Polynomial decay for a hyperbolic-parabolic coupled system*

Jeffrey Rauch[†], Xu Zhang[‡] and Enrique Zuazua[§]

Abstract

This paper analyzes the long time behavior of a linearized model for fluid-structure interaction. The space domain consists of two parts in which the evolution is governed by the heat equation and the wave equation respectively, with transmission conditions at the interface. Based on the construction of ray-like solutions by means of Geometric Optics expansions and a careful analysis of the transfer of the energy at the interface, we show the lack of uniform decay of solutions in general domains. Also, we prove the polynomial decay result for smooth solutions under a suitable Geometric Control Condition. This condition requires that all rays propagating in the wave domain reach the interface in a uniform time after, possibly, bouncing in the exterior boundary.

Résumé

Ce papier étudie le comportement asymptotique en temps d'un modèle linéarisé d'interaction fluide-structure. Le domaine (en espace) est composé de deux parties dans lesquelles l'évolution est gouvernée par l'équation de la chaleur et l'équation des ondes respectivement, avec des conditions de transmission à l'interface. En s'appuyant sur la construction de solutions type rayon obtenues par la méthode des expansions géométriques, et une analyse précise du transfert de l'énergie à l'interface, on montre la perte de décroissance uniforme des solutions définies sur des domaines quelconques. De plus, en supposant une condition de contrôle géométrique, on montre également

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un résultat de décroissance polynomial pour des solutions régulières. Cette condition requiert que tous les rayons se propageant sur la partie du domaine gouverné par l'équation des ondes atteignent l'interface en un temps uniforme, après éventuellement avoir rebondi sur la frontière extérieure.

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Contents

1 Introduction

This paper analyzes a linearized model for fluid-structure interaction. This system consists of a wave and a heat equation coupled through an interface with suitable transmission conditions.

Let us describe this system in detail.

Let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain boundary Γ. Let Ω_1 be a sub-domain of Ω and set $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. Denote by γ the interface, $\Gamma_j = \partial\Omega_j \setminus \overline{\gamma}$ $(j = 1, 2)$, and ν_j the unit outward normal vector of Ω_j $(j = 1, 2)$. We assume $\gamma \neq \emptyset$, and $\Gamma, \Gamma_1, \Gamma_2$, and $\operatorname{Int}\gamma$, the relative interior of γ, to be of C^1 (unless otherwise stated). Denote by $\Box$ the d'Alembert operator $\partial_{tt} - \Delta$.

We consider the following hyperbolic-parabolic coupled system:

$$
\begin{cases}
y_t - \Delta y = 0 & \text{in } (0,\infty) \times \Omega_1, \\
\Box z = 0 & \text{in } (0,\infty) \times \Omega_2, \\
y = 0 & \text{on } (0,\infty) \times \Gamma_1, \\
z = 0 & \text{on } (0,\infty) \times \Gamma_2, \\
y = z, \quad \dfrac{\partial y}{\partial \nu_1} = -\dfrac{\partial z}{\partial \nu_2} & \text{on } (0,\infty) \times \gamma, \\
y(0,x) = y_0(x) & \text{in } \Omega_1, \\
z(0,x) = z_0(x), \quad z_t(0,x) = z_1(x) & \text{in } \Omega_2.
\end{cases}
\tag{1.1}
$$

Here and henceforth $x = (x_1, ..., x_n) = (x_1, x')$.

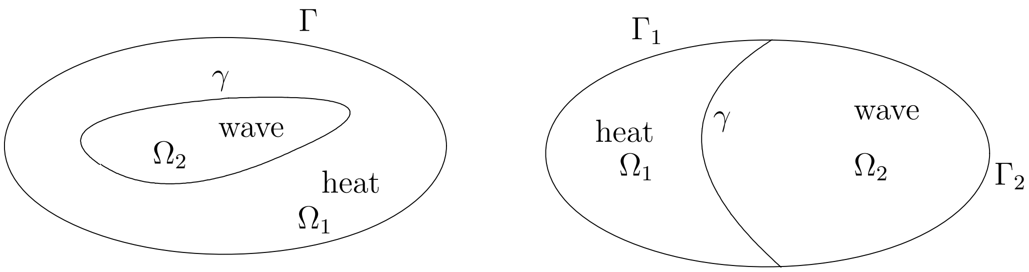

FIGURE 1: A FLUID-STRUCTURE INTERACTION MODEL

This system can be viewed as a simplified and a linearized version of a true fluid-structure interaction model. Of course, more realistic models should combine the system of elasticity for the structure, the Stokes or Navier-Stokes equations for the fluid, and the transmission condition should hold along a moving interface. This would certainly produce important extra technical difficulties for the problems of existence, uniqueness and decay of solutions for large time.

The main purpose of this paper is to analyze the long time behavior of (1.1). Put

$$
H^1_{\Gamma_1}(\Omega_1) \triangleq \left\{ h \in H^1(\Omega_1) \,\middle|\, h = 0 \text{ on } \Gamma_1 \right\}, \qquad H^1_{\Gamma_2}(\Omega_2) \triangleq \left\{ h \in H^1(\Omega_2) \,\middle|\, h = 0 \text{ on } \Gamma_2 \right\}.
$$

It is easy to show that system (1.1) is well-posed in the Hilbert space

$$H \triangleq \Big\{(f_1,f_2)\in H^1_{\Gamma_1}(\Omega_1)\times H^1_{\Gamma_2}(\Omega_2)\;\Big|\; f_1|_\gamma=f_2|_\gamma\Big\}\times L^2(\Omega_2). \tag{1.2}$$

H is the *energy state space* of (1.1), whose norm is given by

$$|f|_H=\sqrt{|\nabla f_1|^2_{(L^2(\Omega_1))^n}+|\nabla f_2|^2_{(L^2(\Omega_2))^n}+|f_3|^2_{L^2(\Omega_2)}},\quad \forall\, f=(f_1,f_2,f_3)\in H. \tag{1.3}$$

Note that the first component of the space H is in fact a pair (f_1,f_2) obtained from any function $f\in H^1_0(\Omega)$ so that $f_1=f|_{\Omega_1}$ and $f_2=f|_{\Omega_2}$.

The energy of system (1.1) is defined by

$$E(t)\triangleq E(y,z,z_t)(t)=\frac{1}{2}|(y(t),z(t),z_t(t))|^2_H. \tag{1.4}$$

By means of the classical energy method, it is easy to check that

$$\frac{d}{dt}E(t)=-\int_{\Omega_1}|y_t|^2dx=-\int_{\Omega_1}|\Delta y|^2dx. \tag{1.5}$$

Formula (1.5) shows that the only dissipative mechanism acting on the system is through the heat equation in Ω_1.

According to the energy dissipation law (1.5) and the well-known unique continuation property for solutions of the wave equation, it is easy to show that there are no nonzero solutions of (1.1) which conserve energy. Hence, using LaSalle's invariance principle ([6, p. 18]), we may conclude that the energy of every solution of (1.1) tends to zero as $t\to\infty$, without any geometric conditions on the domains Ω_1 and Ω_2 other than Ω_1 being a non-empty open set. This paper is devoted to analyze whether or not the energy of solutions of system (1.1) tends to zero exponentially as $t\to\infty$, i.e. whether there exist two positive constants C and α such that

$$E(t)\le CE(0)e^{-\alpha t},\qquad \forall\, t\ge 0 \tag{1.6}$$

for every solution of (1.1).

Recall that for the pure heat equation and the wave equation with an effective damping in a sub-domain satisfying the Geometric Control Condition (GCC for short), the energy decays exponentially (e.g., [3], [22]). On the other hand, for the pure wave equation (in the absence of damping), the energy is conserved. Therefore, the problem we are addressing is that of whether the dissipative mechanism that the heat equation introduces in system (1.1) through the subdomain Ω_1 suffices to produce the uniform decay of the energy of the wave component of the solutions or not.

According to the energy dissipation law (1.5), the uniform decay problem (1.6) is equivalent to showing that: there exists $T>0$ and $C>0$ such that every solution of (1.1) satisfies

$$|(y_0,z_0,z_1)|^2_H\le C\int_0^T\int_{\Omega_1}|y_t|^2dxdt,\qquad \forall\,(y_0,z_0,z_1)\in H. \tag{1.7}$$

Inequality (1.7) can be viewed as an observability estimate for equation (1.1) with observation on the heat subdomain Ω_1.

There is an extensive literature on the observability inequalities of PDEs and its connections with stabilization and control problems ([3], [4], [5], [12], [16], [23] and the references cited therein). However, the techniques that have been developed up to now to obtain such estimates depend heavily on the nature of the equations. In the context of the wave equation one may use multipliers ([12]), Carleman inequalities ([20]), or microlocal analysis ([3]); while, in the context of heat equations, one uses Carleman inequalities ([5] and [4]). Nevertheless, a unified Carleman estimate for those two equations has not been well developed although some partial progress has been made in this respect ([9] and [10]). Consequently, we need to develop new techniques to analyze the observability problem (1.7) for system (1.1).

To begin with, the classical result by Ralston ([15]) on the existence of Gaussian beam solutions of the wave equation concentrated along rays shows immediately that a necessary condition for the observability inequality (1.7) to hold, or equivalently, the exponential decay (1.6) of solutions of (1.1) to be fulfilled is that the heat subdomain Ω_1 controls geometrically the wave domain Ω_2 (*see* Theorem 4.3).

In view of the above negative result, it is natural to address the case where the heat subdomain does satisfy the GCC. A "naive" conjecture would be that this condition will lead to the exponential decay of solutions of (1.1). However, in [21] it is shown that this conjecture is not correct even in one space dimension. Indeed, the high frequency spectral analysis in [21] shows that the spectrum of system (1.1) in $1-d$ is split into two parts: the hyperbolic and the parabolic one. Hyperbolic eigenvalues have an asymptotically vanishing real part and this contradicts exponential decay. The corresponding eigenvectors are mainly concentrated on the wave interval and, consequently, they are very weakly dissipated by the damping mechanism introduced by the heat equation. On the other hand, parabolic eigenvalues are asymptotically real and negative and the corresponding eigenvectors, whose energy is mainly concentrated on the heat interval, are efficiently dissipated. The approach in [21], based on spectral analysis, does not apply to multidimensional situations.

The first topic of this article is to show the lack of exponential decay in several dimensions, as the $1-d$ spectral analysis suggests. For this purpose, we need to analyze carefully the interaction of the wave and heat like solutions on the interface for general geometries. Our method is based on the WKB method of asymptotic expansion in Geometric Optics (*see* for example, [2]). We will show that waves concentrated along rays are almost completely reflected on the interface, which implies that (1.7) fails. The method of Geometric Optics determines the reflection coefficient as a function of the angle of incidence of waves on the interface. This shows the lack of uniform decay of solutions of (1.1) in any geometry, even if the GCC is assumed (*see* Theorem 4.1).

According to the above analysis, it is easy to see that, even under the GCC, one can only expect a polynomial stability property of smooth solutions. This is the second topic addressed in this paper. To do this, we need to derive a weakened observability inequality in Theorem 5.1 by means of the energy method and the existing observability results for the wave equation. Then, based on this theorem, we show in Theorem 5.2 a polynomial decay rate of smooth solutions for system (1.1) under the GCC.

This paper is organized as follows. In Section 2, we give some preliminary results. Section 3 is devoted to the WKB asymptotic analysis for the transmission problem in the whole space with flat interface, which implies the lack of exponential decay for equation (1.1) in polyhedral domains. This case is easier to handle and it allows an easier presentation of the key ideas, since one needs only to use linear real-valued phases for constructing the approximate solutions. This analysis shows that when the wave domain is polyhedral, regardless what the geometry of the heat domain is, the energy of solutions of (1.1) does not decay uniformly. In Section 4 we perform the WKB asymptotic analysis for general interfaces and we conclude the lack of uniform decay for equation (1.1) in general domains. Note that, when treating general interfaces and boundaries, to avoid caustics, we use Gaussian beams to construct the approximate solutions, where the phases are nonlinear and complex-valued. In Section 5, we prove a weakened observability inequality for equation (1.1) under the GCC, and then derive the polynomial decay rate of its smooth solutions.

Notation:

Throughout this paper, for a subset $\omega \subset \mathbb{R}^n$, we denote its characteristic function by χ_ω. For any $\eta > 0$, the η-neighborhood of ω is denoted by $\mathcal{O}_\eta(\omega)$. Also, when writing $(w_1, w_2) \in H^s(\Omega)$ (*resp.* $H^s_0(\Omega)$ for $s \in \mathbb{R}$, we mean that the function $w \triangleq w_1\chi_{\Omega_1} + w_2\chi_{\Omega_2}$ belongs to $H^s(\Omega)$ (*resp.* $H^s_0(\Omega)$). With this convention, it is easy to see that $H = H^1_0(\Omega) \times L^2(\Omega_2)$.

Further, if $M \subset \mathbb{R}^m$ ($m \in \mathbb{N}$) is an open set and f^ε is a family of functions in $C^\infty(M)$ depending on $\varepsilon \in (0, 1)$, we say that $f^\varepsilon \sim 0$ iff for all compact $K \subset M$, $\alpha \in \mathbb{N}^m$, and $N \in \mathbb{N}$ there is a constant $C > 0$ so that

$$\sup_{y\in K} |\partial^\alpha_y f^\varepsilon(y)| \ \leq \ C\,\varepsilon^N$$

holds for all small ε. In this case we also write $f^\varepsilon = O(\varepsilon^\infty)$. For two families $f^\varepsilon, g^\varepsilon$ of smooth functions, $f^\varepsilon \sim g^\varepsilon$ means $f^\varepsilon - g^\varepsilon \sim 0$. We shall also often write $a^\varepsilon \sim \sum_{j=0}^{\infty} a_j\varepsilon^j$. This does not mean that the series converges but rather that for all $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^m$, there is a constant $C > 0$ so that

$$\sup_{y\in M} \Big|\partial^\alpha_y \Big(a^\varepsilon(y) - \sum_{j=0}^{N} a_j(y)\varepsilon^j\Big)\Big| \leq C\varepsilon^{N+1}$$

holds for all small ε. In a similar way, we write $f(t, x_1, x') \sim \sum_{j=0}^{\infty} a_j(t, x')x_1^j$ and $f(t, x_1, x') = O(|x_1|^\infty)$.

Finally, for any nonnegative real functions f^ε and g^ε of $\varepsilon \in (0, 1)$, we write $f^\varepsilon \approx g^\varepsilon$ if for sufficiently small ε, there exist constants $0 < c < C < \infty$ so that

$$cf^\varepsilon \leq g^\varepsilon \leq Cf^\varepsilon.$$

2 Some preliminaries

2.1 Well-posedness

First, we need the following simple result (The proof is standard (*see* for example [11])).

Lemma 2.1 *Let Ω be a bounded domain with C^1 boundary. Then there is a constant $C > 0$ such that for any distribution $z \in \mathcal{D}'(\Omega)$ with $\nabla z \in (L^2(\Omega))^n$ and $\Delta z \in L^2(\Omega)$, it holds*

$$\Big|\frac{\partial z}{\partial \nu}\Big|_{H^{-1/2}(\Gamma)} \leq C\Big[|\nabla z|_{(L^2(\Omega))^n} + |\Delta z|_{L^2(\Omega)}\Big]. \tag{2.1}$$

Next, we prove the well-posedness of system (1.1) in H. For this purpose, we put $X = (y, z, z_t)$ and $X_0 = (y_0, z_0, z_1)$. We define an unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ by

$$\mathcal{A}Y = (\Delta Y_1, Y_3, \Delta Y_2), \tag{2.2}$$

where $Y = (Y_1, Y_2, Y_3) \in D(\mathcal{A})$, and

$$\begin{aligned} D(\mathcal{A}) = \Big\{(Y_1, Y_2, Y_3) \in H \;\Big|\; & (Y_1, Y_2) \in H^2(\Omega), \quad Y_3 \in H^1(\Omega_2), \quad \Delta Y_1 \in H^1(\Omega_1), \\ & \Delta Y_1|_{\Gamma_1} = Y_3|_{\Gamma_2} = 0 \text{ and } \Delta Y_1|_\gamma = Y_3|_\gamma\Big\}. \end{aligned} \tag{2.3}$$

Then, it is easy to see that system (1.1) can be re-written as an abstract Cauchy problem in H as follows:

$$X_t = \mathcal{A}X \text{ for } t > 0, \text{ and } X(0) = X_0. \tag{2.4}$$

We have the following result.

Theorem 2.1 *Denote the resolvent of $\mathcal{A}$ by $\rho(\mathcal{A})$. Then $0 \in \rho(\mathcal{A})$, $\mathcal{A}^{-1}$ is compact, and $\mathcal{A}$ is the generator of a contractive C_0-semigroup $\{S(t)\}_{t\geq 0}$ in H.*

We refer to Appendix A for the proof of Theorem 2.1.

Remark 2.1 *Although the semigroup $\{S(t)\}_{t\geq 0}$ is contractive, Theorem 4.1 in Section 4 will show that, whatever the geometric configuration is, the operator norm of $S(t)$ is equal to 1 for all $t \geq 0$. In other words, there is no uniform decay for the energy of solutions of (1.1).*

Remark 2.2 *$D(\mathcal{A})$ is a Hilbert space with its graph norm. Since $0 \in \rho(\mathcal{A})$, we may define an Hilbert space H_{-1} as the completion of H with respect to the norm $|\cdot|_{H_{-1}} = |\mathcal{A}^{-1}\cdot|_H$.*

As a consequence of Theorem 2.1, system (1.1) is well-posed in its finite energy space H.

Let us now analyze what happens if the initial data (y_0, z_0, z_1) of (1.1) belong to the larger Hilbert space H_{-1}. To see this, assume $X_0 = (y_0, z_0, z_1) \in H_{-1}$. We may solve system (2.4) as follows. Set

$$\widetilde{X}(t) = \mathcal{A}^{-1}X_0 + \int_0^t X(s)ds. \tag{2.5}$$

Then $\widetilde{X}(t)$ solves

$$\widetilde{X}_t = \mathcal{A}\widetilde{X} \text{ for } t > 0, \text{ and } \widetilde{X}(0) = \mathcal{A}^{-1}X_0 (\in H). \tag{2.6}$$

In view of Theorem 2.1, system (2.6) admits a unique mild solution $\widetilde{X} \in C([0,\infty); H)$, and

$$\begin{aligned} |X|_{L^\infty(0,\infty;H_{-1})} &= |\widetilde{X}_t|_{L^\infty(0,\infty;H_{-1})} = |\mathcal{A}\widetilde{X}|_{L^\infty(0,\infty;H_{-1})} \\ &= |\widetilde{X}|_{L^\infty(0,\infty;H)} \le |\mathcal{A}^{-1}X_0|_H = |X_0|_{H_{-1}}. \end{aligned} \tag{2.7}$$

Consequently, we have proved:

Theorem 2.2 *Let $(y_0, z_0, z_1) \in H_{-1}$. Then system (1.1) admits a unique solution in the class: $X = (y, z, z_t) \in C([0,\infty); H_{-1})$, and $|(y, z, z_t)|_{L^\infty(0,\infty;H_{-1})} \le |(y_0, z_0, z_1)|_{H_{-1}}$.*

2.2 Multiply reflected rays

We first need to recall the definition of rays.

Consider the hyperbolic operator on $\mathbb{R}^n$:

$$W = \partial_{tt} - \sum_{j,k=1}^{n} \alpha_{jk}(x)\partial_{x_j}\partial_{x_k}, \tag{2.8}$$

where $(\alpha_{ij})_{n\times n} \in C^2$ is strictly positive definite. Put

$$g(x,\xi) \stackrel{\triangle}{=} \sum_{j,k=1}^{n} \alpha_{jk}(x)\xi_j\xi_k, \qquad \xi = (\xi_1, \cdots, \xi_n). \tag{2.9}$$

A *null bicharacteristic* of W is defined to be a solution $(x(t), \xi(t))$ of the following (generally nonlinear) system of ordinary differential equations:

$$\begin{cases} \dot{x}(t) = \nabla_\xi g(x(t), \xi(t)), \\ \dot{\xi}(t) = -\nabla_x g(x(t), \xi(t)), \\ x(0) = x^0, \quad \xi(0) = \xi^0, \end{cases} \tag{2.10}$$

where the initial data (x^0, ξ^0) are chosen such that $g(x^0, \xi^0) = 1/4$. (Here, the choice of $1/4$ is only for convenience. Indeed, by scaling, one may replace it by any other positive real number). It is easy to check that

$$g(x(t), \xi(t)) = 1/4, \qquad \forall\, t \in \mathbb{R}. \tag{2.11}$$

The projection of the null bicharacteristic to the physical time-space, $(t, x(t))$, traces a curve in $\mathbb{R}^{1+n}$, which is called a *ray* of W. Sometimes, we also refer to $(t, x(t), \xi(t))$ as the ray. Obviously, for any operator W with constant coefficients, its rays are straight lines.

