Chapter 1

Observability of 1-d Waves in Heterogeneous and Semi-Discrete Media

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1.1 Introduction

In the last years important progresses have been made in the problems of boundary observability and controllability of waves. There is by now a well established theory for wave equations with sufficiently smooth coefficients. When waves propagate in highly heterogeneous media much less is known. In this work we address two one-dimensional model problems: the wave equation with highly oscillatory coefficients and the finite-difference space semi-discretizations of the wave equation. As we shall see, in both cases, the interaction of waves with the microstructure of the medium may cause some pathological behaviors of the high frequencies. In particular, the velocity of propagation of waves may tend to zero when the wavelength of solutions is of the order of the size of the microstructure and the latter tends to zero. As a consequence of this fact, the time needed to uniformly observe the waves from the boundary tends to infinity as the microstructure becomes finer

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and finer. Our analysis is based on the Fourier decomposition of solutions. We use the WKB asymptotic expansion method and classical results on the theory of non-harmonic Fourier series.

1.2 The constant coefficient wave equation

In order to motivate the problems we have in mind let us first consider the constant coefficient $1-d$ wave equation:

$$\begin{array}{l} u_{tt} - u_{xx} = 0,\ 0 < x < 1,\ 0 < t < T \\ u(0,t) = u(1,t) = 0,\ 0 < t < T \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x),\ 0 < x < 1. \end{array} \tag{1.1}$$

In (1.1) $u = u(x,t)$ describes the displacement of a vibrating string occupying the interval $(0,1)$.

The energy of solutions of (1.1) is conserved along time, i.e.

$$E(t) = \frac{1}{2}\int_0^1 \left[|u_x(x,t)|^2 + |u_t(x,t)|^2\right] dx = E(0),\ \forall 0 \le t \le T. \tag{1.2}$$

The problem of boundary observability of (1.1) may be formulated roughly as follows: *Give sufficient conditions on the length of the time interval T such that there exists a constant $C(T) > 0$ such that the following inequality holds for all solutions of (1.1):*

$$E(0) \le C(T)\int_0^T |u_x(1,t)|^2\, dt. \tag{1.3}$$

Inequality (1.3), when it holds, guarantees that the total energy of solutions may be "observed" or estimated from the energy concentrated or measured on the extreme $x = 1$ of the string during the time interval $(0,T)$. Of course, (1.3) is just an example of a variety of similar observability problems. Among its possible variants, the following are worth mentioning: (a) one could observe the energy concentrated on the extreme $x = 0$ or in the two extremes $x = 0$ and 1 simultaneously; (b) the $L^2(0,T)-$norm of $u_x(1,t)$ could be replaced by some other norm, etc.

The observability problem above is equivalent to a boundary controllability problem. More precisely, the observability inequality above holds, if and only if, for any $\left(y^0, y^1\right) \in L^2(0,1)\times H^{-1}(0,1)$ there exists $v \in L^2(0,T)$ such that the solution of the controlled wave equation

$$\begin{array}{l} y_{tt} - y_{xx} = 0,\ 0 < x < 1,\ 0 < t < T \\ y(0,t) = 0;\ y(1,t) = v(t),\ 0 < t < T \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x),\ 0 < x < 1 \end{array} \tag{1.4}$$

satisfies

$$y(x,T) = y_t(x,T) = 0,\ 0 < x < 1. \tag{1.5}$$

We refer to (Lions, 1988) for a systematic analysis of the equivalence of controllability and observability through the so called Hilbert Uniqueness Method (HUM).

All along this paper we shall focus on the problem of observability without considering the related controllability problems. However, in view of the equivalence above, all the results we shall present here have immediate consequences in the context of controllability.

System (1.1) is observable if $T \geq 2$. More precisely, the following may be proved:

Proposition 1.1 *For any $T \geq 2$, system* (1.1) *is observable. In other words, for any $T \geq 2$ there exists $C(T) > 0$ such that* (1.3) *holds for any solution of* (1.1)*. Morevoer, if $T < 2$, system* (1.1) *is not observable, or, equivalently,*

$$\sup_{u \text{ solution of } (1.1)} \left[\frac{E(0)}{\displaystyle\int_0^T \mid u_x(1,t) \mid^2 dt} \right] = \infty. \tag{1.6}$$

The proof of the positive statement of Proposition 1.1, i.e. of the fact that observability holds when $T \geq 2$, can be done in several ways. In particular, it can be proved using Fourier series, multipliers (Komornik, 1994; Lions, 1988) and microlocal tools (Bardos et al., 1992; Burq and Gérard, 1997).

Let us explain how it can be proved using Fourier series. Solutions of (1.1) can be written in the form

$$u = \sum_{k \geq 1} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x) \tag{1.7}$$

where a_k, b_k are such that

$$u^0 = \sum_{k \geq 1} a_k \sin(k\pi x);\ u^1 = \sum_{k \geq 1} b_k \sin(k\pi x).$$

It follows that

$$E(0) = \frac{1}{4} \sum_{k \geq 1} \left[|a_k|^2 k^2 \pi^2 + |b_k|^2 \right]$$

On the other hand,

$$u_x(1,t) = \sum_{k\geq 1}(-1)^k k\pi \left[a_k \sin(k\pi t) + b_k \cos(k\pi t)\right].$$

Using the orthogonality of trigonometric functions $\sin(k\pi t)$ and $\cos(k\pi t)$ in $L^2(0,2)$, it follows that

$$\int_0^2 |u_x(1,t)|^2 \, dt = \pi^2 \sum_{k\geq 1} \left(a_k^2 k^2 + b_k^2 k^2\right).$$

The two identities above show that the observability inequality holds when $T = 2$ and therefore for any $T > 2$ as well.

On the other hand, when $T < 2$ it is easy to prove that the observability inequality does not hold. Indeed, assume that $T \leq 2 - 2\delta$ with $\delta > 0$. We solve the wave equation

$$\begin{array}{l} u_{tt} - u_{xx} = 0,\ 0 < x < 1,\ 0 < t < T \\ u(0,t) = u(1,t) = 0,\ 0 < t < T \end{array} \tag{1.8}$$

with data at time $t = T/2$ with support in the subinterval $(0, \delta)$. Note that, in view of the time reversibility of the wave equation, the solution is determined in a unique way both for $t \geq T/2$ and $t \leq T/2$. It is easy to see that this solution is such that $u_x(1,t) = 0$ for $0 < t < T$. This can be seen using the classical fact that the segment $x = 1$, $t \in (0,T)$ remains outside the domain of influence of the segment $t = T/2$, $x \in (0, \delta)$. This is a consequence of the fact that the velocity of propagation in this system is one.

The results in Proposition 1.1 show that a necessary and sufficient condition for the observability to hold is that $T \geq 2$. The necessity is a consequence of the finite speed of propagation. The sufficiency has been proved using Fourier series but, of course, it is also related to the finite speed of propagation. Indeed, when developing solutions of (1.1) in Fourier series the solution is decomposed on particular solutions of the form

$$u_k = \sin(k\pi t)\sin(k\pi x)$$

and

$$\bar{u}_k = \cos(k\pi t)\sin(k\pi x).$$

Observe that both u_k and $\bar{u}_k$ can be written in the form

$$\begin{array}{rcl} u_k & = & \dfrac{\cos(k\pi(t-x)) - \cos(k\pi(t+x))}{2} \\ \bar{u}_k & = & \dfrac{\sin(k\pi(x+t)) - \sin(k\pi(t-x))}{2} \end{array}$$

and therefore they are linear combinations of functions of the form $f(x+t)$ and $g(x-t)$ for suitable profiles f and g.

This shows that, regardless of the frequency of oscillation of the initial datum of the equation, solutions propagate with velocity 1 and therefore may be observed on the extreme $x = 1$ of the string at the latest at time $T = 2$. Note that the observability time is twice the length of the string. This is due to the fact that an initial disturbance concentrated near $x = 1$ may be built so that it propagates to the left (in the space variable) as t increases and only reaches the extreme $x = 1$ of the interval after bouncing at the left extreme $x = 0$. A simple computation shows that this requires the time interval to be $T \geq 2$.

In several space dimensions the problem is much more complex and can not be solved using Fourier series. The velocity of propagation is still one for all solutions. However, in order to guarantee that all characteristics reach the observation subset of the boundary in a uniform time this observation subset has to be selected in an appropriate way and has to be in general large enough (see for instance Bardos et al. 1992 and Burq and Gérard, 1997).

