the plate system associated to the eigenvalue λ for each $u \in \mathcal{V}$.

Thus, there exist constants $c_1(u), \ldots, c_h(u)$ such that

$$y_1'(u) = c_1(u)\, y_1 + \cdots + c_h(u)\, y_h. \tag{5.6}$$

Therefore,

$$\begin{aligned} \frac{\partial^2 y_1'(u)}{\partial n^2} &= -(u\cdot n)\frac{\partial^3 y_1}{\partial n^3} \\ &= c_1(u)\frac{\partial^2 y_1}{\partial n^2} + \cdots + c_h(u)\frac{\partial^2 y_h}{\partial n^2} \\ &= c_2(u)\frac{\partial^2 y_2}{\partial n^2} + \cdots + c_h(u)\frac{\partial^2 y_h}{\partial n^2} \quad \text{on } \Gamma_0. \end{aligned} \tag{5.7}$$

On the other hand,

$$\lambda_i'(u) = -\int_{\Gamma_0} (u\cdot n)\left|\frac{\partial^2 y_i}{\partial n^2}\right|^2 \quad \forall i = 1, \ldots, h. \tag{5.8}$$

From (5.5) we have that

$$\frac{\partial^2 y_1}{\partial n^2} = 0 \quad \text{on } \Gamma_0. \tag{5.9}$$

Since the multiplicity does not decrease in a neighborhood U of $u = 0$, we obtain that

$$\lambda_1'(u) = \cdots = \lambda_h'(u) \tag{5.10}$$

for all $u \in U$ such that $u = 0$ on $\partial\Omega \setminus \Gamma_0$. Moreover, according to (5.7)–(5.9), we also have that $\lambda_1'(u) = 0$. Therefore,

$$\lambda_1'(u) = \cdots = \lambda_h'(u) = 0.$$

Thus

$$\frac{\partial^2 y_i}{\partial n^2} = 0 \quad \text{on } \Gamma_0 \;\; \forall i = 1, \ldots, h. \tag{5.11}$$

From (5.7) and (5.11), we deduce that

$$\frac{\partial^3 y_1}{\partial n^3} = 0 \quad \text{on } \Gamma_0. \tag{5.12}$$

From Lemma 5.1 we have that $y_1 \equiv 0$ on Ω, which is impossible because y_1 is an eigenfunction.

Therefore, there exists a deformation $\widetilde{u} \in \mathcal{V}$ such that the eigenvalue $\lambda_1(\widetilde{u})$ has multiplicity at most $h-1$ or the unique continuation property holds for $y_1(\widetilde{u})$.

If the unique continuation property does not hold in the new domain $\Omega+\widetilde{u}$, we can apply the same argument in an iterative way and obtain a deformation $\widehat{u}$, arbitrarily small, such that the eigenvalue $\lambda(\widehat{u})$ is of multiplicity two or the unique continuation property holds. Since the case where the multiplicity is two was solved before, this completes the proof of the density of A_{n+1} on A_n for all $n \geq 0$.

Applying Baire's lemma (see Lemmas 2.1 and 2.2), we complete the proof of Theorem 1.1. ∎

Remark 5.1. Note that in the proof of Theorem 1.1 above, we consider only deformations u such that $u = 0$ on $\partial\Omega \setminus \Gamma_0$. Consequently, the deformations u we use do deform the subset Γ_0 of the boundary.

If we consider deformations which do not deform the set Γ_0 (that is, such that $u = 0$ on Γ_0), the argument of the proof does not apply. That is because we cannot guarantee anymore that the local variations $y_i'(u)$ are eigenfunctions of (1.1). ∎

6. Proof of Theorem 1.2. The proof of Theorem 1.2 is similar to the proof of the generic simplicity of the eigenvalues of the Stokes system (see [25]). We apply Baire's lemma for a suitable sequence of sets $\{A_n\}_{n\geq 0}$.