In the presence of boundaries, rays, when reaching the boundary are reflected following the usual rules of Geometric Optics. Along this paper we will consider rays in the wave domain Ω_2. Since Ω_2 is obtained from the global domain Ω by cutting it off by means of a $n-1$-dimensional manifold γ, the domain Ω_2 is not necessarily smooth but only piecewise smooth. In particular, the boundary of Ω_2 in $2-d$ could have some corners or cusps. All along this paper we will work with rays in Ω_2 that never meet the boundary at those exceptional points. In view of this, we consider a bounded domain $M \subset \mathbb{R}^n$ with piecewise C^1 boundary ∂M, the singular set being localized on a closed (topological) sub-manifold S with $\dim S \le n-2$. We now introduce the following definition of multiply reflected ray.

Definition 2.1 *A continuous parametric curve:* $[0,T] \ni t \mapsto (t, x(t), \xi(t)) \in C([0,T] \times \overline{M} \times \mathbb{R}^n)$*, with a given* $T > 0$*,* $x(0) \in M$ *and* $x(T) \in M$*, is called a multiply reflected ray for the operator* $\Box$ *in* $[0,T] \times \overline{M}$ *if there exist* $m \in \mathbb{N}$*,* $0 < t_0 < t_1 < \cdots < t_m = T$ *such that each* $(t, x(t), \xi(t))|_{t_i < t < t_{i+1}}$ *is a straight ray for* $\Box$ *(*$i = 0, 1, 2, \cdots, m-1$*), which arrives at* $\partial M \setminus S$ *at time* $t = t_{i+1}$*, and is reflected by* $(t, x(t), \xi(t))|_{t_{i+1} < t < t_{i+2}}$ *by the law of Geometric Optics whenever* $i < m - 1$*.*

We show the following geometric lemma.

Lemma 2.2 *Assume that for each* $T > 0$*, there is a multiply reflected ray for the wave operator* $\Box$ *in* $[0,T] \times \overline{M}$*. Then by a small perturbation, one can find a new multiply reflected ray which always meets* $\partial M \setminus S$ *transversally and non-normally.*

Remark 2.3 *i) The above result holds also for more general hyperbolic operators as* W*. ii) We give below a sketch of the proof of Lemma 2.2 following [7, Vol: III, Section 24.3, p. 441].*

Proof of Lemma 2.2. Recall that, for a given time duration $[0,T]$, the multiply reflected rays for $\Box$ in M are finite ordered sequences of line segments in $\overline{M}$, reflected one by one on $\partial M \setminus S$, and contained in M except the reflected points $s_1, \cdots, s_{n_0}$. Take any such a ray ℓ and assume the direction of its first segment to be $\mathbf{v}$. The reflected points of ℓ is divided into two subsets B_1 and B_2. The first one, B_1, is constituted by those for which the ray ℓ meets the boundary $\partial M \setminus S$ transversally and non-normally. The second one, B_2, is constituted by the reminding points that will be referred as exceptional reflected points. If $B_2 = \emptyset$, then ℓ is exactly what we need. Otherwise, let $B_2 = \{s_{i_1}, \ldots, s_{i_m}\}$ ($m \leq n_0$). Then, by continuity, we may make a very small perturbation on the initial direction $\mathbf{v}$ so that for the new ray, the original reflected points of type B_1 remains to be of the same type, but the first exceptional reflected point s_{i_1} becomes of type B_1. Repeating this procedure, one may remove all the exceptional reflected points. This completes the proof of Lemma 2.2. $\Box$

In view of Lemma 2.2 and Remark 2.3, it is reasonable to introduce the following geometric assumption on Ω_2, which is needed to develop our Geometric Optics analysis.

Assumption 2.1 *Assume that for each* $T > 0$*, there is a multiply reflected ray:* $[0,T] \ni t \mapsto (t, x(t), \xi(t)) \in C([0,T] \times \overline{\Omega_2} \times \mathbb{R}^n)$ *for the wave operator* $\Box$ *in* $[0,T] \times \overline{\Omega_2}$ *which meets* $\partial\Omega_2 \setminus (\overline{\Gamma_2} \cap \overline{\gamma})$ *transversally and non-normally, and* $\partial\Omega_2$ *is of* C^2 *(resp.* C^3*) in some neighborhood of every reflected point in* Γ_2 *(resp.* γ*).*

Remark 2.4 *By Lemma 2.2, ones sees that Assumption 2.1 holds in the following particular cases: 1)* Ω *is of class* C^3*. This is the case, for instance, when* Ω *and* Ω_1 *are of class* C^3 *and* Ω_1 *is included in* Ω*; 2)* Ω_2 *is a piecewise* C^3 *curved polyhedron without cusps. Assumption 2.1 holds as well for piecewise* C^2 *domains in two space dimensions, i.e.,* $n = 2$*. This was pointed to us by I. Mundet. His argument uses the properties of the symplectic form induced by the billiards flow and Poincaré's recurrence theorem (see [18, Lemma 1.7.1]). Whether the same is true for* $n > 2$ *in the presence of cusps is an open problem.*

3 WKB asymptotic analysis for the transmission problem with flat interface

This section is devoted to an heuristic exposition of the key ideas leading to the construction of ray-like solutions for system (1.1).

We consider the case of a flat interface γ, say $\gamma = \{x_1 = 0\}$, Ω being the whole space $\mathbb{R}^n$. In this case, system (1.1) may be written as follows:

$$\begin{cases} y_t - \Delta y = 0 & \text{in } (0,\infty)\times\{x_1<0\}, \\ \Box z = 0 & \text{in } (0,\infty)\times\{x_1>0\}, \\ y = z, \quad y_{x_1} = z_{x_1} & \text{on } (0,\infty)\times\{x_1=0\}. \end{cases} \tag{3.1}$$

The main problem is to describe the behavior of ray-like solutions of the wave equation when reaching the interface. As we will see later, the answer is that the solutions are reflected with reflection coefficient $r = e^{2\theta i}$, where $\theta \in [0, \pi/2)$ is the angle of incidence. Note that the reflection coefficient is of modulus one and consequently, only a negligible high frequency wave enters the heat domain.

Throughout this section, $\tau \in \mathbb{R}\setminus\{0\}$ and $\xi = (\xi_1, \xi') \in \mathbb{R}^n$ with $\xi_1 \neq 0$ are given and are assumed to satisfy the condition

$$\tau^2 - |\xi|^2 = 0, \quad \text{i.e.,} \quad \tau = \pm|\xi|. \tag{3.2}$$

3.1 WKB expansion for the wave equation

We begin by seeking approximate solutions for the wave equation $\Box z = 0$, with an ansatz of WKB type with linear phase

$$z^\varepsilon(t,x) = e^{i(\tau t+\xi\cdot x)/\varepsilon} a^\varepsilon(t,x), \qquad a^\varepsilon(t,x) \sim \sum_{j=0}^{\infty} \varepsilon^j a_j(t,x), \tag{3.3}$$

where the functions a_j $(j = 1, 2, \cdots)$ will be determined later.

Computing $\Box z^\varepsilon$ and setting its $O(1/\varepsilon^2)$ term equal to zero yields the eikonal equation (3.2). By (3.2), one has

$$\Box z^\varepsilon = \varepsilon^{-1} e^{i(\tau t+\xi\cdot x)/\varepsilon}\Big[2i(\tau\partial_t - \xi\cdot\partial_x) + \varepsilon\Box\Big]a^\varepsilon.$$

Define

$$\mathbf{v} \equiv (v_1, v_2, \cdots, v_n) \triangleq -\frac{\xi}{\tau}.$$

Then $\Box z^\varepsilon = O(\varepsilon^\infty)$ is equivalent to

$$\Big[\partial_t + \mathbf{v}\cdot\partial_x + \frac{\varepsilon}{2\tau i}\Box\Big]\sum_{j=0}^{\infty}\varepsilon^j a_j = O(\varepsilon^\infty). \tag{3.4}$$

Setting the leading order term of the left hand side of (3.4) equal to zero yields

$$(\partial_t + \mathbf{v} \cdot \partial_x) a_0 = 0. \tag{3.5}$$

This is a transport equation. It is easy to check that solutions $a_0 \triangleq a_0(t,x) \in C^1(\mathbb{R}^{1+n})$ of (3.5) are of the form: $a_0(t,x) = g_0(x - \mathbf{v}t)$, with $g_0 \triangleq g_0(x) \in C^1(\mathbb{R}^n)$.

We now choose a function $g_0(x) = a_0(0,x) \in C^\infty(\mathbb{R}^n)$ with support in a neighborhood of a point $x_0 \in \mathbb{R}^n$. Then a_0 is, at any $t > 0$, concentrated near the point $x = x_0 + \mathbf{v}t$. Therefore, to leading order,

$$z^\varepsilon(t,x) \sim e^{i(\tau t + \xi \cdot x)/\varepsilon} a_0(t,x) = e^{i(\tau t + \xi \cdot x)/\varepsilon} g_0(x - \mathbf{v}t),$$

and we see that z^ε looks like a localized function transported along the ray $x = x_0 + \mathbf{v}t$, with an oscillating factor $e^{i(\tau t + \xi \cdot x)/\varepsilon}$.

Continuing in this fashion, the profiles a_j of the higher order terms are uniquely determined from their initial data $a_j|_{t=0} = g_j \in C_0^\infty(\mathbb{R}^n)$ by solving recursively the transport equations

$$\Big(\partial_t + \mathbf{v} \cdot \partial_x\Big) a_j + \frac{\Box a_{j-1}}{2\tau i} = 0, \qquad j = 1, 2, \cdots. \tag{3.6}$$

Assume that g_j have support in a compact set independent of j. Then, the profiles a_j are supported in a tube of rays (i.e., characteristic curves of $\partial_t + \mathbf{v} \cdot \partial_x$). Indeed, one may check directly that $a_j(t,x) = g_j(x - \mathbf{v}t) + it(2\tau)^{-1} \Box a_{j-1}(x - \mathbf{v}t)$ is the unique solution of (3.6) such that $a_j|_{t=0} = g_j$.

Remark 3.1 *Instead of the Cauchy problem (3.5) and (3.6) with initial condition imposed at $t = 0$, one may consider the same problem but with initial condition imposed at the interface $x_1 = 0$:*

$$\begin{cases} (\partial_t + \mathbf{v} \cdot \partial_x) a_0 = 0, \\ \Big(\partial_t + \mathbf{v} \cdot \partial_x\Big) a_j + \dfrac{\Box a_{j-1}}{2\tau i} = 0, \\ a_0(t, 0, x') = a_0^0(t, x'), \qquad a_j(t, 0, x') = a_j^0(t, x'), \end{cases} \tag{3.7}$$

for any given functions a_0^0 and a_j^0 ($j = 1, 2, \cdots$) (because the first component v_1 of $\mathbf{v}$ is assumed not to vanish). Similar ray-like solutions can be constructed for system (3.7).

The classical Borel's Theorem (e.g., pp. 16 in [7]) allows one to choose a C^∞−smooth function $a^\varepsilon(t,x)$, which is supported in the above mentioned tube of rays and has the expansion at $\varepsilon = 0$:

$$a^\varepsilon(t,x) \sim \sum_{j=0}^{\infty} \varepsilon^j a_j(t,x).$$

Consequently, we conclude that

Theorem 3.1 *The function*

$$z^\varepsilon \triangleq e^{i(\tau t+\xi\cdot x)/\varepsilon} a^\varepsilon(t,x) \tag{3.8}$$

constructed above is C^∞-smooth and it is an infinitely accurate solution of the wave equation in the sense that $\Box z^\varepsilon = O(\varepsilon^\infty)$ in $\mathbb{R}_t \times \mathbb{R}^n_x$.

Remark 3.2 *The ansatz (3.3) is not unique. Instead, for example, one may also use a different one:*

$$z^\varepsilon(t,x) = e^{i(\tau t+\xi\cdot x)/\varepsilon} a^\varepsilon(t,x), \qquad a^\varepsilon(t,x) \sim \sum_{j=0}^{\infty} \varepsilon^{j/2} a_j(t,x). \tag{3.9}$$

This ansatz will be essentially used in Subsection 3.5 for constructing the reflected rays in the case of normal incidence. Note that (3.3) is a special case of (3.9) in which the odd terms have been chosen to be identically zero. In the present case, one needs to change a little bit the transport equations for $a_0, a_1, a_2, \cdots$, which now read as follows:

$$\begin{cases} \Big(\partial_t + \mathbf{v}\cdot\partial_x\Big) a_0 = 0, \\ \Big(\partial_t + \mathbf{v}\cdot\partial_x\Big) a_{2k-1} = 0, \qquad k = 1,2,\cdots, \\ \Big(\partial_t + \mathbf{v}\cdot\partial_x\Big) a_{2j} + \dfrac{\Box a_{2j-2}}{2\,\tau\, i} = 0, \qquad j = 1,2,\cdots. \end{cases} \tag{3.10}$$

Clearly, system (3.10) is uniquely solvable with initial conditions imposed either at $t = 0$ or at $x_1 = 0$. Moreover, z^ε constructed in this way satisfies $\Box z^\varepsilon = O(\varepsilon^\infty)$, too.

We now need to address the question of the behavior of these localized waves when they reach the interface. More precisely, choose τ, ξ with $\xi_1 > 0$ so that

$$\frac{-\xi_1}{\tau} = v_1 < 0 \qquad \text{and} \qquad \tau^2 > \xi_2^2 + \cdots \xi_n^2.$$

Then the asymptotic solutions constructed in Theorem 3.1 move towards the left on the x_1 direction within the wave domain $x_1 > 0$. Taking initial data $a^\varepsilon(0,x)$ compactly supported in $\{x_1 > 0\}$, z^ε represents a wave which starts on the wave equation side $x_1 > 0$ and approaches the interface $x_1 = 0$. The problem is to describe the behavior of the solution of the transmission problem (3.1) after the wave reaches the interface $x_1 = 0$.

The traces of solutions as in (3.8) are rapidly oscillating on the boundary $x_1 = 0$. The key step is to find solutions of the heat equation in $x_1 < 0$ which oscillate on $x_1 = 0$, too. These solutions will be "matched" to produce a solution of the whole system (3.1), in which the wave component is the sum of an incoming wave and a reflected one.

3.2 WKB expansion for the heat equation, non-normal incidence

We now construct infinitely accurate approximate solutions of the heat equation in the region $x_1 < 0$ whose trace in $x_1 = 0$ oscillates rapidly,

$$y^\varepsilon(t,0,x') = e^{i(\tau t+\xi'\cdot x')/\varepsilon} f^\varepsilon(t,x'), \qquad f^\varepsilon(t,x') \sim \sum_{j=0}^{\infty} \varepsilon^j f_j(t,x'), \tag{3.11}$$

where $\xi' = (\xi_2, \cdots, \xi_n) \in \mathbb{R}^{n-1}$ is assumed to be nonzero. This is the case where incidence is non-normal. The exceptional case $\xi' = 0$ corresponding to normal incidence will be discussed in the next section.

The coefficients f_j are assumed to be smooth and vanish outside a fixed compact set in $\{x_1 = 0\}$. The same is true for the functions f^ε for all ε.

Note that in (3.11) the oscillations are equally in the variables t and in x'. This does not follow the natural heat equation scaling. Consequently, one has to look at the heat equation with the attitude that all the variables are on an equal footing. Hence, the term y_t^ε is a lower order term. It will not intervene in the determination of the phase.

The WKB ansatz with linear phase for for the heat equation $(\partial_t - \Delta)y = 0$ in $x_1 < 0$ is then

$$y^\varepsilon \sim e^{i(\tau t + x\cdot\hat{\xi})/\varepsilon} \sum_{j=0}^{\infty} \varepsilon^j A_j, \tag{3.12}$$

where $\hat{\xi} = (\hat{\xi}_1, \xi')$, $\hat{\xi}_1$ will be determined later; while τ and ξ' are the same as before. Injecting it in the heat equation $(\partial_t - \Delta)y^\varepsilon = O(\varepsilon^\infty)$, the $1/\varepsilon^2$ terms yield the eikonal equation

$$\hat{\xi}_1^2 + \xi_2^2 + \cdots + \xi_n^2 = 0, \qquad \hat{\xi}_1^2 = -|\xi'|^2.$$

The requirement of boundedness in $x_1 < 0$ yields $\operatorname{Im}\hat{\xi}_1 < 0$. Hence we choose

$$\hat{\xi}_1 = -i|\xi'|, \tag{3.13}$$

and the *ansatz* (3.12) becomes

$$y^\varepsilon \sim e^{|\xi'|x_1/\varepsilon} e^{i(\tau t + x'\cdot\xi')/\varepsilon} \sum_{j=0}^{\infty} \varepsilon^j A_j. \tag{3.14}$$

Compute

$$\Big(\partial_t - \Delta\Big) y^\varepsilon \sim \frac{1}{\varepsilon}\, e^{|\xi'|x_1/\varepsilon} e^{i(\tau t + x'\cdot\xi')/\varepsilon} \sum_{j=0}^{\infty} \varepsilon^j B_j, \tag{3.15}$$

where B_j $(j = 1, 2, \cdots)$ will be given later. Note that the $1/\varepsilon^2$ term in (3.15) is absent since we choose the phase to satisfy the eikonal equation.

The leading term of (3.15) is

$$B_0 = \Big(-2|\xi'|\partial_{x_1} - 2i\xi'\cdot\partial_{x'} + i\tau\Big) A_0. \tag{3.16}$$

From then on,

$$B_j = \Big(-2|\xi'|\partial_{x_1} - 2i\xi'\cdot\partial_{x'} + i\tau\Big) A_j + (\partial_t - \Delta)A_{j-1}, \qquad \text{for} \quad j \geq 1. \tag{3.17}$$

This leads to an initial value problem for A_0

$$\begin{cases} \Big(-2|\xi'|\partial_{x_1} - 2i\xi'\cdot\partial_{x'} + i\tau\Big) A_0 = 0, \\ A_0(t, 0, x') = f_0(t, x'). \end{cases} \tag{3.18}$$

Because of the complex coefficient this is an ill posed problem. Thanks to the factor $e^{|\xi'|x_1/\varepsilon}$ in (3.14), one is only interested in the region $x_1 = O(\varepsilon)$ and it suffices to solve the initial value problem (3.18) to infinite order at $x_1 = 0$.

To this end, one uses (3.18) to determine the smooth functions $\partial_{x_1}^j A_0(t, 0, x')$ for all $j \geq 0$. Then we may choose a smooth function $A_0(t, x_1, x')$ which vanishes for t, x' outside the support of f^ε and which has the expansion

$$A_0(t, x_1, x') \sim \sum_{k=0}^{\infty} \frac{\partial_{x_1}^k A_0(t, 0, x')}{k!} x_1^k = \sum_{k=0}^{\infty} \frac{x_1^k}{k!} \left(\frac{i\tau - 2i\xi' \cdot \partial_{x'}}{2|\xi'|} \right)^k f_0(t, x').$$

Then A_0 satisfies the initial condition in (3.18) and for the transport equation one has

$$\Big(- 2|\xi'|\partial_{x_1} - 2i\xi' \cdot \partial_{x'} + i\tau \Big) A_0 = 2|\xi'| \left[-\partial_{x_1} A_0 + \left(\frac{i\tau - 2i\xi' \cdot \partial_{x'}}{2|\xi'|} \right) A_0 \right] = O(|x_1|^\infty). \quad (3.19)$$

Similarly, by induction, we may choose $A_j(t, x)$ so that they vanish for t, x' outside the union of the supports of $f_0, f_1, \cdots, f_j$ for all $j \geq 1$, and

$$\begin{cases} \Big(- 2|\xi'|\partial_{x_1} - 2i\xi' \cdot \partial_{x'} + i\tau \Big) A_j + (\partial_t - \Delta) A_{j-1} = O(|x_1|^\infty), \\ A_j(t, 0, x') = f_j(t, x'). \end{cases} \quad (3.20)$$

Finally, assuming that f_j have support in a compact set independent of j, one may choose a smooth function $\phi^\varepsilon = \phi^\varepsilon(t, x)$ with the same support property and

$$\phi^\varepsilon(t, x) \sim \sum_{j=0}^{\infty} \varepsilon^j A_j(t, x). \quad (3.21)$$

Define an approximate solution for the heat equation $y_t - \Delta y = 0$ in $\{x_1 < 0\}$ by

$$y^\varepsilon = e^{|\xi'|x_1/\varepsilon} e^{i(\tau t + \xi' \cdot x')/\varepsilon} \phi^\varepsilon(t, x). \quad (3.22)$$

Then we have the following result.