As we have mentioned above, in this paper we only discuss observability problems but each result obtained in this context has immediate consequences in the context of controllability. Note however that controllability is not the only application of the observability inequalities that are also of systematic use in the context of Inverse Problems (Isakov, 1998).

In the following section we formulate in detail the problems we shall address all along this paper.

1.3 Problem formulation

In the previous section we have shown how the observability problem for the constant coefficient wave equation may be solved easily using Fourier series expansions. We now address the problem of the continuous dependence of the observability constant $C(T)$ in (1.3) with respect to two natural perturbations of the simple model (1.1):

- Rapidly oscillatory coefficients.
- Finite-difference space semi-discretizations.

These two problems arise naturally. The first one in the context of the homogenization theory and composite materials, the second one in the numerical implementation of the controllability and observability properties.

As we shall see, these two problems, although apparently of very different nature, present two common features:

- The observability constant $C(T)$ in (1.3) tends to infinity for any T as the parameter describing the size of the microstucture or the mesh tends to zero.
- The observability constant may be uniform if the high frequencies are filtered.
- The velocity of propagation of solutions tends to zero as the size of the microstructure/mesh tends to zero and the wavelength of solutions is of the same order as the size of the microstructure/mesh. Let us now formulate these problems in a more precise way.

1.3.1 Rapidly oscillating coefficients

Let us consider the wave equation

$$\begin{array}{l} \rho(x/\varepsilon)u_{tt} - u_{xx} = 0,\, 0 < x < 1,\, 0 < t < T \\ u(0,t) = u(1,t) = 0,\, 0 < t < T \\ u(x,0) = u^0(x),\, u_t(x,0) = u^1(x),\, 0 < x < 1. \end{array} \tag{2.1}$$

Here $\rho \in L^\infty(\mathbb{R})$ is a periodic function of period $\ell > 0$ such that

$$0 < \rho_m \leq \rho(x) \leq \rho_M < \infty \text{ a.e. } x \in \mathbb{R}$$

where ρ_m, ρ_M are two positive constants. The parameter ε ranges in the interval $0 < \varepsilon < 1$ and is devoted to tend to zero. System (2.1) is a simple model for the vibrations of a string with rapidly oscillating density.

Let us denote $\rho_\varepsilon(x) = \rho(x/\varepsilon)$. It is easy to check that

$$\rho_\varepsilon \rightharpoonup \bar{\rho} \text{ weakly } -* \text{ in } L^\infty(0,1); \text{ as } \varepsilon \to 0, \tag{2.2}$$

i.e.,

$$\int_0^1 \rho_\varepsilon(x)\varphi(x)dx \to \bar{\rho}\int_0^1 \varphi(x)dx \text{ as } \varepsilon \to 0,\, \forall \varphi \in L^1(0,1)$$

where

$$\bar{\rho} = \frac{1}{\ell}\int_0^\ell \rho(x)dx \tag{2.3}$$

is the average density of the string.

The energy of solutions of (2.1) is given by

$$E_\varepsilon(t) = \frac{1}{2}\int_0^1 \left[\rho_\varepsilon(x) \mid u_t(x,t) \mid^2 + \mid u_x(x,t) \mid^2\right] dx \tag{2.4}$$

and it is constant in time.

The problem of observability for (2.1) can be formulated as above. More precisely, it consists on finding $T > 0$ and $C_\varepsilon(T) > 0$ such that

$$E_\varepsilon(0) \leq C_\varepsilon(T) \int_0^T \mid u_x(1,t) \mid^2 dt \tag{2.5}$$

holds for any solution of (2.1). As we shall see, if ρ is regular enough, say $\rho \in C^1(\mathbb{R})$, one can show that observability holds for all $T > 2\sqrt{\rho_M}$ and for all $0 < \varepsilon < 1$.

The problem of *uniform observability* may be formulated as follows: Given $T > 2\sqrt{\rho_M}$ is the observability constant $C_\varepsilon(T)$ in (2.5) bounded as $\varepsilon \to 0$?

This question arises naturally since the limit of system (2.1) as $\varepsilon \to 0$, in view of (2.1), is given by the wave equation with constant density $\bar{\rho}$:

$$\begin{array}{l} \bar{\rho} u_{tt} - u_{xx} = 0, \; 0 < x < 1, \; 0 < t < T \\ u(0,t) = u(1,t) = 0, \; 0 < t < T \\ u(x,0) = u^0(x), \; u_t(x,0) = u^1(x), \; 0 < x < 1. \end{array} \tag{2.6}$$

According to the results presented in section 1.2, system (2.6) is observable for all $T > 2\sqrt{\bar{\rho}}$. On the other hand, obviously, $2\sqrt{\rho_M} > 2\sqrt{\bar{\rho}}$. Therefore, the limit system (2.6) being observable, the problem of the uniform observability of the solutions of (2.1) as $\varepsilon \to 0$ arises naturally.

Solutions of (2.1) can be developed in Fourier series. For, we consider the eigenvalue problem

$$\begin{array}{l} -\frac{\partial^2 w_k^\varepsilon}{\partial x^2} = \lambda_k^\varepsilon w_k^\varepsilon, \; 0 < x < 1 \\ w_k^\varepsilon(0) = w_k^\varepsilon(1) = 0. \end{array} \tag{2.7}$$

For any $\varepsilon > 0$, system (2.7) admits a sequence of eigenvalues

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon < \cdots < \lambda_k^\varepsilon < \cdots \to \infty$$

with corresponding eigenfunctions $\{w_k^\varepsilon\}_{k\geq 1}$ that may be chosen to constitute and orthonormal basis of $L^2(0,1)$ with the scalar product

$$(\varphi, \psi)_\varepsilon = \int_0^1 \varphi(x)\psi(x)\rho_\varepsilon(x)dx. \tag{2.8}$$

Then, solutions of (2.1) may be written in the form

$$u_\varepsilon = \sum_{k\geq 1} \left(a_k^\varepsilon \cos\left(\sqrt{\lambda_k^\varepsilon} t\right) + \frac{b_k^\varepsilon}{\sqrt{\lambda_k^\varepsilon}} \sin\left(\sqrt{\lambda_k^\varepsilon} t\right) \right) w_k^\varepsilon(x) \tag{2.9}$$

where a_k^ε and b_k^ε are the Fourier coefficients of the initial data

$$u^0 = \sum_{k\geq 1} a_k^\varepsilon w_k^\varepsilon;\ u^1 = \sum_{k\geq 1} b_k^\varepsilon w_k^\varepsilon. \tag{2.10}$$

On the other hand, solutions of (2.6) can be written as

$$u = \sum_{k\geq 1} \left[a_k \cos\left(\frac{k\pi t}{\sqrt{\bar\rho}}\right) + \frac{\sqrt{\bar\rho} b_k}{k\pi} \sin\left(\frac{k\pi t}{\sqrt{\bar\rho}}\right)\right] \sin(k\pi x). \tag{2.11}$$

Using the mini-max characterisation of the eigenvalues of (2.7) one can deduce that, for each $k \geq 1$,

$$\lambda_k^\varepsilon \to \frac{k^2\pi^2}{\bar\rho} \text{ as } \varepsilon \to 0, \tag{2.12}$$

$$w_k^\varepsilon \to \sqrt{\frac{2}{\bar\rho}} \sin(k\pi x) \text{ in } H_0^1(0,1), \text{ as } \varepsilon \to 0. \tag{2.13}$$

In a first approximation one may think that convergences (2.12)-(2.13) should suffice to analyze the problem of uniform observability under consideration. However, this is far from being the case. Indeed, in order to attack the problem of observability, one has to know how uniform the convergences (2.12)-(2.13) are with respect to the index k.

Classical results in the theory of homogenization provide convergence rates for (2.12)-(2.13). However, they are still insufficient to provide an answer to the uniform observability problem. For instance, according to Oleinick et al. (1992) we know that

$$\left| \sqrt{\lambda_k^\varepsilon} - \frac{k\pi}{\sqrt{\bar\rho}} \right| \leq C\varepsilon k^2 \tag{2.14}$$

for some $C > 0$ which is independent of $0 < \varepsilon < 1$ and $k \geq 1$. This allows to show that for any $\delta > 0$ there exists $c_\delta > 0$ such that

$$\left| \sqrt{\lambda_k^\varepsilon} - \frac{k\pi}{\sqrt{\bar\rho}} \right| \leq \delta,\ \forall k \leq c_\delta/\sqrt{\varepsilon}. \tag{2.15}$$

However, this result is far from being sufficient as well. Indeed, the critial scale for the problem under consideration is $k \sim 1/\varepsilon$ which corresponds to the case where the wavelength of the solutions is of the order of the microstructure. Obviously, this critical size is much beyond the range $k \leq C/\sqrt{\varepsilon}$ in which the results above apply.