Let $n = 1, 2, \ldots$, and define the sets

$$A_0 = W_0 = \left\{u \in W^{5,\infty}(\Omega, \mathbb{R}^d) \,:\, u = 0 \text{ on } \partial\Omega \setminus \Gamma_0\right\}, \tag{6.1}$$

and

$$A_n = \left\{u \in W^{5,\infty}(\Omega, \mathbb{R}^d) \colon u = 0 \text{ on } \partial\Omega \setminus \Gamma_0, \text{ and the first } n \text{ branches } \lambda_1(u), \ldots, \lambda_n(u) \text{ of eigenvalues of (3.24) are simple}\right\}. \tag{6.2}$$

As in the proof of Theorem 1.1, we need to check that A_n is an open subset of W_0 and $A_{n+1}$is a dense subset of A_n for all $n \geq 0$. Then, applying Lemma 2.2, we conclude the proof.

(a) A_n is an open set in $W^{5,\infty}(\Omega, \mathbb{R}^d) \cap W_0$.

It is clear that A_0 is an open set. Let $u \in A_n$ for $n \geq 1$, and then

$$\lambda_i(u) \neq \lambda_j(u) \quad \forall i, j = 1, \ldots, n+1, i \neq j.$$

Let

$$\delta = \min\{|\lambda_i(u) - \lambda_j(u)| : i, j = 1, \ldots, n, i \neq j\}.$$

Let $u' \in U$, where

$$U = \left\{w \in \mathcal{W} : \|u - w\|_{5,\infty} < \frac{\delta}{2c}\right\}$$

and c is the maximum of the Lipschitz constants for the functions $u' \to \lambda_i(u+u')$, $i = 1, \ldots, n+1$.

Then

$$\begin{aligned} \delta &\leq |\lambda_i(u) - \lambda_j(u)| \\ &\leq |\lambda_i(u+u') - \lambda_i(u)| + |\lambda_i(u+u') - \lambda_j(u+u')| \\ &+ |\lambda_j(u+u') - \lambda_j(u)| \\ &< \delta + |\lambda_i(u+u') - \lambda_j(u+u')|. \end{aligned}$$

Thus

$$|\lambda_i(u') - \lambda_j(u')| > 0, \quad i, j = 1, \ldots, n+1,$$

which proves that $u + u' \in A_n$, and the set A_n is open.

(b) A_{n+1} is dense in A_n for all $n \geq 0$.

Let $w \in A_n \setminus A_{n+1}$. Without loss of generality we can assume that $u = 0$. Then, either λ_{n+1} remains to be of multiplicity h in a neighborhood of $u = 0$, or there exists $u \neq 0$ arbitrarily small such that the multiplicity is at most $h - 1$. (In the case of $n = 0$, we have that the first eigenvalue has multiplicity $h \geq 2$; that is, $\lambda = \lambda_1(u) = \cdots = \lambda_h(u)$.) Iterating this argument, it can be shown that the $(n+1)$th eigenvalue becomes simple for suitable arbitrarily small perturbations u or it remains of constant multiplicity $h \geq 2$ in a neighborhood of $u = 0$. If the eigenvalue becomes simple, the proof of the density of A_{n+1} is concluded.

Thus, we can assume that λ_{n+1} is of constant multiplicity $h \geq 2$; that is,

$$\lambda = \lambda_{n+1}(u) = \cdots = \lambda_{n+h}(u)$$

for any u in a neighborhood of $u = 0$.

Let $y_1(u), \dots, y_h(u)$ be the eigenfunctions associated to λ normalized in $L^2(\Omega + u)$.

In view of the generic unique continuation result of Theorem 1.1, we can also assume that the spectral uniqueness holds in Ω.

Then, from (4.2) we have that

$$\lambda'_{n+i}(\Omega; u)\,\delta_{ij} = -\int_{\partial\Omega} (u \cdot n)\,\frac{\partial^2 y_i}{\partial n^2}\frac{\partial^2 y_j}{\partial n^2}. \tag{6.3}$$

We will prove that there exists a deformation u such that

$$\lambda'_{n+i}(\Omega; u) \neq \lambda'_{n+j}(\Omega; u) \quad \forall\, i \neq j. \tag{6.4}$$

Assuming for the moment that this holds, we deduce that

$$\lambda_{n+i}(\varepsilon u) \neq \lambda_{n+j}(\varepsilon u) \quad \forall i, j = 1, \dots, n+h+1, i \neq j,$$

for $\varepsilon > 0$ small enough and then $\varepsilon u \in A_{n+1}$.