Theorem 3.2 *The function y^ε constructed in (3.22) satisfies*

$$x_1^\beta \Big(\partial_t - \Delta \Big) y^\varepsilon = O(\varepsilon^\infty) \quad \text{in } \{x_1 < 0\}, \qquad \forall\, \beta \in \mathbb{N};$$
$$y^\varepsilon(t, 0, x') \sim f^\varepsilon(t, x') e^{i(\tau t + \xi' \cdot x')/\varepsilon}.$$

3.3 WKB expansion for the heat equation, normal incidence

This subsection analyzes the case $\xi' = 0$. In this case, we assume the incoming wave to be of the form (3.9) and then the boundary condition (3.11) at $x_1 = 0$ reads

$$y^\varepsilon(t, 0, x') = f^\varepsilon(t, x') e^{i\tau t/\varepsilon}, \qquad f^\varepsilon(t, x') \sim \sum_{j=0}^{\infty} \varepsilon^{j/2} f_j(t, x'), \quad (3.23)$$

where $\tau \in \mathbb{R}$ is the same as before, i.e., given by (3.2) with $\xi = (\xi_1, 0)$. Here, we choose the power $\varepsilon^{j/2}$ instead of ε^j to match the solutions of the heat equation with the alternative ansatz in Remark 3.2 for that of the wave one.

In this case the oscillations of the data are in a direction which is characteristic for the heat operator. It is no longer true that the oscillations in x are dominant and there is an even competition between spacial and temporal oscillations with the classical scaling for the heat equation. The WKB ansatz with linear phase for the heat equation $(\partial_t - \Delta)y = 0$ in $x_1 < 0$ is now:

$$y^\varepsilon \sim e^{i(\tau t/\varepsilon + \eta_1 x_1/\sqrt{\varepsilon})} \sum_{j=0}^{\infty} \varepsilon^{j/2} B_j, \tag{3.24}$$

where η_1 will be determined later. To avoid the square roots let $\varepsilon \triangleq \mu^2$. The ansatz (3.24) becomes

$$y^\mu \sim e^{i(\tau t/\mu^2 + \eta_1 x_1/\mu)} \sum_{j=0}^{\infty} \mu^j B_j. \tag{3.25}$$

The leading order, $O(1/\mu^2)$ term, in the expression obtained when applying the heat operator to y^μ, yields the eikonal equation

$$i\tau + \eta_1^2 = 0, \qquad \mathrm{Re}\, \eta_1 > 0, \tag{3.26}$$

which uniquely determines η_1 from τ.

Using the ansatz (3.25) and the eikonal equation (3.26), one finds

$$(\partial_t - \Delta)y^\mu \sim \mu^{-1} e^{i(\tau t/\mu^2 + \eta_1 x_1/\mu)} \Big[-2i\eta_1 \partial_{x_1} + \mu(\partial_t - \Delta) \Big] \sum_{j=0}^{\infty} \mu^j B_j. \tag{3.27}$$

The $O(\mu^{-1})$ term in the right hand side of (3.27) yields the equations

$$\partial_{x_1} B_0 = 0, \qquad B_0|_{x_1=0} = f_0.$$

Thus

$$B_0(t, x_1, x') = f_0(t, x'). \tag{3.28}$$

Similarly, the order $O(\mu^0)$ term in the right hand side of (3.27) yields

$$-2i\eta_1 \partial_{x_1} B_1 + (\partial_t - \Delta) B_0 = 0, \qquad B_1|_{x_1=0} = f_1. \tag{3.29}$$

This determines B_1 uniquely in an obvious way.

Continuing this argument, we may find smooth functions B_j, which vanish for t, x' outside the support of f^ε, and are uniquely determined from their initial values $B_j(t, 0, x') = f_j(t, x')$. Choose

$$\psi^\mu(t, x) \sim \sum_{j=0}^{\infty} \mu^j B_j(t, x), \qquad \text{for} \quad \mu \to 0. \tag{3.30}$$

The approximate solution for the heat equation $(\partial_t - \Delta)y = 0$ in $\{x_1 < 0\}$ is now chosen as

$$y^\mu \triangleq e^{\eta_1 x_1/\mu} e^{i\tau t/\mu^2} \psi^\mu(t, x). \tag{3.31}$$

Then we have the following result.

Theorem 3.3 *The function y^μ constructed in (3.31) satisfies*

$$x_1^\beta\Big(\partial_t - \Delta\Big)y^\mu = O(\mu^\infty) \text{ in } \{x_1 < 0\}, \qquad \forall\, \beta \in \mathbb{N};$$
$$y^\mu(t,0,x') \sim f^\mu(t,x')e^{i\tau t/\mu^2}.$$

Remark 3.3 *Note that the qualitative behavior is different for the cases of normal incidence and non-normal incidence. The "skin thickness" at $x_1 = 0$ is $O(\mu) = O(\sqrt{\varepsilon})$ in the case of normal incidence; while in the case of non-normal incidence, the "thickness" was $O(\varepsilon)$. This is in agreement with the intuition that the normal incident wave should penetrate more into the heat domain.*

3.4 Derivation of the reflection law, non-normal incidence

Let $e^{i(\tau t+\xi\cdot x)/\varepsilon}\sum_{j=0}^{\infty}\varepsilon^j a_j$ be the *incoming wave*, with $|\tau| > |\xi'| > 0$, or equivalently, $\xi_1 \neq 0$. Put $\tilde{\xi} = (-\xi_1, \xi')$ and $\tilde{\mathbf{v}} \triangleq -\tilde{\xi}/|\xi|$. There are two linear phases, $e^{i(\tau t+\xi\cdot x)/\varepsilon}$ and $e^{i(\tau t+\tilde{\xi}\cdot x)/\varepsilon}$, which yield the same oscillatory factor $e^{i(\tau t+\xi'\cdot x')/\varepsilon}$ when restricted to the interface $x_1 = 0$. We now seek a solution to the transmission problem (3.1) which in $x_1 > 0$ is a solution z^ε of the wave equation of the form

$$z^\varepsilon(t,x) = e^{i(\tau t+\xi\cdot x)/\varepsilon}\sum_{j=0}^{\infty}\varepsilon^j a_j + e^{i(\tau t+\tilde{\xi}\cdot x)/\varepsilon}\sum_{j=0}^{\infty}\varepsilon^j b_j. \tag{3.32}$$

The second component in the right hand side of (3.32) is refereed to as the *outgoing wave*. The strategy is to glue this to an approximate solution y^ε of the heat equation given by (3.22).

Put $a_j^0 \equiv a_j^0(t,x') \triangleq a_j(t,0,x')$ $(j = 0,1,2,\cdots)$, which are determined by the incoming wave. We now need to determine all f_j entering in (3.11) for the construction of the heat solution, and $b_j^0 \equiv b_j^0(t,x') \triangleq b_j(t,0,x')$ corresponding to the outgoing wave. The transmission condition at the interface $x_1 = 0$ is

$$y^\varepsilon(t,0,x') = z^\varepsilon(t,0,x'), \qquad \partial_{x_1}y^\varepsilon(t,0,x') = \partial_{x_1}z^\varepsilon(t,0,x'). \tag{3.33}$$

Clearly, the first condition in (3.33) holds if and only if $f^\varepsilon \sim a^\varepsilon + b^\varepsilon$, which is equivalent to

$$f_j = a_j^0 + b_j^0, \qquad \forall\, j = 0,1,2,\cdots. \tag{3.34}$$

On the other hand, from (3.32), we see that

$$\partial_{x_1}z^\varepsilon(t,x_1,x') \sim \frac{1}{\varepsilon}\Bigg[e^{i(\tau t+\xi\cdot x)/\varepsilon}\left(i\xi_1 a_0 + \sum_{j=1}^{\infty}\varepsilon^j(i\xi_1 a_j + \partial_{x_1}a_{j-1})\right)$$
$$+e^{i(\tau t+\tilde{\xi}\cdot x)/\varepsilon}\left(-i\xi_1 b_0 + \sum_{j=1}^{\infty}\varepsilon^j(-i\xi_1 b_j + \partial_{x_1}b_{j-1})\right)\Bigg].$$

Similarly from (3.21) and (3.22), we have

$$\begin{aligned}\partial_{x_1} y^\varepsilon(t,x_1,x') &\sim \frac{1}{\varepsilon} e^{|\xi'|x_1/\varepsilon} e^{i(\tau t+\xi'\cdot x')/\varepsilon}\left(|\xi'|\phi^\varepsilon + \varepsilon\partial_{x_1}\phi^\varepsilon\right)\\ &\sim \frac{1}{\varepsilon} e^{|\xi'|x_1/\varepsilon} e^{i(\tau t+\xi'\cdot x')/\varepsilon}\left[|\xi'|A_0 + \sum_{j=1}^\infty \varepsilon^j\left(|\xi'|A_j + \partial_{x_1}A_{j-1}\right)\right].\end{aligned}$$

Therefore the second condition in (3.33) holds if and only if at $x_1 = 0$

$$\begin{aligned}&|\xi'|f_0 + \sum_{j=1}^\infty \varepsilon^j\Big(|\xi'|f_j + \partial_{x_1}A_{j-1}(t,0,x')\Big)\\ &\sim i\xi_1(a_0^0 - b_0^0) + \sum_{j=1}^\infty \varepsilon^j\Big[i\xi_1(a_j^0 - b_j^0) + \partial_{x_1}a_{j-1}(t,0,x') + \partial_{x_1}b_{j-1}(t,0,x')\Big].\end{aligned} \tag{3.35}$$

Equation (3.35) is equivalent to

$$\begin{cases} |\xi'|f_0 = i\xi_1(a_0^0 - b_0^0),\\ |\xi'|f_j + \partial_{x_1}A_{j-1}(t,0,x') = i\xi_1(a_j^0 - b_j^0) + \partial_{x_1}a_{j-1}(t,0,x') + \partial_{x_1}b_{j-1}(t,0,x'),\\ \qquad \forall\, j = 1,2,\cdots. \end{cases} \tag{3.36}$$

Note that from (3.18) and (3.20), one may express $\partial_{x_1}A_{j-1}(t,0,x')$ in terms of $f_0,\cdots,f_{j-1}$. Also, by Remark 3.1, $\partial_{x_1}b_{j-1}(t,0,x')$ can be expressed in terms of $b_0^0,\cdots,b_{j-1}^0$. Consequently, by induction, equations (3.34) and (3.36) uniquely determine all b_j^0 and f_j in terms of the incoming coefficients $a_0^0,\cdots,a_j^0$. This gives an infinitely accurate solution of the transmission problem corresponding to the incoming wave of geometric optics type.

Let us now analyze some of the qualitative properties of the solutions constructed as above. Recall that

$$(\partial_t + \mathbf{v}\cdot\partial_x)a_0 = 0, \quad \text{and} \quad (\partial_t + \tilde{\mathbf{v}}\cdot\partial_x)b_0 = 0.$$

The condition $\xi_1 > 0$ guarantees that $v_1 < 0$ and the a_0 wave moves towards $\{x_1 = 0\}$ with velocity $\mathbf{v}$ while the b_0 wave moves away with velocity $\tilde{\mathbf{v}}$. *The former is called the incoming wave and the latter is the reflected one.* The angle of incidence $\theta \in [0,\pi/2)$ and reflection coefficient r are defined by

$$\tan\theta = |\xi'|/\xi_1, \qquad r \triangleq b_0^0/a_0^0. \tag{3.37}$$

The leading order transmission conditions in (3.34) and (3.36) reads

$$f_0 = a_0^0 + b_0^0 \quad \text{and} \quad |\xi'|f_0 = i\xi_1(a_0^0 - b_0^0). \tag{3.38}$$

Hence,

$$b_0^0 = \frac{i\xi_1 - |\xi'|}{i\xi_1 + |\xi'|}a_0^0, \qquad f_0 = \frac{2i\xi_1}{i\xi_1 + |\xi'|}a_0^0.$$

From (3.37) and (3.38), we find

$$\tan\theta = \frac{|\xi'|}{\xi_1} = \frac{ia_0^0 - ib_0^0}{a_0^0 + b_0^0} = \frac{i(1-r)}{1+r}. \tag{3.39}$$

Solving (3.39) for r yields

$$r = \frac{1 + i\tan\theta}{1 - i\tan\theta} = \frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta} = e^{2\theta i}. \tag{3.40}$$

In particular, from (3.40), we see that for incidence near normal, i.e., θ close to 0, the reflection coefficient r is close to 1.

3.5 Derivation of the reflection law, normal incidence

Assume now $e^{i(\tau t+\xi_1 x_1)/\varepsilon}\sum_{j=0}^{\infty}\varepsilon^j a_j$ to be the incoming wave. This is the case for which $\xi' = 0$. We need to construct the reflected wave in $x_1 > 0$ and the approximate solution for the heat equation in $x_1 < 0$.

The heat equation solutions y^ε constructed by (3.30)–(3.31) satisfies at $x_1 = 0$ (recall $\varepsilon = \mu^2$)

$$y^\varepsilon(t,0,x') \sim e^{i\tau t/\varepsilon}\sum_{j=0}^{\infty}\varepsilon^{j/2}f_j(t,x') = e^{i\tau t/\varepsilon}\left(\sum_{k=0}^{\infty}\varepsilon^k f_{2k}(t,x') + \sum_{j=0}^{\infty}\varepsilon^{j+1/2}f_{2j+1}(t,x')\right) \tag{3.41}$$

and

$$\begin{aligned}\partial_{x_1}y^\varepsilon(t,0,x') &\sim \frac{1}{\sqrt{\varepsilon}}e^{i\tau t/\varepsilon}\left\{\eta_1 f_0 + \sum_{j=1}^{\infty}\varepsilon^{j/2}\Big[\eta_1 f_j + \partial_{x_1}B_{j-1}(t,0,x')\Big]\right\} \\ &= \frac{1}{\sqrt{\varepsilon}}e^{i\tau t/\varepsilon}\left\{\eta_1 f_0 + \sum_{k=1}^{\infty}\varepsilon^k\Big[\eta_1 f_{2k} + \partial_{x_1}B_{2k-1}(t,0,x')\Big]\right. \\ &\qquad \left. + \sum_{j=1}^{\infty}\varepsilon^{j-1/2}\Big[\eta_1 f_{2j-1} + \partial_{x_1}B_{2j-2}(t,0,x')\Big]\right\},\end{aligned} \tag{3.42}$$

where η_1 is determined from τ by (3.26). In order to match the $\varepsilon^{j-1/2}$-terms in (3.42) with those of the wave equation, we modify the reflected wave by seeking a solution z^ε to the wave equation in $x_1 > 0$ of the following form

$$z^\varepsilon(t,x) = e^{i(\tau t+\xi_1 x_1)/\varepsilon}\sum_{j=0}^{\infty}\varepsilon^j a_j(t,x) + e^{i(\tau t-\xi_1 x_1)/\varepsilon}\sum_{j=0}^{\infty}\varepsilon^{j/2}b_j(t,x). \tag{3.43}$$

In view of Remark 3.2, this modification is compatible with z^ε being solutions of the wave equation. As before, we set $a_j^0 \equiv a_j^0(t,x') \triangleq a_j(t,0,x')$ and $b_j^0 \equiv b_j^0(t,x') \triangleq b_j(t,0,x')$. Then, from (3.43), it is easy to check that

$$z^\varepsilon(t,0,x') = e^{i\tau t/\varepsilon}\sum_{j=0}^{\infty}\left(\varepsilon^j a_j^0 + \varepsilon^{j/2}b_j^0\right) = e^{i\tau t/\varepsilon}\left[\sum_{k=0}^{\infty}\varepsilon^k(a_k^0 + b_{2k}^0) + \sum_{j=0}^{\infty}\varepsilon^{j+1/2}b_{2j+1}^0\right] \tag{3.44}$$

and

$$
\begin{aligned}
&\partial_{x_1} z^\varepsilon(t,0,x') \\
&= \frac{1}{\sqrt{\varepsilon}} e^{i\tau t/\varepsilon} \left\{ \frac{i\xi_1}{\sqrt{\varepsilon}}(a_0^0 - b_0^0) - i\xi_1 b_1^0 + \sum_{j=1}^{\infty} \varepsilon^{j-1/2} \Big[i\xi_1 a_j^0 + \partial_{x_1} a_{j-1}(t,0,x') \Big] \right. \\
&\qquad \left. + \sum_{j=1}^{\infty} \varepsilon^{j/2} \Big[- i\xi_1 b_{j+1}^0 + \partial_{x_1} b_{j-1}(t,0,x') \Big] \right\} \\
&= \frac{1}{\sqrt{\varepsilon}} e^{i\tau t/\varepsilon} \left\{ \frac{i\xi_1}{\sqrt{\varepsilon}}(a_0^0 - b_0^0) - i\xi_1 b_1^0 + \sum_{k=1}^{\infty} \varepsilon^{k} \Big[- i\xi_1 b_{2k+1}^0 + \partial_{x_1} b_{2k-1}(t,0,x') \Big] \right. \\
&\qquad \left. + \sum_{j=1}^{\infty} \varepsilon^{j-1/2} \Big[i\xi_1 (a_j^0 - b_{2j}^0) + \partial_{x_1} a_{j-1}(t,0,x') + \partial_{x_1} b_{2j-2}(t,0,x') \Big] \right\}.
\end{aligned}
\tag{3.45}
$$

Now, from (3.41), (3.42), (3.44) and (3.45), we conclude that the transmission conditions at the interface $x_1 = 0$, i.e., $y^\varepsilon(t,0,x') = z^\varepsilon(t,0,x')$ and $\partial_{x_1} y^\varepsilon(t,0,x') = \partial_{x_1} z^\varepsilon(t,0,x')$, are equivalent to

$$
\begin{cases}
f_0 = a_0^0 + b_0^0, \quad 0 = a_0^0 - b_0^0, \\
\eta_1 f_0 = -i\xi_1 b_1^0, \quad f_1 = b_1^0, \\
f_{2k} = a_k^0 + b_{2k}^0, \\
f_{2j+1} = b_{2j+1}^0, \\
\eta_1 f_{2k} + \partial_{x_1} B_{2k-1}(t,0,x') = -i\xi_1 b_{2k+1}^0 + \partial_{x_1} b_{2k-1}(t,0,x'), \\
\eta_1 f_{2j-1} + \partial_{x_1} B_{2j-2}(t,0,x') = i\xi_1 (a_j^0 - b_{2j}^0) + \partial_{x_1} a_{j-1}(t,0,x') + \partial_{x_1} b_{2j-2}(t,0,x'),
\end{cases}
\tag{3.46}
$$

where $k, j = 1, 2, \cdots$.

From (3.46), one may determine uniquely all f_j and b_k^0 in terms of the incoming coefficients $a_0^0, \cdots, a_j^0$, by which we obtain an infinitely accurate approximate solution of system (3.1). Indeed, it is easy to get b_0^0, f_0, b_1^0 and f_1. By the last four equations in (3.46), one has

$$
\begin{aligned}
b_{2k+1}^0 &= -i[\partial_{x_1} b_{2k-1}(t,0,x') - \eta_1 (a_k^0 + b_{2k}^0) - \partial_{x_1} B_{2k-1}(t,0,x')]/\xi_1, \\
b_{2j}^0 &= a_j^0 - i[\partial_{x_1} a_{j-1}(t,0,x') + \partial_{x_1} b_{2j-2}(t,0,x') - \eta_1 b_{2j-1}^0 - \partial_{x_1} B_{2j-2}(t,0,x')]/\xi_1.
\end{aligned}
$$

By induction, this gives all b_j^0. Finally, from $f_{2k} = a_k^0 + b_{2k}^0$ and $f_{2j+1} = b_{2j+1}^0$, we get all f_j.

For the leading amplitudes a_0^0 and b_0^0, one finds

$$
a_0^0 + b_0^0 = f_0, \qquad a_0^0 - b_0^0 = 0.
$$

The second relation yields the reflection coefficient $r = 1$.

Thus, though the asymptotic description at normal incidence is qualitatively different from that at non-normal incidence, the reflection coefficient is continuous as $\theta \to 0$.

3.6 The energy absorbed upon reflection

The fact that the reflection coefficients have modulus one, shows that, to leading order, there is conservation of energy upon reflection. Since the correction term is smaller by a factor ε in the case of non-normal incidence and by a factor $\sqrt{\varepsilon}$ in the case of normal incidence it is reasonable to expect that upon reflection about ε% (*resp.* $\sqrt{\varepsilon}$%) of the energy is absorbed. This expectation is confirmed by the following estimate of the energy absorbed in the heat region.

In the case of non-normal incidence, the solution is localized in a boundary layer of width $O(\varepsilon)$ as $\varepsilon \to 0$. The time derivatives $\partial_t y^\varepsilon = O(1/\varepsilon)$. Therefore, the energy dissipated between $t = 0$ and $t = T$ is of the order of

$$\int_0^T \int_{\{x_1<0\}} |\partial_t y^\varepsilon|^2 dtdx \approx \int_0^T \int_{\{-C\varepsilon<x_1<0\}} \frac{1}{\varepsilon^2} dtdx \approx \frac{1}{\varepsilon}.$$

On the other hand, the total energy is $O(1/\varepsilon^2)$. Hence, the energy dissipated is negligible, and the negligible loss can be quantified as ε% of the total energy.

In the case of normal incidence, on the heat equation side $\partial_t y^\varepsilon = O(1/\varepsilon)$ in a boundary layer of thickness $O(\sqrt{\varepsilon})$. Therefore the dissipation is $O(\sqrt{\varepsilon}(1/\varepsilon^2))$ which is again negligible compared to the initial energy which is $O(1/\varepsilon^2)$. Also, the negligible loss can be quantified as $\sqrt{\varepsilon}$% of the total energy.