In section 1.4 we shall give a complete answer to the problem. We shall first show that, whatever $T>0$ is, uniform observability fails because "spurious" oscillations occur when $k \sim C/\varepsilon$ if C is large enough. Then, using the WKB method (Bender and Orszag, 1978) we shall exhibit a complete asymptotic description of the spectrum in the range $k \leq c/\varepsilon$ with $\varepsilon > 0$ small enough. Finally, we shall show how uniform observability results may be obtained provided the high frequencies are filtered in an appropriate way. We shall also see that the velocity of propagation decreases as the frequency of oscillation approaches C/ε with an explicit value of $C>0$. The results of this section are proved in detail in (Castro and Zuazua, 1997, 1998). We refer to (Castro, 1998a) for a complete analysis of the limit behavior of the controllability properties as $\varepsilon \to 0$.

1.3.2 Finite-difference approximations

Given $N \in \mathbb{N}$ we define $h = 1/(N+1) > 0$. We consider the mesh

$$x_0 = 0;\ x_j = jh,\ j = 1, \cdots, N;\ x_{N+1} = 1 \tag{2.16}$$

of the interval $(0,1)$.

Consider the following finite-difference approximation of the wave equation (1.1):

$$\begin{array}{l} u_j'' - \frac{[u_{j+1}+u_{j-1}-2u_j]}{h^2} = 0,\ 0 < t < T,\ j = 1, \cdots, N \\ u_j(t) = 0,\ j = 0,\ N+1 \\ u_j(0) = u_j^0,\ u_j^1(0) = u_j^1,\ j = 1, \cdots, N. \end{array} \tag{2.17}$$

Observe that (2.17) is a coupled system of N linear differential equations of second order. The function $u_j(t)$ provides an approximation of $u(x_j,t)$ for all $j = 1, \cdots, N$, u being the solution of the continuous wave equation (1.1). The conditions $u_0 = u_{N+1} = 0$ take account of the homogeneous Dirichlet boundary conditions. In (2.17) the second order derivation with respect to x has been replaced by the finite-difference $[u_{j+1} + u_{j-1} - 2u_j]/h^2$.

The energy of solutions of (2.17) is as follows:

$$E_h(t) = \frac{h}{2} \sum_{j=1}^{N} \left[|\ u_j'\ |^2 + \left| \frac{u_{j+1} - u_j}{h} \right|^2 \right]. \tag{2.18}$$

It is easy to see that the energy is constant in time. Obviously (2.18) is the natural discretization of the continuous energy (1.2).

The problem of the observability of system (2.17) can be formulated as follows: To find $T>0$ and $C_h(T)>0$ such that

$$E_h(0) \leq C_h(T) \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \tag{2.19}$$

holds for all solutions of (2.17).

Observe that $\mid u_N/h \mid^2$ is a natural approximation of $\mid u_x(1,t) \mid^2$ for the solution of the continuous system (1.1). Indeed $u_x(1,t) \sim [u_{N+1}(t) - u_N(t)]/h$ and, taking into account that $u_{N+1} = 0$, it follows that $u_x(1,t) \sim -u_N(t)/h$.

System (2.17) is finite-dimensional. Therefore, if observability holds for some $T > 0$, then it holds for all $T > 0$. But we are interested mainly on the uniformity of the constant $C_h(T)$ as $h \to 0$. Taking into account that the observability of the limit system (1.1) only holds for $T \geq 2$, it seems natural to expect $T \geq 2$ to be a necessary condition for the uniform observability of the system (2.17) as $h \to 0$. However, as we shall see, this condition is far from being sufficient. In order to explain this fact it is convenient to analyze the spectrum of system (2.17).

Let us consider the eigenvalue problem

$$\begin{array}{c} -\left[w_{j+1} + w_{j-1} - 2w_j\right]/h^2 = \lambda w_j,\ j = 1, \cdots, N \\ w_0 = w_{N+1} = 0. \end{array} \tag{2.20}$$

The spectrum can be computed explicitly in this case (Isaacson and Keller, 1966). The eigenvalues

$$0 < \lambda_1(h) < \lambda_2(h) < \cdots < \lambda_N(h)$$

are as follows

$$\lambda_k^h = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right). \tag{2.21}$$

The corresponding eigenvectors are

$$w_k^h = (w_{k,1}, \cdots, w_{k,N}) : w_{k,j} = \sin(k\pi j h),\ k, j = 1, \cdots, N. \tag{2.22}$$

It is easy to see that

$$\lambda_k^h \to \lambda_k = k^2\pi^2,\ \text{as } h \to 0 \tag{2.23}$$

for each $k \geq 1$, $\lambda_k = k^2\pi^2$ being the $k-th$ eigenvalue of the continuous wave equation (1.1). On the other hand we see that the eigenvectors w_k^h of the discrete system (2.20) in fact coincide with the eigenfunctions $w_k(x) = \sin(k\pi x)$ of the continuous wave equation (1.1).

According to (2.21) we have $\sqrt{\lambda_k^h} = \frac{2}{h}\sin\left(\frac{k\pi h}{2}\right)$ and therefore, in a first approximation, we have

$$\left|\sqrt{\lambda_k^h} - k\pi\right| \sim \frac{k^3\pi h^2}{4}. \tag{2.24}$$

This indicates that the convergence in (2.23) is only uniform in the range $k << h^{-2/3}$. Thus, as in the context of the rapidly oscillating coefficients, one can not expect to solve completely the problem of uniform observability by analyzing the uniform convergence of the spectrum. A more careful analysis of the behavior of the eigenvalues and eigenvectors at high frequencies is needed. This problem will be addresed in section 1.5 following Infante and Zuazua (1998a, 1998b).

1.4 Rapidly oscillating density

This section is devoted to the analysis of the observability for the equation

$$\begin{array}{ll} \rho(x/\varepsilon)u_{tt} - u_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ u(0,t) = u(1,t) = 0, & 0 < t < T \\ u(x,0) = u^0,\ u_t(x,0) = u^1(x), & 0 < x < 1 \end{array} \tag{3.1}$$

where the density ρ satisfies the assumptions of section 1.3, and $\varepsilon \to 0$.

In the first paragraph we shall describe the lack of uniform observability. In section 1.4.2 we present the asymptotic results on the spectrum obtained by the WKB method. In section 1.4.3 we give the uniform observability result of the low frequencies whose proof is based in Ingham's inequality for non-harmonic Fourier series (Young, 1980). Finally, in section 1.4.4 we summarize the main results and discuss some
related issues.

1.4.1 Non-uniform observability

We recall that the eigenvalue problem associated to system (3.1) is

$$\begin{array}{l} -w'' = \lambda\rho(x/\varepsilon)w, \quad 0 < x < 1 \\ w(0) = w(1) = 0 \end{array} \tag{3.2}$$

and the eigenvalues and eigenfunctions are denoted respectively by $\{\lambda_k^\varepsilon\}_{k\geq 1}$ and $\{w_k^\varepsilon\}_{k\geq 1}$ for each $0 < \varepsilon < 1$.

In order to analyze the behavior of the eigenvalues of the order of $\lambda \sim 1/\varepsilon^2$ it is natural to introduce the change of variables $y = x/\varepsilon$, so that equation (3.2) becomes

$$\begin{array}{l} -w_{yy} = \mu\rho(y)w, \quad 0 < y < 1/\varepsilon \\ w(0) = w(1/\varepsilon) = 0 \end{array} \tag{3.3}$$

with

$$\mu = \lambda\varepsilon^2. \tag{3.4}$$

Consider the eigenvalue problem in the whole line associated to (3.3):

$$-w_{yy} = \mu\rho(y)w,\ y \in I\!R. \tag{3.5}$$

Taking into account that ρ is periodic, the behavior of solutions of (3.5) depending on the different values of μ may be analyzed by means of Floquet Theorem (see Eastham, 1973). Indeed, the following holds:

- There exist values

$$0 = \mu_0 < \mu_1' \le \mu_2' < \mu_1 \le \mu_2 < \mu_3' \le \mu_4' < \mu_3 < \cdots \to \infty$$

  such that when μ belongs to one of the so called stability intervals $(\mu_0, \mu_1') \cup (\mu_2', \mu_1) \cup (\mu_2, \mu_3') \cup \cdots$ solutions of (3.5) are bounded in the whole line. However, if μ belongs to one of the instability intervals $(\mu_1', \mu_2') \cup (\mu_1, \mu_2) \cup (\mu_3', \mu_4') \cup \cdots$, all solutions of (3.5) are unbounded.