We proceed by contradiction. Suppose that (6.4) does not hold. Then there exist $i \neq j$, $i, j \in \{1, \dots, h\}$, such that

$$\lambda'_{n+i}(\Omega; u) = \lambda'_{n+j}(\Omega; u) \tag{6.5}$$

for all u in a neighborhood of $u = 0$.

Thus, from (4.2) we have that

$$\int_{\partial\Omega} (u \cdot n)\left[\frac{\partial^2 y_i}{\partial n^2}\frac{\partial^2 y_j}{\partial n^2}\right] = 0,$$

and from (6.3) and (6.5) we have that

$$\int_{\partial\Omega} (u \cdot n)\left[\left|\frac{\partial^2 y_i}{\partial n^2}\right|^2 - \left|\frac{\partial^2 y_j}{\partial n^2}\right|^2\right] = 0$$

for all u in a neighborhood of $u = 0$.

Since $u = 0$ on $\partial\Omega \setminus \Gamma_0$, we deduce that

$$\frac{\partial^2 y_i}{\partial n^2}\frac{\partial^2 y_j}{\partial n^2} = 0 \quad \text{on } \Gamma_0 \tag{6.6}$$

and

$$\left|\frac{\partial^2 y_i}{\partial n^2}\right| = \left|\frac{\partial^2 y_j}{\partial n^2}\right| \quad \text{on } \Gamma_0. \tag{6.7}$$

Therefore, we obtain that

$$\frac{\partial^2 y_i}{\partial n^2} = \frac{\partial^2 y_j}{\partial n^2} = 0 \quad \text{on } \Gamma_0. \tag{6.8}$$

Since the spectral uniqueness property holds in the domain Ω, we obtain that

$$y_i = y_j = 0 \quad \text{in } \Omega.$$

But this is impossible because y_i and y_j are eigenfunctions of the plate system.

Applying Baire's lemma (see Lemmas 2.1 and 2.2) to the sets A_n on the space W_0, we complete the proof.

7. Proof of Theorem 1.3. In this section we analyze the evolution dissipative plate equation (1.3). First, we prove the existence and uniqueness of solutions. Then we derive a generic unique continuation result in an infinite time interval. Finally, we complete the proof of Theorem 1.3.

7.1. Existence and uniqueness of solutions for the evolution plate system. First, we analyze the variational formulation of system (1.3).

Recall that $X_1 = \{\varphi \in H^2(\Omega) \cap H_0^1(\Omega) \ : \ \frac{\partial \varphi}{\partial n} = 0 \ \text{ on } \partial\Omega \setminus \Gamma_0, \}$, $X_2 = L^2(\Omega)$, and $X = X_1 \times X_2$. Then $X = X_1 \times X_2$ is a Hilbert space endowed with the norm of $H^2(\Omega) \times L^2(\Omega)$.

We also introduce the space $X_3 = \{\varphi \in X_1 \ : \ \triangle^2 \varphi \in L^2(\Omega)\}$.

Let $B : X_1 \to X_1'$ be the map

$$\langle Bz, v\rangle_{X_1', X_1} = \int_{\Gamma_0} \frac{\partial z}{\partial n}\frac{\partial v}{\partial n}. \tag{7.1}$$

We can see that the linear map B is continuous and accretive.