3.7 Non-uniform decay in polyhedral wave domains

As a consequence of the above analysis, we conclude that

Theorem 3.4 *Let the wave domain Ω_2 be a polyhedral in $\mathbb{R}^n$. Then*

i) For any given $T > 0$, there is no constant $C > 0$ such that (1.7) holds for all solutions of (1.1);

ii) The energy $E(t)$ of solutions of system (1.1) does not decay exponentially.

Proof. It suffices to show the first assertion. Since more general cases will be considered in the next section, we only give here a sketch of the proof.

The main idea is that, whatever $T > 0$ is, one can find a sequence of solutions of (1.1), concentrated along a multiply reflected ray (*see* Definition 2.1), for which (1.7) fails. Since Ω_2 is a polyhedron in $\mathbb{R}^n$, in view of Remark 2.4, one may choose a multiply reflected ray ℓ to be finite ordered sequences of line segments in $\overline{\Omega_2}$, reflected one by one at the smooth points of $\partial\Omega_2$, and contained in Ω_2 except the reflected points.

For simplicity, we choose a multiply reflected ray ℓ with only two line segments, an incoming ray ℓ_0 and a reflected ray ℓ_1.

There are two cases. The first one is $P \in \gamma$. Since $\partial\Omega_2$ is polyhedral, one may find a hyperplane, say $\{x_1 = 0\}$, containing a (small) neighborhood of P in $\partial\Omega_2$, such that at leat a small neighborhood of P in the wave domain Ω_2 is located in $\{x_1 > 0\}$, while a small neighborhood of P in the heat domain Ω_1 is located in $\{x_1 < 0\}$. Assume the incoming wave is given by $e^{i(\tau t+\xi\cdot x)/\varepsilon}\sum_{j=0}^{\infty}\varepsilon^j a_j$ with $\xi_1 \neq 0$ and $\xi' \neq 0$. In this case, we construct the approximate heat and wave solutions y^ε and z^ε as in (3.22) and (3.32), respectively; while the initial data for determining the amplitudes of y^ε and the reflected wave $e^{i(\tau t+\tilde{\xi}\cdot x)/\varepsilon}\sum_{j=0}^{\infty}\varepsilon^j b_j$ are given by (3.36). Choosing the support of the initial data of the incoming ray to be smaller if necessary, one sees immediately that z^ε are in fact defined globally in $(0,T)\times\Omega_2$. On the other hand, since the localization effect of the fact $e^{|\xi'|x_1/\varepsilon}$, multiplying y^ε by a cut-off function (with respect to x_1) if necessary, one may also assume y^ε is defined globally in $(0,T)\times\Omega_1$. From Theorems 3.1 and 3.2, one concludes that $(y^\varepsilon, z^\varepsilon)$ solves

$$\begin{cases} \partial_t y^\varepsilon - \Delta y^\varepsilon = O(\varepsilon^\infty) & \text{in } (0,T)\times\Omega_1,\\ \Box\, z^\varepsilon = O(\varepsilon^\infty) & \text{in } (0,T)\times\Omega_2,\\ y^\varepsilon = 0 & \text{on } (0,T)\times\Gamma_1,\\ z^\varepsilon = 0 & \text{on } (0,T)\times\Gamma_2,\\ y^\varepsilon = z^\varepsilon + O(\varepsilon^\infty), \quad \dfrac{\partial y^\varepsilon}{\partial\nu_1} = -\dfrac{\partial z^\varepsilon}{\partial\nu_2} + O(\varepsilon^\infty) & \text{on } (0,T)\times\gamma. \end{cases} \tag{3.47}$$

However, according to the analysis in Subsection 3.6, we have

$$|\partial_t y^\varepsilon|^2_{L^2((0,T)\times\Omega_1)} = O\Big(\frac{1}{\varepsilon}\Big), \qquad E^\varepsilon(0) = E(y^\varepsilon, z^\varepsilon, \partial_t z^\varepsilon)(0) \approx \frac{1}{\varepsilon^2}. \tag{3.48}$$

The second case is $P \in \Gamma_2$, the exterior boundary of Ω_2. This is an even easier case. Choosing a hyperplane, say $\{x_1 = 0\}$, such that Ω is located in $\{x_1 > 0\}$, we seek the solutions of the wave equation in the form of (3.32). To guarantee that z^ε satisfy the Dirichlet boundary condition on Γ_2, we choose initial data b_j^0 of b_j to be equal to a_j^0 for all j. On the other hand, we choose $y^\varepsilon \equiv 0$. Choosing the support of the initial data of the incoming ray to be smaller if necessary, by Theorem 3.1, one sees that $(y^\varepsilon, z^\varepsilon)$ solves

$$\begin{cases} \partial_t y^\varepsilon - \Delta y^\varepsilon = 0 & \text{in } (0,T)\times\Omega_1,\\ \Box\, z^\varepsilon = O(\varepsilon^\infty) & \text{in } (0,T)\times\Omega_2,\\ y^\varepsilon = 0 & \text{on } (0,T)\times\Gamma_1,\\ z^\varepsilon = O(\varepsilon^\infty) & \text{on } (0,T)\times\Gamma_2,\\ y^\varepsilon = z^\varepsilon, \quad \dfrac{\partial y^\varepsilon}{\partial\nu_1} = -\dfrac{\partial z^\varepsilon}{\partial\nu_2} & \text{on } (0,T)\times\gamma. \end{cases} \tag{3.49}$$

Clearly, in this case, we have

$$|\partial_t y^\varepsilon|^2_{L^2((0,T)\times\Omega_1)} = 0, \qquad E^\varepsilon(0) = E(y^\varepsilon, z^\varepsilon, \partial_t z^\varepsilon)(0) \approx \frac{1}{\varepsilon^2}. \tag{3.50}$$

In both cases, we may correct the approximate solutions $(y^\varepsilon, z^\varepsilon)$ to become exact solutions $(y_\varepsilon, z_\varepsilon)$ of (1.1) (except the initial data) such that

$$|\partial_t y_\varepsilon|^2_{L^2((0,T)\times\Omega_1)} = O\Big(\frac{1}{\varepsilon}\Big), \qquad E_\varepsilon(0) = E(y_\varepsilon, z_\varepsilon, \partial_t z_\varepsilon)(0) \approx \frac{1}{\varepsilon^2}. \tag{3.51}$$

Consequently, (1.7) fails to be true for all solutions of (1.1).

As for the general case that ℓ is consisted by finite ordered sequences of line segments, one may repeat the above constructions to build a family of wave solutions z_ε along the ray and a family of heat solutions y_ε with negligible energies for all finite T. This completes the proof of Theorem 3.4. □

4 Non-uniform decay in general domains via Gaussian Beams

In this section we perform a careful analysis of the interaction of waves at a general interface by means of a Gaussian Beam approach.

4.1 Statement of the main result

The main non-uniform decay result in this paper is stated as follows:

Theorem 4.1 *Let Assumption 2.1 hold. Then*

i) For any given $T > 0$, there is no constant $C > 0$ such that the observability inequality (1.7) holds for all solutions of (1.1);

ii) The energy $E(t)$ of solutions of system (1.1) does not satisfy the uniform decay property (1.6);

iii) Accordingly, $||S(t)||_{\mathcal{L}(H)} \stackrel{\triangle}{=} \sup_{|h|_H=1} |S(t)h|_H \equiv 1, \quad \forall\, t \geq 0.$

The main statement in Theorem 4.1 is the first assertion. Indeed, as we mentioned before, once we know that the observability inequality (1.7) fails, we deduce immediately that the exponential decay property (1.6) fails and that the corresponding semigroup is of unit norm for all $t \geq 0$.

Similar to the proof of Theorem 3.4, the first assertion in Theorem 4.1 is a consequence of Theorem 4.4 at the end of Subsection 4.5. The proof of Theorem 4.4 uses Gaussian Beams to construct solutions of system (1.1) which are supported near rays. The main idea is similar to the one we have developed in the previous section. However, the construction of approximate solutions in the general case is more sophisticated since the boundary and the interface are not flat. For any given $T > 0$, and any ray of Geometric Optics in the wave domain which for $0 \leq t \leq T$ reflects transversally and non-normally at the exterior boundary Γ_2 or at the interface γ, we construct a family of solutions $(y_\varepsilon, z_\varepsilon)$ of system (1.1) such that the un-dissipated energy of $(y_\varepsilon, z_\varepsilon)$ is concentrated in the wave domain Ω_2, and more precisely located in a very small neighborhood of the ray, for which a negligible part of the whole un-dissipated energy enters the heat domain Ω_1. The analysis of the previous section is not sufficient for this purpose. Indeed, when the boundary of the wave subdomain is not flat, the phase is generally not longer linear after reflection. Hence the solutions of the involved eikonal equations may have singularities and may not be globally well-defined. In order to overcome this difficulty, we adopt the Gaussian beam approach by Ralston in [15] in which the phase is taken to be complex. We then need to adapt the analysis in the previous section to understand how much of the energy of these ray-like solutions enters the heat domain through the interface. The conclusion is the same as the flat case: Only a negligible percent of the total energy enters the heat domain and, accordingly the observability inequality (1.7) fails.

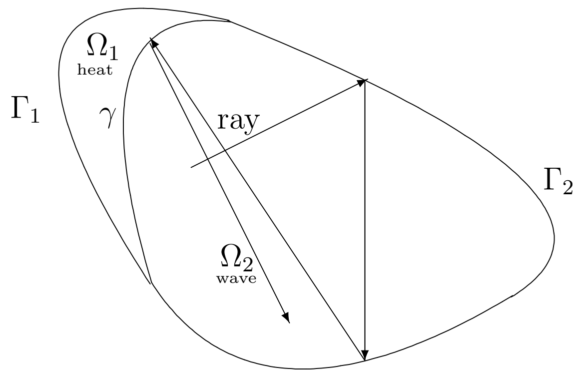

FIGURE 2: RAY TRAPPED IN THE WAVE DOMAIN

Theorem 4.1 shows that one cannot have uniform decay of the energy even if the heat domain satisfies the GCC. As we see from Theorem 4.4, this is due to the fact that even when the wave rays intersect the interface most of the energy bounces back in the wave domain without entering the heat one. This explains the inefficiency of the heat equation to dissipate the energy of the system.

According to this negative result, in order to have uniform decay, an additional dissipation mechanism has to be added in the wave domain to dissipate the energy. The detailed analysis of this problem remains to be done. Nevertheless, as we shall see in Section 5, when the heat domain satisfies the GCC, smooth solutions of (1.1) decay polynomially.

4.2 Gaussian Beams for general wave equations

Given a ray $(t, x(t))$ for the operator W defined by (2.8), one may construct a family of highly localized approximate solutions of the equation

$$Wu = 0, \qquad \text{in } (0,T) \times \mathbb{R}^n \tag{4.1}$$

in the following form

$$u^\varepsilon(t,x) = \varepsilon^{1-n/4} a(t) e^{i\phi(t,x)/\varepsilon}, \qquad \varepsilon > 0. \tag{4.2}$$

In (4.2), we take the phase function ϕ to be of the form

$$\phi(t,x) = \xi(t)^T (x - x(t)) + \frac{1}{2}\Big(x - x(t)\Big)^T M(t)\Big(x - x(t)\Big), \tag{4.3}$$

where $M(t)$ is a $n \times n$ complex symmetric matrix with positive definite imaginary part. The construction of approximate solutions (4.2) requires an appropriate selection of $a(t)$ and $M(t)$.

Denote the energy of (4.1) by

$$e(t) \equiv e(u)(t) = \frac{1}{2} \int_{\mathbf{R}^n} \Big[|u(t,x)|^2 + \sum_{j,k=1}^{n} \alpha_{jk}(x) \partial_{x_j} u(t,x) \partial_{x_k} u(t,x) + |u_t(t,x)|^2 \Big] dx. \qquad (4.4)$$

The following result can be found in [15] and [13]:

Theorem 4.2 *Let $T > 0$ be given, $\alpha_{jk} \in C^2(\mathbb{R}^n)$, $\beta_j \in C^1(\mathbb{R}^n)$, and $(t, x(t))$ be a ray for W. Then for any $n \times n$ complex symmetric matrix M_0 with $\operatorname{Im} M_0 > 0$ and any $a_0 \in \mathbb{C} \setminus \{0\}$, there is a complex-valued symmetric matrix $M(\cdot) \in C^2([0,T];\mathbb{C}^{n\times n})$ and a complex-valued function $a(\cdot) \in C^2([0,T];\mathbb{C} \setminus \{0\})$ with*

$$M(0) = M_0, \qquad \operatorname{Im} M(t) > 0, \qquad a(0) = a_0, \qquad (4.5)$$

such that

1) The u^ε are approximate solutions of (4.1):

$$\sup_{t\in(0,T)} |W u^\varepsilon(t,\cdot)|_{L^2(\mathbf{R}^n)} = O(\varepsilon^{1/2}); \qquad (4.6)$$

2) The initial energy of u^ε is bounded below as $\varepsilon \to 0$, i.e.,

$$e^\varepsilon(0) \equiv e(u^\varepsilon)(0) \approx 1; \qquad (4.7)$$

3) The energy of u^ε is exponentially small off the ray $(t, x(t))$:

$$\sup_{t\in(0,T)} \int_{\mathbf{R}^n \setminus B_{\varepsilon^{1/4}}(t)} \Big[|u_t^\varepsilon(t,x)|^2 + |\nabla u^\varepsilon(t,x)|^2 \Big] dx = O(e^{-\beta/\varepsilon}), \qquad (4.8)$$

where $B_{\varepsilon^{1/4}}(t)$ is the ball centered at $x(t)$ with radius $\varepsilon^{1/4}$ and $\beta > 0$ is a constant, independent of ε.

Remark 4.1 *We recall that, by [15] and [13], $a(t)$ in Theorem 4.2 is determined by the ODE:*

$$\begin{cases} \dfrac{d}{dt} a(t) = a(t) W\phi(t, x(t)), \\ a(t_0) = a_0, \end{cases}$$

To simplify the presentation, we choose $a(t)$ in (4.2) such that it only depends on t. If assuming further that the coefficients of W are infinitely smooth and choosing the amplitude in (4.2) to be $a^\varepsilon(t,x)$, one may obtain infinitely accurate approximate solutions to $Wu = 0$, *i.e., instead of (4.6), one may get* $Wu^\varepsilon = O(\varepsilon^\infty)$*. On the other hand,* $M(t)$ *in Theorem 4.2 is determined by the Riccati equation:*

$$\begin{cases} \dfrac{dM(t)}{dt} + M(t)C(t)M(t) + B(t)M(t) + M(t)B(t)^T + A(t) = 0, \\ M(0) = M_0, \end{cases}$$

where $C(t)$*,* $B(t)$ *and* $A(t)$ *are* $n \times n$ *matrices whose coefficients are determined by the first and second derivatives of* g *evaluated along the ray* $(t, x(t), \xi(t))$ *(recall (2.9) for* g*). We refer to [15] for the global existence of solutions to this nonlinear ODE with initial datum* M_0 *so that* $\operatorname{Im} M_0 > 0$*.*

Remark 4.2 *Let* $a_{ijk} \in \mathbb{C}$*,* $i, j, k = 1, 2, \cdots, n$*, be given so that* $a_{ijk} = a_{i'j'k'}$ *for any permutation* i', j', k' *of* i, j, k*. Put* $(x_1(t), \cdots, x_n(t)) \equiv x(t)$*. From [15] and [13], we see that, the conclusions in Theorem 4.2 hold if one replaces the phase function* ϕ *given by (4.3) by*

$$\begin{aligned} \phi(t, x) = \xi(t)^T(x - x(t)) + \frac{1}{2}\Big(x - x(t)\Big)^T M(t)\Big(x - x(t)\Big) \\ + \sum_{i,j,k=1}^{n} a_{ijk}(x_i - x_i(t))(x_j - x_j(t))(x_k - x_k(t)). \end{aligned} \tag{4.9}$$

Indeed, this follows from the fact that the difference between those two phases is of order $|x - x(t)|^3$*. This observation will play a key role in the sequel.*

Remark 4.3 *For any give* $b \in \mathbb{C}^n$*, from [15] and [13], we see that, the conclusions in Theorem 4.2 hold if one replaces the amplitude* $a(t)$ *by* $a(t) + b^T(x - x(t))$*. Indeed, this follows from the fact that the difference between those two amplitudes is of order* $|x - x(t)|$*. This observation will play a key role in the sequel, too.*

Remark 4.4 *Let* $\chi \in C_0^\infty(\mathbb{R}^{1+n})$ *be any given cut-off function which is identically equal to 1 in a neighborhood of the ray* $\Big\{(t, x(t)) \ \Big| \ t \in [0, T]\Big\}$*. Then the functions* χu^ε *also satisfies (4.6)–(4.8). In view of this, we may choose* u^ε *such that they are supported in any given (small) neighborhood of the ray.*

4.3 Gaussian Beams for the wave equation with curved wavefronts

From now on to the rest of this section, we construct highly localized solutions to system (1.1).

Assume $(t, x^-(t), \xi^-(t))$ is a ray for $\Box \equiv \partial_{tt} - \Delta$ starting from Ω_2 at time $t = 0$, i.e., $x^-(0) \in \Omega_2$, and arriving at the boundary $\partial\Omega_2$ at time $t = t_0$, i.e., $x_0 \stackrel{\triangle}{=} x^-(t_0) \in \partial\Omega_2$. We must distinguish two cases, i.e., either x_0 belongs to the exterior boundary Γ_2 of the wave domain Ω_2, or to the interface γ. By Assumption 2.1, we exclude the rarely case $x_0 \in \overline{\Gamma_2} \cap \overline{\gamma}$. The case where $x_0 \in \gamma$ will be analyzed in the next two subsections. In this subsection, we focus on the first case, i.e., $x_0 \in \Gamma_2$. The construction of reflected beams is in the spirit of [14], [15] and [17] but our presentation is more elementary.

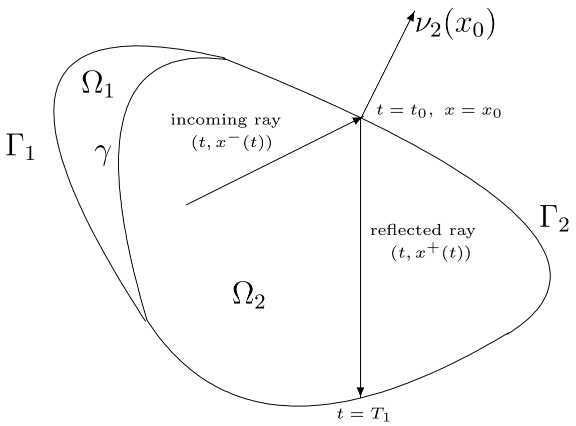

FIGURE 3: INCOMING AND REFLECTED RAYS

- **Ansatz of the incoming and reflected waves**

Theorem 4.2 enables us to construct a family of approximate solutions $z_\varepsilon^- = z_\varepsilon^-(t,x)$ to the second equation in (1.1). However, z_ε^- may not satisfy the homogeneous Dirichlet boundary condition $z_\varepsilon^-(t,x_0) = 0$. One then has to superpose z_ε^- with another approximate solution z_ε^+. The later is constructed from the ray $(t, x^+(t), \xi^+(t))$, which reflects the original one, $(t, x^-(t), \xi^-(t))$, at the boundary. Recall that the rays of $\Box$ are simply straight lines in $\mathbb{R}^n$. Hence, both the incoming and reflected rays $(t, x^\pm(t), \xi^\pm(t))$ are globally defined. The point is to select approximate solutions z_ε^+ to the second equation in (1.1), concentrated in a small neighborhood of the reflected ray $(t, x^+(t), \xi^+(t))$, such that $z_\varepsilon^- + z_\varepsilon^+$ satisfies approximately the homogeneous Dirichlet boundary condition.