- If ρ is non constant at least one of the instability intervals is non empty.

- When μ lies in an instability interval, there exist two linearly independent solutions of (3.5) of the form $w_1(y) = e^{-\alpha y}p_1(y)$, $w_2(y) = e^{\alpha y}p_2(y)$, with $\alpha > 0$ real and p_1, p_2 ℓ-periodic functions with an infinite number of zeroes.

These facts suffice to show that the observability inequality

$$E_\varepsilon(0) \le C_\varepsilon(T) \int_0^t |u_x(1,t)|^2\, dt \tag{3.6}$$

may not hold with a uniform constant as $\varepsilon \to 0$, for any positive time. Indeed, the following holds:

Theorem 3.1 *Assume that $\rho \in L^\infty(I\!R)$ is ℓ-periodic and such that*

$$0 < \rho_m \le \rho(x) \le \rho_M < \infty, \text{a.e. } x \in I\!R. \tag{3.7}$$

Then, there exists a sequence $\varepsilon_j \to 0$ and a sequence of indexes $k_j \to \infty$ of the order of ε_j^{-1} such that the corresponding eigenfunctions $w_{k_j}^{\varepsilon_j}$ of (3.2) satisfy

$$\int_0^1 \left|\partial_x w_{k_j}^{\varepsilon_j}(x)\right|^2 dx \Big/ \left|\partial_x w_{k_j}^{\varepsilon_j}(1)\right|^2 \ge \exp(C/\varepsilon_j) \text{ as } \varepsilon_j \to 0 \tag{3.8}$$

for some $C > 0$.

Remark 3.1 This result was proved by Castro and Zuazua (1997, 1998) under the assumption $\rho \in C^2(\mathbb{R})$ following the ideas in (Avellaneda et al., 1992) and, in particular, the fact (3.5) can be transformed into a Hill's equation by a simple change of variables. The result in its present form was proved in (Castro, 1998a) applying directly Floquet's Theorem to equation (3.5).

As an immediate corollary the following holds:

Corollary 3.1 *Under the assumptions of Theorem 3.1, there exists a sequence $\varepsilon_j \to 0$ such that*

$$\sup_{u \, \text{solution of (3.1)}} \left[\frac{E_\varepsilon(0)}{\displaystyle\int_0^T |u_x(1,t)|^2 \, dt} \right] \to \infty, \text{as } \varepsilon_j \to \infty \tag{3.9}$$

for all $T > 0$.

Proof of Corollary 3.1. Assuming for the moment that Theorem 3.1 holds we consider solutions of (3.1) of the form

$$u_{\varepsilon_j}(x,t) = \cos\left(\sqrt{\lambda_{k_j}^{\varepsilon_j}}\, t\right) w_{k_j}^{\varepsilon_j}(x)$$

where the sequence $\varepsilon_j \to 0$ and the sequence of eigenvalues $\lambda_{k_j}^{\varepsilon_j}$ and eigenfunctions $w_{k_j}^{\varepsilon_j}$ are as in Theorem 3.1. We have

$$E_\varepsilon(0) = \frac{1}{2} \int_0^1 \left| \partial_x w_{k_j}^{\varepsilon_j}(x) \right|^2 dx \tag{3.10}$$

and

$$\int_0^T \left| \partial_x w_{k_j}^{\varepsilon_j}(1,t) \right|^2 dt = \left| \partial_x w_{k_j}^{\varepsilon_j}(1) \right|^2 \int_0^T \cos^2\left(\sqrt{\lambda_{k_j}^{\varepsilon_j}}\, t\right) dt. \tag{3.11}$$

Since

$$\lambda_{k_j}^{\varepsilon_j} \to \infty \text{ as } \varepsilon_j \to 0, \tag{3.12}$$

we have

$$\int_0^T \cos^2\left(\sqrt{\lambda_{k_j}^{\varepsilon_j}}\, t\right) dt \to 1/2. \tag{3.13}$$

Note that (3.12) holds since, by using the min-max characterization of eigenvalues one has

$$\frac{k^2\pi^2}{\rho_M} \le \lambda_k^\varepsilon \le \frac{k^2\pi^2}{\rho_m}$$

for all $0 < \varepsilon < 1$ and $k \ge 1$. Combining (3.8), (3.10), (3.11) and (3.13) we deduce that (3.9) holds.

Remark 3.2 The convergence in (3.9) holds at an exponential rate.

Remark 3.3 Note that the Floquet Theorem guarantees that $\mu_1' > 0$. This indicates that the solutions of the Cauchy problem (3.5) are bounded in $\mathbb{R}$ for μ in the interval $\mu \in (0, \mu_1')$. Therefore, $\mu_1' > 0$ is, roughly speaking, the lowest value of μ for which instabilities arise in (3.5). Therefore the proof of Theorem 3.1 provides necessarily sequences of eigenvalues $\lambda_{k_j}^{\varepsilon_j}$ such that $\lambda_{k_j}^{\varepsilon_j} \geq \mu_1'/\varepsilon_j^2$. As we shall see in the following section this is a sharp estimate since uniform observability estimates hold for eigenfunctions corresponding to eigenvalues λ such that $1 \leq C/\varepsilon^2$ with $C > 0$ small enough.

1.4.2 Asymptotic analysis of the spectrum

In this section we present the results of Castro and Zuazua (1997, 1998) on the asymptotic behavior of the spectrum as $\varepsilon \to 0$ in the range $\lambda \leq C/\varepsilon^2$.

Theorem 3.2 *For any $\delta > 0$ there exists $C(\delta) > 0$ such that*

$$\sqrt{\lambda_{k+1}^{\varepsilon}} - \sqrt{\lambda_k^{\varepsilon}} \geq \frac{\pi}{\sqrt{\bar{\rho}}} - \delta \tag{3.14}$$

for all $k \leq C(\delta)/\varepsilon$ and $0 < \varepsilon < 1$. Moreover, there exist $C, c > 0$ such that

$$\frac{1}{C} \left|\partial_x w_k^{\varepsilon}(1)\right|^2 \leq \int_0^1 \left|\partial_x w_k^{\varepsilon}\right|^2 dx \leq C \left|\partial_x w_k^{\varepsilon}(1)\right|^2 \tag{3.15}$$

for all $k \leq c/\varepsilon$ and $0 < \varepsilon < 1$.

Remark 3.4 The first statement of this theorem guarantees that the *gap* between consecutive eigenvalues remains uniformly bounded below in the range $k \leq C/\varepsilon$ for $C > 0$ small enough (which is equivalent to $\lambda \leq C'/\varepsilon^2$ for a suitable C'). In fact the gap corresponding to the limit spectrum $\lambda_k = k^2\pi^2/\bar{\rho}$ is precisely $\pi/\sqrt{\bar{\rho}}$, i.e.

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = \frac{(k+1)\pi}{\sqrt{\bar{\rho}}} - \frac{k\pi}{\sqrt{\bar{\rho}}} = \frac{\pi}{\sqrt{\bar{\rho}}}.$$

According to (3.14) the gap may be made to be arbitrarily close to $\pi/\sqrt{\bar{\rho}}$ for all $0 < \varepsilon < 1$ provided $k \leq C(\delta)/\varepsilon$ with $C(\delta)$ small enough.

The second statement of Theorem 3.2 guarantees the uniform observability of the eigenfunctions from the extreme $x = 1$ provided $k \leq c/\varepsilon$ with $c > 0$ small enough. This result is sharp since, according to Theorem 3.1, there exists $\varepsilon_j \to 0$ and a sequence of eigenvalues of the order of $\lambda_{k_j}^{\varepsilon_j} \sim C/\varepsilon_j^2$ for a sufficiently large C such that

$$\int_0^1 \left|\partial_x w_{k_j}^{\varepsilon_j}(x)\right|^2 dx \Big/ \left|\partial_x w_{k_j}^{\varepsilon_j}(1)\right|^2 \to 0, \text{ as } \varepsilon_j \to 0.$$

Let us now briefly comment the proof of Theorem 3.2.