Let $z \in X_1$. Then, multiplying (1.3) by z, we have that

$$\begin{aligned} 0 &= \left\langle y'' + \triangle^2 y, z\right\rangle_{X_1 \times X_1'} \\ &= \int_\Omega (y'' z + \triangle y \triangle z) + \int_{\partial\Omega} \left(z \frac{\partial(\triangle y)}{\partial n} - \triangle y \frac{\partial z}{\partial n}\right) \\ &= \int_\Omega (y'' z + \triangle y \triangle z) + \int_{\Gamma_0} \frac{\partial y'}{\partial n}\frac{\partial z}{\partial n}. \end{aligned} \tag{7.2}$$

We also define the map $A : X_1 \to X_1'$ as

$$\langle Az, v\rangle_{X_1', X_1} = \int_\Omega \triangle z \triangle v. \tag{7.3}$$

Then, (7.2) is equivalent to

$$\langle y'' + Ay + By', v\rangle_{X_1', X_1} = 0 \ \forall v \in X_1. \tag{7.4}$$

We define the operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ by

$$\mathcal{A}U = (-U_2, AU_1 + BU_2), \quad \text{where } U = (U_1, U_2),$$

and $D(\mathcal{A}) = \{(y, z) \in X_3 \times X_1 \ : \ \triangle y = -\frac{\partial z}{\partial n} \ \text{ on } \Gamma_0\}$.

Thus, the problem (1.3) is equivalent to solve

$$\begin{aligned} U' + \mathcal{A}U &= 0 \qquad \text{in } \mathbb{R}^+, \\ U(0) &= (y_0, y_1), \end{aligned}$$

with $U = (y, y')$.

PROPOSITION 7.1. *The operator $\mathcal{A}$ is m-accretive.* □

Proof. Let $U, V \in D(\mathcal{A})$. Then

$$\begin{aligned} \langle \mathcal{A}U - \mathcal{A}V, U - V \rangle_X &= \langle V_2 - U_2, U_1 - V_1 \rangle_{X_1} \\ &\quad + \langle \mathcal{A}U_1 - \mathcal{A}V_1 + BU_2 - BV_2, U_2 - V_2 \rangle_{X_1', X_1} \\ &= \int_\Omega \triangle (V_2 - U_2) \triangle (U_1 - V_1) + \int_\Omega \triangle (U_1 - V_1) \triangle (U_2 - V_2) \\ &= + \int_{\Gamma_0} \frac{\partial (U_2 - U_1)}{\partial n} \frac{(\partial (U_2 - V_2)}{\partial n} \\ &= \int_{\Gamma_0} \left| \frac{\partial (U_2 - V_2)}{\partial n} \right|^2 \geq 0, \end{aligned}$$

which proves that $\mathcal{A}$ is an accretive operator.

We must prove that $I + \mathcal{A} : D(\mathcal{A}) \to X$ is onto.

Let $W = (W_1, W_2) \in X$. We must show that there exists $U = (U_1, U_2) \in D(\mathcal{A})$, such that

$$(U_1 - U_2, U_2 + AU_1 + BU_2) = (W_1, W_2). \tag{7.5}$$

System (7.5) is equivalent to

$$U_2 = U_1 - W_1, \qquad U_1 + AU_1 + BU_1 = W_2 + BW_1 + W_1. \tag{7.6}$$

To prove the existence of a solution of (7.6), it is enough to show that for every $f \in X_1'$, there exists $U_1 \in X_1$ such that

$$(I + A + B) U_1 = f. \tag{7.7}$$

We first observe that

$$\langle (I + A + B) v, v \rangle_{X_1', X_1} = \int_\Omega v^2 + |\triangle v|^2 + \int_{\Gamma_0} \left| \frac{\partial v}{\partial n} \right|^2 \geq \alpha \|v\|_{X_1}^2 \quad \forall v \in X_1, \tag{7.8}$$

which shows that the bilinear form associated to $(I + A + B)$ is coercive.

On the other hand, the embedding $H^2(\Omega) \hookrightarrow H^1(\Gamma_0)$ is continuous. Thus, the bilinear form associated to $(I + A + B)$ is continuous. Therefore, from Lax–Milgram's lemma we have that there exists a unique $U_1 \in X_1$ such that (7.7) holds.

Therefore, there exists $(U_1, U_2) = (U_1, U_1 - W_1,) \in X_1 \times X_1$ which satisfies (7.5). Moreover, $U_1 \in X_3$ and $(U_1, U_2) \in D(A)$.