It is convenient to introduce geodesic normal coordinates near the reflected point $x_0 \in \Gamma_2$, called henceforth $\tilde{x} \equiv \tilde{x}(x) = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \equiv (\tilde{x}_1, \tilde{x}')$, centered at the reflected point $\tilde{x}_0 \equiv (0, \tilde{x}_0')$, the new coordinate of x_0, such that Ω_2 is locally given by $\tilde{x}_1 \geq 0$, Γ_2 is flat near $\tilde{x}_0$. Denote the inverse Jacobian matrix of $\tilde{x} = \tilde{x}(x)$ by $J(\tilde{x})$, i.e.

$$J(\tilde{x}) \equiv \Big(g_{ij}(\tilde{x})\Big)_{1\leq i,j\leq n} \triangleq \frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)}. \tag{4.10}$$

In this subsection, we only need to assume Γ_2 to be C^2 near the reflected points. This means that $J(\tilde{x}) \in C^1$. Let $\nu_2(x_0)$ be the unit outward normal vector of Ω_2 at x_0. It is easy to see that, in the new coordinates the unit outward normal vector at the reflected point becomes $(-1, 0, \cdots, 0)$. Hence

$$(J(\tilde{x}_0))^{-1}\nu_2(x_0) = |(J(\tilde{x}_0))^{-1}\nu_2(x_0)|(-1, 0, \cdots, 0). \tag{4.11}$$

According to (2.10), $(x^-(t), \xi^-(t))$ satisfies

$$\begin{cases} \dot{x}^-(t) = 2\xi^-(t), \\ \dot{\xi}^-(t) = 0, \\ x^-(t_0) = x_0, \quad \xi^-(t_0) = \xi^-(t_0). \end{cases} \tag{4.12}$$

Also, one assumes that $(x^+(t), \xi^+(t))$ satisfies

$$\begin{cases} \dot{x}^+(t) = 2\xi^+(t), \\ \dot{\xi}^+(t) = 0, \\ x^+(t_0) = x_0, \\ \xi^+(t_0) = \xi^-(t_0) - 2\xi^-(t_0) \cdot \nu_2(x_0)\nu_2(x_0). \end{cases} \tag{4.13}$$

The choice of the initial data $\xi^+(t_0)$ is such that the directions of the incoming and reflected rays satisfy the "*geometric optics law*", i.e.,

$$\dot{x}^+(t_0) = \dot{x}^-(t_0) - 2\dot{x}^-(t_0) \cdot \nu_2(x_0)\nu_2(x_0).$$

On the other hand, from (2.11), one has $|\xi^-(t_0)| = 1/2$. Hence, noting $|\nu_2(x_0)| = 1$, it is easy to check that $|\xi^+(t_0)| = 1/2$. Therefore,

$$|\xi^\pm(t)| = \frac{1}{2}, \qquad \forall\, t \in \mathbb{R}. \tag{4.14}$$

By Assumption 2.1, we may assume $\xi^-(t)$ is transversal and non-normal to the boundary $\partial\Omega_2$ at time $t = t_0$, i.e.,

$$\xi^-(t_0) \cdot \nu_2(x_0) \neq 0 \quad \text{ and } \quad \xi^-(t_0) \not\parallel \nu_2(x_0). \tag{4.15}$$

Finally, according to (4.2) and Theorem 4.2, and noting Remarks 4.2 and 4.3, we may assume the incoming wave to be of the form

$$z_\varepsilon^-(t,x) = \varepsilon^{1-n/4}[a^-(t) + (b^-)^T(x - x^-(t))]e^{i\phi^-(t,x)/\varepsilon} \tag{4.16}$$

where

$$\begin{aligned} \phi^-(t,x) = {} & \xi^-(t)^T(x - x^-(t)) + \frac{1}{2}\Big(x - x^-(t)\Big)^T M^-(t)\Big(x - x^-(t)\Big) \\ & + \sum_{i,j,k=1}^n a_{ijk}^-(x_i - x_i^-(t))(x_j - x_j^-(t))(x_k - x_k^-(t)). \end{aligned} \tag{4.17}$$

In (4.17), $M^-(t)$ is some $n \times n$ complex symmetric matrix with positive definite imaginary part, while $b^- \in \mathbb{C}^n$ and a_{ijk}^-, $i, j, k = 1, 2, \cdots, n$, are any given complex numbers so that $a_{ijk}^- = a_{i'j'k'}^-$ for any permutation i', j', k' of i, j, k.

Denote by $T_1 > 0$ the instant when the reflected ray arrives at $\partial\Omega_2$, i.e., $x^+(T_1) \in \partial\Omega_2$ (Note that, of course, $0 < t_0 < T_1$). Fix any

$$T^* \in (t_0, T_1). \tag{4.18}$$

Our aim is to find another approximate solution

$$z_\varepsilon^+(t,x) = \varepsilon^{1-n/4}[a^+(t) + (b^+)^T(x - x^+(t))]e^{i\phi^+(t,x)/\varepsilon} \tag{4.19}$$

of $\Box u = 0$, which is concentrated in a small neighborhood of the reflected ray $(t, x^+(t), \xi^+(t))$ such that

$$|z_\varepsilon^- + z_\varepsilon^+|_{H^1((0,T^*)\times\partial\Omega_2)} = O(\varepsilon^{1/2}). \tag{4.20}$$

The constructions of $\phi^+(t, x)$ and $a^+(t)$ are close to that used in [15] and [13]. However, for the reader's convenience, we recall a sketch of the constructions. First, by Remark 4.1, $a^+(t)$ is determined from its initial value at $t = t_0$, which is given by

$$a^+(t_0) = -a^-(t_0). \tag{4.21}$$

Next, choose

$$\begin{aligned}\phi^+(t, x) = \xi^+(t)^T(x - x^+(t)) + \frac{1}{2}\Big(x - x^+(t)\Big)^T M^+(t)\Big(x - x^+(t)\Big) \\ + \sum_{i,j,k=1}^{n} a^+_{ijk}(x_i - x^+_i(t))(x_j - x^+_j(t))(x_k - x^+_k(t)),\end{aligned} \tag{4.22}$$

where $M^+(t)$ is a suitable $n \times n$ complex symmetric matrix with positive definite imaginary part, while a^+_{ijk}, $i, j, k = 1, 2, \cdots, n$, are any given complex numbers so that $a^+_{ijk} = a^+_{i'j'k'}$ for any permutation i', j', k' of i, j, k. According to Remark 4.1, $M^+(t)$ is determined by its initial data $M^+(t_0)$ and the reflected ray $(t, x^+(t), \xi^+(t))$. We emphasize that in this subsection the "reflected coefficients" b^+ in (4.19) and a^+_{ijk} in (4.22) may be chosen arbitrarily. Therefore, it remains to assign $M^+(t_0)$.

- **Assignment of** $M^+(t_0)$

Write the expression of $\phi^\pm(t, x)$ in the $\tilde{x}$-coordinates as

$$\begin{aligned}\tilde{\phi}^\pm(t, \tilde{x}) = \xi^\pm(t)^T\Big(x(\tilde{x}) - x^\pm(t)\Big) + \frac{1}{2}\Big(x(\tilde{x}) - x^\pm(t)\Big)^T M^\pm(t)\Big(x(\tilde{x}) - x^\pm(t)\Big) \\ + \sum_{i,j,k=1}^{n} a^\pm_{ijk}\Big(x_i(\tilde{x}) - x^\pm_i(t)\Big)\Big(x_j(\tilde{x}) - x^\pm_j(t)\Big)\Big(x_k(\tilde{x}) - x^+_k(t)\Big).\end{aligned} \tag{4.23}$$

Put

$$\begin{aligned}\sigma^\pm \equiv (\sigma_1^\pm, \sigma'_\pm) \triangleq (J(\tilde{x}_0))^T\xi^\pm(t_0), \\ \eta^\pm \equiv (\eta_1^\pm, \eta_2^\pm, \cdots, \eta_n^\pm) \equiv (\eta_1^\pm, \eta'_\pm) \triangleq (J(\tilde{x}_0))^{-1}\xi^\pm(t_0),\end{aligned} \tag{4.24}$$

where $\sigma'_\pm, \eta'_\pm \in \mathbb{R}^{n-1}$. Both $\sigma_\pm$ and $\eta^\pm$ will be needed to compute the derivatives of $\tilde{\phi}^\pm(t, 0, \tilde{x}')$ at $(t_0, \tilde{x}_0)$ up to second order.

The following holds.

Proposition 4.1 *Under the assumption (4.15), it holds:*

$$\eta_1^\pm \neq 0, \qquad \eta'_+ = \eta'_-, \qquad \sigma_1^\pm \neq 0, \qquad \sigma'_+ = \sigma'_- \neq 0. \tag{4.25}$$

Proof. First, we claim that

$$((J(\tilde{x}_0))^{-1})^T(J(\tilde{x}_0))^{-1}\nu_2(x_0) \parallel \nu_2(x_0). \tag{4.26}$$

Indeed, denote by $\mathcal{T}$ the tangent space to $\partial\Omega_2$ at the reflected point x_0. Then, it is obvious that $(J(\tilde{x}_0))^{-1}\mathcal{T} = \{\tilde{x}_1 = 0\}$. Noting (4.11), this means that $(J(\tilde{x}_0))^{-1}\nu_2(x_0) \perp (J(\tilde{x}_0))^{-1}\mathcal{T}$. Hence, $((J(\tilde{x}_0))^{-1})^T(J(\tilde{x}_0))^{-1}\nu_2(x_0) \perp \mathcal{T}$. This yields (4.26).

Next, by the last equation in (4.13) and (4.15), we deduce that

$$\xi^+(t_0)\cdot\nu_2(x_0) = -\xi^-(t_0)\cdot\nu_2(x_0) \neq 0. \tag{4.27}$$

Hence, from (4.11), (4.26) and (4.27), we obtain

$$\begin{aligned} \eta_1^\pm &= -\left((J(\tilde{x}_0))^{-1}\xi^\pm(t_0), \frac{(J(\tilde{x}_0))^{-1}\nu_2(x_0)}{|(J(\tilde{x}_0))^{-1}\nu_2(x_0)|}\right)_{\mathbf{R}^n} \\ &= -\left(\xi^\pm(t_0), \frac{((J(\tilde{x}_0))^{-1})^T(J(\tilde{x}_0))^{-1}\nu_2(x_0)}{|(J(\tilde{x}_0))^{-1}\nu_2(x_0)|}\right)_{\mathbf{R}^n} \\ &= -\frac{|((J(\tilde{x}_0))^{-1})^T(J(\tilde{x}_0))^{-1}\nu_2(x_0)|}{|(J(\tilde{x}_0))^{-1}\nu_2(x_0)|}\xi^\pm(t_0)\cdot\nu_2(x_0) \neq 0. \end{aligned} \tag{4.28}$$

Also, by the last equation in (4.13), we have

$$(J(\tilde{x}_0))^{-1}\xi^+(t_0) = (J(\tilde{x}_0))^{-1}\xi^-(t_0) - 2\xi^-(t_0)\cdot\nu_2(x_0)(J(\tilde{x}_0))^{-1}\nu_2(x_0). \tag{4.29}$$

Hence, in view of (4.11) and (4.29), we see that the j-th component of $(J(\tilde{x}_0))^{-1}\xi^+(t_0)$ is equal to that of $(J(\tilde{x}_0))^{-1}\xi^-(t_0)$ for $j = 2, \cdots, n$. This means $\eta'_+ = \eta'_-$.

Similarly, from (4.11) and (4.27), we obtain

$$\sigma_1^\pm = -\left((J(\tilde{x}_0))^T\xi^\pm(t_0), \frac{(J(\tilde{x}_0))^{-1}\nu_2(x_0)}{|(J(\tilde{x}_0))^{-1}\nu_2(x_0)|}\right)_{\mathbf{R}^n} = \frac{-\xi^\pm(t_0)\cdot\nu_2(x_0)}{|(J(\tilde{x}_0))^{-1}\nu_2(x_0)|} \neq 0. \tag{4.30}$$

Also, by the last equation in (4.13), we have

$$(J(\tilde{x}_0))^T\xi^+(t_0) = (J(\tilde{x}_0))^T\xi^-(t_0) - 2\xi^-(t_0)\cdot\nu_2(x_0)(J(\tilde{x}_0))^T\nu_2(x_0). \tag{4.31}$$

Clearly, (4.26) yields

$$(J(\tilde{x}_0))^T\nu_2(x_0) \parallel (J(\tilde{x}_0))^{-1}\nu_2(x_0). \tag{4.32}$$

Note that (4.32) and (4.11) imply that $(J(\tilde{x}_0))^T\nu_2(x_0) \parallel (-1, 0, \cdots, 0)$. Hence, in view of (4.31), we see that the j-th component of $(J(\tilde{x}_0))^T\xi^+(t_0)$ is equal to that of $(J(\tilde{x}_0))^T\xi^-(t_0)$ for $j = 2, \cdots, n$. This means $\sigma'_+ = \sigma'_-$. It remains to show that $\sigma'_- \neq 0$. Assume this is not correct, i.e., $\sigma'_- = 0$. Then noting (4.11) and (4.24), we have $(J(\tilde{x}_0))^T\xi^-(t_0) \parallel (J(\tilde{x}_0))^{-1}\nu_2(x_0)$. This, combined with (4.32), implies that $\xi^-(t_0) \parallel ((J(\tilde{x}_0))^T)^{-1}(J(\tilde{x}_0))^{-1}\nu_2(x_0) \parallel \nu_2(x_0)$, which contradicts our assumption (4.15). This completes the proof of Proposition 4.1. □

Denote

$$\widetilde{M}^\pm(t_0) \triangleq (J(\tilde{x}_0))^T M^\pm(t_0) J(\tilde{x}_0). \tag{4.33}$$

Obviously, determining $M^+(t_0)$ is equivalent to chose $\widetilde{M}^+(t_0)$. For this purpose, put

$$\tilde{x}^\pm(t) = \tilde{x}(x^\pm(t)). \tag{4.34}$$

Then, one has

$$x^\pm(t) = x(\tilde{x}^\pm(t)). \tag{4.35}$$

We need several useful technical propositions.

Proposition 4.2 *As $(t, \tilde{x}')$ tends to $(t_0, \tilde{x}'_0)$, the following estimates hold*

$$(0, \tilde{x}') - \tilde{x}^\pm(t) = (-2\eta_1^\pm(t-t_0), \tilde{x}' - \tilde{x}'_0 - 2\eta'_\pm(t-t_0)) + O(|t-t_0|^2 + |\tilde{x}' - \tilde{x}'_0|^2), \tag{4.36}$$

$$\tilde{\phi}^\pm(t, 0, \tilde{x}') = O(|t-t_0| + |\tilde{x}' - \tilde{x}'_0|), \tag{4.37}$$

$$\partial_t \tilde{\phi}^\pm(t, 0, \tilde{x}') = -\frac{1}{2} + O(|t-t_0| + |\tilde{x}' - \tilde{x}'_0|), \tag{4.38}$$

$$\nabla_{\tilde{x}} \tilde{\phi}^\pm(t, 0, \tilde{x}') = \sigma^\pm + O(|t-t_0| + |\tilde{x}' - \tilde{x}'_0|), \tag{4.39}$$

$$\partial_{tt} \tilde{\phi}^\pm(t, 0, \tilde{x}') = 4(\eta^\pm)^T \widetilde{M}^\pm(t_0)\eta^\pm + O(|t-t_0| + |\tilde{x}' - \tilde{x}'_0|), \tag{4.40}$$

$$\partial_t \nabla_{\tilde{x}} \tilde{\phi}^\pm(t, 0, \tilde{x}') = -2\widetilde{M}^\pm(t_0)\eta^\pm + O(|t-t_0| + |\tilde{x}' - \tilde{x}'_0|), \tag{4.41}$$

$$\nabla^2_{\tilde{x}} \tilde{\phi}^\pm(t, 0, \tilde{x}') = \nabla_{\tilde{x}}\Big((J(\tilde{x}_0))^T \xi^\pm(t_0)\Big) + \widetilde{M}^\pm(t_0) + O(|t-t_0| + |\tilde{x}' - \tilde{x}'_0|). \tag{4.42}$$

The proof of Proposition 4.2 will be given in Appendix B.

Denote

$$\nabla_{\tilde{x}}\Big((J(\tilde{x}_0))^T \xi^\pm(t_0)\Big) \equiv \Big(h^\pm_{ij}\Big)_{1\le i,j\le n}, \qquad \widetilde{M}^\pm(t_0) \equiv \Big(m^\pm_{ij}\Big)_{1\le i,j\le n} \equiv \begin{pmatrix} m^\pm_{11} & \vartheta^T_\pm \\ \vartheta_\pm & \hat{M}^\pm \end{pmatrix}, \tag{4.43}$$

where $\vartheta_\pm = (m^\pm_{21}, \cdots, m^\pm_{n1})^T$ and $\hat{M}^\pm = \Big(m^\pm_{ij}\Big)_{2\le i,j\le n}$. Note that all m^-_{ij} are known. We now assign all m^+_{ij} to then obtain $\widetilde{M}^+(t_0)$ in (4.24). First of all, we choose

$$m^+_{ij} = h^-_{ij} + m^-_{ij} - h^+_{ij}, \qquad 2 \le i, j \le n. \tag{4.44}$$

This determines $\hat{M}^+$. By (4.43)–(4.44), we see that

$$\nabla_{\tilde{x}}\Big((J(\tilde{x}_0))^T \xi^+(t_0)\Big) + \widetilde{M}^+(t_0) = \nabla_{\tilde{x}}\Big((J(\tilde{x}_0))^T \xi^-(t_0)\Big) + \widetilde{M}^-(t_0). \tag{4.45}$$

Next, by (4.24) and (4.43), we see that

$$\widetilde{M}^\pm(t_0)\eta^\pm = \begin{pmatrix} m^\pm_{11}\eta^\pm_1 + \vartheta^T_\pm \eta'_\pm \\ \eta^\pm_1 \vartheta_\pm + \hat{M}^\pm \eta'_\pm \end{pmatrix}. \tag{4.46}$$

Hence, we choose

$$\vartheta_+ = \frac{\eta^-_1 \vartheta_- + \hat{M}^- \eta'_- - \hat{M}^+ \eta'_+}{\eta^+_1}. \tag{4.47}$$

This determines $m_{j1}^+ = m_{1j}^+$ for $j = 2, \cdots, n$. From (4.46)–(4.47), we get

$$\widetilde{M}^+(t_0)\eta^+ = \widetilde{M}^-(t_0)\eta^-. \tag{4.48}$$

Finally, from by (4.24) and (4.46), we have

$$(\eta^\pm)^T\widetilde{M}^\pm(t_0)\eta^\pm = m_{11}^\pm|\eta_1^\pm|^2 + 2\eta_1^\pm\vartheta_\pm^T\eta'_\pm + (\eta'_\pm)^T\hat{M}^\pm\eta'_\pm. \tag{4.49}$$

Therefore, we choose

$$m_{11}^+ = \frac{m_{11}^-|\eta_1^-|^2 + 2\eta_1^-\vartheta_-^T\eta'_- + (\eta'_-)^T\hat{M}^-\eta'_- - 2\eta_1^+\vartheta_+^T\eta'_+ - (\eta'_+)^T\hat{M}^+\eta'_+}{|\eta_1^+|^2}. \tag{4.50}$$

In view of (4.49)–(4.50), we get

$$(\eta^+)^T\widetilde{M}^+(t_0)\eta^+ = (\eta^-)^T\widetilde{M}^-(t_0)\eta^-. \tag{4.51}$$

This completes the assignment of $\widetilde{M}^+(t_0)$, and hence $M^+(t_0)$.

- **Properties of $\tilde{\phi}^\pm(t,\tilde{x})$ and $\widetilde{M}^+(t_0)$**

With the above choice on $\widetilde{M}^+(t_0)$, noting (4.40) and (4.51), (4.41) and (4.48), and (4.42) and (4.45), respectively, one concludes easily that

Proposition 4.3 *As $(t,\tilde{x}')$ tends to $(t_0,\tilde{x}'_0)$, the following estimates hold*

$$\partial_{tt}\tilde{\phi}^+(t,0,\tilde{x}') - \partial_{tt}\tilde{\phi}^-(t,0,\tilde{x}') = O(|t-t_0| + |\tilde{x}' - \tilde{x}'_0|), \tag{4.52}$$

$$\partial_t\nabla_{\tilde{x}'}\tilde{\phi}^+(t,0,\tilde{x}') - \partial_t\nabla_{\tilde{x}'}\tilde{\phi}^-(t,0,\tilde{x}') = O(|t-t_0| + |\tilde{x}' - \tilde{x}'_0|), \tag{4.53}$$

$$\nabla^2_{\tilde{x}'}\tilde{\phi}^+(t,0,\tilde{x}') - \nabla^2_{\tilde{x}'}\tilde{\phi}^-(t,0,\tilde{x}') = O(|t-t_0| + |\tilde{x}' - \tilde{x}'_0|). \tag{4.54}$$

Remark 4.5 *Combining Propositions 4.1–4.3, it is easy to see that the choice of $\widetilde{M}^+(t_0)$, and hence of $M^+(t_0)$, is such that*

$$D^s_{(t,\tilde{x}')}\tilde{\phi}^+(t_0,\tilde{x}_0) = D^s_{(t,\tilde{x}')}\tilde{\phi}^-(t_0,\tilde{x}_0) \tag{4.55}$$

for $s = 0, 1, 2$. In other words, all derivatives of order $s \le 2$ in the $(t,\tilde{x}')$ variables coincide at $(t_0,\tilde{x}_0)$. This is enough in this subsection. Note that in this case, the values of $a_{ijk}^\pm$ may be chosen arbitrarily. However, in Subsection 4.5 when the ray-like solutions of (1.1) for the case that the incoming ray arrives at the interface γ at time $t = t_0$ will be constructed, we need to choose a_{ijk}^+ suitably such that (4.55) holds for the third order derivatives as well.