We perform the change of variables $y = x/\varepsilon$ so that the differential equation corresponding to the eigenfunctions becomes

$$\partial_y^2 w + \mu\rho(y)w = 0 \tag{3.16}$$

with $\mu = \varepsilon^2\lambda$. We employ the *shooting method*. Thus we solve (3.16) under the "initial conditions"

$$w(1/\varepsilon) = 0; \partial_y w(1/\varepsilon) = 1. \tag{3.17}$$

Finding the eigenvalues λ_k^ε is then equivalent to find the values of μ_k^ε such that the solution of (3.16)-(3.17) satisfies

$$w(0) = 0. \tag{3.18}$$

We employ the WKB asymptotic expansion method (see Bender and Orszag, 1978) to analyze the structure of the solutions of (3.16)-(3.17). This asymptotic expansion turns out to converge in the interval $(0, 1/\varepsilon)$ when $\mu > 0$ is small enough, i.e. when $\lambda \le C/\varepsilon^2$ for $C > 0$ small enough. This allows us to rewrite equation (3.17) as finding the zeroes of a infinite series. More precisely, we conclude that in the range $\lambda \le c/\varepsilon^2$, with $c > 0$ sufficiently small, λ_k^ε is an eigenvalue if and only if λ_k^ε is the root of

$$\sqrt{\lambda_k^\varepsilon \bar\rho} + \sum_{n\ge1}\left(\varepsilon^{2n}d_{2n-1} + \varepsilon^{2n+1}c_{2n}\left(\varepsilon^{-1}\right)\right)(\lambda_k^\varepsilon)^{(2n+1)/2} = k\pi \tag{3.19}$$

where $\{d_{2j-1}\}_{j\ge1}$ are constants and $\{c_{2j}\}_{j\ge1}$ are ℓ-periodic functions that may be computed explicitly. The same method provides and asymptotic expansion of the eigenfuncitons w_k^ε as well.

In order to illustrate how the gap condition (3.14) arises let us consider the second order approximation of λ_k^ε. According to (3.19) it follows that

$$\sqrt{\lambda_k^\varepsilon} \sim \frac{k\pi}{\sqrt{\bar\rho}} - \varepsilon^2(k\pi)^3\frac{d_1}{\bar\rho^2}. \tag{3.20}$$

Therefore

$$\begin{aligned}\sqrt{\lambda_{k+1}^\varepsilon} - \sqrt{\lambda_k^\varepsilon} &\sim \frac{\pi}{\sqrt{\bar\rho}} - \frac{\varepsilon^2\pi^3 d_1}{\bar\rho^2}\left((k+1)^3 - k^3\right) \\ &= \frac{\pi}{\sqrt{\bar\rho}} - \frac{\varepsilon^2\pi^3 d_1}{\bar\rho^2}\left(\varepsilon k^2 + 3k + 1\right) \\ &\sim \frac{\pi}{\sqrt{\bar\rho}} - \frac{3\pi^3 d_1}{\bar\rho^2}(\varepsilon k)^2.\end{aligned} \tag{3.21}$$

Thus, according to the second order approximation, (3.14) holds if

$$\varepsilon k \leq \frac{\bar{\rho}\sqrt{\delta}}{\sqrt{3\pi^3 d_1}}. \tag{3.22}$$

The second order approximation allows also to analyze rigorously the problem of the uniform convergence of $\sqrt{\lambda_k^\varepsilon}$ towards the limit spectrum $k\pi/\sqrt{\bar{\rho}}$. Indeed, we have

$$\sqrt{\lambda_k^\varepsilon} - \frac{k\pi}{\sqrt{\bar{\rho}}} \sim \frac{\pi^3 d_1}{\bar{\rho}^2}\varepsilon^2 k^3. \tag{3.23}$$

Therefore, in order to guarantee that

$$\left| \sqrt{\lambda_k^\varepsilon} - \frac{k\pi}{\sqrt{\bar{\rho}}} \right| \leq \delta \tag{3.24}$$

we need to restrict the index k to the case

$$k \leq C\varepsilon^{-2/3} \tag{3.25}$$

with $C > 0$ small enough.

Observe that condition (3.25) is weaker than the one we got in (2.15) from the classical homogenization theory. Condition (3.25) is sharp and proves that the uniform convergence of the spectrum only holds in the range $k \leq C\varepsilon^{-2/3}$ which is much narrower that the range $k \leq C\varepsilon^{-1}$ under consideration.

To summarize, one can say that the WKB expansion method allows to prove the uniform convergence of $\{\sqrt{\lambda_k^\varepsilon}\}$ towards $\{k\pi/\sqrt{\bar{\rho}}\}$ as $\varepsilon \to 0$ in the range $k \leq C\varepsilon^{-2/3}$ but that the uniform gap condition (3.14) is guaranteed up to the critical level $k \leq C\varepsilon^{-1}$ with $C > 0$ small enough.

As we shall see in the following section, this uniform gap condition together with the uniform observability of the eigenfunctions (3.15) is sufficient to prove the uniform observability of the solutions whose spectrum lies in the range $\lambda \leq C\varepsilon^{-2}$ with $C > 0$ small enough.

1.4.3 Uniform observability

In this section we prove the main uniform observability result for system (3.1). In addition to the sharp spectral results of the previous section we shall use a classical result due to Ingham in the theory of non-harmonic Fourier series (see Young, 1980).

Ingham's Theorem. *Let $\{\mu_k\}_{k\in Z}$ be a sequence of real numbers such that*

$$\mu_{k+1}-\mu_k \geq \gamma > 0,\ \forall k\in Z. \tag{3.26}$$

Then, for any $T>2\pi/\gamma$ there exists a positive constant $C(T,\gamma)>0$ such that

$$\frac{1}{C(T,\gamma)}\sum_{k\in Z}\mid a_k\mid^2 \leq \int_0^T \left|\sum_{k\in Z} a_k e^{i\mu_k t}\right|^2 dt \leq C(T,\gamma)\sum_{k\in Z}\mid a_k\mid^2 \tag{3.27}$$

for all sequence of complex numbers $a_k\in\ell^2$.

Remark 3.5 Ingham's inequality may be viewed as a generalization of the orthogonality property of trigonometric functions. Indeed, assume that

$$\mu_k = k\gamma,\ k\in Z$$

for some $\gamma>0$. Then (3.26) holds with equality for all k. We set $T=2\pi/\gamma$. Then

$$\int_0^{2\pi/\gamma}\left|\sum_{k\in Z} a_k e^{i\gamma k t}\right|^2 dt = \frac{2\pi}{\gamma}\sum_{k\in Z}\mid a_k\mid^2 . \tag{3.28}$$

Note that under the weaker gap condition (3.26) we obtain upper and lower bounds instead of identity (3.28). Observe also that Ingham's inequality does not apply at the minimal time $2\pi/\gamma$.

Theorem 3.3 *Assume that $\rho\in L^\infty(\mathbb{R})$ is $\ell-$periodic and such that*

$$0<\rho_m\leq\rho(x)\leq\rho_M<\infty,\ a.e.\ x\in\mathbb{R}.$$

Then, for any $T>2\sqrt{\bar\rho}$ there exist positive constant $c(T)$, $C(T)>0$ such that

$$\frac{1}{C(T)}\int_0^T\left|\partial_x u(1,t)\right|^2 dt \leq E_\varepsilon(0)\leq C(T)\int_0^T\left|\partial_x u(1,t)\right|^2 dt \tag{3.29}$$

for all $0<\varepsilon<1$ and all solution u of (3.1) in the class

$$u\in \operatorname{span}\left\{w_k^\varepsilon : k\leq C(T)\varepsilon^{-1}\right\}. \tag{3.30}$$

Remark 3.6 Observe that the minimal time needed to apply Theorem 3.3 is $2\sqrt{\bar\rho}$ which is the observability time for the limit wave equation

$$\begin{cases} \bar\rho u_{tt}-u_{xx}=0, & 0<x<1,\quad 0<t<T\\ u(0,t)=u(1,t)=0, & 0<t<T. \end{cases} \tag{3.31}$$

Note however that the range of frequencies $k \leq C(T)\varepsilon^{-1}$ for which Theorem 3.3 applies depends on T. As we shall see along the proof of Theorem 3.3,

- If $T \searrow 2\sqrt{\rho}$, then $C(T) \searrow 0$;
- If $T \nearrow \infty$, then $C(T) \nearrow C^* > 0$, where C^* is the critical constant at which the gap condition is lost.