This proves that the map $I + \mathcal{A}$ is onto, and thus the operator $\mathcal{A}$ is an m-accretive operator on $X_3 \times X_1$. □

Remark 7.1. Note that since the operator $\mathcal{A}$ is m-accretive in X, in view of [5, Proposition VII.1, p. 101], we deduce that $D(\mathcal{A})$ is dense on X. □

We have the following result on the existence, uniqueness, and regularity of solutions of (1.3).

PROPOSITION 7.2. *Let* $(y_0, y_1) \in X$. *Then there exists a unique solution* y *of* (1.3) *which verifies*

$$y \in C_b([0,+\infty[; X_1) \cap C_b^1([0,+\infty[; X_2). \tag{7.9}$$

Furthermore, if $(y_0, y_1) \in D(\mathcal{A})$, *the solution* y *verifies*

$$(y, y_t) \in C_b([0,+\infty[; D(A)). \tag{7.10}$$

Moreover, when $(y_0, y_1) \in D(\mathcal{A})$, *the energy* $E : \mathbb{R}^+ \to \mathbb{R}^+$, *defined as*

$$E(t) = \frac{1}{2}\int_\Omega \left[|y_t(x,t)|^2 + |\triangle y(x,t)|^2\right] dx,$$

is a decreasing and differentiable function on $\mathbb{R}^+$. □

Remark 7.2. Here and in what follows we denote by $C_b^k([0,+\infty[; X)$ the space of functions $C^k([0,+\infty[; X) \cap W^{k,\infty}(0,+\infty; X)$. □

Proof of Proposition 7.2. Since the operator $\mathcal{A}$ m-accretive in X, from the Hille–Yosida theorem we have the following.

1. If $(y_0, y_1) \in X = X_1 \times X_2$, there exists a unique solution $U = (y, y_t)$ of

$$\begin{cases} \dfrac{dU}{dt} + \mathcal{A}U = 0 & \text{on } (0,+\infty), \\ U(0) = (y_0, y_1), \end{cases} \tag{7.11}$$

such that (see, for instance, [5, Theorem VII.5, p. 111])

$$U \in C_b([0,+\infty[; X). \tag{7.12}$$

2. If $(y_0, y_1) \in D(\mathcal{A})$, there exists a unique solution $U = (y, y_t)$ of (7.11) such that (see, for instance, [5, Theorem VII.4, p. 105])

$$U \in C_b^1([0,+\infty[; X) \cap C_b([0,+\infty[; D(\mathcal{A})). \tag{7.13}$$

On the other hand,

$$\frac{dE}{dt}(t) = \int_\Omega [y_{tt}(x,t)\, y_t(x,t) + \triangle y_t(x,t)\, \triangle y(x,t)]\, dx. \tag{7.14}$$

Taking $z = y_t$ in (7.2) and integrating by parts, we have that

$$\frac{dE}{dt}(t) = -\int_{\Gamma_0} \left|\frac{\partial y_t}{\partial n}\right|^2 \le 0, \tag{7.15}$$

which shows us that the energy E is decreasing. □

7.2. A generic unique continuation result. In order to prove Theorem 1.3, we need a nonstandard unique continuation result for the evolution plate system. This is the object of the following proposition.

PROPOSITION 7.3. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set of class C^4, and let $\Gamma_0 \subset \partial\Omega$ be a nonempty open set such that the spectral uniqueness property holds.*

If $(y_0, y_1) \in D(\mathcal{A})$ and the corresponding solution of (1.3) is such that

$$\frac{\partial y_t}{\partial n} = 0 \text{ on } \Gamma_0 \times (0,\infty), \tag{7.16}$$

then necessarily $y \equiv 0$. □

Proof. We set $v = y_t$. Then v is a weak solution of the system

$$\begin{cases} v_{tt} + \triangle^2 v &= 0 & \text{in } \Omega \times (0,\infty), \\ v &= 0 & \text{on } \partial\Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial n} &= 0 & \text{on } \partial\Omega \times (0,\infty), \\ v(x,0) &= v_0 = y_1 & \text{in } \Omega, \\ v_t(x,0) &= v_1 = -\triangle^2 y_0 & \text{in } \Omega, \end{cases} \tag{7.17}$$