Thanks to Taylor's formula, it follows from Remark 4.5 that

Proposition 4.4 *As $(t,\tilde{x}')$ tends to $(t_0,\tilde{x}'_0)$, it holds*

$$\tilde{\phi}^+(t,0,\tilde{x}') - \tilde{\phi}^-(t,0,\tilde{x}') = O(|t-t_0|^3 + |\tilde{x}' - \tilde{x}'_0|^3). \tag{4.56}$$

As mentioned before, it is crucial to show the following result:

Proposition 4.5 *Both $\widetilde{M}^+(t_0)$ constructed above and the desired $M^+(t_0)$, and hence $M^+(t)$, are $n \times n$ complex symmetric matrices with positive definite imaginary part.*

Proof. First, from (4.44) and noting that $h_{ij}^{\pm} \in \mathbb{R}$, one finds

$$\operatorname{Im} \hat{M}^+ = \operatorname{Im} \hat{M}^-. \tag{4.57}$$

Next, by (4.47) and (4.57), and noting that $\eta'_+ = \eta'_- \in \mathbb{R}^{n-1}$ (by Proposition 4.1), we get

$$\operatorname{Im} \vartheta_+ = \frac{\eta_1^-}{\eta_1^+} \operatorname{Im} \vartheta_-. \tag{4.58}$$

Finally, by (4.50), (4.57) and (4.58), we see that

$$\operatorname{Im} m_{11}^+ = \frac{|\eta_1^-|^2}{|\eta_1^+|^2} \operatorname{Im} m_{11}^-. \tag{4.59}$$

Now, combining (4.57)–(4.59), we arrive at

$$\operatorname{Im} \widetilde{M}^+(t_0) = \operatorname{diag}\left[\frac{\eta_1^-}{\eta_1^+}, 1, \cdots, 1\right] \operatorname{Im} \widetilde{M}^-(t_0) \operatorname{diag}\left[\frac{\eta_1^-}{\eta_1^+}, 1, \cdots, 1\right].$$

Recalling that $\operatorname{Im} \widetilde{M}^-(t_0) > 0$, we conclude the desired result. This completes the proof of Proposition 4.5. □

We also need the following result.

Proposition 4.6 *As $(t, \tilde{x}')$ tends to $(t_0, \tilde{x}'_0)$, the following estimate*

$$\operatorname{Im} \tilde{\phi}^{\pm}(t, 0, \tilde{x}') \geq c(|t - t_0|^2 + |\tilde{x}' - \tilde{x}'_0|^2) \tag{4.60}$$

holds for some constant $c > 0$.

The proof of Proposition 4.6 is given in Appendix B.

Remark 4.6 *Proposition 4.6 shows that the factors $e^{i\tilde{\phi}^{\pm}(t,0,\tilde{x}')/\varepsilon}$ localize $\tilde{z}_\varepsilon^{\pm}(t, 0, \tilde{x}')$ in the region $|t - t_0|^2 + |\tilde{x}' - \tilde{x}'_0|^2 = O(\varepsilon)$.*

- **Verification of (4.20)**

Now, we are in the position to show that

Lemma 4.1 *The approximate solutions $z_\varepsilon^{\pm}(t, x)$ of $\Box u = 0$, constructed by (4.16) and (4.19) with $a^+(t_0)$ and $M^+(t_0)$ chosen above (but for arbitrary $b^{\pm}$ and $a_{ijk}^{\pm}$), satisfy (4.20).*

Proof. Let $\tilde{z}^{\pm}(t,\tilde{x})$ be the new coordinate expressions of $z^{\pm}(t,x)$. According to Remark 4.4, the support of $z_{\varepsilon}^{\pm}|_{(0,T^*)\times\partial\Omega_2}$ being very small, we can use the change of variable $x \mapsto \tilde{x}$ to get

$$|z_{\varepsilon}^{-}+z_{\varepsilon}^{+}|_{H^1((0,T^*)\times\partial\Omega_2)} \le C|\tilde{z}_{\varepsilon}^{-}(t,0,\tilde{x}')+\tilde{z}_{\varepsilon}^{+}(t,0,\tilde{x}')|_{H^1((0,T^*)\times\mathbf{R}^{n-1}_{\tilde{x}'})}. \tag{4.61}$$

Since the $O(|x-x(t)|)$ terms in the amplitudes $a^{\pm}(t)+(b^{\pm})^T(x-x^{\pm}(t))$ play no role in this subsection, without loss of generality we may simply take $b^{\pm}=0$. It is easy to see that

$$\tilde{z}_{\varepsilon}^{-}(t,0,\tilde{x}')+\tilde{z}_{\varepsilon}^{+}(t,0,\tilde{x}') = \varepsilon^{1-n/4}\left[a^{-}(t)e^{i\tilde{\phi}^{-}(t,0,\tilde{x}')/\varepsilon}+a^{+}(t)e^{i\tilde{\phi}^{+}(t,0,\tilde{x}')/\varepsilon}\right], \tag{4.62}$$

Hence, by (4.62), and noting (4.21) and (4.39), we get

$$\begin{aligned}
&\nabla_{\tilde{x}'}\Big(\tilde{z}_{\varepsilon}^{-}(t,0,\tilde{x}')+\tilde{z}_{\varepsilon}^{+}(t,0,\tilde{x}')\Big)\\
&= i\varepsilon^{-n/4}\left[a^{-}(t)e^{i\tilde{\phi}^{-}(t,0,\tilde{x}')/\varepsilon}\nabla_{\tilde{x}'}\tilde{\phi}^{-}(t,0,\tilde{x}')+a^{+}(t)e^{i\tilde{\phi}^{+}(t,0,\tilde{x}')/\varepsilon}\nabla_{\tilde{x}'}\tilde{\phi}^{+}(t,0,\tilde{x}')\right]\\
&= i\varepsilon^{-n/4}\Big\{a^{-}(t_0)\nabla_{\tilde{x}'}\tilde{\phi}^{-}(t_0,\tilde{x}'_0)\left[e^{i\tilde{\phi}^{-}(t,0,\tilde{x}')/\varepsilon}-e^{i\tilde{\phi}^{+}(t,0,\tilde{x}')/\varepsilon}\right]\\
&\quad +e^{i\tilde{\phi}^{-}(t,0,\tilde{x}')/\varepsilon}O(|t-t_0|+|\tilde{x}'-\tilde{x}'_0|)+e^{i\tilde{\phi}^{+}(t,0,\tilde{x}')/\varepsilon}O(|t-t_0|+|\tilde{x}'-\tilde{x}'_0|)\Big\}.
\end{aligned} \tag{4.63}$$

Also, by Proposition 4.4, we see that

$$\begin{aligned}
&e^{i\tilde{\phi}^{-}(t,0,\tilde{x}')/\varepsilon}-e^{i\tilde{\phi}^{+}(t,0,\tilde{x}')/\varepsilon}\\
&= \frac{i(\tilde{\phi}^{-}(t,0,\tilde{x}')-\tilde{\phi}^{+}(t,0,\tilde{x}'))}{\varepsilon}\int_0^1 e^{i[\tilde{\phi}^{+}(t,0,\tilde{x}')+s(\tilde{\phi}^{-}(t,0,\tilde{x}')-\tilde{\phi}^{+}(t,0,\tilde{x}'))]/\varepsilon}ds\\
&= \frac{i}{\varepsilon}\int_0^1 e^{i[\tilde{\phi}^{+}(t,0,\tilde{x}')+s(\tilde{\phi}^{-}(t,0,\tilde{x}')-\tilde{\phi}^{+}(t,0,\tilde{x}'))]/\varepsilon}dsO(|t-t_0|^3+|\tilde{x}'-\tilde{x}'_0|^3).
\end{aligned} \tag{4.64}$$

By Proposition 4.6, we see that the factors $e^{i\tilde{\phi}^{\pm}(t,0,\tilde{x}')/\varepsilon}$ and $e^{i[\tilde{\phi}^{+}(t,0,\tilde{x}')+s(\tilde{\phi}^{-}(t,0,\tilde{x}')-\tilde{\phi}^{+}(t,0,\tilde{x}'))]/\varepsilon}$ localize the integrant in the region

$$|t-t_0|^2+|\tilde{x}'-\tilde{x}'_0|^2 = O(\varepsilon).$$

Therefore, by (4.63) and (4.64), we conclude that, for some positive constant C it holds

$$\begin{aligned}
&\Big|\nabla_{\tilde{x}'}\Big(\tilde{z}_{\varepsilon}^{-}(t,0,\tilde{x}')+\tilde{z}_{\varepsilon}^{+}(t,0,\tilde{x}')\Big)\Big|^2_{L^2((0,T^*)\times\mathbf{R}^{n-1}_{\tilde{x}'})}\\
&\le C\varepsilon^{-n/2}\int_{|t-t_0|^2+|\tilde{x}'-\tilde{x}'_0|^2=O(\varepsilon)}\Big[\varepsilon^{-2}O(|t-t_0|^6+|\tilde{x}'-\tilde{x}'_0|^6)+O(|t-t_0|^2+|\tilde{x}'-\tilde{x}'_0|^2)\Big]dtd\tilde{x}'\\
&= O(\varepsilon).
\end{aligned} \tag{4.65}$$

Similarly, one shows that

$$|\tilde{z}_{\varepsilon}^{-}(t,0,\tilde{x}')+\tilde{z}_{\varepsilon}^{+}(t,0,\tilde{x}')\Big|_{H^1(0,T^*)\times\mathbf{R}^{n-1}_{\tilde{x}'})} = O(\varepsilon^{1/2}). \tag{4.66}$$

Finally, combining (4.61), (4.65) and (4.66), we arrive at the desired result (4.20). This completes the proof of Lemma 4.1. □

- **Highly localized solutions of (1.1) without GCC**

Summing up the above analysis, we arrive at the following conclusion:

Proposition 4.7 *Let $(t, x^-(t), \xi^-(t))$, with $x^-(0) \in \Omega_2$, be an incoming ray for $\Box$, which arrives transversely at Γ_2 at time $t = t_0$, i.e., the first assumption in (4.15) holds. Let $(t, x^+(t), \xi^+(t))$ be the (global) reflected ray constructed above, with the reflected point $x_0 \in \Gamma_2$. Then there is a family of solutions $\{(y_\varepsilon, z_\varepsilon)\}_{\varepsilon>0}$ of system (1.1) in $(0, T^*) \times \Omega$ (the initial conditions being excepted) (recall (4.18) for T^*), such that*

$$|\partial_t y_\varepsilon|^2_{L^2((0,T^*)\times\Omega_1)} = O(\varepsilon), \qquad E_\varepsilon(0) = E(y_\varepsilon, z_\varepsilon, \partial_t z_\varepsilon)(0) \approx 1. \tag{4.67}$$

Proof. Put

$$\hat{y}_\varepsilon \equiv 0, \qquad \hat{z}_\varepsilon \stackrel{\triangle}{=} z_\varepsilon^+ + z_\varepsilon^-.$$

By Remark 4.4, one may assume the supports of z^- and z^+ are away from the interface γ. Hence, in view of Theorem 4.2, we deduce that $(\hat{y}_\varepsilon, \hat{z}_\varepsilon)$ are approximate solutions of system (1.1) in $(0, T^*) \times \Omega$ (the initial conditions and the boundary conditions for the hyperbolic component being excepted), in the sense that $(\hat{y}_\varepsilon, \hat{z}_\varepsilon)$ satisfy the heat equation, the boundary conditions for the parabolic component, the transmission conditions on the interface of (1.1), and

$$\sup_{t\in(0,T^*)} |\Box\, \hat{z}_\varepsilon|_{L^2(\Omega_2)} = O(\varepsilon^{1/2}). \tag{4.68}$$

Moreover,

$$E(\hat{y}_\varepsilon, \hat{z}_\varepsilon, \partial_t \hat{z}_\varepsilon)(0) \approx 1. \tag{4.69}$$

We may correct $\{(\hat{y}_\varepsilon, \hat{z}_\varepsilon)\}$ to become a family of exact solutions of equation (1.1). For this, let

$$y_\varepsilon = \hat{y}_\varepsilon + v_\varepsilon, \qquad z_\varepsilon = \hat{z}_\varepsilon + w_\varepsilon, \tag{4.70}$$

where $(v_\varepsilon, w_\varepsilon)$ solves

$$\begin{cases} \partial_t v_\varepsilon - \Delta v_\varepsilon = 0 & \text{in } (0,T^*)\times\Omega_1, \\ \Box\, w_\varepsilon = -\Box\, \hat{z}_\varepsilon = O(\varepsilon^{1/2}) & \text{in } (0,T^*)\times\Omega_2, \\ v_\varepsilon = 0 & \text{on } (0,T^*)\times\Gamma_1, \\ w_\varepsilon = -\hat{z}_\varepsilon & \text{on } (0,T^*)\times\Gamma_2, \\ v_\varepsilon = w_\varepsilon, \quad \dfrac{\partial v_\varepsilon}{\partial \nu_1} = -\dfrac{\partial w_\varepsilon}{\partial \nu_2} & \text{on } (0,T^*)\times\gamma, \\ v_\varepsilon(0) = 0 & \text{in } \Omega_1, \\ w_\varepsilon(0) = \partial_t w_\varepsilon(0) = 0 & \text{in } \Omega_2. \end{cases} \tag{4.71}$$

Then, noting (4.20), applying the classical energy method to system (4.71), similar to [13], it is easy to show that

$$|\partial_t v_\varepsilon|^2_{L^2((0,T^*)\times\Omega_1)} = O(\varepsilon). \tag{4.72}$$

Therefore, $(y_\varepsilon, z_\varepsilon)$ satisfy the conclusion of Proposition 4.7. □

As a direct consequence of Proposition 4.7, we have

Corollary 4.1 *Let Assumption 2.1 hold and $x(t) \notin \overline{\gamma}$ for any $t \in [0,T]$. Then there is a family of solutions $\{(y_\varepsilon, z_\varepsilon, \partial_t z_\varepsilon)\}_{\varepsilon>0}$ of system (1.1) in $[0,T]\times\Omega$ (the initial conditions being excepted), such that*

$$|\partial_t y_\varepsilon|^2_{L^2((0,T)\times\Omega_1)} = O(\varepsilon), \qquad E_\varepsilon(0) = E(y_\varepsilon, z_\varepsilon, \partial_t z_\varepsilon)(0) \approx 1. \tag{4.73}$$

From Corollary 4.1, one concludes that

Theorem 4.3 *Let $T>0$. Suppose the heat subdomain Ω_1 does not control geometrically the wave domain Ω_2 in time interval $[0,T]$, and the boundary $\partial\Omega_2$ of Ω_2 belongs to C^2. Then there exists a sequence of initial data $\{(y_0^m, z_0^m, z_1^m)\}_{m=1}^{\infty} \subset H$ such that:*

i)

$$|(y_0^m, z_0^m, z_1^m)|_H = 1; \tag{4.74}$$

ii) the corresponding solutions (y^m, z^m, z_t^m) of (1.1) satisfy

$$\lim_{m\to\infty} |y_t^m|_{L^2((0,T)\times\Omega_1)} = 0. \tag{4.75}$$

As for the proof of Theorem 4.3, we may proceed as in the second case in the proof of Theorem 3.4. Indeed, if the GCC fails, then there exists a multiply reflected ray that does not reach the interface γ. In view of Lemma 2.2, the multiply reflected ray can be assumed to have only transversal and non-normal reflections.

Remark 4.7 *By assuming that the boundary $\partial\Omega_2$ of Ω_2 belongs to C^∞ and changing the above argument by adding more terms on the wave constructions (more precisely on the constructions of the amplitude $a^\pm(t,x)$ and the phases $\phi^\pm(t,x)$), one sees that the Gaussian beam construction of approximate solutions for the wave equations allows proving that when the GCC fails one may strength (4.75) to get*

$$\lim_{m\to\infty} |y_t^m|_{H^s((0,T)\times\Omega_1)} = 0, \qquad \forall\, s \geq 0,$$

which means that in this case one can not even expect a weakened version of (1.7) to hold when the norm on its right side is replaced by any stronger Sobolev norm.

Remark 4.8 *The non-normal incidence assumption in (4.15) has not been used in the above construction. However, we will use it in the next subsection for constructing the approximate solutions for the heat equation when the ray intersects the interface γ.*

4.4 Highly localized approximate solutions for the heat equation with curved wavefronts

We now analyze what happens when the incoming ray $(t, x^-(t))$ for $\square$ arrives at the interface γ at time t_0, i.e., $x_0 \in \gamma$.

We still use the normal coordinates near the reflected point $x_0 \in \gamma$, called also $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \equiv (\tilde{x}_1, \tilde{x}')$, centered at the reflected point $\tilde{x}_0$, i.e., the new coordinate of x_0, such that:

i) γ is flat near $\tilde{x}_0$ in the new coordinates;

ii) Ω_1 is locally given by $\tilde{x}_1 < 0$, and the heat operator is locally as follows:

$$\widetilde{H} = \partial_t - \sum_{i,j=1}^{n} \tilde{\alpha}_{ij}(\tilde{x}) \partial_{\tilde{x}_i} \partial_{\tilde{x}_j} - \sum_{i=1}^{n} \tilde{\beta}_i(\tilde{x}) \partial_{\tilde{x}_i}; \tag{4.76}$$

iii) Ω_2 is locally given by $\tilde{x}_1 > 0$, and the operator $\Box$ takes the following form

$$\widetilde{W} = \partial_{tt} - \sum_{j,k=1}^{n} \tilde{\alpha}_{jk}(\tilde{x}) \partial_{\tilde{x}_j} \partial_{\tilde{x}_k} - \sum_{j=1}^{n} \tilde{\beta}_j(\tilde{x}) \partial_{\tilde{x}_j}; \tag{4.77}$$

iv) For $\tilde{x}$ belonging to a small neighborhood of $\tilde{x}_0$, $(\tilde{\alpha}_{ij})_{n \times n} \in C^2$ is strictly positive definite and symmetric and $\tilde{\beta}_j$ are C^1 functions. Moreover one has

$$\tilde{\alpha}_{11}(\tilde{x}) \equiv 1, \qquad \tilde{\alpha}_{1k}(\tilde{x}) \equiv 0 \text{ for any } k = 2, \cdots, n. \tag{4.78}$$

Put

$$F \triangleq \Big(\tilde{\alpha}_{ij}(\tilde{x}_0) \Big)_{2 \le i,j \le n}. \tag{4.79}$$

Obviously F is a positive definite matrix.

Assume $\tilde{z}_\varepsilon^-(t, \tilde{x})$ to be the $\tilde{x}$-coordinate expression of the incoming wave $z_\varepsilon^-(t, x)$ given in (4.16), i.e.,

$$\tilde{z}_\varepsilon^-(t, \tilde{x}) = \varepsilon^{1-n/4} [a^-(t) + (b^-)^T (x(\tilde{x}) - x^-(t))] e^{i\tilde{\phi}^-(t,\tilde{x})/\varepsilon} \tag{4.80}$$

where $\tilde{\phi}^-(t, \tilde{x})$ is as in (4.23). To construct approximate solutions of system (1.1), we will seek reflected waves $z_\varepsilon^+(t, x)$ as in the previous subsection and approximate solutions

$$\tilde{y}_\varepsilon = \tilde{y}_\varepsilon(t, \tilde{x}) = \varepsilon^{1-n/4} \widetilde{A}(t, \tilde{x}) e^{i\tilde{\psi}(t,\tilde{x})/\varepsilon} \tag{4.81}$$

of the heat equation:

$$\widetilde{H} \tilde{y} = 0. \tag{4.82}$$

The goal of this subsection is to describe the construction of the parabolic approximate solutions $\tilde{y}_\varepsilon$, which will be glued with the hyperbolic approximate solutions $z_\varepsilon^\pm(t, x)$ to produce approximate solutions of our transmission problem (1.1) in the next subsection.

To construct the approximate solutions $\tilde{y}_\varepsilon$ of (4.82), the idea is to match such solutions given by (4.81) to that of the wave equation such that

$$\tilde{\psi}(t, 0, \tilde{x}') = \tilde{\phi}^-(t, 0, \tilde{x}'). \tag{4.83}$$

This idea is the same as when the interface is flat. The case of linear phases suggests us that the case $\nabla_{\tilde{x}'} \tilde{\phi}^-(t, 0, \tilde{x}') \neq 0$ corresponding to non-normal incidence is the most natural one.

This is guaranteed by Assumption 2.1. Our treatment for the "flat interface" also indicates that by continuity $\tilde{\psi}$ should have positive imaginary part in $\tilde{x}_1 < 0$ near the interface.

