Uniform exponential long time decay for the space semi-discretization of a locally damped wave equation via an artificial numerical viscosity

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Abstract. We consider the finite-difference space semi-discretization of a locally damped wave equation, the damping being supported in a suitable subset of the domain under consideration, so that the energy of solutions of the damped wave equation decays exponentially to zero as time goes to infinity. The decay rate of the semi-discrete systems turns out to depend on the mesh size h of the discretization and tends to zero as h goes to zero. We prove that adding a suitable vanishing numerical viscosity term leads to a uniform (with respect to the mesh size) exponential decay of the energy of solutions. This numerical viscosity term damps out the high frequency numerical spurious oscillations while the convergence of the scheme towards the original damped wave equation is kept. We discuss this problem in $1D$ and $2D$ in the interval and the square respectively. Our method of proof relies on discrete multiplier techniques.

1. Introduction and statement of the main results: the 1-D case.

Consider the 1-D damped wave equation

$$\tag{1.1} \begin{cases} y'' - y_{xx} + ay' = 0 \text{ in } (0,1)\times(0,\infty) \\ y(0,t) = y(1,t) = 0 \text{ in } (0,\infty) \\ y(x,0) = y^0 \text{ in } (0,1) \\ y'(0) = y^1 \text{ in } (0,1) \end{cases}$$

where $\{y^0, y^1\} \in H_0^1(0,1)\times L^2(0,1)$ and a is a bounded nonnegative function on (0,1) satisfying

$$\tag{1.2} \exists \alpha_0,\ a_M > 0:\ a(x) \le a_M \text{ a.e. } x\in(0,1), \quad a(x)\ge \alpha_0 \text{ a.e. } x \in \omega,$$

ω being a nonempty open subset of $(0,1)$.

Here and in the sequel $'$ denotes partial differentiation with respect to time.

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The energy

$$E(t) = \frac{1}{2}\int_0^1 \{|y'(x,t)|^2 + |y_x(x,t)|^2\}dx, \qquad \forall t \geq 0 \tag{1.3}$$

of solutions of the damped wave equation (1.1) obeys the following dissipation law:

$$\frac{dE(t)}{dt} = -\int_0^1 a(x)|y'|^2 dx. \tag{1.4}$$

According to (1.2) we see that the damping term $a(x)y'$ of the equation is effective in the open nonempty subinterval ω. As (1.4) shows, the energy of each solution decreases as time increases.

The question of whether solutions of (1.1) decays to zero as time goes to infinity is more subtle. However, by now it is well-known (cf. [BLR], [H2], [La], [TT1], [Z1]) that, under assumption (1.2), the energy of solutions of (1.1) satisfies, for some $t_0 > 0$, the estimate

$$E(t) \leq \left[\exp(1 - \frac{t}{t_0})\right] E(0), \quad \forall t \geq t_0. \tag{1.5}$$

Moreover, according to [H2], the exponential decay property of solutions of (1.1) is equivalent to an Observability Inequality (O.I.) for the solutions of the undamped system with $a \equiv 0$. Namely, it is equivalent to the existence of a positive time T and a positive constant C such that

$$E(0) \leq C\int_0^T \int_0^1 a(x)|y_t|^2 dxdt \tag{O.I.}$$

for every solution of (1.1) with $a \equiv 0$. This inequality guarantees that the energy of solutions of the undamped equation is captured uniformly for all solutions by the damping mechanism in time T.

There is by now an extensive literature on the subject and it is well known that, in the general multi-dimensional setting, observability inequalities of the form (O.I.) are valid if and only if a suitable Geometric Control Condition is satisfied (see [BLR]).

Our main purpose in this paper is to analyze whether some of the most classical numerical approximation schemes for (1.1) do possess the same decay rate and whether it is uniform with respect to the mesh size.

We consider the simplest of such numerical approximation models: the finite-difference space semi-discretization.

Let N be a nonnegative integer. Set $h = 1/(N+1)$ and consider the subdivision of $(0,1)$ given by

$$0 = x_0 < x_1 < ... < x_N < x_{N+1} = 1,$$

where $x_j = jh$.

The finite-difference space semi-discretization of System (1.1) is as follows

$$\begin{cases} y_j'' - \dfrac{y_{j+1} - 2y_j + y_{j-1}}{h^2} + a_j y_j' = 0 \text{ in } (0,\infty), \quad j = 1,\ 2, ...,\ N \\ y_0(t) = y_{N+1}(t) = 0 \text{ in } (0,\infty) \\ y_j(0) = y_j^0, \quad y_j'(0) = y_j^1, \quad j = 1,\ 2, ...,\ N \end{cases} \tag{1.6}$$

where $a_j,\ y_j^0,\ y_j^1,\ j = 0,\ 1, ...,\ N+1$ are approximations of the functions a, y^0 and y^1 respectively.

The energy of System (1.6) is given by

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \{(y_j'(t))^2 + \big(\frac{y_{j+1} - y_j}{h}\big)^2\} \tag{1.7}$$

and it is a nonincreasing function of the time t. In fact its derivative is given by

$$E_h'(t) = -h \sum_{j=1}^{N} a_j (y_j'(t))^2. \tag{1.8}$$

Observe that E_h is a natural semi-discrete version of the energy E of system (1.1) and that (1.8) is the natural semi-discrete analogue of the energy dissipation law (1.4).

It is then natural to wonder whether the energies E_h decay exponentially and uniformly (with respect to $h \to 0$) to zero as the time t approaches infinity. The answer to this question is negative, with the exception of the trivial case in which the damping coefficient a is uniformly positive all along the space interval $(0,1)$. Indeed, let us consider the conservative system associated with (1.6):

$$\begin{cases} u_j'' - \dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = 0 \text{ in } (0,\infty), \quad j = 1,\ 2, ...,\ N \\ u_0(t) = u_{N+1}(t) = 0 \text{ in } (0,\infty) \\ u_j(0) = u_j^0, \quad u_j'(0) = u_j^1, \quad j = 1,\ 2, ...,\ N. \end{cases} \tag{1.9}$$

In order to prove the exponential decay of solutions of the damped semi-discrete system (1.6) it is necessary and sufficient to prove the existence of a time $T > 0$ and a constant $C > 0$ such that the following observability inequality (O.I$_h$.) holds

$$E_h(0) \leq C \int_0^T \left[h \sum_{j=1}^{N} a_j |u_j'|^2 \right] dt \tag{O.I$_h$.}$$

for every solution u of (1.9). Moreover, the exponential decay rate is uniform with respect to the mesh size h, if and only if the time T and the constant C in this inequality are also independent of h (see section 2.1 below).

However, as it was proved in [G] and [IZ1, 2], these type of inequalities are not uniform in h as $h \to 0$. In [IZ1, 2] this fact was established in the case where the observation is done on the boundary of the interval $(0,1)$, i. e. replacing the right hand side of inequality $(\mathrm{O.I}_h.)$ by, for instance, $\int_0^T |u_N/h|^2 dt$ (which is a natural semi-discrete version of the tension of the string at the extreme $x = 1$). More recently, it has been proved in [Ma] using Wiener measures techniques that the observability inequality $(\mathrm{O.I}_h.)$ is not uniform (with respect to the net-spacing h), the trivial case where a is uniformly positive on $(0,1)$ being excepted. Thus, the exponential decay of E_h as $t \to \infty$ is not uniform. The reason for $(\mathrm{O.I}_h.)$ (or its boundary counterpart discussed in [IZ1,2]) not to be uniform as $h \to 0$ is the existence of high frequency solutions whose (group) velocity of propagation is of the order of h. If the initial data for those solutions is localized away from the support of a, in order for $(\mathrm{O.I}_h.)$ to be true with a uniform constant C, the time T needs to be of the order of $1/h$. Thus, for a fixed finite time T the constant C in $(\mathrm{O.I}_h.)$ necessarily blows up as h tends to zero.

Therefore, solutions of (1.6) do not decay exponentially to zero as $t \to \infty$ with a uniform (with respect to the mesh-size h) rate of decay.

One of the most common tools to obtain exponential decay rates for evolution PDE's is to use multipliers. A discrete multiplier version adapted to (1.9) was introduced in [IZ1,2]. However when using discrete multipliers, in agreement with the obstruction above for the uniform observability inequality to hold, one gets crossed terms whose absorption requires ruling out the high frequency spurious modes in order to obtain a uniform inequality (cf. [G], [IZ2]).

Therefore, in order to get a uniform decay, it seems natural to add in System (1.6) a suitable extra numerical viscosity term. When doing it at the right scale, the new system we obtain is as follows

$$(1.10) \qquad \begin{cases} y_j'' - \dfrac{y_{j+1} - 2y_j + y_{j-1}}{h^2} - h^2\left(\dfrac{y'_{j+1} - 2y'_j + y'_{j-1}}{h^2}\right) + a_j y'_j = 0 \text{ in } (0,\infty) \\ y_0(t) = y_{N+1}(t) = 0 \text{ in } (0,\infty) \\ y_j(0) = y_j^0, \quad y'_j(0) = y_j^1, \quad j = 1,\ 2, ...,\ N. \end{cases}$$

The energy of this new system is still E_h, and its derivative is now

$$(1.11) \qquad E'_h(t) = -h^3 \sum_{j=0}^{N} \left(\frac{y'_{j+1} - y'_j}{h}\right)^2 - h \sum_{j=1}^{N} a_j (y'_j(t))^2.$$

For System (1.10), we prove:

i) a decay rate of type (1.5) which is uniform with respect to the net-spacing h;

ii) the convergence of its solutions towards those of the original wave equation (1.1) as $h \to 0$, in a suitable topology.

These two results show that the discretization (1.10) of System (1.1), in which a suitable artificial numerical viscosity term is introduced, is a good approximate scheme for (1.1) because not only it guarantees the convergence of solutions as $h \to 0$ (which is also true for the simpler approximation (1.6)) but because it also provides a uniform (with respect to $h \to 0$) decay rate of solutions as $t \to \infty$. This second fact shows that the viscous damping term in (1.10) correctly captures the long time asymptotic properties of System (1.1), and that it efficiently rules out the aforementioned high frequency numerical spurious oscillations.

The suitability of this numerical damping mechanism to restore the uniform exponential decay is closely connected to the efficiency of the Tychonoff regularization techniques developed in [GLL] when building up numerical schemes for the controllability of the wave equation.

We are now in the position to state our main results. The uniform stabilization result may be stated as follows:

THEOREM 1.1. *Let a satisfy (1.2) with* $\omega = (l_1, l_2)$ *where* $0 \le l_1 < l_2 \le 1$. *Then there exist positive constants M and* t_1 *independent of h such that the energy* E_h *of System (1.10) satisfies*

$$E_h(t) \le M[\exp -t_1 t] E_h(0), \quad \forall t \ge 0 \tag{1.12}$$

for every solution of system (1.10) and every $0 < h < 1$.

According to Theorem 1.1, the numerical viscosity term that we add, i.e

$$-h^2 \left(\frac{y'_{j+1} - 2y'_j + y'_{j-1}}{h^2}\right) = -(y'_{j+1} - 2y'_j + y'_{j-1}),$$

is the correct one in order to restore the uniform (with respect to $h \to 0$) exponential decay.

To see this, it is convenient to write System (1.10) in the following simplified form:

$$Y'' + A_h Y + h^2 A_h Y' + a_h Y' = 0, \quad t > 0$$

where Y stands for the column vector $(y_1, ..., y_N)$, A_h denotes the matrix

$$A_h = -\frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & & 0 \\ -1 & 2 & -1 & & \\ . & . & . & & . \\ & & -1 & 2 & -1 \\ 0 & & 0 & -1 & 2 \end{pmatrix}$$

entering in the finite difference discretization of the Laplacian and $a_h Y'$ denotes the vector $(a_1 y_1', ..., a_N y_N')$.

It is not hard to see that the numerical viscosity term $h^2 A_h Y'$ has the right scaling properties. Indeed, as we mentioned above, the lack of uniform decay is due to the high frequency components of the solutions corresponding to the eigenvectors of A_h of the order of $\lambda \sim \frac{c}{h^2}$ with $0 < c < 4$. Note however that, for those high frequencies, $h^2 A_h Y'$ is precisely of the order of cY'. Thus, it is a damping mechanism of unit order that suffices to produce a uniform exponential decay.