Proof of Theorem 3.3. As we have seen in (2.9), solutions of (3.1) may be developed in Fourier series as follows:

$$u = \sum_{k\geq 1} \left(a_k^\varepsilon \cos\left(\sqrt{\lambda_k^\varepsilon} t\right) + \frac{b_k^\varepsilon}{\sqrt{\lambda_k^\varepsilon}} \sin\left(\sqrt{\lambda_k^\varepsilon} t\right) \right) w_k^\varepsilon$$

where $\{a_k^\varepsilon\}$ and $\{b_k^\varepsilon\}$ are the Fourier coefficients of the initial data:

$$u^0 = \sum_{k\geq 1} a_k^\varepsilon w_k^\varepsilon;\ u^1 = \sum_{k\geq 1} b_k^\varepsilon w_k^\varepsilon.$$

We set

$$\mu_k^\varepsilon = \sqrt{\lambda_k^\varepsilon};\ \mu_{-k}^\varepsilon = -\mu_k^\varepsilon;\ w_{-k}^\varepsilon = w_k^\varepsilon,\ \text{for } k \geq 1.$$

Then, u can also be written as follows

$$u = \sum_{k\in Z\setminus\{0\}} c_k^\varepsilon e^{i\mu_k^\varepsilon t} w_k^\varepsilon \tag{3.32}$$

where

$$c_k^\varepsilon = \frac{a_k^\varepsilon - ib_k^\varepsilon/\mu_k^\varepsilon}{2}, \quad a_k^\varepsilon = a_{-k}^\varepsilon, \quad b_k^\varepsilon = b_{-k}^\varepsilon.$$

According to (3.32) we have

$$\partial_x u(1,t) = \sum_{k\in Z\setminus\{0\}} c_k^\varepsilon e^{i\mu_k^\varepsilon t} \partial_x w_k^\varepsilon(1). \tag{3.33}$$

We now consider solutions with frequencies in the range $k \leq c\varepsilon^{-1}$. Then

$$u = \sum_{\substack{|k|\leq c/\varepsilon \\ k\neq 0}} c_k^\varepsilon e^{i\mu_k^\varepsilon t} w_k^\varepsilon \tag{3.34}$$

and

$$\partial_x u(1,t) = \sum_{\substack{|k|\leq c/\varepsilon \\ l\neq 0}} c_k^\varepsilon e^{i\mu_k^\varepsilon t} \partial_x w_k^\varepsilon(1). \tag{3.35}$$

Given $T > 2\sqrt{\overline{\rho}}$ we have

$$T > 2\pi/\gamma \tag{3.36}$$

for some $\gamma < \pi/\sqrt{\overline{\rho}}$. Let $\delta > 0$ be such that

$$\frac{\pi}{\sqrt{\overline{\rho}}} - \delta \geq \gamma > 0. \tag{3.37}$$

According to Theorem 3.2, there exists $c(\delta) > 0$ such that

$$\sqrt{\lambda_{k+1}^{\varepsilon}} - \sqrt{\lambda_k^{\varepsilon}} \geq \frac{\pi}{\sqrt{\overline{\rho}}} - \delta \geq \gamma \tag{3.38}$$

for all $0 < \varepsilon < 1$ and $k \leq C(\delta)\varepsilon^{-1}$.

In view of (3.38) we can apply Ingham's Theorem to the series in (3.35) provided $c \leq c(\delta)$. It then follows that there exists $C > 0$ such that

$$\begin{aligned} \frac{1}{C} \sum_{\substack{|k| \leq c/\varepsilon \\ k \neq 0}} | c_k^{\varepsilon} |^2 | \partial_x w_k^{\varepsilon}(1) |^2 &\leq \int_0^T \left| \sum_{\substack{|k| \leq c/\varepsilon \\ k \neq 0}} c_k^{\varepsilon} e^{i\mu_k^{\varepsilon} t} w_k^{\varepsilon} \right|^2 dt \\ &\leq C \sum_{\substack{|k| \leq c/\varepsilon \\ k \neq 0}} | c_k^{\varepsilon} |^2 | \partial_x w_k^{\varepsilon}(1) |^2 . \end{aligned} \tag{3.39}$$

On the other hand, taking into account that

$$\int_0^1 |\partial_x w_k^{\varepsilon}(x)|^2 \, dx = \lambda_k^{\varepsilon} \int_0^1 | w_k^{\varepsilon} | \rho(x/\varepsilon) dx = \lambda_k^{\varepsilon},$$

according to (3.15) we deduce that, for $k \leq c/\varepsilon$,

$$\frac{\lambda_k^{\varepsilon}}{C} \leq |\partial_x w_k^{\varepsilon}(1)|^2 \leq C\lambda_k^{\varepsilon}. \tag{3.40}$$

Combining (3.39) and (3.40) and choosing a possibly smaller $c > 0$ such that both (3.39) and (3.40) apply in the range $k \leq c\varepsilon^{-1}$ we deduce that

$$\begin{aligned} \frac{1}{C} \sum_{\substack{|k| \leq c/\varepsilon \\ k \neq 0}} \lambda_k^{\varepsilon} | c_k^{\varepsilon} |^2 &\leq \int_0^T \left| \sum_{\substack{|k| \leq c/\varepsilon \\ k \neq 0}} c_k^{\varepsilon} e^{i\mu_k^{\varepsilon} t} \partial_x w_k^{\varepsilon}(1) \right|^2 dt \qquad (3.41) \\ &\leq C \sum_{\substack{|k| \leq c/\varepsilon \\ k \neq 0}} \lambda_k^{\varepsilon} | c_k^{\varepsilon} |^2 . \end{aligned}$$

On the other hand,

$$\sum_{\substack{|k|\leq c/\varepsilon \\ k\neq 0}} \lambda_k^\varepsilon \mid c_k^\varepsilon \mid^2 = \frac{1}{2} \sum_{1\leq k\leq c/\varepsilon} \left[\lambda_k^\varepsilon \mid a_k^\varepsilon \mid^2 + \mid b_k^\varepsilon \mid^2\right] = E_\varepsilon(0). \tag{3.42}$$

Combining (3.41)-(3.42) inequality (3.29) follows.

1.4.4 Final remarks

We have shown that the uniform observability does not hold when the wavelength of solutions is of the order ε of the microstructure. We have also shown that uniform observability holds in the class of solutions whose spectrum is in the range $\lambda \leq c\varepsilon^{-2}$ for $c > 0$ small enough. More precisely, we have shown that if

$$k \leq c\varepsilon^{-1} \tag{3.43}$$

with

$$0 < c < c^* \tag{3.44}$$

and

$$c^* = \left(\frac{\bar{\rho}^{3/2}}{3\pi^2 d_1}\right)^{1/2} \tag{3.45}$$

uniform observability holds for $T > T(c)$ where $T(c)$ is such that

- $T(c) \searrow 2\sqrt{\bar{\rho}}$ as $c \searrow 0$;
- $T(c) \nearrow \infty$ as $c \nearrow c^*$.

This result shows that the observability inequality for the limit wave equation (3.31) may be viewed as the limit when $\varepsilon \to 0$ of observability inequalities for systems (3.1) in the range (3.43) for an appropriate $c > 0$.

As we mentioned in the introduction, the lack of uniform observability is related to the fact that the speed of propagation may tend to zero as $\varepsilon \to 0$ and the wavelength of solutions approaches a critical size. However, the analysis above shows that the lack of observability arises even if the velocity of propagation as described in section 1.2 is bounded below. To see this it is convenient to consider again the second order approximation of the eigenvalues (3.20). We recall that

$$\sqrt{\lambda_k^\varepsilon} \sim \frac{k\pi}{\sqrt{\bar{\rho}}} - \varepsilon^2 (k\pi)^3 \frac{d_1}{\bar{\rho}2}.$$

Thus

$$\sqrt{\lambda_k^\varepsilon}/k \sim \frac{\pi}{\sqrt{\bar{\rho}}} - \frac{\pi^3 d_1}{\bar{\rho}^2}\varepsilon^2 k^2.$$

Therefore, the velocity of propagation vanishes when

$$\varepsilon k \sim \left(\frac{\bar{\rho}^{3/2}}{\pi^2 d_1}\right)^{1/2}$$

while the gap between consecutive eigenvalues vanishes when

$$\varepsilon k \sim \left(\frac{\bar{\rho}^{3/2}}{3\pi^2 d_1}\right)^{1/2} < \left(\frac{\bar{\rho}^{3/2}}{\pi^2 d_1}\right)^{1/2}$$

as observed above. Clearly the latter happens before the velocity of propagation vanishes. However, these two phenomena arise when $k \sim c/\varepsilon$ for suitable critical constant c.