We introduce the complex first order partial differential operator

$$\widetilde{\mathcal{H}} \triangleq \partial_t \tilde{\psi} - 2 \sum_{i,j=1}^{n} \tilde{\alpha}_{ij}(\tilde{x}) \partial_{\tilde{x}_i} \tilde{\psi} \partial_{\tilde{x}_j} - \sum_{i,j=1}^{n} \tilde{\alpha}_{ij}(\tilde{x}) \partial^2_{\tilde{x}_i \tilde{x}_j} \tilde{\psi} - \sum_{j=1}^{n} \tilde{\beta}_j \partial_{\tilde{x}_j} \tilde{\psi}. \tag{4.84}$$

Then, from (4.76) and (4.81), we get

$$\widetilde{H} \tilde{y}_{\varepsilon} = \varepsilon^{1-n/4} e^{i\tilde{\psi}/\varepsilon} \left[\frac{\widetilde{A}}{\varepsilon^2} \sum_{i,j=1}^{n} \tilde{\alpha}_{ij}(\tilde{x}) \partial_{\tilde{x}_i} \tilde{\psi} \partial_{\tilde{x}_j} \tilde{\psi} + \frac{i}{\varepsilon} \widetilde{\mathcal{H}} \widetilde{A} + \widetilde{H} \widetilde{A} \right]. \tag{4.85}$$

First, setting the leading term in the right hand side of (4.85) to be equal to zero yields the eikonal equation

$$\sum_{i,j=1}^{n} \tilde{\alpha}_{ij}(\tilde{x}) \partial_{\tilde{x}_i} \tilde{\psi} \partial_{\tilde{x}_j} \tilde{\psi} = 0, \tag{4.86}$$

or, by (4.78)), equivalently,

$$(\partial_{\tilde{x}_1} \tilde{\psi})^2 = - \sum_{i,j=2}^{n} \tilde{\alpha}_{ij}(\tilde{x}) \partial_{\tilde{x}_i} \tilde{\psi} \partial_{\tilde{x}_j} \tilde{\psi}. \tag{4.87}$$

Since $(\tilde{\alpha}_{ij})_{n\times n}$ is positive definite, there are no real solutions for (4.87). To insure that $\operatorname{Im} \tilde{\psi} > 0$ in $\tilde{x}_1 < 0$ one determines $\partial_{\tilde{x}_1} \tilde{\psi}(t, 0, \tilde{x}')$ as the solution of (4.83) and (4.87) with $\operatorname{Im} \partial_{\tilde{x}_1} \tilde{\psi} < 0$. This is possible since by Propositions 4.1 and 4.2, one has

$$\nabla_{\tilde{x}'} \tilde{\psi}(t_0, \tilde{x}_0) = \nabla_{\tilde{x}'} \tilde{\phi}^-(t_0, \tilde{x}_0) = \sigma'_- \neq 0.$$

Note that equation (4.87) is a complex equation and the initial value problem (4.83) and (4.87) is in general not solvable. However, similar to the case of flat interface, we do not need to solve (4.87) exactly. Instead, as we see later, it suffices to solve it up to order $O(|\tilde{x}_1|^2)$ near the interface. For this purpose, we determine uniquely from (4.87) the derivatives $\partial^j_{\tilde{x}_1} \tilde{\psi}(t, 0, \tilde{x}')$ for $j = 0, 1, 2$. More precisely, put

$$\begin{aligned}
&\tilde{\alpha}^0_{ij} \triangleq \tilde{\alpha}_{ij}(0, \tilde{x}'), \qquad \tilde{\alpha}^1_{ij} \triangleq \partial_{\tilde{x}_1} \tilde{\alpha}_{ij}(t, 0, \tilde{x}'), \qquad f_0 \equiv f_0(t, 0, \tilde{x}') \triangleq \tilde{\phi}^-(t, 0, \tilde{x}'), \\
&f_1 \equiv f_1(t, 0, \tilde{x}') \triangleq -i \sqrt{\sum_{i,j=2}^{n} \tilde{\alpha}^0_{ij} \partial_{\tilde{x}_i} \tilde{\phi}^-(t, 0, \tilde{x}') \partial_{\tilde{x}_j} \tilde{\phi}^-(t, 0, \tilde{x}')}\,, \\
&f_2 \equiv f_2(t, 0, \tilde{x}') \triangleq -\frac{1}{2f_1} \sum_{i,j=2}^{n} \Big(2\tilde{\alpha}^0_{ij} \partial_{\tilde{x}_i} f_0 \partial_{\tilde{x}_j} f_1 + \tilde{\alpha}^1_{ij} \partial_{\tilde{x}_i} f_0 \partial_{\tilde{x}_j} f_0 \Big).
\end{aligned} \tag{4.88}$$

We choose

$$\tilde{\psi}(t, \tilde{x}_1, \tilde{x}') \triangleq \sum_{j=0}^{2} \frac{f_j(t, 0, \tilde{x}')}{j!} \tilde{x}_1^j. \tag{4.89}$$

Then the following result holds:

Proposition 4.8 *For any given $(t, \tilde{x}')$, the function $\tilde{\psi}$ in (4.89) satisfies (4.83) and*

$$\sum_{i,j=1}^{n} \tilde{\alpha}_{ij}(\tilde{x})\partial_{\tilde{x}_i}\tilde{\psi}\partial_{\tilde{x}_j}\tilde{\psi} = O(|\tilde{x}_1|^2) \qquad \text{as} \quad \tilde{x}_1 \to 0. \tag{4.90}$$

Proof. From (4.88), we see that

$$f_1^2 + \sum_{i,j=2}^{n} \tilde{\alpha}_{ij}^0 \partial_{\tilde{x}_i} f_0 \partial_{\tilde{x}_j} f_0 \equiv 0, \qquad 2f_1 f_2 + \sum_{i,j=2}^{n} \Big(2\tilde{\alpha}_{ij}^0 \partial_{\tilde{x}_i} f_0 \partial_{\tilde{x}_j} f_1 + \tilde{\alpha}_{ij}^1 \partial_{\tilde{x}_i} f_0 \partial_{\tilde{x}_j} f_0\Big) \equiv 0.$$

Therefore, by (4.78) and (4.89), one sees that

$$\begin{aligned}
&\sum_{i,j=1}^{n} \tilde{\alpha}_{ij}(\tilde{x})\partial_{\tilde{x}_i}\tilde{\psi}\partial_{\tilde{x}_j}\tilde{\psi} \\
&= (f_1 + f_2\tilde{x}_1)^2 + \sum_{i,j=2}^{n} \tilde{\alpha}_{ij}(\tilde{x})\Big(\partial_{\tilde{x}_i} f_0 + \partial_{\tilde{x}_i} f_1 \tilde{x}_1 + O(|\tilde{x}_1|^2)\Big)\Big(\partial_{\tilde{x}_j} f_0 + \partial_{\tilde{x}_j} f_1 \tilde{x}_1 + O(|\tilde{x}_1|^2)\Big) \\
&= f_1^2 + 2f_1 f_2 \tilde{x}_1 + \sum_{i,j=2}^{n} \tilde{\alpha}_{ij}(\tilde{x})\Big(\partial_{\tilde{x}_i} f_0 + \partial_{\tilde{x}_i} f_1 \tilde{x}_1\Big)\Big(\partial_{\tilde{x}_j} f_0 + \partial_{\tilde{x}_j} f_1 \tilde{x}_1\Big) + O(|\tilde{x}_1|^2) \\
&= f_1^2 + 2f_1 f_2 \tilde{x}_1 + \sum_{i,j=2}^{n} (\tilde{\alpha}_{ij}^0 + \tilde{\alpha}_{ij}^1 \tilde{x}_1)\Big(\partial_{\tilde{x}_i} f_0 + \partial_{\tilde{x}_i} f_1 \tilde{x}_1\Big)\Big(\partial_{\tilde{x}_j} f_0 + \partial_{\tilde{x}_j} f_1 \tilde{x}_1\Big) + O(|\tilde{x}_1|^2) \\
&= f_1^2 + \sum_{i,j=2}^{n} \tilde{\alpha}_{ij}^0 \partial_{\tilde{x}_i} f_0 \partial_{\tilde{x}_j} f_0 + \Big[2f_1 f_2 + \sum_{i,j=2}^{n} \Big(2\tilde{\alpha}_{ij}^0 \partial_{\tilde{x}_i} f_0 \partial_{\tilde{x}_j} f_1 + \tilde{\alpha}_{ij}^1 \partial_{\tilde{x}_i} f_0 \partial_{\tilde{x}_j} f_0\Big)\Big]\tilde{x}_1 + O(|\tilde{x}_1|^2) \\
&= O(|\tilde{x}_1|^2),
\end{aligned}$$

which gives (4.90). □

We next compute the leading term of $\tilde{\psi}(t, \tilde{x}_1, \tilde{x}')$.

Proposition 4.9 *The function $\tilde{\psi}$ satisfies, for some constant $c > 0$,*

$$\operatorname{Im} \tilde{\psi}(t, \tilde{x}_1, \tilde{x}') \geq c(|t - t_0|^2 + |\tilde{x}_1| + |\tilde{x}' - \tilde{x}_0'|^2), \qquad \text{as} \quad (t, \tilde{x}') \to (t_0, \tilde{x}_0') \text{ and } \tilde{x}_1 \to 0^-, \tag{4.91}$$

and

$$\operatorname{Im} \partial_{\tilde{x}_1} \tilde{\psi}(t_0, \tilde{x}_0) < 0. \tag{4.92}$$

Proof. Using (4.88) and (4.89), we get

$$\tilde{\psi}(t, \tilde{x}_1, \tilde{x}') = \tilde{\phi}^-(t, 0, \tilde{x}') - i\sqrt{\sum_{i,j=2}^{n} \tilde{\alpha}_{ij}^0 \partial_{\tilde{x}_i} \tilde{\phi}^-(t, 0, \tilde{x}') \partial_{\tilde{x}_j} \tilde{\phi}^-(t, 0, \tilde{x}')}\ \tilde{x}_1 + O(|\tilde{x}_1|^2). \tag{4.93}$$

By Proposition 4.1 and (4.39) in Proposition 4.2, we see that (recall (4.24) for σ_-')

$$\nabla_{\tilde{x}'} \tilde{\phi}^-(t_0, \tilde{x}_0) = \sigma_-' \neq 0.$$

Hence, we conclude that (recall (4.79) for the positive definite matrix F)

$$
\begin{aligned}
&\mathrm{Re}\sqrt{\sum_{i,j=2}^{n}\tilde{\alpha}^0_{ij}\partial_{\tilde{x}_i}\tilde{\phi}^-(t,0,\tilde{x}')\partial_{\tilde{x}_j}\tilde{\phi}^-(t,0,\tilde{x}')}\\
&=\mathrm{Re}\sqrt{\sum_{i,j=2}^{n}\tilde{\alpha}_{ij}(\tilde{x}_0)\partial_{\tilde{x}_i}\tilde{\phi}^-(t_0,\tilde{x}_0)\partial_{\tilde{x}_j}\tilde{\phi}^-(t_0,\tilde{x}_0)+O(|t-t_0|+|\tilde{x}'-\tilde{x}_0'|)}\\
&=\mathrm{Re}\sqrt{(\sigma_-')^TF\sigma_-'+O(|t-t_0|+|\tilde{x}'-\tilde{x}_0'|)}=\mathrm{Re}\left[\sqrt{(\sigma_-')^TF\sigma_-'}+O(|t-t_0|+|\tilde{x}'-\tilde{x}_0'|)\right]\\
&\geq c>0,\qquad \text{as } t\to t_0.
\end{aligned}
\tag{4.94}
$$

Combining (4.93), (4.60) and (4.94), we arrive at the desired estimates (4.91) and (4.92). □

Remark 4.9 *From Proposition 4.9, it is easy to see that for any $T>t_0$, the factor $e^{i\tilde{\psi}/\varepsilon}$ localizes $\tilde{y}_\varepsilon$ as in (4.81) in the region*

$$
Q^\varepsilon \triangleq \left\{(t,\tilde{x}_1,\tilde{x}')\in(0,T)\times(-\infty,0)\times\mathbb{R}^{n-1}\;\Big|\;|t-t_0|^2+|\tilde{x}_1|+|\tilde{x}'-\tilde{x}_0'|^2=O(\varepsilon)\right\}, \tag{4.95}
$$

and as we will show later, (4.90) is a sufficient approximation of the original eikonal equation (4.86) in that region. Further, this proposition also tells us that the energy of the parabolic component $\tilde{y}_\varepsilon$ is localized in a small neighborhood of the reflected point $(t_0,\tilde{x}_0)$, of size $O(\varepsilon)$ in $\tilde{x}_1$ but $O(\sqrt{\varepsilon})$ in t and $\tilde{x}'$.

Next, setting the second term in the right hand side of (4.85) to be equal to 0 yields the equation

$$
\widetilde{\mathcal{H}}\widetilde{A}=0. \tag{4.96}
$$

Matching with the incoming wave will lead to the initial conditions for $\widetilde{A}$ at $\tilde{x}_1=0$, i.e., for a suitable function $g_0(t,\tilde{x}')$, we will have

$$
\widetilde{A}(t,0,\tilde{x}')=g_0(t,\tilde{x}'). \tag{4.97}
$$

This is carried out in the next subsection. The resulting initial value problem (4.96)–(4.97) for $\widetilde{A}$ is ill posed, but similar to solving the eikonal equation (4.87), it suffices to solve (4.96) approximately. For this, we choose

$$
\begin{aligned}
g_1(t,\tilde{x}')\triangleq{}&\frac{1}{2\partial_{\tilde{x}_1}\tilde{\psi}(t,0,\tilde{x}')}\Bigg[-2\sum_{i,j=2}^{n}\tilde{\alpha}_{ij}(0,\tilde{x}')\partial_{\tilde{x}_i}\tilde{\psi}(t,0,\tilde{x}')\partial_{\tilde{x}_j}g_0\\
&+\left(\partial_t\tilde{\psi}(t,0,\tilde{x}')-\sum_{i,j=1}^{n}\tilde{\alpha}_{ij}(0,\tilde{x}')\partial^2_{\tilde{x}_i\tilde{x}_j}\tilde{\psi}(t,0,\tilde{x}')-\sum_{j=1}^{n}\tilde{\beta}_j(0,\tilde{x}')\partial_{\tilde{x}_j}\tilde{\psi}(t,0,\tilde{x}')\right)g_0\Bigg].
\end{aligned}
\tag{4.98}
$$

Now, put

$$
\widetilde{A}(t,\tilde{x})\triangleq g_0(t,\tilde{x}')+g_1(t,\tilde{x}')\tilde{x}_1. \tag{4.99}
$$

Then, similar to the proof of Proposition 4.8, we conclude that

Proposition 4.10 *For $\widetilde{A}$ as in (4.99), it holds*

$$\widetilde{\mathcal{H}}\widetilde{A} = O(|\tilde{x}_1|) \qquad \text{as} \quad \tilde{x}_1 \to 0. \tag{4.100}$$

We conclude that:

Proposition 4.11 *For any $T > t_0$ and any $g_0 \in C^2$, the function $\tilde{y}_\varepsilon$ defined by (4.81), with $\tilde{\psi}$ and $\widetilde{A}$ given by (4.89) and (4.99) respectively, satisfies*

$$\int_0^T \int_{\{\tilde{x}_1<0\}} |\widetilde{H}\tilde{y}_\varepsilon|^2 dt d\tilde{x} = O(\varepsilon^3). \tag{4.101}$$

Proof. By (4.85) and Propositions 4.8 and 4.10, we find

$$\int_0^T \int_{\{\tilde{x}_1<0\}} |\widetilde{H}\tilde{y}_\varepsilon|^2 dt d\tilde{x} \leq C\varepsilon^{2-n/2} \int_0^T \int_{\{\tilde{x}_1<0\}} e^{i2\mathrm{Im}\,\tilde{\psi}/\varepsilon} \Big[\varepsilon^{-4} O(|\tilde{x}_1|^4) + \varepsilon^{-2} O(|\tilde{x}_1|^2) + 1\Big] dt d\tilde{x}.$$

From Proposition 4.9, it is easy to see that the factor $e^{i2\mathrm{Im}\,\tilde{\psi}/\varepsilon}$ localizes the above integrant in the region Q^ε defined by (4.95). Hence,

$$\int_0^T \int_{\{\tilde{x}_1<0\}} |\widetilde{H}\tilde{y}_\varepsilon|^2 dt d\tilde{x} \leq C\varepsilon^{2-n/2} \int_{Q^\varepsilon} \Big[\varepsilon^{-4} O(|\tilde{x}_1|^4) + \varepsilon^{-2} O(|\tilde{x}_1|^2) + 1\Big] dt d\tilde{x} = O(\varepsilon^3),$$

which gives the desired result (4.101). □

Remark 4.10 *By (4.95) in Remark 4.9, one sees that $\tilde{y}_\varepsilon$ is mainly located in a small neighborhood of the reflected point $(t_0, \tilde{x}_0)$. Fix any two neighborhoods $\mathcal{O}_j$ (j=1,2) of $(t_0, \tilde{x}_0)$ in the half space $\Big\{(t, \tilde{x}) \;\Big|\; \tilde{x}_1 < 0\Big\}$ small enough such that estimate (4.91) holds and $\mathcal{O}_1$ is a proper subset of $\mathcal{O}_2$. For any given C^2 function $\theta = \theta(t, \tilde{x})$, with $\theta \equiv 1$ in $\mathcal{O}_1$ and $\theta \equiv 0$ in $\Big\{(t, \tilde{x}) \;\Big|\; \tilde{x}_1 < 0\Big\} \setminus \mathcal{O}_2$, replacing $\tilde{y}_\varepsilon$ by $\theta\tilde{y}_\varepsilon$, using Proposition 4.9 and noting (4.95) in Remark 4.9 again, one finds that Proposition 4.11 remains true for the new function $\theta\tilde{y}_\varepsilon$. In this way, $\theta\tilde{y}_\varepsilon$ is still a family of approximate solutions of the heat equation (4.82), which are supported in a small neighborhood of $(t_0, \tilde{x}_0)$ but may be defined globally for all $\tilde{x}_1 < 0$, even if, as we do in the next subsection, $\tilde{y}_\varepsilon$ is first only constructed in a small neighborhood of the reflected point $(t_0, \tilde{x}_0)$.*

4.5 Highly concentrated solutions of the transmission problem

Assume $z_\varepsilon^-(t, x)$ given by (4.16) to be the incoming wave. First, we seek approximate solutions $(\tilde{y}_\varepsilon, \tilde{z}_\varepsilon)$ of the transmission problem (1.1) but in the new coordinates $(\tilde{x}_1, \cdots, \tilde{x}_n)$, in which $\tilde{y}_\varepsilon$ is an approximate solution given by (4.81) to the heat equation, and $\tilde{z}_\varepsilon$ is a Gaussian Beam solution of the wave equation of the form

$$\begin{aligned}\tilde{z}_\varepsilon = \tilde{z}_\varepsilon^-(t, \tilde{x}) + \tilde{z}_\varepsilon^+(t, \tilde{x}) = \varepsilon^{1-n/4} \Big\{&[a^-(t) + (b^-)^T (x(\tilde{x}) - x^-(t))] e^{i\tilde{\phi}^-(t,\tilde{x})/\varepsilon} \\ &+ [a^+(t) + (b^+)^T (x(\tilde{x}) - x^+(t))] e^{i\tilde{\phi}^+(t,\tilde{x})/\varepsilon}\Big\},\end{aligned} \tag{4.102}$$

where $\tilde{z}^\pm(t,\tilde{x})$ and $\tilde{\phi}^\pm(t,\tilde{x})$ are the new coordinate expressions of $z^\pm(t,x)$ and $\phi^\pm(t,x)$ given by (4.16), (4.17), (4.19) and (4.22).

According to the Gaussian beam construction for approximate solutions of the wave equation in Subsection 4.3, and the construction of approximate solutions for the heat equation in Subsection 4.4, one needs to determine $a^+(t_0)$ and $\widetilde{A}(t,0,\tilde{x}')$, the initial value of $a^+(t)$ at $t=t_0$ and $\widetilde{A}(t,\tilde{x}_1,\tilde{x}')$ at $\tilde{x}_1=0$, respectively.

First, before determining $a^+(t_0)$ and $\widetilde{A}(t,0,\tilde{x}')$, we recall that in the phases $\phi^\pm$, defined by (4.17) and (4.22), we introduce extra terms of order $|x-x^\pm(t)|^3$, which play no roles in the Subsection 4.3 (since their coefficients $a^\pm_{ijk}$ can be chosen arbitrarily there). In this subsection, however, as mentioned in Remark 4.5, we need to choose a^+_{ijk} suitably such that (4.55) holds for the third order derivatives as well. In other words, we have

Proposition 4.12 *For any given complex numbers a^-_{ijk}, $i,j,k=1,2,\cdots,n$ so that $a^-_{ijk}=a^-_{i'j'k'}$ for any permutation i',j',k' of i,j,k, there are complex numbers a^+_{ijk}, $i,j,k=1,2,\cdots,n$ with $a^+_{ijk}=a^+_{i'j'k'}$ for any permutation i',j',k' of i,j,k, such that the phases $\tilde{\phi}^\pm$ defined by (4.23) satisfy*

$$\tilde{\phi}^+(t,0,\tilde{x}')-\tilde{\phi}^-(t,0,\tilde{x}')=O(|t-t_0|^4+|\tilde{x}'-\tilde{x}_0'|^4),\qquad \text{as } (t,\tilde{x}')\to(t_0,\tilde{x}_0'). \tag{4.103}$$

The proof of Proposition 4.12 is given in Appendix C.