Summarizing, this numerical damping mechanism has the following two properties:

- it damps out efficiently the high frequency components;

- the amount of damping added by this mechanism tends to zero as h approaches zero over a given frequency band.

This explains the efficiency of scheme (1.10) in restoring the uniform (as $h \to 0$) decay property and also, the fact that for given initial data, the solution of (1.10) converges to the solution of (1.1) as $h \to 0$.

To prove Theorem 1.1, we use a discrete version of the multiplier method such as presented in [Lio2]. As we mentioned above, it has been observed in [IZ1, IZ2] in the context of the boundary observability that this method introduces crossed terms in the estimates which must be absorbed in order to get a uniform inequality. In order to get rid of these crossed terms in [IZ1,2] the Fourier expansion of the solution and the filtering of the high frequencies were used. In our case, the introduction of the aforementioned numerical viscosity term enables us to successfully get rid of these crossed terms without any filtering of the high frequencies.

Before stating our second result, we need some additional notations. Set $a_h = (a_j)_j$, $y_h = (y_j)_j$, $y_h^0 = (y_j^0)_j$, and $y_h^1 = (y_j^1)_j$. Introduce the extension operators defined by (see [Lio1]):

$$(1.13) \qquad p_h v_h = \begin{cases} \text{the continuous function, linear in each interval } [jh, (j+1)h], \\ \text{such that } p_h v_h(jh) = v_j, \ j = 0, \ 1, \ ..., \ N+1, \end{cases}$$

$$(1.14) \quad q_h v_h = \begin{cases} \text{the step function defined in each interval } ((j - \frac{1}{2})h, (j + \frac{1}{2})h) \cap (0,1) \\ \text{by } q_h v_h(x) = v_j, \ j = 0, \ 1, \ ..., \ N+1. \end{cases}$$

It is not hard to check that

$$(1.15) \qquad \begin{aligned} &\int_0^1 (p_h v_h)_x (p_h w_h)_x dx = h \sum_{j=0}^{N} \big(\frac{v_{j+1} - v_j}{h}\big)\big(\frac{w_{j+1} - w_j}{h}\big) \\ &\int_0^1 (q_h v_h)(q_h w_h) dx = h \sum_{j=0}^{N} v_j w_j. \end{aligned}$$

We are now in the position to state our convergence result:

THEOREM 1.2. *Let y_h denote the solution of (1.10). Assume that a_h, y_h^0 and y_h^1 are such that there is a nonnegative constant c independent of h with*

$$(1.16)\qquad \begin{cases} E_h(0) \le c, \qquad q_h a_h \to a \text{ weakly * in } L^\infty(0,1) \\ p_h y_h^0 \to y^0 \text{ weakly in } H_0^1(0,1), \quad q_h y_h^1 \to y^1 \text{ weakly in } L^2(0,1), \end{cases}$$

as h tends to zero.

Then we have

$$(1.17)\qquad \begin{cases} p_h y_h \to y \text{ weakly * in } L^\infty(0,\infty; H_0^1(0,1)) \\ q_h y_h' \to y' \text{ weakly * in } L^\infty(0,\infty; L^2(0,1)) \end{cases}$$

where y is the solution of System (1.1).

The convergence hypotheses contained in (1.16), are plausible. Indeed, for every function u, continuous on [0,1], if we set $u_h = (u_j)_j = (u(jh))_j$, it follows that

$$(1.18)\qquad q_h u_h \to u \text{ strongly in } L^\infty(0,1).$$

Moreover, if $u \in H^1(0,1)$, then

$$(1.19)\qquad \|p_h u_h - q_h u_h\|_{L^2(0,1)}^2 = h^2 \left[\frac{h}{12} \sum_{j=0}^{N} \left(\frac{u_{j+1} - u_j}{h}\right)^2 \right] = O(h^2).$$

so that $p_h u_h$ is bounded in $H^1(0,1)$. Thus, according to (1.18)-(1.19) we deduce that

$$(1.20)\qquad p_h u_h \to u \text{ weakly in } H^1(0,1).$$

Now let $v \in L^2(0,1)$. For all $j = 0, 1, ..., N$, set

$$(1.21)\qquad v_j = \frac{1}{h} \int_{jh}^{(j+1)h} v(x)dx, \quad v_h = (v_j)_j.$$

Then

$$(1.22)\qquad q_h v_h \to v \text{ strongly in } L^2(0,1).$$

Indeed, let (φ_n) be a sequence of continuous functions on [0,1] satisfying

$$(1.23)\qquad \varphi_n \to v \text{ strongly in } L^2(0,1).$$

If we define (φ_{nh}) as we defined v_h, it easily follows that

$$\|q_h v_h - v\|^2_{L^2(0,1)} \leq 6\|\varphi_n - v\|^2_{L^2(0,1)} + 4\|q_h\varphi_{nh} - \varphi_n\|^2_{L^2(0,1)} \tag{1.24}$$

since, from the definition of q_h,

$$\|q_h\varphi_{nh} - q_h v_h\|^2_{L^2(0,1)} \leq \|\varphi_n - v\|^2_{L^2(0,1)}. \tag{1.25}$$

Therefore (1.22) follows from (1.24), (1.23) and the uniform continuity of each φ_n.

The rest of the paper is organized as follows:

Section 2 deals with the proofs of Theorem 1.1 (in the particular case where $\omega = (l, 1)$) and Theorem 1.2.

In Section 3 we address the same problem in a square of $\mathbf{R}^2$.

The results of this section can be easily adapted to more general $2 - D$ domains, whose boundary is constituted by points of a finite-difference mesh. Indeed, to do that, it is sufficient to apply the discrete multipliers of this section which are an adaptation at the discrete level of those in [Lio2] and [Z1].

Then in Appendix A, we provide guidelines for the proof of the general case of Theorem 1.1 and, finally, in Appendix B we give details on some identities established and used along the proofs of Theorems 1.1 and 3.1.

2. Proofs of Theorems 1.1 and 1.2. The proof of Theorem 1.1 provided here involves the case where ω is a neighborhood of the endpoint $x = 1$; this proof is easily adaptable to the case where the damping is located in a neighborhood of the endpoint $x = 0$. The proof provided here is a bit simpler than in the general case where ω is an arbitrary nonempty open subset of $(0, 1)$. The latter proof follows by an additional cut-off argument that allows to reduce the general case to the previous one in which the damping is localized in a neighborhood of the boundary. Details are given in Appendix A at the end of the paper. We choose this presentation to highlight the fact that, even in the simpler situation where the damping is located near one endpoint, the introduction of a suitable viscous term is essential to obtain a uniform decay of the energy E_h; the main difficulty is definitely not the location of the damping mechanism but the high frequency spurious oscillations mentioned above.

2.1. Proof of Theorem 1.1.

We proceed in several steps. We first reduce the problem to obtaining a suitable uniform (with respect to h) observability inequality for the undamped system. We then establish a first estimate in which the normal derivative (in the discrete version) of solutions enters. Finally, using a compactness-uniqueness argument, we obtain the desired inequality.

Step 1. We claim that it is sufficient to prove the following estimate for solutions of the conservative system (1.9): There exists $T > 0$ and $C > 0$ such that

$$E_h(0) \leq Ch \sum_{j=0}^{N} \int_0^T \left[a_j |u_j'|^2 + h^2 \left| \frac{u_{j+1}' - u_j'}{h} \right|^2 \right] dt, \tag{2.1}$$

for every solution u of (1.9) and for all $0 < h < 1$.

Here and in the sequel C denotes a generic positive constant that may vary from line to line but is independent of h and the solution under consideration.

Indeed, let us assume that (2.1) holds. We decompose the solution y_h of (1.10) as $y_h = u_h + z_h$ where u_h solves (1.9) with initial data (y_h^0, y_h^1) and z_h is the solution of

$$\begin{cases} z_j'' - \dfrac{z_{j+1} - 2z_j + z_{j-1}}{h^2} = h^2\left(\dfrac{y'_{j+1} - 2y'_j + y'_{j-1}}{h^2}\right) - a_j y'_j \text{ in } (0,\infty) \\ z_0(t) = z_{N+1}(t) = 0 \text{ in } (0,\infty) \\ z_j(0) = z'_j(0) = 0, \quad j = 1,\ 2, ...,\ N. \end{cases} \tag{2.2}$$

According to (2.1) we have

$$E_h(0) \le Ch \sum_{j=0}^{N} \int_0^T \left[a_j |y'_j|^2 + h^2 \left| \frac{y'_{j+1} - y'_j}{h} \right|^2 + a_j |z'_j|^2 + h^2 \left| \frac{z'_{j+1} - z'_j}{h} \right|^2 \right] dt. \tag{2.3}$$

Let us now assume for the moment that the following inequality holds:

$$\begin{aligned} Ch \sum_{j=0}^{N} \int_0^T \left[a_j |z'_j|^2 + h^2 \left| \frac{z'_{j+1} - z'_j}{h} \right|^2 \right] dt \le Ch \sum_{j=1}^{N} \int_0^T a_j |y'_j|^2 dt \\ + Ch^3 \sum_{j=0}^{N} \int_0^T \left| \frac{y'_{j+1} - y'_j}{h} \right|^2 dt. \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), it follows then that

$$E_h(T) \le E_h(0) \le Ch \sum_{j=0}^{N} \int_0^T \left[a_j |y'_j|^2 + h^2 \left| \frac{y'_{j+1} - y'_j}{h} \right|^2 \right] dt. \tag{2.5}$$

Now, integrating in $(0,T)$ the energy dissipation law (1.11) for solutions of (1.10) it follows that

$$E_h(T) - E_h(0) = -h \sum_{j=0}^{N} \int_0^T \left[a_j |y'_j|^2 + h^2 \left| \frac{y'_{j+1} - y'_j}{h} \right|^2 \right] dt. \tag{2.6}$$

Combining (2.5)-(2.6), we deduce that

$$E_h(T) \le \gamma E_h(0) \tag{2.7}$$

with $\gamma = C/(C+1)$. Inequality (2.7) together with the semigroup property imply that

$$E_h(t) \leq \frac{1}{\gamma} \exp^{\frac{t \log(\gamma)}{T}} E_h(0), \quad \forall t > 0, \quad \forall 0 < h < 1. \tag{2.8}$$

with $0 < \gamma < 1$ independent of h.

This is the uniform, with respect to h, exponential decay rate we were looking for.

Let us check (2.4) now. Multiplying the equation satisfied by z_h by z_j', and adding with respect to $j = 1, ..., N$, we deduce that

$$\begin{aligned} F_h'(t) &= h^2 \sum_{j=0}^{N} \Big(\frac{y_{j+1}' - y_j'}{h} \Big) z_j' - h^2 \sum_{j=0}^{N} \Big(\frac{y_{j+1}' - y_j'}{h} \Big) z_{j+1}' - h \sum_{j=1}^{N} a_j y_j'(t) z_j'(t) \\ &\leq F_h(t) + C \Big(h^3 \sum_{j=0}^{N} (\frac{y_{j+1}' - y_j'}{h})^2 + h \sum_{j=1}^{N} a_j (y_j'(t))^2 \Big), \end{aligned} \tag{2.9}$$

where F_h denotes the energy of System (2.2).

Integrating (2.9) in $[0, t]$, taking into account that $F_h(0) = 0$, using the Gronwall lemma and the fact that

$$h^3 \sum_{j=0}^{N} \int_0^T (\frac{z_{j+1}' - z_j'}{h})^2 dt + h \sum_{j=1}^{N} \int_0^T a_j (z_j'(t))^2 dt \leq C \int_0^T F_h(t) dt,$$

inequality (2.4) immediately follows. QED

In the sequel we focus on obtaining inequality (2.1) for solutions of (1.9).