with initial datum $(v_0, v_1) \in H_0^2(\Omega) \times L^2(\Omega)$, and, furthermore, satisfies the condition

$$\triangle v = 0 \quad \text{on } \Gamma_0 \times (0,\infty). \tag{7.18}$$

We will see that $v = 0$. Thus $y(x,t) \equiv y(x)$. Therefore, $y(x) = y_0$ solves

$$\begin{cases} \triangle^2 y &= 0 & \text{in } \Omega, \\ y &= 0 & \text{on } \partial\Omega, \\ \dfrac{\partial y}{\partial n} &= 0 & \text{on } \partial\Omega \setminus \Gamma_0, \\ \triangle y &= 0 & \text{on } \Gamma_0 \times (0,\infty). \end{cases} \tag{7.19}$$

Multiplying (7.19) by y and integrating by parts, we deduce that $\int_\Omega |\triangle y|^2\, dx = 0$; that is, $y \equiv 0$, since $y = 0$ on $\partial\Omega$.

Thus, the problem is reduced to show that

$$v \equiv 0. \tag{7.20}$$

To prove (7.20), first we observe that $(v, v_t) \in C_b([0,+\infty); H_0^2(\Omega) \times L^2(\Omega))$.

This can be proved easily by analyzing the well posedness of the conservative plate system (7.17).

In what follows we shall use the notation $X = H_0^2(\Omega) \times L^2(\Omega)$.

The solution v of (7.17) may be developed in a Fourier series. Indeed, let $\{\lambda_n\}_n$ be the eigenvalues of the bilaplacian operator with clamped boundary conditions, and let w_n be the associated eigenfunctions normalized in $L^2(\Omega)$:

$$\begin{cases} \triangle^2 w_n &= \lambda_n w_n & \text{in } \Omega, \\ w_n = \dfrac{\partial w_n}{\partial n} &= 0 & \text{on } \partial\Omega. \end{cases} \tag{7.21}$$

We decompose the initial data in Fourier series:

$$v_0(x) = \sum_{n=1}^{\infty} a_n w_n, \qquad v_1(x) = \sum_{n=1}^{\infty} b_n w_n. \tag{7.22}$$

It is easy to check that

$$\|v_0\|^2_{H_0^2(\Omega)} = \sum_{n=1}^{\infty} \lambda_n |a_n|^2, \qquad \|v_1\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty} |b_n|^2. \tag{7.23}$$

The solution v of (7.17) may be written as

$$v(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\sqrt{\lambda_n} t\right) + \frac{b_n}{\lambda_n} \sin\left(\sqrt{\lambda_n} t\right) \right] w_n(x). \tag{7.24}$$

Let us define

$$\mu_k = \sqrt{\lambda_k}, \quad \mu_{-k} = -\sqrt{\lambda_k}, \quad w_{-k}(x) = w_k(x). \tag{7.25}$$

If we define the complex coefficients

$$c_k = \frac{1}{2}\left(a_k - i\frac{b_k}{\sqrt{\lambda_k}}\right), \quad c_{-k} = \frac{1}{2}\left(a_k + i\frac{b_k}{\sqrt{\lambda_k}}\right), \tag{7.26}$$

we can write

$$v(x,t) = \sum_{k \in \mathbb{Z}} c_k e^{i\mu_k t} w_k(x). \tag{7.27}$$

Taking into account that $\triangle v = 0$ on $\Gamma_0 \times (0,\infty)$ and applying Bohr's transform to $\triangle v(x,t)$ on Γ_0 (see [3]), we deduce that

$$c_k \triangle w_k = 0 \quad \text{on } \Gamma_0 \ \forall k \geq 1. \tag{7.28}$$

Therefore, for every $k \in \mathbb{Z}$, we have that $c_k = 0$ or w_k is a solution of the problem

$$\begin{cases} \triangle^2 w_k &= \lambda_k^2 w_k & \text{in } \Omega, \\ w_k &= 0 & \text{on } \partial\Omega, \\ \dfrac{\partial w_k}{\partial n} &= 0 & \text{on } \partial\Omega, \\ \triangle w_k &= 0 & \text{on } \Gamma_0. \end{cases} \tag{7.29}$$

Since, by assumption, the spectral uniqueness holds in Ω, (7.29) implies that $w_k = 0$ in Ω, which is a contradiction. Thus $c_k = 0$ for all $k \in \mathbb{Z}$ and consequently $v \equiv 0$.