All this section has been devoted to the analysis of the low frequencies $k \leq c/\varepsilon$. Note however that, under suitable regularity assumptions on the density ρ, the WKB method allows to obtain an asymptotic expansion for the high frequencies $k >> \varepsilon^{-1}$ as well. We refer to (Castro and Zuazua, 1997) for a careful analysis of this problem. At this respect, it is worth mentioning that, in a first approximation, the effective wave equation for the high frequencies is

$$\begin{cases} \rho^* u_{tt} - u_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ u(0,t) = u(1,t) = 0, & 0 < t < T \end{cases}$$

where

$$\rho^* = \left(\frac{1}{\ell}\int_0^\ell \sqrt{\rho}\,dx\right)^2.$$

Therefore, the effective speed of propagation is different for the high frequencies. Note that, actually, $\rho^* < \bar{\rho}$. This indicates that the time of observability for the high frequencies $k >> \varepsilon^{-1}$ is smaller than for the low ones.

We refer to (Castro, 1998b) for a detailed discussion of the behavior of eigenvalues and eigenfunctions corresponding to $\lambda \sim c\varepsilon^{-2}$.

1.5 Finite-difference approximations

This section is devoted to the analysis of the observability problem for the finite-difference space semi-discretization of the wave equation (1.1) introduced in section 1.3.2. We follow (Infante and Zuazua, 1998a, 1998b).

We recall that, given $N \in \mathbb{N}$ we set $h = 1/(N+1)$ and

$$x_j = jh,\ j = 0, \cdots, N+1.$$

The finite-difference space semi-discretization is then as follows:

$$\begin{array}{l} u_j'' - [u_{j+1} + u_{j-1} - 2u_j]/h^2 = 0,\ 0 < t < T,\ j = 1, \cdots, N \\ u_0 = u_{N+1} = 0,\ 0 < t < T \\ u_j(0) = u_j^0,\ u = 1, \cdots, N. \end{array} \quad . \tag{4.1}$$

In section 1.5 below we analyze the spectrum of the system and prove that the uniform observability as $h \to 0$ does not hold for any $T > 0$. In section 1.5.2 we establish uniform observability inequalities for solutions whose high frequencies have been cut-off in a suitable way. In section 1.5.3 we make some final comments.

1.5.1 Spectral analysis. Non uniform observability

As indicated in section 1.3.2 the spectrum of system (4.1) may be computed explicitly. We have

$$\lambda_k^h = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2}\right),\ k = 1, \cdots, N \tag{4.2}$$

and

$$w_k^h = \left(w_{k,1}^h, \cdots, w_{k,N}^h\right),\ w_{k,j}^h = \sin(\pi k_j h),\ k, j = 1, \cdots, N. \tag{4.3}$$

The following identity holds:

Lemma 4.1 *For any $h > 0$ and any eigenvector of* (4.1) *associated with the eigenvalue λ the following identity holds:*

$$h \sum_{j=0}^{N} \left| \frac{w_{j+1} - w_j}{h} \right|^2 = \frac{2}{4 - \lambda h^2} \left| \frac{w_N}{h} \right|^2 . \tag{4.4}$$

We now observe that the largest eigenvalue λ_N^h is such that

$$\lambda_N^h h^2 \to 4 \text{ as } h \to 0. \tag{4.5}$$

Indeed

$$\begin{aligned} \lambda_N^h h^2 &= 4\sin^2\left(\frac{\pi N h}{2}\right) = 4\sin^2\left(\frac{\pi(1-h)}{2}\right) \\ &= 4\cos^2(h/2) \to 4 \text{ as } h \to \infty. \end{aligned}$$

Combining (4.4) and (4.5) the following result on non-uniform observability holds:

Theorem 4.1 *For any $T > 0$ it follows that*

$$\sup_{u \text{ solution of } (4.1)} \left[\frac{E_h(0)}{\displaystyle\int_0^T \mid u_N/h \mid^2 dt} \right] \to \infty \tag{4.6}$$

as $h \to 0$.

Proof of Theorem 4.1. We consider solutions of (4.1) of the form

$$u^h = \cos\left(\sqrt{\lambda_N^h} t\right) w_N^h,$$

where λ_N^h and w_N^h are the $N-$th eigenvalue and eigenvector of (4.1). We have

$$E_h(0) = \frac{h}{2} \sum_{j=0}^{N} \left| \frac{w_{N,j+1}^h - w_{N,j}^h}{h} \right|^2 \tag{4.7}$$

and

$$\int_0^T \left| \frac{u_N^h}{h} \right|^2 dt = \left| \frac{w_{N,N}^h}{h} \right|^2 \int_0^T \cos^2\left(\sqrt{\lambda_N^h} t\right) dt. \tag{4.8}$$

Taking into account that $\lambda_N^h \to \infty$ as $h \to 0$ it follows that

$$\int_0^T \cos^2\left(\sqrt{\lambda_N^h} t\right) dt \to T/2 \text{ as } h \to 0. \tag{4.9}$$

Combining (4.4), (4.7), (4.8) and (4.9), (4.6) follows immediately.

Remark 4.1 Note that the construction above applies to any sequence of eigenvalues $\lambda_{j(h)}^h$ such that

$$h^2 \lambda_{j(h)}^h \to 4, \text{ as } h \to 0. \tag{4.10}$$

Observe that (4.10) is equivalent to

$$\sin^2\left(\frac{\pi j(h) h}{2}\right) \to 1$$

or, in other words, to

$$j(h)h \to 1. \tag{4.11}$$

As we shall see in section 1.5.2 below, as soon as we work in classes of solutions where $\lambda \leq \gamma h^{-2}$ with $0 < \gamma < 4$ or with indexes $0 < j < \delta h^{-1}$ with $0 < \delta < 1$, the observability inequality is uniform.

1.5.2 Uniform observability

We recall that solutions of (4.1) can be developed in Fourier series as follows:

$$u = \sum_{k=1}^{N} \left(a_k \cos\left(\sqrt{\lambda_k^h} t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h} t\right) \right) w_k^h \tag{4.12}$$

where a_k, b_k are the Fourier coefficients of the initial data, i.e.

$$u_j^0 = \sum_{k=1}^{N} a_k w_{k,j}^h; \; u_j^1 = \sum_{k=1}^{N} b_k w_{k,j}^h.$$

Given $0 < \delta < 1$ we introduce the following classes of solutions of (4.1):

$$\begin{aligned} \mathcal{C}_\delta(h) &= \Bigg\{ u \text{ solution of (4.1) of the form} \qquad (4.13) \\ u &= \sum_{k=1}^{\delta/h} \left(a_k \cos\left(\sqrt{\lambda_k^h} t\right) + b_k \sin\left(\sqrt{\lambda_k^h} t\right) \right) w_k^h \quad \Bigg\}. \end{aligned}$$

Note that in the class $\mathcal{C}_\delta(h)$ the high frequencies corresponding to the indexes $j > \delta N$ have been cut-off. The following result holds:

Theorem 4.2 *For any $\delta > 0$ there exists $T(\delta) > 0$ such that the following holds. For all $T > T(\delta)$ there exists $C(T, \delta) > 0$ so that*

$$\frac{1}{C} E_h(0) \leq \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \leq C E_h(0) \tag{4.14}$$

for every solution u of (4.1) in the class $\mathcal{C}_\delta(h)$, and for all $h > 0$. Moreover

- $T(\delta) \to 2$ *as* $\delta \to 0$;
- $T(\delta) \to \infty$ *as* $\delta \to 1$.

Remark 4.2 Theorem 4.2 guarantees the uniform observability of (4.1) in each class $\mathcal{C}_\delta(h)$, for all $0 < \delta < 1$. Note however that the minimal time $T(\delta)$ for the uniform observability depends on δ.