Step 2. For this step, we use the multiplier $j(\frac{u_{j+1} - u_{j-1}}{2})$ which corresponds to a discrete version of the multiplier $x u_x$ introduced in [IZ1,2]. Multiplying the first equation of (1.9) by $j(\frac{u_{j+1} - u_{j-1}}{2})$ and taking the sum over j, we obtain for $0 < T < \infty$

$$\begin{aligned} & h \sum_{j=1}^{N} u_j' j (\frac{u_{j+1} - u_{j-1}}{2}) \Big]_0^T - h \sum_{j=1}^{N} j \int_0^T u_j' (\frac{u_{j+1}' - u_{j-1}'}{2}) dt \\ & - h \sum_{j=1}^{N} j \int_0^T (\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}) (\frac{u_{j+1} - u_{j-1}}{2}) dt = 0. \end{aligned} \tag{2.10}$$

Some elementary calculations show that

$$
\begin{aligned}
(2.11)\qquad & h\sum_{j=1}^{N} u_j' j\Big(\frac{u_{j+1}-u_{j-1}}{2}\Big)\Big]_0^T = h^2\sum_{j=1}^{N} u_j' j\Big(\frac{u_{j+1}-u_j}{2h}\Big)\Big]_0^T \\
& + h^2\sum_{j=1}^{N} u_j' j\Big(\frac{u_j-u_{j-1}}{2h}\Big)\Big]_0^T \\
& = h^2\sum_{j=0}^{N} u_j' j\Big(\frac{u_{j+1}-u_j}{2h}\Big)\Big]_0^T + h^2\sum_{j=0}^{N} u_{j+1}'(j+1)\Big(\frac{u_{j+1}-u_j}{2h}\Big)\Big]_0^T \\
& = h^2\sum_{j=0}^{N}\big(ju_j'+(j+1)u_{j+1}'\big)\Big(\frac{u_{j+1}-u_j}{2h}\Big)\Big]_0^T,
\end{aligned}
$$

$$
\begin{aligned}
(2.12)\qquad & -h\sum_{j=1}^{N} j\int_0^T \Big(\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}\Big)\Big(\frac{u_{j+1}-u_{j-1}}{2}\Big)dt = -\frac{1}{2}\int_0^T \Big(\frac{u_N}{h}\Big)^2 dt \\
& + \frac{h}{2}\sum_{j=0}^{N}\int_0^T \Big(\frac{u_{j+1}-u_j}{h}\Big)^2 dt
\end{aligned}
$$

and

$$
(2.13)\qquad -h\sum_{j=1}^{N}\int_0^T ju_j'\left(\frac{u_{j+1}'-u_{j-1}'}{2}\right)dt = \frac{h}{2}\sum_{j=0}^{N}\int_0^T u_{j+1}'u_j' dt =
$$

$$
= -\frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left(\frac{u_{j+1}'-u_j'}{h}\right)^2 dt + \frac{h}{2}\sum_{j=1}^{N}\int_0^T (u_j')^2 dt.
$$

Taking the sums in (2.11)-(2.13) side by side, using (1.7) and reporting the result in (2.10), we find

$$
\begin{aligned}
(2.14)\qquad & TE_h(0) = \int_0^T E_h(t)dt = -h^2\sum_{j=0}^{N}\big(ju_j'+(j+1)u_{j+1}'\big)\Big(\frac{u_{j+1}-u_j}{2h}\Big)\Big]_0^T \\
& + \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \Big(\frac{u_{j+1}'-u_j'}{h}\Big)^2 dt + \frac{1}{2}\int_0^T \Big(\frac{u_N}{h}\Big)^2 dt.
\end{aligned}
$$

Before carrying out the remaining estimates, we would like to draw the reader's attention to the fact that in the right hand side of (2.14) the term associated with the energy dissipation rate of the viscous damping term appears naturally. As pointed out in [IZ1, 2], that term cannot be bounded above in terms of the energy, uniformly for all solutions. For that, high frequencies have to be filtered. As explained in the introduction, in the context of stabilization it is better to keep that term as it is, and absorb it through the viscous damping mechanism.

Using Young inequality as well as (1.7), (1.11) and the conservation of the energy, we easily obtain the inequality

$$\Big| - h^2 \sum_{j=0}^{N} \big(ju'_j + (j+1)u'_{j+1}\big)\big(\frac{u_{j+1}-u_j}{2h}\big)\Big]_0^T \Big| \leq 2E_h(0). \tag{2.15}$$

Summarizing, we have

$$(T-2)E_h(0) \leq \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \big(\frac{u'_{j+1}-u'_j}{h}\big)^2 dt + \frac{1}{2}\int_0^T \big(\frac{u_N}{h}\big)^2 dt. \tag{2.16}$$

Step 3. We recall that $\omega = (l, 1)$, where $0 \leq l < 1$, and we set $n = \mathcal{E}(l/h) + 1$, where $\mathcal{E}(x)$ denotes the greatest integer less than or equal to x.

Introduce the function $f \in W^{1,\infty}(0,1)$ such that

$$f(1) = 1, \quad f(x) = 0 \quad \text{in} \quad [0,1) \setminus (l+\theta, 1) \tag{2.17}$$

where θ is a positive constant such that $l + \theta < 1$. Set $n_1 = \mathcal{E}(\frac{l+\theta}{h})$ and denote by $(f_j)_j$ an approximation of the function f. It is clear that

$$f_{N+1} = 1, \quad f_j = 0 \text{ for } j \leq n_1. \tag{2.18}$$

We also introduce a smooth nonnegative function $g = g(t)$ such that $g(0) = g(T) = 0$ and $g(t) = 1$ in $[\delta, T-\delta]$, where $\delta > 0$, sufficiently small, will be chosen later.

Now, multiply the first equation of (1.9) by $g(t)f_j(\frac{u_{j+1}-u_{j-1}}{2h})$ and proceed as in the second step. We find

$$\begin{aligned} &-\frac{h}{2}\sum_{j=n_1}^{N}\int_0^T g'(t)(f_j u'_j + f_{j+1}u'_{j+1})\frac{u_{j+1}-u_j}{h}dt \\ &+\frac{h}{2}\sum_{j=n_1}^{N}\frac{f_{j+1}-f_j}{h}\int_0^T g(t)u'_{j+1}u'_j dt - \frac{1}{2}\int_0^T g(t)\big(\frac{u_N}{h}\big)^2 dt \\ &+\frac{h}{2}\sum_{j=n_1}^{N}\frac{f_{j+1}-f_j}{h}\int_0^T g(t)\big(\frac{u_{j+1}-u_j}{h}\big)^2 dt = 0. \end{aligned} \tag{2.19}$$

It follows from (2.19) and (1.2) that

$$\begin{aligned}\int_\delta^{T-\delta} (\frac{u_N}{h})^2 dt &\le Ch \sum_{j=n_1}^{N} \int_0^T \Big[g(t)(\frac{u_{j+1}-u_j}{h})^2 + (u_j')^2 \Big] dt \\ &\le Ch \Big[\sum_{j=n_1}^{N} \int_0^T g(t)(\frac{u_{j+1}-u_j}{h})^2 dt + \sum_{j=1}^{N} \int_0^T a_j (u_j')^2 dt \Big].\end{aligned} \tag{2.20}$$

Reporting (2.20) in (2.16) with $T > 2$ and $\delta > 0$ small enough so that $T - 2\delta > 0$, we obtain

$$\begin{aligned} E_h(0) &\le \frac{h^3}{4} \sum_{j=0}^{N} \int_0^T (\frac{u_{j+1}' - u_j'}{h})^2 dt \\ &+ Ch \Big[\sum_{j=n_1}^{N} \int_0^T g(t)(\frac{u_{j+1}-u_j}{h})^2 dt + \sum_{j=1}^{N} \int_0^T a_j (u_j')^2 dt \Big].\end{aligned} \tag{2.21}$$

At this stage, we observe that we only need a further estimate on the second term in the right hand side of (2.21).

Step 4. Introduce the function η which satisfies

$$\eta \in W^{1,\infty}(0,1), \quad 0 \le \eta \le 1, \quad \eta = 1 \quad in \quad (l+\theta, 1), \quad \eta = 0, \ \text{ in } \ (0,1) \setminus \omega. \tag{2.22}$$

Denote by $(\eta_j)_j$, an approximation of η having properties similar to those of η. Multiplying the first equation of (1.9) by $g(t)\eta_j^2 u_j$ and proceeding as in the previous step lead to

$$\begin{aligned} &- h \sum_{j=n}^{N} \eta_j^2 \int_0^T g'(t) u_j' u_j dt - h \sum_{j=n}^{N} \eta_j^2 \int_0^T g(t)(u_j')^2 dt \\ &+ h \sum_{j=n}^{N} \frac{\eta_{j+1}^2 + \eta_j^2}{2} \int_0^T g(t)(\frac{u_{j+1}-u_j}{h})^2 dt \\ &+ h \sum_{j=n}^{N} \frac{\eta_{j+1}^2 - \eta_j^2}{2h} \int_0^T g(t) u_j (\frac{u_{j+1}-u_j}{h}) dt \\ &+ h \sum_{j=n}^{N} \frac{\eta_{j+1}^2 - \eta_j^2}{2h} \int_0^T g(t) u_{j+1} (\frac{u_{j+1}-u_j}{h}) dt = 0,\end{aligned} \tag{2.23}$$

and we easily obtain

$$h\sum_{j=n}^{N}\frac{\eta_{j+1}^2+\eta_j^2}{2}\int_0^T g(t)(\frac{u_{j+1}-u_j}{h})^2dt \le Ch\sum_{j=1}^{N}\int_0^T a_j(u_j')^2dt + \sum_{j=n}^{N}\int_0^T g(t)u_j^2dt. \tag{2.24}$$

The combination of (2.21) and (2.24) gives, for any $T>2$,

$$E_h(0)\le \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T (\frac{u_{j+1}'-u_j'}{h})^2dt + Ch\sum_{j=n}^{N}\int_0^T g(t)(u_j)^2dt + \sum_{j=1}^{N}\int_0^T a_j(u_j')^2dt. \tag{2.25}$$

Step 5. To conclude the proof of (2.1) it is sufficient to show that there exists a constant $C>0$ such that

$$h\sum_{j=n}^{N}\int_0^T g(t)(u_j)^2dt \le Ch^3\sum_{j=0}^{N}\int_0^T (\frac{u_{j+1}'-u_j'}{h})^2dt + Ch\sum_{j=1}^{N}\int_0^T a_j(u_j')^2dt. \tag{2.26}$$

To do this we may argue by means of a classical compactness uniqueness argument as in Proposition 4.1 in [Z2]. Indeed, assume (2.26) is not true. Then, there exists a sequence $h\to 0$ and a sequence of solutions u_h (that by simplicity we shall simply denote by u) such that

$$h^3\sum_{j=0}^{N}\int_0^T (\frac{u_{j+1}'-u_j'}{h})^2dt + \sum_{j=1}^{N}\int_0^T a_j(u_j')^2dt \to 0, \tag{2.27}$$

and

$$h\sum_{j=1}^{N}\int_0^T (u_j)^2dt = 1, \tag{2.28}$$

as $h\to 0$.

In view of inequality (2.25) this sequence is such that the energies $E_h(0)$ are uniformly bounded.