This completes the proof. ∎

7.3. Proof of Theorem 1.3. Now we prove the stabilization result for system (1.3).

Proof. We distinguish two cases.

1. $(y_0, y_1) \in D(\mathcal{A})$.
2. $(y_0, y_1) \in X$.

Step 1. Regular initial data.

Let $\{S_t\}_{t\geq 0}$ be the contraction semigroup associated to (1.3). Then, given $(y_0, y_1) \in D(\mathcal{A})$, we have that $(y, y_t) = S_t\,(y_0, y_1)$ and $\|y, y_t\|_{D(\mathcal{A})} \leq \|y_0, y_1\|_{D(\mathcal{A})}$.

Therefore, the trajectory $\{y\,(t)\,, y_t\,(t)\}_{t\geq 0}$ is bounded in $D(\mathcal{A})$, and, according to the compactness of the imbedding $D(\mathcal{A}) \hookrightarrow X$, $\{y\,(t)\,, y_t\,(t)\}_{t\geq 0}$ is relatively compact in X.

Moreover, the energy E is a strict Lyapunov functional for the semigroup $\{S_t\}_{t\geq 0}$. Indeed, suppose that $(y_0, y_1) \in D\,(\mathcal{A})$ is such that $E\,(t)$ is constant for all $t \geq 0$. Then y solves (1.3) with

$$\frac{\partial y_t}{\partial n} = 0 \quad \text{on } \Gamma_0 \times (0, \infty)\,. \tag{7.30}$$

From Proposition 7.3, we have that $y \equiv 0$. Therefore, $(y_0, y_1) = (0, 0)\,,$ which is the unique equilibrium point for system (1.3).

Thus, from the La Salle invariance principle (see [6, Theorem 9.2.3, p. 122]), we deduce that the $\omega-$limit of the trajectory $\{y, y_t\}_{t\geq 0}$ in X has a unique point $(y_0, y_1) = (0, 0)\,.$

Therefore,

$$\lim_{t\to+\infty} E\,(t) = 0. \tag{7.31}$$

Step 2. Initial data in *X*.

Let $(y_0, y_1) \in X$. Since $D\,(\mathcal{A})$ is dense in X, there exists a sequence of initial data $(y_0^n, y_1^n) \in D\,(\mathcal{A})$ which converges to (y_0, y_1) in X as $n \to \infty$.

Let y be the solution of (1.3) with initial datum (y_0, y_1), and let y^n be the solution of (1.3) with initial datum $(y_0^n, y_1^n)\,.$

Let

$$E_y\,(t) = \int_\Omega \left[|y_t|^2 + |\triangle y|^2\right].$$

Then

$$0 \leq E_y\,(t) \leq 2E_{y_n}\,(t) + 2E_{y-y_n}\,(t) \leq 2E_{y_n}\,(t) + 2E_{y-y_n}\,(0)\,.$$

Since $(y_0^n, y_1^n) \to (y_0, y_1)$ in X as $n \to \infty$, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$E_{y-y_n}\,(0) < \varepsilon \qquad \text{for } n \geq n_0.$$

Moreover, since $(y_0^{n_0}, y_1^{n_0}) \in D\,(\mathcal{A})\,,$ according to the result of the first step, there exists $t_0 \geq 0$ such that

$$E_{y_{n_0}}\,(t) < \varepsilon, \quad t \geq t_0.$$

Therefore, if $t \geq t_0$, we have that

$$E_y\,(t) \leq 2E_{y_{n_0}}\,(t) + 2E_{y-y_{n_0}}\,(t) < 4\varepsilon.$$

Thus

$$\lim_{t\to+\infty} E_y\,(t) = 0.$$

This completes the proof of Theorem 1.3. $\square$

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