Proof of Theorem 4.2. Let us analyze the gap between consecutive eigenvalues. We have

$$\begin{aligned} \sqrt{\lambda_k^h} - \sqrt{\lambda_{k-1}^h} &= \frac{2}{h} \left[\sin\left(\frac{\pi k h}{2}\right) - \sin\left(\frac{\pi (k-1) h}{2}\right) \right] \\ &= \pi \cos\left(\frac{\pi (k - 1 + \eta) h}{2}\right) \end{aligned}$$

for some $0 < \eta < 1$. Observe that

$$\cos\left(\frac{\pi(k-1+\eta)h}{2}\right) \geq \cos\left(\frac{\pi kh}{2}\right).$$

Therefore

$$\sqrt{\lambda_k^h} - \sqrt{\lambda_{k-1}^h} \geq \pi\cos\left(\frac{\pi kh}{2}\right).$$

It follows that

$$\sqrt{\lambda_k^h} - \sqrt{\lambda_{k-1}^h} \geq \pi\cos\left(\frac{\pi\delta}{2}\right), \text{ for } k \leq \delta h^{-1}. \tag{4.15}$$

We are now in conditions to apply Ingham's Theorem. We rewrite the solution $u \in \mathcal{C}_\delta(h)$ of (4.1) as follows:

$$u = \sum_{\substack{|k|\leq\delta/h \\ k\neq 0}} c_k e^{i\mu_k^h t} w_k \tag{4.16}$$

where

$$\mu_{-k}^h = -\mu_k^h;\ \mu_k = \sqrt{\lambda_k^h};\ w_{-k} = w_k;\ c_k = \frac{a_k - ib_k/\mu_k^h}{2}.$$

Then,

$$u_N(t) = \sum_{\substack{|k|\leq\delta/h \\ k\neq 0}} c_k e^{i\mu_k^h t} w_{k,N}.$$

Therefore

$$\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt = \int_0^T \left|\sum_{\substack{|k|\leq\delta/h \\ k\neq 0}} c_k e^{i\mu_k^h t}\frac{w_{k,N}}{h}\right|^2 dt. \tag{4.17}$$

We now apply Ingham's inequality. In view of (4.15) it follows that if $T > T(\delta)$ with

$$T(\delta) = 2/\cos(\pi\delta/2) \tag{4.18}$$

there exists a constant $C > 0$ such that

$$\frac{1}{C}\sum_{\substack{|k|\leq\delta/h \\ k\neq 0}} |\ c_k\ |^2 \left|\frac{w_{k,N}}{h}\right|^2 \leq \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \leq C\sum_{\substack{|k|\leq\delta/h \\ k\neq 0}} |\ c_k\ |^2 \left|\frac{w_{k,N}}{h}\right|^2, \tag{4.19}$$

for every solution of (4.1) in the class $\mathcal{C}_\delta(h)$. On the other hand, we observe that

$$\lambda_k^h h^2 = 4\sin^2\left(\frac{\pi k h}{2}\right) \leq 4\sin^2\left(\frac{\pi\delta}{2}\right) \tag{4.20}$$

for all $k \leq \delta/h$. Therefore, according to Lemma 4.1 it follows that

$$\frac{1}{2}\left|\frac{w_N}{h}\right|^2 \leq h\sum_{j=0}^{N}\left|\frac{w_{j+1}-w_j}{h}\right|^2 \leq \frac{1}{2\cos^2(\pi\delta/2)}\left|\frac{w_N}{h}\right|^2 \tag{4.21}$$

for all eigenvalue with index $k \leq \delta/h$.

Combining (4.19) and (4.21) we deduce that for all $T > T(\delta)$ there exists $C > 0$ such that

$$\frac{1}{C}\sum_{\substack{|k|\leq\delta/h\\ k\neq 0}} | c_k |^2 \leq \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \leq C \sum_{\substack{|k|\leq\delta/h\\ k\neq 0}} | c_k |^2 . \tag{4.22}$$

Finally we observe that

$$\sum_{\substack{|k|\leq\delta/h\\ k\neq 0}} | c_k |^2 \sim E_h(0).$$

This concludes the proof of Theorem 4.2.

Remark 4.3 Observe that the proof of Theorem 4.2 provides an explicit estimate on the minimal observability time in the class $\mathcal{C}_\delta(h)$. Namely, $T(\delta) = 2/\cos(\pi\delta/2)$.

1.5.3 Conclusion

We have shown that the uniform observability of the finite difference approximations (4.1) fails for any $T > 0$. On the other hand, we have proved that filtering the high frequencies or, in other words, considering solutions in the classes $\mathcal{C}_\delta(h)$ with $0 < \delta < 1$ the uniform observability holds in a minimal time $T(\delta)$ that satisfies

- $T(\delta) \to \infty$, as $\delta \to 1$;
- $T(\delta) \to 2$, as $\delta \to 0$.

Observe that, as $\delta \to 0$, we recover the minimal observabiity time $T = 2$ of the continuous wave equation (1.1). This allows us to obtain the observability property of the continuous wave equation (1.1) as limit when

$h \to 0$ of uniform observability inequalities for the semi-discrete systems (4.1).

On the other hand, we have seen that when $\lambda \sim 4/h^2$ or, equivalently, $k \sim 1/h$, the gap between consecutive eigenvalues vanishes. This forces the uniform observability time $T(\delta)$ in the classes $\mathcal{C}_\delta(h)$ to tend to infinity as $\delta \to 1$.

Note that, for any $h > 0$ fixed, system (4.1) is a linear system of dimension N. Therefore, in view of Lemma 4.1 and the fact that the eigenvalues $\lambda_1^h, \cdots, \lambda_N^h$ are all distinct, it follows that system (4.1) is observable for all $T > 0$ (see Lee and Markus, 1967). This indicates that a singular phenomenon arises when letting $h \to 0$. Indeed, system (4.1) is observable for all $h > 0$ and $T > 0$. But the limit wave equation (1.1) is only observable when $T > 2$. The analysis we have carried out in this section explains this fact. If we want to obtain uniform observability inequalities as $h \to 0$ we have to filter the high frequencies and moreover to consider time intervals of length greater than 2.

Observe that in this problem, like in the first one, the velocity of propagation of waves does not vanish. Indeed we have

$$\frac{\sqrt{\lambda_k^h}}{k} = \frac{2}{kh} \sin\left(\frac{k\pi h}{2}\right) \geq \pi \left(1 - \frac{\pi^2}{12}\right), \; \forall h > 0, \; \forall k = 1, \cdots, N.$$

Note that the velocity of propagation would vanish at $k = 2/h$ but the spectrum of the system is localized on the range $k \leq 1/h$. However, as observed above, the gap vanishes when $k \sim 1/h$.

Similar results have been obtained by Infante and Zuazua (1998a, 1998b) using discrete multiplier techniques. In (Infante and Zuazua, 1998a, 1998b) the same results are proved for the finite-element space semi-discretization of the wave equation (1.1) as well. We refer to (Zuazua, 1998b) for the extension of these results to the square in two space dimensions.

1.6 Comparison and final comments

We have considered two "singular perturbations" of a constant coefficient $1-d$ wave equation. Namely, the wave equation with rapidly oscillatory density and the $1-d$ finite-difference space semi-discretization of the wave equation. We have addressed the problem of boundary observability and, more precisely, the problem of whether the observability estimates are uniform when the singular parameter (size of the microstructure or mesh-size) tends to zero.

We have proved the same qualitative results in both problems:

- Uniform observability does not hold for any time T.
- Uniform observability holds if the time T is large enough provided we filter conveniently the high frequencies.

In the case of the finite-difference approximations our analysis is exhaustive. In the case of highly oscillatory coefficients, the spectrum is unbounded for each $\varepsilon > 0$, and therefore the high frequencies have to be analyzed independently (see Castro and Zuazua, 1997 and Castro, 1998b).

The analogies we have found in these two (apparently) completely different problems can be explained analyzing carefully the dependence of the spectrum with respect to the singular parameter.

As we mentioned in section 1.4, in the context of the highly oscillatory coefficients, in the range $\lambda \leq c\varepsilon^{-2}$ with $c > 0$ sufficiently small, the second order approximation formula for the eigenvalues is as follows:

$$\sqrt{\lambda_k^\varepsilon} \sim \frac{k\pi}{\sqrt{\bar{\rho}}} - \varepsilon^2 (k\pi)^3 \frac{d_1}{\bar{\rho}^2}. \tag{5.1}$$

On the other hand, we have the explicit value of the eigenvalues of the finite-difference approximation:

$$\sqrt{\lambda_k^h} = \frac{2}{h} \sin\left(\frac{k\pi h}{2}\right).$$

Using Taylor's expansion to the second order we have

$$\sqrt{\lambda_k^h} \sim k\pi - \frac{h^2 (\pi k)^3}{4}. \tag{5.2}$$

By simple inspection of formulas (5.1)-(5.2) we see that the qualitative behavior as ε and $h \to 0$ is the same in both cases.

The analysis we have carried out in these two examples illustrates the main features of the wave propagation in $1-d$ highly heterogeneous media.The problem is widely open in several space variables.

In the context of the numerical computation of the boundary control for the wave equation the need of an appropriate filtering of the high frequencies was observed by Glowinski (1992). Two other methods were also introduced in that work for taking care of the spurious high frequency oscillations: Tychonoff regularizations and multigrid techniques.

The same problems have been addressed by López and Zuazua (1997) in the context of the $1-d$ heat equation. In this case the strong dissipative effect of the heat equation on the high frequencies allows to obtain uniform results without any filtering. Actually, in (López and Zuazua, 1997) it is proved that both, the heat equation with rapidly oscillatory coefficients and its finite-difference space semi-discretizations are uniformly observable in any time interval.

5.7 References

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