Arguing as in Proposition 4.1 in [Z2], extracting subsequences and letting $h \to 0$ we obtain a solution $u = u(x,t)$ of the conservative wave equation

$$
(2.29) \qquad \begin{cases} u'' - u_{xx} = 0 \text{ in } (0,1) \times (0,\infty) \\ u(0,t) = u(1,t) = 0 \text{ in } (0,\infty) \end{cases}
$$

such that, by compactness and (2.28),

$$
(2.30) \qquad \int_0^T \int_0^1 u^2(x,t)dxdt = 1,
$$

and, according to (2.27),

$$
(2.31) \qquad a(x)u_t \equiv 0 \quad \text{in} \quad (0,1) \times (0,T).
$$

In view of (2.31), by unique continuation for the solution of the wave equation (2.28) we deduce that $u \equiv 0$ which is in contradiction with (2.30). This completes the proof of Theorem 1.1. □

2.2. Proof of Theorem 1.2. Observe that by the definitions of p_h and q_h, we have for every $t \geq 0$

$$
(2.35) \qquad E_h(t) = \frac{1}{2}\big(\|p_h y_h(t)\|^2_{H^1_0(0,1)} + \|q_h y_h'(t)\|^2_{L^2(0,1)}\big).
$$

Thanks to (1.16), (2.35) and the decrease of E_h, we know that $p_h y_h$ is bounded in $L^\infty(0,\infty; H^1_0(0,1)) \cap W^{1,\infty}(0,\infty; L^2(0,1))$, while $q_h y_h$ is bounded in $L^\infty(0,\infty; L^2(0,1))$. Thus, up to the extraction of a subsequence, we have

$$
(2.36) \qquad \begin{cases} p_h y_h \to y \text{ weakly * in } L^\infty(0,\infty; H^1_0(0,1)) \\ p_h y_h' \to y' \text{ weakly * in } L^\infty(0,\infty; L^2(0,1)) \\ p_h y_h \to y \text{ strongly in } L^2_{loc}(0,\infty; L^2(0,1)) \\ q_h y_h \to y \text{ weakly * in } L^\infty(0,\infty; L^2(0,1)) \\ q_h y_h' \to y' \text{ weakly * in } L^\infty(0,\infty; L^2(0,1)) \\ h p_h y_h' \to 0 \text{ weakly in } L^2(0,\infty; H^1_0(0,1)). \end{cases}
$$

The last convergence in (2.36) follows from the second one and the boundedness of the sequence $\{h p_h y_h'\}$ in the space $L^2(0,\infty; H^1_0(0,1))$.

Note that in (2.36) we implicitly claim that the limits of $p_h y_h$ and $q_h y_h$ are the same. To see this, it is sufficient to observe that thanks to (1.16), (2.36), and the definitions of p_h and q_h, we have for every $t \geq 0$

$$
(2.37) \qquad \int_0^1 |(p_h y_h - q_h y_h)(x,t)|^2 dx = \frac{h^3}{12} \sum_{j=0}^{N} \big(\frac{y_{j+1} - y_j}{h}\big)^2 \leq \frac{h^2}{6} E_h(t) \leq ch^2.
$$

We have to show now that the limit y is the solution of (1.1). To this end, let $w \in \mathcal{D}((0,1)\times(0,\infty))$, and set $w_h = (w_j)_j$ where $w_j = w(jh,.)$. Multiplying the first equation of (1.10) by w_j, integrating by parts on $(0,\infty)$ and taking the sum over j, we find

$$\begin{aligned} &h\sum_{j=1}^{N}\int_0^\infty y_j w_j'' dt + h\sum_{j=0}^{N}\int_0^\infty (\frac{y_{j+1}-y_j}{h})(\frac{w_{j+1}-w_j}{h})dt \\ &+ h^3\sum_{j=0}^{N}\int_0^\infty (\frac{y'_{j+1}-y'_j}{h})(\frac{w_{j+1}-w_j}{h})dt - h\sum_{j=1}^{N} a_j\int_0^\infty y_j w'_j dt = 0. \end{aligned} \tag{2.38}$$

Using the definitions of p_h and q_h, it is easy to check that (2.38) is equivalent to

$$\begin{aligned} &\int_0^\infty\int_0^1 (q_h y_h)(q_h w_h'')dxdt + \int_0^\infty\int_0^1 (p_h y_h)_x (p_h w_h)_x dxdt \\ &+ h^2\int_0^\infty\int_0^1 (p_h y'_h)_x (p_h w_h)_x dxdt - \int_0^\infty\int_0^1 (q_h y_h)(q_h w'_h)(q_h a_h)dxdt = 0. \end{aligned} \tag{2.39}$$

At this stage, we observe that if we show that for every $w \in \mathcal{D}((0,1)\times(0,\infty))$

$$\begin{aligned} &p_h w_h \to w \text{ strongly in } L^2(0,\infty; H_0^1(0,1)), \\ &q_h w_h \to w \text{ strongly in } L^2(0,\infty; L^2(0,1)), \end{aligned} \tag{2.40}$$

then we will be done. In fact, with (2.40) and (2.36), we can pass to the limit in all the terms in (2.39) getting

$$\int_0^\infty\int_0^1 y w'' dxdt + \int_0^\infty\int_0^1 y_x w_x dxdt - \int_0^\infty\int_0^1 y w' a dxdt = 0. \tag{2.41}$$

If we show that $y(0) = y^0$, and $y'(0) = y^1$, then these identities added to (2.36), and (2.41) will clearly show that y solves (1.1). For this purpose, let $v \in \mathcal{D}((0,1))$ and $l \in \mathcal{D}([0,\infty))$, and set $v_h = (v_j)_j$ where $v_j = v(jh)$. Multiplying the first equation of (1.10) by $v_j l$, integrating by parts on $[0,\infty)$ and taking the sum over j, we find

$$\begin{aligned} &-h\sum_{j=1}^{N} y_j^1 v_j l(0) + h\sum_{j=1}^{N} y_j^0 v_j l'(0) - h\sum_{j=1}^{N} a_j y_j^0 v_j l(0) \\ &+ h\sum_{j=1}^{N}\int_0^\infty y_j v_j l'' dt + h\sum_{j=0}^{N}\int_0^\infty (\frac{y_{j+1}-y_j}{h})(\frac{v_{j+1}-v_j}{h}) l dt \\ &+ h^3\sum_{j=0}^{N}\int_0^\infty (\frac{y'_{j+1}-y'_j}{h})(\frac{v_{j+1}-v_j}{h}) l dt - h\sum_{j=1}^{N} a_j\int_0^\infty y_j v_j l' dt = 0. \end{aligned} \tag{2.42}$$

Using the definitions of p_h and q_h, it is easy to check that (2.42) is equivalent to

$$
\begin{aligned}
& - l(0) \int_0^1 (q_h y_h^1)(q_h v_h) dx + l'(0) \int_0^1 (q_h y_h^0)(q_h v_h) dx \\
& - l(0) \int_0^1 (q_h y_h^0)(q_h v_h)(q_h a_h) dx \\
& + \int_0^\infty \int_0^1 (q_h y_h)(q_h v_h) l'' dx dt + \int_0^\infty \int_0^1 (p_h y_h)_x (p_h v_h)_x l dx dt \\
& + h^2 \int_0^\infty \int_0^1 (p_h y_h')_x (p_h v_h)_x l dx dt - \int_0^\infty \int_0^1 (q_h y_h)(q_h v_h)(q_h a_h) l' dx dt = 0.
\end{aligned} \tag{2.43}
$$

Passing to the limit as $h \to 0$ in (2.43), we get

$$
\begin{aligned}
& - l(0) \int_0^1 y^1 v dx + l'(0) \int_0^1 y^0 v dx - l(0) \int_0^1 y^0 v a dx \\
& + \int_0^\infty \int_0^1 y v l'' dx dt + \int_0^\infty \int_0^1 y_x v_x l dx dt - \int_0^\infty \int_0^1 y v a l' dx dt = 0
\end{aligned} \tag{2.44}
$$

from which we easily deduce $y(0) = y^0$, and $y'(0) = y^1$.

To complete the proof of Theorem 1.2, it remains to prove (2.40); we recall that this result is well-known. However, for the sake of completeness, we provide a proof here. To this end, first observe that for every $w \in \mathcal{D}((0,1) \times (0,\infty))$, we have

$$
\begin{aligned}
& \int_0^\infty \int_0^1 |(p_h w_h)_x - w_x|^2 dx dt = h \sum_{j=0}^N \int_0^\infty \Big(\frac{w_{j+1} - w_j}{h}\Big)^2 dt \\
& + \int_0^\infty \int_0^1 |w_x|^2 dx dt - 2 \int_0^\infty h \sum_{j=0}^N \Big(\frac{w_{j+1} - w_j}{h}\Big) \int_0^1 w_x(jh + sh, t) ds dt,
\end{aligned} \tag{2.45}
$$

so that using the properties of Riemann sums, it follows that

$$
\lim_{h \to 0} \int_0^\infty \int_0^1 |(p_h w_h)_x - w_x|^2 dx dt = 0. \tag{2.46}
$$

Consequently, the first convergence in (2.40) holds. Let us turn now to the second convergence. For this purpose, observe that

$$
\begin{aligned}
& \int_0^\infty \int_0^1 (q_h w_h - w)^2 dx dt = \int_0^\infty h \sum_{j=1}^N |w(jh,t)|^2 dt + \int_0^\infty \int_0^1 w^2 dx dt \\
& - 2h \sum_{j=1}^N \int_0^\infty \Big(\int_0^{\frac{1}{2}} w_j w(jh + sh, t) ds + \int_{\frac{1}{2}}^1 w_{j+1} w(jh + sh, t) ds \Big) dt.
\end{aligned} \tag{2.47}
$$

Using the properties of Riemann sums once more, we conclude that the second convergence in (2.40) holds too. This completes the proof of Theorem 1.2. □

3. The 2-D case.

3.1. Position of the problem and statements of the main results. Let Ω be the square $(0,1)\times(0,1)$ of $\mathbf{R}^2$ and consider the following damped wave equation

$$\begin{cases} y'' - \Delta y + ay' = 0 \text{ in } \Omega\times(0,\infty) \\ y = 0 \text{ on } \Sigma = \partial\Omega\times(0,\infty) \\ y(0) = y^0 \text{ in } \Omega \\ y'(0) = y^1 \text{ in } \Omega \end{cases} \tag{3.1}$$

where the function a is a bounded nonnegative function. System (3.1) is then well-posed in the space $H_0^1(\Omega)\times L^2(\Omega)$. Indeed, given $\{y^0, y^1\}\in H_0^1(\Omega)\times L^2(\Omega)$, there exists a unique solution of (3.1) with

$$y \in \mathcal{C}([0,\infty); H_0^1(\Omega)) \cap \mathcal{C}^1([0,\infty); L^2(\Omega)). \tag{3.2}$$

Introduce the energy

$$E_1(t) = \frac{1}{2}\int_\Omega \{|y'(x,t)|^2 + |\nabla y(x,t)|^2\}dx, \qquad \forall t \geq 0. \tag{3.3}$$

The energy E_1 is a nonincreasing function of the time variable t; in fact for every $t\geq 0$, we have

$$E_1'(t) = -\int_\Omega a|y'(x,t)|^2 dx. \tag{3.4}$$

Let Γ_0 be defined by

$$\Gamma_0 = \{(x_1,1); x_1\in(0,1)\}\cup\{(1,x_2); x_2\in(0,1)\} \tag{3.5}$$

and ω denote a neighborhood of Γ_0. It is well-known that if the function a is bounded from below by a positive constant α_0 in ω, then the energy E_1 satisfies (1.5) (cf. [BLR], [H2], [La], [TT1, TT2, TT3], [Z1]) .

Our purpose in this section is to prove that a similar estimate is valid for the "finite-difference space semi-discretization"; of (3.1), uniformly with respect to the mesh size, provided a suitable numerical viscosity term is added, and to show that the solution of the discretized system converges to the solution of (3.1).

For nonnegative integers N and M, we set

$$h_1 = \frac{1}{N+1}, \quad h_2 = \frac{1}{M+1} \tag{3.6}$$

and consider the discretized system

$$\begin{cases} y''_{j,k} - \dfrac{y_{j+1,k} - 2y_{j,k} + y_{j-1,k}}{h_1^2} - \dfrac{y_{j,k+1} - 2y_{j,k} + y_{j,k-1}}{h_2^2} \\ - h_1^2(\dfrac{y'_{j+1,k} - 2y'_{j,k} + y'_{j-1,k}}{h_1^2}) - h_2^2(\dfrac{y'_{j,k+1} - 2y'_{j,k} + y'_{j,k-1}}{h_2^2}) \\ + a_{j,k} y'_{j,k} = 0 \text{ in } (0,\infty) \\ y_{j,0}(t) = y_{j,M+1}(t) = y_{0,k}(t) = y_{N+1,k}(t) = 0 \text{ in } (0,\infty) \\ y_{j,k}(0) = y^0_{j,k}, \quad y'_{j,k}(0) = y^1_{j,k}, \quad j = 1,\ 2, ...,\ N, \quad k = 1,\ 2, ...,\ M \end{cases} \tag{3.7}$$

where $a_{j,k}$, $y^0_{j,k}$, and $y^1_{j,k}$, $j = 0,\ 1, ...,\ N+1$, $k = 0,\ 1, ...,\ M+1$ are approximations of the functions a, y^0 and y^1.

The energy of system (3.7) is given by

$$E_{h_1,h_2}(t) = \frac{h_1 h_2}{2} \sum_{j=0}^{N} \sum_{k=0}^{M} \{(y'_{j,k}(t))^2 + (\frac{y_{j+1,k} - y_{j,k}}{h_1})^2 + (\frac{y_{j,k+1} - y_{j,k}}{h_2})^2\} \tag{3.8}$$

and it is a nonincreasing function of time. More precisely, one has

$$\begin{aligned} E'_{h_1,h_2}(t) = \frac{d}{dt} E_{h_1,h_2}(t) = -h_1 h_2 \sum_{j=0}^{N} \sum_{k=0}^{M} \{a_{j,k}(y'_{j,k}(t))^2 + h_1^2(\frac{y'_{j+1,k} - y'_{j,k}}{h_1})^2\} \\ - h_1 h_2^3 \sum_{j=0}^{N} \sum_{k=0}^{M} (\frac{y'_{j,k+1} - y'_{j,k}}{h_2})^2. \end{aligned} \tag{3.9}$$

Set $\omega = [(l_1, 1) \times (0,1)] \cup [(0,1) \times (l_2, 1)]$ where $0 \le l_1, l_2 < 1$, $y_{h_1,h_2} = (y_{j,k})_{j,k}$, and $\bar{n} = \mathcal{E}(l_1/h_1) + 1$, $\quad m = \mathcal{E}(l_2/h_2)$, where $\mathcal{E}(x)$ denotes the greatest integer less than or equal to x. In the sequel, the generic constant C denotes different nonnegative constants independent of h_1 and h_2.

Now, we are in the position to state our main results. Our stabilization result states as follows:

THEOREM 3.1. *Let ω and a be given as above. Then there exists a positive constant t_2 independent of h_1 and h_2 such that the energy E_{h_1,h_2} of System (3.7) satisfies*

$$E_{h_1,h_2}(t) \le [\exp(1 - \frac{t}{t_2})] E_{h_1,h_2}(0), \quad \forall t \ge 0, \quad \forall 0 < h_1 < 1, \quad \forall 0 < h_2 < 1. \tag{3.10}$$

Before stating the convergence result, we need some additional notations. Following what we did for the one-dimensional case, we introduce the extension operators defined by:

$$
(3.11)\qquad P_{h_1,h_2}v_{h_1,h_2} = \begin{cases} \text{the continuous function linear in each triangle of corners} \\ (jh_1, kh_2), (jh_1, (k+1)h_2), ((j+1)h_1, kh_2), \text{ such that} \\ P_{h_1,h_2}v_{h_1,h_2}(jh_1, kh_2) = v_{j,k}, \\ P_{h_1,h_2}v_{h_1,h_2}(jh_1, (k+1)h_2) = v_{j,k+1}, \\ P_{h_1,h_2}v_{h_1,h_2}((j+1)h_1, kh_2) = v_{j+1,k}, \\ j = 0,\ 1,\ ...,\ N+1,\ k = 0,\ 1,\ ...,\ M+1 \end{cases}
$$

$$
(3.12)\qquad Q_{h_1,h_2}v_{h_1,h_2} = \begin{cases} \text{the step function defined in each square} \\ ((j-\frac{1}{2})h_1, (j+\frac{1}{2})h_1) \times ((k-\frac{1}{2})h_2, (k+\frac{1}{2})h_2) \cap \Omega \\ \text{by } Q_{h_1,h_2}v_{h_1,h_2}(x) = v_{j,k}, \\ j = 0,\ 1,\ ...,\ N+1,\ k = 0,\ 1,\ ...,\ M+1. \end{cases}
$$

We are now in the position to state our convergence result:

THEOREM 3.2. *Let y_{h_1,h_2} denote the solution of (3.7). Assume that a_{h_1,h_2}, $y^0_{h_1,h_2}$ and $y^1_{h_1,h_2}$ are such that there is a nonnegative constant c independent of h_1 and h_2 with*

$$
(3.13)\qquad \begin{cases} E_{h_1,h_2}(0) \le c, \qquad Q_{h_1,h_2}a_{h_1,h_2} \to a \ \textit{weakly * in } L^\infty(\Omega) \\ P_{h_1,h_2}y^0_{h_1,h_2} \to y^0 \ \textit{weakly in } H^1_0(\Omega), \quad Q_{h_1,h_2}y^1_{h_1,h_2} \to y^1 \ \textit{weakly in } L^2(\Omega). \end{cases}
$$

Then we have

$$
(3.14)\qquad \begin{cases} P_{h_1,h_2}y_{h_1,h_2} \to y \ \textit{weakly * in } L^\infty(0,\infty; H^1_0(\Omega)) \\ Q_{h_1,h_2}y'_{h_1,h_2} \to y' \ \textit{weakly * in } L^\infty(0,\infty; L^2(\Omega)) \end{cases}
$$

where y is the solution of System (3.1).

As we mentioned in the introduction in Section 1, these results can be easily adapted to more general $2-D$ domains, whose boundary is constituted by points of a finite-difference mesh. The details of this will be given elsewhere.

The proof of Theorem 3.2 is very similar to that of Theorem 1.2, so we omit it. The remaining part of this section is devoted to the proof of Theorem 3.1.

3.2. Proof of Theorem 3.1. From the proof of Theorem 1.1, we learned that proving Theorem 3.1 amounts to showing that the solutions of the undamped system associated with

(3.7), namely

$$
\begin{cases}
u''_{j,k} - \dfrac{u_{j+1,k} - 2u_{j,k} + u_{j-1,k}}{h_1^2} - \dfrac{u_{j,k+1} - 2u_{j,k} + u_{j,k-1}}{h_2^2} = 0 \text{ in } (0,\infty) \\
u_{j,0}(t) = u_{j,M+1}(t) = u_{0,k}(t) = u_{N+1,k}(t) = 0 \text{ in } (0,\infty) \\
u_{j,k}(0) = y^0_{j,k}, \quad u'_{j,k}(0) = y^1_{j,k}, \quad j = 1,\ 2, ...,\ N, \quad k = 1,\ 2, ...,\ M,
\end{cases}
\tag{3.15}
$$

satisfy a suitable uniform (with respect to $h_1,\ h_2$) observability inequality. More precisely, it will be enough to prove that there are positive constants C and T independent of h_1, h_2 and the initial data such that the solutions of (3.15) satisfy

$$
\begin{aligned}
E_{h_1,h_2}(0) \leq{}& Ch_1h_2 \sum_{j=0}^{N}\sum_{k=0}^{M} \int_0^T \{a_{j,k}(u'_{j,k}(t))^2 + h_1^2(\frac{u'_{j+1,k} - u'_{j,k}}{h_1})^2\}dt \\
&+ Ch_1h_2^3 \sum_{j=0}^{N}\sum_{k=0}^{M} \int_0^T (\frac{u'_{j,k+1} - u'_{j,k}}{h_2})^2 dt.
\end{aligned}
\tag{3.16}
$$

Step 1. For this step, we use the multiplier $M_{j,k}$ given by

$$
j(\frac{u_{j+1,k} - u_{j-1,k}}{2}) + k(\frac{u_{j,k+1} - u_{j,k-1}}{2})
\tag{3.17}
$$

which corresponds to a discrete version of the multiplier $x_1u_{x_1} + x_2u_{x_2}$. Multiplying the first equation of (3.15) by $M_{j,k}$, taking the sum over $j,\ k$, and integrating by parts in $[0,T]$, we obtain

$$
\begin{aligned}
&h_1h_2 \sum_{j=1}^{N}\sum_{k=1}^{M} u'_{j,k}M_{j,k}\Big]_0^T - h_1h_2 \sum_{j=1}^{N}\sum_{k=1}^{M} \int_0^T u'_jM'_{j,k}dt \\
&- h_1h_2 \sum_{j=1}^{N}\sum_{k=1}^{M} \int_0^T (\frac{u_{j+1,k} - 2u_{j,k} + u_{j-1,k}}{h_1^2})M_{j,k}dt \\
&- h_1h_2 \sum_{j=1}^{N}\sum_{k=1}^{M} \int_0^T (\frac{u_{j,k+1} - 2u_{j,k} + u_{j,k-1}}{h_2^2})M_{j,k}dt = 0.
\end{aligned}
\tag{3.18}
$$

Some elementary calculations show that

$$
\begin{aligned}
&h_1h_2 \sum_{j=1}^{N}\sum_{k=1}^{M} u'_{j,k}M_{j,k}\Big]_0^T \\
&= \frac{h_1h_2}{2} \sum_{j=0}^{N}\sum_{k=1}^{M} jh_1u'_{j,k}(\frac{u_{j+1,k} - u_{j-1,k}}{h_1})\Big]_0^T \\
&+ \frac{h_1h_2}{2} \sum_{j=1}^{N}\sum_{k=0}^{M} kh_2u'_{j,k}(\frac{u_{j,k+1} - u_{j,k-1}}{h_2})\Big]_0^T,
\end{aligned}
\tag{3.19}
$$

and

$$
\tag{3.20}
\begin{aligned}
&- h_1h_2\sum_{j=1}^{N}\sum_{k=1}^{M}\int_0^T u_j' M_{j,k}' dt = -\frac{h_1^3h_2}{4}\sum_{j=0}^{N}\sum_{k=1}^{M}\int_0^T (\frac{u_{j+1,k}'-u_{j,k}'}{h_1})^2dt\\
&- \frac{h_1h_2^3}{4}\sum_{j=1}^{N}\sum_{k=0}^{M}\int_0^T (\frac{u_{j,k+1}'-u_{j,k}'}{h_2})^2dt + h_1h_2\sum_{j=1}^{N}\sum_{k=1}^{M}\int_0^T (u_{j,k}')^2dt,
\end{aligned}
$$

and

$$
\tag{3.21}
\begin{aligned}
&- h_1h_2\sum_{j=1}^{N}\sum_{k=1}^{M}\int_0^T (\frac{u_{j+1,k}-2u_{j,k}+u_{j-1,k}}{h_1^2})M_{j,k}dt\\
&= -\frac{h_2}{2}\sum_{k=1}^{M}\int_0^T (\frac{u_{N,k}}{h_1})^2dt + \frac{h_1h_2}{2}\sum_{j=0}^{N}\sum_{k=1}^{M}\int_0^T (\frac{u_{j+1,k}-u_{j,k}}{h_1})^2dt\\
&- \frac{h_1h_2}{2}\sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T (\frac{u_{j+1,k+1}-u_{j,k+1}}{h_1})(\frac{u_{j+1,k}-u_{j,k}}{h_1})dt,
\end{aligned}
$$

and

$$
\tag{3.22}
\begin{aligned}
&- h_1h_2\sum_{j=1}^{N}\sum_{k=1}^{M}\int_0^T (\frac{u_{j,k+1}-2u_{j,k}+u_{j,k-1}}{h_2^2})M_{j,k}dt\\
&= -\frac{h_1}{2}\sum_{j=1}^{N}\int_0^T (\frac{u_{j,M}}{h_2})^2dt + \frac{h_1h_2}{2}\sum_{j=1}^{N}\sum_{k=0}^{M}\int_0^T (\frac{u_{j,k+1}-u_{j,k}}{h_2})^2dt\\
&- \frac{h_1h_2}{2}\sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T (\frac{u_{j+1,k+1}-u_{j+1,k}}{h_2})(\frac{u_{j,k+1}-u_{j,k}}{h_2})dt,
\end{aligned}
$$

In order to be able to absorb the crossed terms in the last lines of (3.21) and (3.22), we multiply the first equation of (3.15) by $\frac{u_{j,k}}{2}$, take the sums over $j,\ k$ and integrate by parts on the interval $[0,T]$. This procedure yields

$$
\tag{3.23}
\begin{aligned}
&\frac{h_1h_2}{2}\sum_{j=1}^{N}\sum_{k=1}^{M}u_{j,k}'u_{j,k}\Big]_0^T - \frac{h_1h_2}{2}\sum_{j=1}^{N}\sum_{k=1}^{M}\int_0^T (u_{j,k}')^2dt\\
&+ \frac{h_1h_2}{2}\sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T \Big\{(\frac{u_{j+1,k}-u_{j,k}}{h_1})^2 + (\frac{u_{j,k+1}-u_{j,k}}{h_2})^2\Big\}dt = 0.
\end{aligned}
$$

Adding the equalities (3.19)-(3.23) side by side, using (3.8) and reporting the result in (3.18), we find

$$
\begin{aligned}
TE_{h_1,h_2}(0) &= \int_0^T E_{h_1,h_2}(t)dt \\
&= -\frac{h_1h_2}{2}\sum_{j=0}^{N}\sum_{k=1}^{M} u'_{j,k}\Big(jh_1(\frac{u_{j+1,k}-u_{j-1,k}}{h_1}) + \frac{u_{j,k}}{2}\Big)\Big]_0^T \\
&- \frac{h_1h_2}{2}\sum_{j=1}^{N}\sum_{k=0}^{M} u'_{j,k}\Big(kh_2(\frac{u_{j,k+1}-u_{j,k-1}}{h_2}) + \frac{u_{j,k}}{2}\Big)\Big]_0^T \\
&+ \frac{h_1h_2}{4}\sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T \Big\{h_1^2(\frac{u'_{j+1,k}-u'_{j,k}}{h_1})^2 + h_2^2(\frac{u'_{j,k+1}-u'_{j,k}}{h_2})^2\Big\}dt \\
&+ \frac{h_2}{2}\sum_{k=1}^{M}\int_0^T (\frac{u_{N,k}}{h_1})^2 dt + \frac{h_1}{2}\sum_{j=1}^{N}\int_0^T (\frac{u_{j,M}}{h_2})^2 dt \\
&- \frac{h_1h_2}{2}\sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T \Big\{(\frac{u_{j+1,k}-y_{j,k}}{h_1})^2 + (\frac{u_{j,k+1}-u_{j,k}}{h_2})^2\Big\}dt \\
&+ \frac{h_1h_2}{2}\sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T (\frac{u_{j+1,k+1}-u_{j,k+1}}{h_1})(\frac{u_{j+1,k}-u_{j,k}}{h_1})dt \\
&+ \frac{h_1h_2}{2}\sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T (\frac{u_{j+1,k+1}-u_{j+1,k}}{h_2})(\frac{u_{j,k+1}-u_{j,k}}{h_2})dt.
\end{aligned}
\tag{3.24}
$$

Now, we are going to estimate the quantities in the right hand side of (3.24). Using Young inequality as well as (3.8) and (3.9), we obtain the inequalities

$$
\begin{aligned}
&\Big| -\frac{h_1h_2}{2}\sum_{j=1}^{N}\sum_{k=1}^{M} u'_{j,k}\Big(jh_1(\frac{u_{j+1,k}-u_{j-1,k}}{h_1}) + \frac{u_{j,k}}{2}\Big)\Big]_0^T\Big| \\
&+ \Big| -\frac{h_1h_2}{2}\sum_{j=1}^{N}\sum_{k=0}^{M} u'_{j,k}\Big(kh_2(\frac{u_{j,k+1}-u_{j,k-1}}{h_2}) + \frac{u_{j,k}}{2}\Big)\Big]_0^T\Big| \\
&\le \Big(2\sqrt{2} + \frac{\sqrt{2}}{4}Max(h_1^2,h_2^2)\Big)E_{h_1,h_2}(0),
\end{aligned}
\tag{3.25}
$$

and

$$
\begin{aligned}
&-\frac{h_1h_2}{2}\sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T\Big\{\big(\frac{u_{j+1,k}-u_{j,k}}{h_1}\big)^2+\big(\frac{u_{j,k+1}-u_{j,k}}{h_2}\big)^2dt\Big\}dt\\
&+\frac{h_1h_2}{2}\sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T\big(\frac{u_{j+1,k+1}-u_{j,k+1}}{h_1}\big)\big(\frac{u_{j+1,k}-u_{j,k}}{h_1}\big)dt\\
&+\frac{h_1h_2}{2}\sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T\big(\frac{u_{j+1,k+1}-u_{j+1,k}}{h_2}\big)\big(\frac{u_{j,k+1}-u_{j,k}}{h_2}\big)dt\le 0.
\end{aligned}
\tag{3.26}
$$

Before we go any further, we would like to draw the reader's attention to the fact that estimate (3.25) improves the one established in [Z2] where in particular, a compactness-uniqueness argument was used in order to get the observability optimal time (cf. [Z2], Section 4). For a detailed proof of (3.25), we refer the reader to Appendix B.

Reporting (3.25) and (3.26) in (3.24), we get

$$
\begin{aligned}
&\Big(T-2\sqrt{2}-\frac{\sqrt{2}}{4}Max(h_1^2,h_2^2)\Big)E_{h_1,h_2}(0)\\
&\le\frac{h_2}{2}\sum_{k=1}^{M}\int_0^T\big(\frac{u_{N,k}}{h_1}\big)^2dt+\frac{h_1}{2}\sum_{j=1}^{N}\int_0^T\big(\frac{u_{j,M}}{h_2}\big)^2dt\\
&+\frac{h_1h_2}{4}\sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T\Big\{h_1^2\big(\frac{u'_{j+1,k}-u'_{j,k}}{h_1}\big)^2+h_2^2\big(\frac{u'_{j,k+1}-u'_{j,k}}{h_2}\big)^2\Big\}dt.
\end{aligned}
\tag{3.27}
$$

As in the proof of Theorem 1.1, the next steps are devoted to the absorption of the first two terms in the righthand side of (3.27).

Step 2. Introduce the function $f=(f^1,f^2)\in(W^{1,\infty}(\Omega))^2$ such that

$$
f^1(1,x_2)=f^2(x_1,1)=1,\quad f(x_1,x_2)=0\quad in\quad \Omega\setminus\hat{\omega}
\tag{3.28}
$$

where $\hat{\omega}=\Omega\setminus(0,l_1+\gamma)\times(0,l_2+\theta)$ with γ and θ being positive constants such that $l_1+\gamma<1$ and $l_2+\theta<1$. Set $\bar{n}_1=\mathcal{E}(\frac{l_1+\gamma}{h_1})$ and $\bar{m}_1=\mathcal{E}(\frac{l_2+\theta}{h_2})$, and denote by $(f_{j,k})_{j,k}$ an approximation of the function f. It is clear that

$$
f^1_{N+1,k}=f^2_{j,M+1}=1,\forall j,k;\quad f_{j,k}=0 \text{ for } j\le\bar{n}_1,\ k\le\bar{m}_1.
\tag{3.29}
$$

The function g being given as in Step 3 of the proof of Theorem 1.1, multiply the first equation of (3.15) by $f^1_{j,k}(\frac{u_{j+1,k}-u_{j-1,k}}{2h_1})g(t)+f^2_{j,k}(\frac{u_{j,k+1}-u_{j,k-1}}{2h_2})g(t)$ and proceed as in the first step.

We find

$$
\begin{aligned}
(3.30)\qquad & \frac{h_1h_2}{2}\sum_j^*\sum_k^* \frac{f^1_{j+1,k}-f^1_{j,k}}{h_1}\int_0^T g(t)u'_{j+1,k}u'_{j,k}dt - \frac{h_2}{2}\sum_{k=1}^M\int_0^T g(t)\big(\frac{u_{N,k}}{h_1}\big)^2dt \\
& - h_1h_2\sum_j^*\sum_k^* f^1_{j,k}\int_0^T g'(t)u'_{j,k}\big(\frac{u_{j+1,k}-u_{j-1,k}}{2h_2}\big)dt \\
& + \frac{h_1h_2}{2}\sum_j^*\sum_k^* \frac{f^1_{j+1,k}-f^1_{j,k}}{h_1}\int_0^T g(t)\big(\frac{u_{j+1,k}-u_{j,k}}{h_1}\big)^2dt \\
& + \frac{h_1h_2}{2}\sum_j^*\sum_k^* \frac{f^2_{j,k+1}-f^2_{j,k}}{h_2}\int_0^T g(t)u'_{j,k+1}u'_{j,k}dt - \frac{h_1}{2}\sum_{j=1}^N\int_0^T g(t)\big(\frac{u_{j,M}}{h_2}\big)^2dt \\
& + \frac{h_1h_2}{2}\sum_j^*\sum_k^* \frac{f^2_{j,k+1}-f^2_{j,k}}{h_2}\int_0^T g(t)\big(\frac{u_{j,k+1}-u_{j,k}}{h_2}\big)^2dt \\
& - h_1h_2\sum_j^*\sum_k^* f^2_{j,k}\int_0^T g'(t)u'_{j,k}\big(\frac{u_{j,k+1}-u_{j,k-1}}{2h_2}\big)dt \\
& - h_1h_2\sum_j^*\sum_k^* f^2_{j,k}\int_0^T g(t)\big(\frac{u_{j+1,k}-2u_{j,k}+u_{j-1,k}}{h_1^2}\big)\big(\frac{u_{j,k+1}-u_{j,k-1}}{2h_2}\big)dt \\
& - h_1h_2\sum_j^*\sum_k^* f^1_{j,k}\int_0^T g(t)\big(\frac{u_{j,k+1}-2u_{j,k}+u_{j,k-1}}{h_2^2}\big)\big(\frac{u_{j+1,k}-u_{j-1,k}}{2h_1}\big)dt = 0,
\end{aligned}
$$

where $\sum_j^*\sum_k^*$ stands for $\sum_{j=0}^{\bar n_1}\sum_{k=\bar m_1}^{M} + \sum_{j=\bar n_1}^{N}\sum_{k=0}^{M}$. It follows from (3.30) that

$$
\begin{aligned}
(3.31)\qquad & h_1\sum_{j=1}^N\int_\delta^{T-\delta}\big(\frac{u_{j,M}}{h_2}\big)^2dt + h_2\sum_{k=1}^M\int_\delta^{T-\delta}\big(\frac{u_{N,k}}{h_1}\big)^2dt \\
& \le Ch_1h_2\sum_{j=0}^N\sum_{k=0}^M a_{j,k}\int_0^T (u'_{j,k}(t))^2dt \\
& + Ch_1h_2\sum_j^*\sum_k^*\int_0^T g(t)\Big\{\big(\frac{u_{j+1,k}-u_{j,k}}{h_1}\big)^2 + \big(\frac{u_{j,k+1}-u_{j,k}}{h_2}\big)^2\Big\}dt \\
& + Ch_1h_2\sum_j^*\sum_k^*\int_0^T g(t)(u_{j,k}(t))^2dt.
\end{aligned}
$$

Reporting (3.31) in (3.27) with $T > 2\sqrt{2} + \frac{\sqrt{2}}{4} Max(h_1^2, h_2^2)$ and $\delta > 0$ small enough, we obtain

$$
\begin{aligned}
E_{h_1,h_2}(0) &\le Ch_1h_2 \sum_{j=0}^{N}\sum_{k=0}^{M} \int_0^T \{a_{j,k}(u'_{j,k}(t))^2 + h_1^2 (\frac{u'_{j+1,k} - u'_{j,k}}{h_1})^2\} dt \\
&+ Ch_1h_2^3 \sum_{j=0}^{N}\sum_{k=0}^{M} \int_0^T (\frac{u'_{j,k+1} - u'_{j,k}}{h_2})^2 dt + Ch_1h_2 \sum_j^{*}\sum_k^{*} \int_0^T g(t)(u_{j,k}(t))^2 dt \\
&+ Ch_1h_2 \sum_j^{*}\sum_k^{*} \int_0^T g(t)\Big\{ (\frac{u_{j+1,k} - u_{j,k}}{h_1})^2 + (\frac{u_{j,k+1} - u_{j,k}}{h_2})^2 \Big\} dt.
\end{aligned}
\tag{3.32}
$$

At this stage, we observe that if the last two terms in the right hand side of (3.32) can be uniformly absorbed, then (3.16) will hold as desired.

Step 3. Introduce the function η satisfying

(3.33). $$\eta \in W^{1,\infty}(\Omega), \quad 0 \le \eta \le 1, \quad \eta = 1 \text{ in } \hat{\omega}, \quad \eta = 0 \text{ in } \Omega \setminus \omega.$$

Denote by $(\eta_{j,k})_{j,k}$, an approximation of η having properties similar to those of η. Multiplying the first equation of (3.15) by $g(t)\eta_{j,k}^2 u_{j,k}$ and proceeding as in the first step, we find

$$
\begin{aligned}
&- h_1h_2 \sum_j^{*}\sum_k^{*} \eta_{j,k}^2 \int_0^T g(t)(u'_{j,k})^2 dt - h_1h_2 \sum_j^{*}\sum_k^{*} \eta_{j,k}^2 \int_0^T g'(t)u'_{j,k}u_{j,k} dt \\
&+ h_1h_2 \sum_j^{*}\sum_k^{*} \frac{\eta_{j+1,k}^2 + \eta_{j,k}^2}{2} \int_0^T g(t)\Big\{ (\frac{u_{j+1,k} - u_{j,k}}{h_1})^2 + (\frac{u_{j,k+1} - u_{j,k}}{h_2})^2 \Big\} dt \\
&+ h_1h_2 \sum_j^{*}\sum_k^{*} \frac{\eta_{j+1,k}^2 - \eta_{j,k}^2}{h_1} \int_0^T g(t)(\frac{u_{j+1,k} + u_{j,k}}{2})(\frac{u_{j+1,k} - u_{j,k}}{h_1}) dt \\
&+ h_1h_2 \sum_j^{*}\sum_k^{*} \frac{\eta_{j,k+1}^2 - \eta_{j,k}^2}{h_2} \int_0^T g(t)(\frac{u_{j,k+1} + u_{j,k}}{2})(\frac{u_{j,k+1} - u_{j,k}}{h_2}) dt = 0,
\end{aligned}
\tag{3.34}
$$

where $\sum_j^{*}\sum_k^{*}$ stands for $\sum_{j=0}^{\bar{n}}\sum_{k=m}^{M} + \sum_{j=\bar{n}}^{N}\sum_{k=0}^{M}$. Some calculations give

$$
\begin{aligned}
&h_1h_2 \sum_j^{*}\sum_k^{*} \frac{\eta_{j+1,k}^2 + \eta_{j,k}^2}{2} \int_0^T g(t)\Big\{ (\frac{u_{j+1,k} - u_{j,k}}{h_1})^2 + (\frac{u_{j,k+1} - u_{j,k}}{h_2})^2 \Big\} dt \\
&\le Ch_1h_2 \sum_{j=0}^{N}\sum_{k=0}^{M} \int_0^T a_{j,k}(u'_{j,k}(t))^2 dt + Ch_1h_2 \sum_j^{*}\sum_k^{*} \int_0^T (u_{j,k})^2 dt.
\end{aligned}
\tag{3.35}
$$

The combination of (3.32) and (3.35) yields

$$
(3.36)\qquad \begin{aligned}
E_{h_1,h_2}(0) \le C h_1 h_2 \sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T \{a_{j,k}(u'_{j,k}(t))^2 + h_1^2\big(\frac{u'_{j+1,k}-u'_{j,k}}{h_1}\big)^2\}dt \\
+ C h_1 h_2^3 \sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T \big(\frac{u'_{j,k+1}-u'_{j,k}}{h_2}\big)^2 dt + C h_1 h_2 \sum_{j}^{*}\sum_{k}^{*}\int_0^T g(t)(u_{j,k}(t))^2 dt.
\end{aligned}
$$

Step 4. To absorb the last term of the right hand side of (3.36), we proceed as in Step 5 of the proof of Theorem 1.1 using a compactness-uniqueness argument as in the proof of Proposition 4.1 in [Z2]. We omit the details. This completes the proof of Theorem 3.1. □

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Appendix A. Proof of Theorem 1.1: The general case.

In this section, we show how the proof of Theorem 1.1 presented in Section 2.1 can be adapted and completed to prove it for an arbitrary nonempty open subset ω. The basic idea is to use appropriate cut-off functions to transform the problem where the damping is located in an arbitrary nonempty subinterval into two problems in each of which the damping is located in a neighborhood of one endpoint.

We recall that proving Theorem 1.1 amounts to proving an observability inequality- uniform with respect to h- for the conservative system (1.9). Our purpose here is to show how to establish (2.1) when ω is an arbitrary open subset of $(0,1)$.

Remember that $\omega = (l_1, l_2)$, where $0 \leq l_1 < l_2 \leq 1$. Now, set $\hat{\omega} = (\hat{l}_1, \hat{l}_2)$, and $\check{\omega} = (\check{l}_1, \check{l}_2)$ with $l_1 < \check{l}_1 < \hat{l}_1 < \hat{l}_2 < \check{l}_2 < l_2$, $n = \mathcal{E}(l_1/h)+1$, $\quad m = \mathcal{E}(l_2/h)$, $\hat{n} = \mathcal{E}(\hat{l}_1/h)$, $\hat{m} = \mathcal{E}(\hat{l}_2/h)$, $\check{n} = \mathcal{E}(\check{l}_1/h)$, and $\check{m} = \mathcal{E}(\check{l}_2/h)$. Introduce the infinitely differentiable functions g^1 and g^2 satisfying

$$(A.1) \qquad \begin{aligned} &0 \leq g^1(x) \leq 1, \text{ in } [0,1], \quad g^1(x) = 1 \text{ in } [0, \check{l}_1], \quad g^1(x) = 0 \text{ in } [\hat{l}_1, 1] \\ &0 \leq g^2(x) \leq 1, \text{ in } [0,1], \quad g^2(x) = 1 \text{ in } [\check{l}_2, 1], \quad g^2(x) = 0 \text{ in } [0, \hat{l}_2]. \end{aligned}$$

Now, for $i = 1,\ 2$ and $j = 0,\ 1,\ \ldots,\ N+1$, set $g^i_j = g^i(jh)$, $\varphi_j = g^1_j u_j$ and $v_j = g^2_j u_j$, where $(u_j)_j$ is the solution of (1.9). One easily checks that $(\varphi_j)_j$ satisfies the equations

$$(A.2) \qquad \begin{cases} \varphi_j'' - \dfrac{\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}}{h^2} = k_j \text{ in } (0,\infty) \\ \varphi_0(t) = \varphi_{\hat{n}+1}(t) = 0 \text{ in } (0,\infty) \\ \varphi_j(0) = g^1_j y^0_j, \quad \varphi_j'(0) = g^1_j y^1_j, \quad j = 1,\ 2, \ldots,\ \hat{n}, \end{cases}$$

where

$$(A.3) \qquad k_j = -\Big(\frac{g^1_{j+1} - 2g^1_j + g^1_{j-1}}{h^2}\Big)u_{j+1} - \Big(\frac{g^1_j - g^1_{j-1}}{h}\Big)\Big(\frac{u_{j+1} - u_{j-1}}{h}\Big).$$

It is not hard to check that $(v_j)_j$ satisfies a similar system. Now set

$$(A.4) \qquad \begin{aligned} E(\varphi_h; t) &= \frac{h}{2}\sum_{j=0}^{\hat{n}} \Big\{(\varphi_j'(t))^2 + \Big(\frac{\varphi_{j+1} - \varphi_j}{h}\Big)^2\Big\}, \\ E(v_h; t) &= \frac{h}{2}\sum_{j=\hat{m}}^{N} \Big\{(v_j'(t))^2 + \Big(\frac{v_{j+1} - v_j}{h}\Big)^2\Big\}. \end{aligned}$$

Thanks to the proof in Section 2.1, and the definitions of φ_j and v_j, we know that there are positive constants C and T independent of h such that

$$
\begin{aligned}
(A.5)\qquad & E(\varphi_h;0) \leq Ch^3 \sum_{j=0}^{N} \int_0^T (\frac{u'_{j+1}-u'_j}{h})^2 dt + Ch \sum_{j=1}^{N} \int_0^T a_j (u'_j)^2 dt \\
& + Ch \sum_{j=\check{n}}^{\hat{n}} \int_0^T \{ (\frac{u_{j+1}-u_j}{h})^2 + (u_{j+1}(t))^2 \} dt
\end{aligned}
$$

and

$$
\begin{aligned}
(A.6)\qquad & E(v_h;0) \leq Ch^3 \sum_{j=0}^{N} \int_0^T (\frac{u'_{j+1}-u'_j}{h})^2 dt + Ch \sum_{j=1}^{N} \int_0^T a_j (u'_j)^2 dt \\
& + Ch \sum_{j=\hat{m}}^{\check{m}} \int_0^T \{ (\frac{u_{j+1}-u_j}{h})^2 + (u_{j+1}(t))^2 \} dt
\end{aligned}
$$

Using the Gronwall lemma, one easily checks that

$$
\begin{aligned}
(A.7)\qquad & E(\varphi_h;t) \leq CE(\varphi_h;0) + Ch \sum_{j=\check{n}}^{\hat{n}} \int_0^T \{ (u'_j(t))^2 + \big(\frac{u_{j+1}-u_j}{h}\big)^2 + (u_{j+1}(t))^2 \} dt \\
& E(v_h;t) \leq CE(v_h;0) + Ch \sum_{j=\hat{m}}^{\check{m}} \int_0^T \{ (u'_j(t))^2 + \big(\frac{u_{j+1}-u_j}{h}\big)^2 + (u_{j+1}(t))^2 \} dt
\end{aligned}
$$

Moreover, it is not hard to check that the energies $E(\varphi_h;.)$ and $E(v_h;.)$ satisfy, for all $t \geq 0$, the estimates

$$
\begin{aligned}
& E(\varphi_h;t) + E(v_h;t) \leq CE(u_h;t) = CE_h(0) \\
(A.8)\qquad & E_h(0) = E(u_h;t) \leq E(\varphi_h;t) + E(v_h;t) \\
& + Ch \sum_{j=\check{n}}^{\check{m}} \{ (u'_j(t))^2 + \big(\frac{u_{j+1}-u_j}{h}\big)^2 + (u_{j+1}(t))^2 \}
\end{aligned}
$$

where $E(u_h;.)$ is the energy of the undamped system (1.9).

Integrating the last inequality of (A.8) in $(0,T)$, we get

$$
\begin{aligned}
&E_h(0) \le C\int_0^T \Big(E(\varphi_h;t)+E(v_h;t)\Big)dt \\
&+Ch\sum_{j=\check{n}}^{\check{m}}\int_0^T \big\{(u_j'(t))^2+\big(\frac{u_{j+1}-u_j}{h}\big)^2+(u_{j+1}(t))^2\big\}dt \\
&\le C(E(\varphi_h;0)+E(v_h;0))+Ch\sum_{j=1}^{N}\int_0^T a_j(u_j')^2dt \\
(A.9)\qquad &+Ch\sum_{j=\check{n}}^{\check{m}}\int_0^T \big\{\big(\frac{u_{j+1}-u_j}{h}\big)^2+(u_{j+1}(t))^2\big\}dt, \quad \text{(by A.7)} \\
&\le Ch^3\sum_{j=0}^{N}\int_0^T \big(\frac{u_{j+1}'-u_j'}{h}\big)^2dt+\sum_{j=1}^{N}\int_0^T a_j(u_j')^2dt \\
&+Ch\sum_{j=\hat{m}}^{\check{m}}\int_0^T \big\{\big(\frac{u_{j+1}-u_j}{h}\big)^2+(u_{j+1}(t))^2\big\}dt, \quad \text{(by A.5) and (A.6).}
\end{aligned}
$$

Following Steps 4 and 5 in the proof of Theorem 1.1 presented in Section 2.1, and using (A.9), one gets (2.1). □

Appendix B.

In this section, we provide details on obtaining some of the identities and estimates used in the proofs of Theorems 1.1, and 3.1.

Details on obtaining the third, fourth and fifth terms in (2.23):

On one hand, we have

$$
\begin{aligned}
& - h \sum_{j=n}^{N} \eta_j^2 \int_0^T g(t) u_j \big(\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \big) dt = \\
& - h \sum_{j=n}^{N} \eta_j^2 \int_0^T g(t) u_j \big(\frac{u_{j+1} - u_j}{h^2} \big) dt + h \sum_{j=n}^{N} \eta_j^2 \int_0^T g(t) u_j \big(\frac{y_j - u_{j-1}}{h^2} \big) dt \\
& = -h \sum_{j=n}^{N} \eta_j^2 \int_0^T g(t) u_j \big(\frac{u_{j+1} - u_j}{h^2} \big) dt \\
& + h \sum_{j=n}^{N} \eta_{j+1}^2 \int_0^T g(t) u_{j+1} \big(\frac{u_{j+1} - u_j}{h^2} \big) dt \\
& = h \sum_{j=n}^{N} \eta_j^2 \int_0^T g(t) \big(\frac{u_{j+1} - u_j}{h} \big)^2 dt \\
& + h \sum_{j=n}^{N} \frac{\eta_{j+1}^2 - \eta_j^2}{h} \int_0^T g(t) u_{j+1} \big(\frac{u_{j+1} - u_j}{h} \big) dt
\end{aligned}
\tag{2.23.0}
$$

and on the other hand

$$
\begin{aligned}
& - h \sum_{j=n}^{N} \eta_j^2 \int_0^T g(t) u_j \big(\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \big) dt \\
& = -h \sum_{j=n}^{N} \eta_j^2 \int_0^T g(t) u_j \big(\frac{u_{j+1} - u_j}{h^2} \big) dt \\
& + h \sum_{j=n}^{N} \eta_j^2 \int_0^T g(t) u_j \big(\frac{y_j - u_{j-1}}{h^2} \big) dt = -h \sum_{j=n}^{N} \eta_j^2 \int_0^T g(t) u_j \big(\frac{u_{j+1} - u_j}{h^2} \big) dt \\
& + h \sum_{j=n}^{N} \eta_{j+1}^2 \int_0^T g(t) u_{j+1} \big(\frac{u_{j+1} - u_j}{h^2} \big) dt \\
& = h \sum_{j=n}^{N} \frac{\eta_{j+1}^2 - \eta_j^2}{h} \int_0^T g(t) u_j \big(\frac{u_{j+1} - u_j}{h} \big) dt \\
& + h \sum_{j=n}^{N} \eta_{j+1}^2 \int_0^T g(t) \big(\frac{u_{j+1} - u_j}{h} \big)^2 dt.
\end{aligned}
\tag{2.23.1}
$$

Hence

$$
\begin{aligned}
&- h\sum_{j=n}^{N}\eta_j^2\int_0^T g(t)u_j\big(\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}\big)dt\\
(2.23.2)\qquad &= h\sum_{j=n}^{N}\frac{\eta_{j+1}^2+\eta_j^2}{2}\int_0^T g(t)\big(\frac{u_{j+1}-u_j}{h}\big)^2dt\\
&+ h\sum_{j=n}^{N}\frac{\eta_{j+1}^2-\eta_j^2}{h}\int_0^T g(t)\frac{u_{j+1}+u_j}{2}\big(\frac{u_{j+1}-u_j}{h}\big)dt
\end{aligned}
$$

Details on obtaining the last term in (3.21):

$$
\begin{aligned}
&- h_1h_2\sum_{j=1}^{N}\sum_{k=1}^{M}k\int_0^T\big(\frac{u_{j+1,k}-2u_{j,k}+u_{j-1,k}}{h_1^2}\big)\frac{u_{j,k+1}-u_{j,k-1}}{2}dt\\
&= -h_1h_2\sum_{j=1}^{N}\sum_{k=1}^{M}k\int_0^T\big(\frac{u_{j+1,k}-u_{j,k}}{h_1^2}\big)\frac{u_{j,k+1}-u_{j,k-1}}{2}dt\\
(3.21)\qquad &+ h_1h_2\sum_{j=1}^{N}\sum_{k=1}^{M}k\int_0^T\big(\frac{u_{j,k}-u_{j-1,k}}{h_1^2}\big)\frac{u_{j,k+1}-u_{j,k-1}}{2}dt\\
&= -h_1h_2\sum_{j=1}^{N}\sum_{k=1}^{M}k\int_0^T\big(\frac{u_{j+1,k}-u_{j,k}}{h_1^2}\big)\frac{u_{j,k+1}-u_{j,k-1}}{2}dt\\
&+ h_1h_2\sum_{j=0}^{N}\sum_{k=1}^{M}k\int_0^T\big(\frac{u_{j+1,k}-u_{j,k}}{h_1^2}\big)\frac{u_{j+1,k+1}-u_{j+1,k-1}}{2}dt =
\end{aligned}
$$

$$
\begin{aligned}
&= -h_1h_2\sum_{j=0}^{N}\sum_{k=1}^{M}k\int_0^T\big(\frac{u_{j+1,k}-u_{j,k}}{h_1^2}\big)\big(\frac{u_{j+1,k-1}-u_{j,k-1}}{2}-\frac{u_{j+1,k+1}-u_{j,k+1}}{2}\big)dt\\
&= -h_1h_2\sum_{j=0}^{N}\sum_{k=0}^{M}(k+1)\int_0^T\big(\frac{u_{j+1,k+1}-u_{j,k+1}}{h_1^2}\big)\big(\frac{u_{j+1,k}-u_{j,k}}{2}\big)dt\\
&+ h_1h_2\sum_{j=0}^{N}\sum_{k=0}^{M}k\big(\frac{u_{j+1,k}-u_{j,k}}{h_1^2}\big)\big(\frac{u_{j+1,k+1}-u_{j,k+1}}{2}\big)dt\\
&= -\frac{h_1h_2}{2}\sum_{j=0}^{N}\sum_{k=0}^{M}\int_0^T\big(\frac{u_{j+1,k+1}-u_{j,k+1}}{h_1}\big)\big(\frac{u_{j+1,k}-u_{j,k}}{h_1}\big)dt.
\end{aligned}
$$

Detailed proof of (3.25):

First observe that for all $t \geq 0$

$$
\begin{aligned}
& h_1h_2\sum_{j=1}^{N}\sum_{k=1}^{M}\Big(jh_1\big(\frac{u_{j+1,k}-u_{j-1,k}}{h_1}\big)+\frac{u_{j,k}}{2}\Big)^2 \\
& = h_1h_2\sum_{j=1}^{N}\sum_{k=1}^{M} j^2h_1^2\big(\frac{u_{j+1,k}-u_{j-1,k}}{h_1}\big)^2 \\
& + h_1h_2\sum_{j=1}^{N}\sum_{k=1}^{M} jh_1\big(\frac{u_{j+1,k}-u_{j-1,k}}{h_1}\big)u_{j,k}+\frac{h_1h_2}{4}\sum_{j=1}^{N}\sum_{k=1}^{M}(u_{j,k})^2 \\
& \leq 4h_1h_2\sum_{j=0}^{N}\sum_{k=1}^{M} j^2h_1^2\big(\frac{u_{j+1,k}-u_{j,k}}{h_1}\big)^2+h_1h_2\sum_{j=0}^{N}\sum_{k=1}^{M} ju_{j+1,k}u_{j,k} \\
& - h_1h_2\sum_{j=0}^{N}\sum_{k=1}^{M}(j+1)u_{j+1,k}u_{j,k}+\frac{h_1h_2}{4}\sum_{j=1}^{N}\sum_{k=1}^{M}(u_{j,k})^2 \\
& \leq 4h_1h_2\sum_{j=0}^{N}\sum_{k=1}^{M}\big(\frac{u_{j+1,k}-u_{j,k}}{h_1}\big)^2-h_1h_2\sum_{j=0}^{N}\sum_{k=1}^{M}u_{j+1,k}u_{j,k} \\
& + \frac{h_1h_2}{4}\sum_{j=1}^{N}\sum_{k=1}^{M}(u_{j,k})^2 \\
& \leq 4h_1h_2\sum_{j=0}^{N}\sum_{k=1}^{M}\big(\frac{u_{j+1,k}-u_{j,k}}{h_1}\big)^2+\frac{h_1^3h_2}{2}\sum_{j=0}^{N}\sum_{k=1}^{M}\big(\frac{u_{j+1,k}-u_{j,k}}{h_1}\big)^2 \\
& - \frac{3h_1h_2}{4}\sum_{j=1}^{N}\sum_{k=1}^{M}(u_{j,k})^2 \\
& \leq 4h_1h_2\sum_{j=0}^{N}\sum_{k=1}^{M}\big(\frac{u_{j+1,k}-u_{j,k}}{h_1}\big)^2+\frac{h_1^3h_2}{2}\sum_{j=0}^{N}\sum_{k=1}^{M}\big(\frac{u_{j+1,k}-u_{j,k}}{h_1}\big)^2.
\end{aligned}
\tag{3.25.0}
$$

Similarly, we have

$$
\begin{aligned}
& h_1h_2\sum_{j=1}^{N}\sum_{k=1}^{M}\Big(jh_1\big(\frac{u_{j,k+1}-u_{j,k-1}}{h_2}\big)+\frac{u_{j,k}}{2}\Big)^2 \\
& \leq 4h_1h_2\sum_{j=1}^{N}\sum_{k=0}^{M}\big(\frac{u_{j,k+1}-u_{j,k}}{h_2}\big)^2+\frac{h_1h_2^3}{2}\sum_{j=1}^{N}\sum_{k=0}^{M}\big(\frac{u_{j,k+1}-u_{j,k}}{h_2}\big)^2.
\end{aligned}
\tag{3.25.1}
$$

Using Young inequality as well as (3.25.0) and (3.25.1) , we find, for all $t \geq 0$

$$
\begin{aligned}
& \Big| -\frac{h_1 h_2}{2}\sum_{j=1}^{N}\sum_{k=1}^{M} u'_{j,k}\Big(jh_1(\frac{u_{j+1,k}-u_{j-1,k}}{h_1}) + \frac{u_{j,k}}{2}\Big)\Big| \\
& + \Big| -\frac{h_1 h_2}{2}\sum_{j=1}^{N}\sum_{k=1}^{M} u'_{j,k}\Big(kh_2(\frac{u_{j,k+1}-u_{j,k-1}}{h_2}) + \frac{u_{j,k}}{2}\Big)\Big| \\
& \leq \frac{\varepsilon h_1 h_2}{2}\sum_{j=0}^{N}\sum_{k=1}^{M}(u'_{j,k})^2 + \frac{h_1 h_2}{4\varepsilon}\sum_{j=1}^{N}\sum_{k=1}^{M}\Big(jh_1(\frac{u_{j+1,k}-u_{j-1,k}}{h_1}) + \frac{u_{j,k}}{2}\Big)^2 \\
& + \frac{h_1 h_2}{4\varepsilon}\sum_{j=0}^{N}\sum_{k=1}^{M}\Big(kh_2(\frac{u_{j,k+1}-u_{j,k-1}}{h_2}) + \frac{u_{j,k}}{2}\Big)^2, \quad (\forall \varepsilon > 0) \\
& \leq \frac{\sqrt{2}h_1 h_2}{2}\sum_{j=1}^{N}\sum_{k=1}^{M}(u'_{j,k})^2 \\
& + \frac{\sqrt{2}h_1 h_2}{2}\sum_{j=0}^{N}\sum_{k=0}^{M}\Big\{(\frac{u_{j+1,k}-u_{j,k}}{h_1})^2 + (\frac{u_{j,k+1}-u_{j,k}}{h_2})^2\Big\} \\
& + \frac{h_1 h_2}{8\sqrt{2}}\sum_{j=0}^{N}\sum_{k=0}^{M}\Big\{h_1^2(\frac{u_{j+1,k}-u_{j,k}}{h_1})^2 + h_2^2(\frac{u_{j,k+1}-u_{j,k}}{h_2})^2\Big\} \\
& \leq (\sqrt{2} + \frac{\sqrt{2}}{8}Max(h_1^2, h_2^2))E_{h_1,h_2}(0), (\text{ by choosing } \varepsilon = \sqrt{2})
\end{aligned} \tag{3.25.2}
$$

hence (3.25) and we are done. QED