Next, from the transmission condition on the reflected point $(t_0,\tilde{x}_0)$, one concludes that

$$\begin{cases} \widetilde{A}(t_0,\tilde{x}_0)=a^-(t_0)+a^+(t_0), \\ \partial_{\tilde{x}_1}\tilde{\psi}(t_0,\tilde{x}_0)\widetilde{A}(t_0,\tilde{x}_0)=\partial_{\tilde{x}_1}\tilde{\phi}^-(t_0,\tilde{x}_0)a^-(t_0)+\partial_{\tilde{x}_1}\tilde{\phi}^+(t_0,\tilde{x}_0)a^+(t_0). \end{cases} \tag{4.104}$$

However, by (4.39) in Proposition 4.2, noting Proposition 4.1, and recalling (4.24), we see that $\partial_{\tilde{x}_1}\tilde{\phi}^\pm(t_0,\tilde{x}_0)=\sigma_1^\pm\neq0$. Hence, equation (4.104) determines uniquely $\widetilde{A}(t_0,\tilde{x}_0)$ and $a^+(t_0)$ in terms of $a^-(t_0)$, $\partial_{\tilde{x}_1}\tilde{\psi}(t_0,\tilde{x}_0)$ and $\partial_{\tilde{x}_1}\tilde{\phi}^\pm(t_0,\tilde{x}_0)$. In particular, we get

$$a^+(t_0)=\frac{\left[\partial_{\tilde{x}_1}\tilde{\phi}^-(t_0,\tilde{x}_0)-\partial_{\tilde{x}_1}\tilde{\psi}(t_0,\tilde{x}_0)\right]a^-(t_0)}{\partial_{\tilde{x}_1}\tilde{\psi}(t_0,\tilde{x}_0)-\partial_{\tilde{x}_1}\tilde{\phi}^+(t_0,\tilde{x}_0)}. \tag{4.105}$$

Choosing the initial data

$$\widetilde{A}(t,0,\tilde{x}')\equiv\widetilde{A}(t_0,\tilde{x}_0), \tag{4.106}$$

one determines $\widetilde{A}(t,\tilde{x})$ according to (4.98)–(4.99) with $g_0(t,\tilde{x}')=\widetilde{A}(t_0,\tilde{x}_0)$. In view of Subsection 4.4, this completes the construction of $\tilde{y}_\varepsilon$.

Finally, note that in the amplitude $a^\pm(t)+(b^\pm)^T(x(\tilde{x})-x^\pm(t))$ of $\tilde{z}^\pm_\varepsilon$ in (4.102), we also introduced an extra terms of $O(|x-x^\pm(t)|)$, which play no roles in the Subsection 4.3, either. However, as we shall see, this gives us one more freedom to achieve higher order of precision for approximate solutions to the transmission problem. We have

Proposition 4.13 *There is a $b^\pm\in\mathbb{C}^n$ such that*

$$a^\pm(t)+(b^\pm)^T(x(0,\tilde{x}')-x^\pm(t))=a^\pm(t_0)+O(|t-t_0|^2+|\tilde{x}'-\tilde{x}_0'|^2) \tag{4.107}$$

as $(t,\tilde{x}_0')\to(t_0,\tilde{x}_0')$.

Proof. By (4.35) and (4.36), one has

$$\begin{aligned} &a^{\pm}(t) + (b^{\pm})^T(x(0,\tilde{x}') - x^{\pm}(t)) \\ &= a^{\pm}(t_0) + a_t^{\pm}(t_0)(t-t_0) + (b^{\pm})^T J(\tilde{x}_0)(-2\eta_1^{\pm}(t-t_0), \tilde{x}' - \tilde{x}_0' - 2\eta_{\pm}'(t-t_0)) \\ &\quad + O(|t-t_0|^2 + |\tilde{x}' - \tilde{x}_0'|^2). \end{aligned} \tag{4.108}$$

We choose

$$b^{\pm} = ((J(\tilde{x}_0)^{-1})^T\Big(\frac{a_t^{\pm}(t_0)}{2\eta_1^{\pm}}, 0, \cdots, 0\Big)^T. \tag{4.109}$$

Then, from (4.108) we conclude that (4.107) holds. □

This completes the construction of the (local) approximate solutions $(\tilde{y}_\varepsilon, \tilde{z}_\varepsilon)$ of the transmission problem (1.1) (in the new coordinates).

Now, putting

$$\hat{z}_\varepsilon \equiv \hat{z}_\varepsilon \stackrel{\triangle}{=} z^-(t,x) + z^+(t,x),$$

and by Remark 4.10, returning the approximate solutions $\theta\tilde{y}_\varepsilon$ to the original coordinates, called henceforth $\hat{y}_\varepsilon$, we actually obtain global approximate solutions $(\hat{y}_\varepsilon, \hat{z}_\varepsilon)$ of the transmission problem (1.1).

We arrive at the following conclusion:

Lemma 4.2 *Let $(t, x^-(t), \xi^-(t))$, with $x^-(0) \in \Omega_2$, be an incoming ray, which arrives transversely and non-normally at γ at time $t = t_0$. Let $(t, x^+(t), \xi^+(t))$ be the reflected ray. Then $(\hat{y}_\varepsilon, \hat{z}_\varepsilon)$ constructed above satisfy (recall (4.18) for T^*):*

$$\begin{cases} (\partial_t - \Delta)\hat{y}_\varepsilon = r_1 & \text{in } (0,T^*) \times \Omega_1, \\ \Box\, \hat{z}_\varepsilon = r_2 & \text{in } (0,T^*) \times \Omega_2, \\ \hat{y}_\varepsilon = 0 & \text{on } (0,T^*) \times \Gamma_1, \\ \hat{z}_\varepsilon = r_3 & \text{on } (0,T^*) \times \Gamma_2, \\ \hat{y}_\varepsilon = \hat{z}_\varepsilon + r_4, \quad \dfrac{\partial \hat{y}_\varepsilon}{\partial \nu_1} = -\dfrac{\partial \hat{z}_\varepsilon}{\partial \nu_2} + r_5 & \text{on } (0,T^*) \times \gamma, \end{cases} \tag{4.110}$$

with

$$\begin{aligned} &|r_1|_{L^2((0,T^*)\times\Omega_1)} = O(\varepsilon^{3/2}), \quad |r_2|_{L^2((0,T^*)\times\Omega_2)} = O(\varepsilon^{1/2}), \quad |r_3|_{H^1((0,T^*)\times\Gamma_2)} = O(\varepsilon^{1/2}), \\ &|r_4|_{H^{3/2}((0,T^*)\times\gamma)} = O(\varepsilon^{1/4}), \qquad |r_5|_{H^{3/2}(0,T^*)\times\gamma)} = O(\varepsilon^{1/4}), \\ &|\partial_t \hat{y}_\varepsilon|^2_{L^2((0,T^*)\times\Omega_1)} = O(\varepsilon), \qquad E_\varepsilon(0) = E(\hat{y}_\varepsilon, \hat{z}_\varepsilon, \partial_t \hat{z}_\varepsilon)(0) \approx 1. \end{aligned} \tag{4.111}$$

Moreover, $\hat{y}_\varepsilon$ can be chosen so that its support is located in any given small neighborhood of the reflected point; while $\hat{z}_\varepsilon$ can be chosen so that its support is located in any given neighborhood of the incoming and reflected rays. In particular, one may have

$$\operatorname{supp} r_j \subset (0,T^*) \times \operatorname{Int}\gamma, \qquad j = 4,5. \tag{4.112}$$

Proof. We only prove the estimate on $|r_4|_{H^1((0,T^*)\times\Gamma_2)}$. The other estimates in (4.111) can be either treated similarly or known from the analysis in Subsections 4.3–4.4. Let $\tilde{r}_4$ be the expressions of r_4 in the new coordinate $\tilde{x}$. Then, by (4.81), (4.83), (4.99), (4.102) and (4.106), we see that

$$\begin{aligned}\tilde{r}_4(t,0,\tilde{x}') = \varepsilon^{1-n/4}\Big\{&\widetilde{A}(t_0,\tilde{x}_0)e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}\\ &-[a^-(t)+(b^-)^T(x(0,\tilde{x}')-x^-(t))]e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}\\ &-[a^+(t)+(b^+)^T(x(0,\tilde{x}')-x^+(t))]e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}\Big\}.\end{aligned} \tag{4.113}$$

Hence, using Proposition 4.6, one gets

$$|r_4|_{L^2((0,T^*)\times\gamma)} = O(\varepsilon). \tag{4.114}$$

From (4.113), one has

$$\begin{aligned}\partial_t\tilde{r}_4(t,0,\tilde{x}') = i\varepsilon^{-n/4}\Big[&\partial_t\tilde{\phi}^-(t,0,\tilde{x}')\widetilde{A}(t_0,\tilde{x}_0)e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}\\ &-[a^-(t)+(b^-)^T(x(0,\tilde{x}')-x^-(t))]\partial_t\tilde{\phi}^-(t,0,\tilde{x}')e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}\\ &-[a^+(t)+(b^+)^T(x(0,\tilde{x}')-x^+(t))]\partial_t\tilde{\phi}^+(t,0,\tilde{x}')e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}\Big]\\ &+e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}O(\varepsilon^{1-n/4})+e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}O(\varepsilon^{1-n/4}).\end{aligned} \tag{4.115}$$

Noting (4.38) and the first condition in (4.104), we get

$$\begin{aligned}&\partial_t\tilde{\phi}^-(t,0,\tilde{x}')\widetilde{A}(t_0,\tilde{x}_0)e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}\\ &\quad-[a^-(t)+(b^-)^T(x(0,\tilde{x}')-x^-(t))]\partial_t\tilde{\phi}^-(t,0,\tilde{x}')e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}\\ &\quad-[a^+(t)+(b^+)^T(x(0,\tilde{x}')-x^+(t))]\partial_t\tilde{\phi}^+(t,0,\tilde{x}')e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}\\ &=-\tfrac{a^+(t_0)}{2}\Big[e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}-e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}\Big]\\ &\quad+e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}O(|t-t_0|+|\tilde{x}'-\tilde{x}_0'|)+e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}O(|t-t_0|+|\tilde{x}'-\tilde{x}_0'|).\end{aligned} \tag{4.116}$$

Also, by Proposition 4.12, we see that

$$\begin{aligned}&e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}-e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}\\ &=\frac{i}{\varepsilon}\int_0^1 e^{i[\tilde{\phi}^+(t,0,\tilde{x}')+s(\tilde{\phi}^-(t,0,\tilde{x}')-\tilde{\phi}^+(t,0,\tilde{x}'))]/\varepsilon}ds\,O(|t-t_0|^4+|\tilde{x}'-\tilde{x}_0'|^4).\end{aligned} \tag{4.117}$$

Now, by (4.115)–(4.117) and Proposition 4.6, and similar to (4.65) in the proof of Lemma 4.1, we arrive at

$$|\partial_t r_4|_{L^2((0,T^*)\times\gamma)} = O(\varepsilon^{1/2}). \tag{4.118}$$

We have the same estimate for $\nabla_{\tilde{x}'}\tilde{r}_4$, which combined with (4.114) and (4.118), yields

$$|r_4|_{H^1((0,T^*)\times\gamma)} = O(\varepsilon^{1/2}). \tag{4.119}$$

Further, from (4.115) and using (4.107) in Proposition 4.13, we get

$$
\begin{aligned}
&\partial_{tt}\tilde{r}_4(t,0,\tilde{x}')\\
&= -\varepsilon^{-1-n/4}\Big[\Big(\partial_t\tilde{\phi}^-(t,0,\tilde{x}')\Big)^2\widetilde{A}(t_0,\tilde{x}_0)e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}\\
&\qquad\qquad -[a^-(t_0)+O(|t-t_0|^2+|\tilde{x}'-\tilde{x}_0'|^2)]\Big(\partial_t\tilde{\phi}^-(t,0,\tilde{x}')\Big)^2 e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}\\
&\qquad\qquad -[a^+(t_0)+O(|t-t_0|^2+|\tilde{x}'-\tilde{x}_0'|^2)]\Big(\partial_t\tilde{\phi}^+(t,0,\tilde{x}')\Big)^2 e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}\Big]\\
&\quad +e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}O(\varepsilon^{-n/4})+e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}O(\varepsilon^{-n/4}).
\end{aligned}
\tag{4.120}
$$

Using (4.38), (4.40) and (4.41), and noting (4.55) in Remark 4.5, we conclude that there are $\tau_0 \in \mathbb{C}$ and $\tau \in \mathbb{C}^{n-1}$ such that

$$
\partial_t\tilde{\phi}^\pm(t,0,\tilde{x}') = -\frac{1}{2}+\tau_0(t-t_0)+\tau^T(\tilde{x}'-\tilde{x}_0')+O(|t-t_0|^2+|\tilde{x}'-\tilde{x}_0'|^2). \tag{4.121}
$$

Hence, by (4.83), (4.121) and the first condition in (4.104), one gets

$$
\begin{aligned}
&\Big(\partial_t\tilde{\phi}^-(t,0,\tilde{x}')\Big)^2\widetilde{A}(t_0,\tilde{x}_0)e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}\\
&\quad -[a^-(t_0)+O(|t-t_0|^2+|\tilde{x}'-\tilde{x}_0'|^2)]\Big(\partial_t\tilde{\phi}^-(t,0,\tilde{x}')\Big)^2 e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}\\
&\quad -[a^+(t_0)+O(|t-t_0|^2+|\tilde{x}'-\tilde{x}_0'|^2)]\Big(\partial_t\tilde{\phi}^+(t,0,\tilde{x}')\Big)^2 e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}\\
&= \tfrac{a^+(t_0)}{4}\Big[e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}-e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}\Big]-\tau_0 a^+(t_0)\Big[e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}-e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}\Big](t-t_0)\\
&\quad -a^+(t_0)\Big[e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}-e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}\Big]\tau^T(\tilde{x}'-\tilde{x}_0')\\
&\quad +e^{i\tilde{\phi}^-(t,0,\tilde{x}')/\varepsilon}O(|t-t_0|^2+|\tilde{x}'-\tilde{x}_0'|^2)+e^{i\tilde{\phi}^+(t,0,\tilde{x}')/\varepsilon}O(|t-t_0|^2+|\tilde{x}'-\tilde{x}_0'|^2).
\end{aligned}
\tag{4.122}
$$

Consequently, combining (4.120), (4.122) and (4.117), and similar to (4.118), we end up with

$$
|\partial_{tt}r_4|_{L^2((0,T^*)\times\gamma)} = O(1). \tag{4.123}
$$

We have the same estimate for $\nabla^2_{t,\tilde{x}'}\tilde{r}_4$, which combined with (4.114), (4.119) and (4.123), yields

$$
|r_4|_{H^2((0,T^*)\times\gamma)} = O(1). \tag{4.124}
$$

Applying the usual interpolation method to (4.119) and (4.124), we obtain the desired estimate for r_4.

Finally, noting Remarks 4.4 and 4.10, by multiplying suitable cut-off functions if necessary, one obtains the desired highly concentrated approximate solutions. This completes the proof of Lemma 4.2. □

The above analysis yields the following result:

Lemma 4.3 *Let the assumptions in Lemma 4.2 hold. Then there is a family of solutions* $\{(y_\varepsilon, z_\varepsilon)\}_{\varepsilon>0}$ *of system (1.1) in* $(0,T^*)$ *(the initial conditions being excepted), such that*

$$
|\partial_t y_\varepsilon|^2_{L^2((0,T^*)\times\Omega_1)} = O(\varepsilon^{1/2}), \qquad E_\varepsilon(0)=E(y_\varepsilon,z_\varepsilon,\partial_t z_\varepsilon)(0)\approx 1. \tag{4.125}
$$

Proof. We now correct the approximate solutions $\{(\hat{y}_\varepsilon, \hat{z}_\varepsilon)\}$ of system (1.1), given in Lemma 4.2, to become a family of exact solutions of equation (1.1). For this, let

$$y_\varepsilon = \hat{y}_\varepsilon + v_\varepsilon^1 + v_\varepsilon^2, \qquad z_\varepsilon = \hat{z}_\varepsilon + w_\varepsilon^1 + w_\varepsilon^2, \tag{4.126}$$

where $(v_\varepsilon^i, w_\varepsilon^i)$ $(i = 1, 2)$ solve respectively (recall Lemma 4.2 for r_j, $j = 1, 2, 3, 4, 5$)

$$\begin{cases} \partial_t v_\varepsilon^1 - \Delta v_\varepsilon^1 = -r_1 & \text{in } (0, T^*) \times \Omega_1, \\ \Box\, w_\varepsilon^1 = -r_2 & \text{in } (0, T^*) \times \Omega_2, \\ v_\varepsilon^1 = 0 & \text{on } (0, T^*) \times \Gamma_1, \\ w_\varepsilon^1 = -r_3 & \text{on } (0, T^*) \times \Gamma_2, \\ v_\varepsilon^1 = w_\varepsilon^1, \quad \dfrac{\partial v_\varepsilon^1}{\partial \nu_1} = -\dfrac{\partial w_\varepsilon^1}{\partial \nu_2} & \text{on } (0, T^*) \times \gamma, \\ v_\varepsilon^1(0) = 0 & \text{in } \Omega_1, \\ w_\varepsilon^1(0) = \partial_t w_\varepsilon^1(0) = 0 & \text{in } \Omega_2 \end{cases} \tag{4.127}$$

and

$$\begin{cases} \partial_t v_\varepsilon^2 - \Delta v_\varepsilon^2 = 0 & \text{in } (0, T^*) \times \Omega_1, \\ \Box\, w_\varepsilon^2 = 0 & \text{in } (0, T^+) \times \Omega_2, \\ v_\varepsilon^2 = 0 & \text{on } (0, T^*) \times \Gamma_1, \\ w_\varepsilon^2 = 0 & \text{on } (0, T^*) \times \Gamma_2, \\ v_\varepsilon^2 = w_\varepsilon^2 - r_4, \quad \dfrac{\partial v_\varepsilon^2}{\partial \nu_1} = -\dfrac{\partial w_\varepsilon^2}{\partial \nu_2} - r_5 & \text{on } (0, T^*) \times \gamma, \\ v_\varepsilon^2(0) = 0 & \text{in } \Omega_1, \\ w_\varepsilon^2(0) = \partial_t w_\varepsilon^2(0) = 0 & \text{in } \Omega_2, \end{cases} \tag{4.128}$$

Similar to (4.72) in the proof of Proposition of 4.7, it is easy to show that

$$|\partial_t v_\varepsilon^1|^2_{L^2((0,T^*)\times\Omega_1)} = O(\varepsilon). \tag{4.129}$$

Then, applying the classical energy method to system (4.128), we conclude that for any $s' \in [0, T^*]$, it holds

$$\begin{aligned} &\int_0^{s'} \int_{\Omega_1} |\partial_t v_\varepsilon^2|^2 dxdt + \frac{1}{2} \int_{\Omega_1} [|\nabla v_\varepsilon^2(s', x)|^2 + |v_\varepsilon^2(s', x)|^2] dx \\ &\quad + \frac{1}{2} \int_{\Omega_2} (|\partial_t w_\varepsilon^2(s', x)|^2 + |\nabla w_\varepsilon^2(s', x)|^2 + |w_\varepsilon^2(s', x)|^2) dx \\ &= -\int_0^{s'} \int_\gamma \Big[\partial_t w_\varepsilon^2 r_5 + \frac{\partial v_\varepsilon^2}{\partial \nu_1} \partial_t r_4\Big] d\gamma dt + \int_0^{s'} \int_{\Omega_1} v_\varepsilon^2 \partial_t v_\varepsilon^2 dxdt + \int_0^{s'} \int_{\Omega_2} w_\varepsilon^2 \partial_t w_\varepsilon^2 dxdt. \end{aligned} \tag{4.130}$$

For any $s \in [0, T^*]$, put

$$F(s) \triangleq \frac{1}{2} \int_0^s \Big[\int_{\Omega_1} [|\nabla v_\varepsilon^2|^2 + |v_\varepsilon^2|^2] dx + \int_{\Omega_2} (|\partial_t w_\varepsilon^2|^2 + |\nabla w_\varepsilon^2|^2 + |w_\varepsilon^2|^2) dx\Big] dt. \tag{4.131}$$

For any $\lambda > 1$, multiplying both sides of (4.130) by $e^{-\lambda s'}$, integrating from 0 to s (with respect to s'), exchanging the order of integration, using integration by parts, and noting (4.131) and (4.112), we get

$$\begin{aligned}
&\int_0^s \frac{e^{-\lambda t} - e^{-\lambda s}}{\lambda} \int_{\Omega_1} |\partial_t v_\varepsilon^2|^2 dxdt + e^{-\lambda s} F(s) + \lambda \int_0^s e^{-\lambda t} F(t) dt \\
&\le - \int_0^s \frac{e^{-\lambda t} - e^{-\lambda s}}{\lambda} \int_\gamma \Big[\partial_t w_\varepsilon^2 r_5 + \frac{\partial v_\varepsilon^2}{\partial \nu_1} \partial_t r_4\Big] d\gamma dt + \int_0^s e^{-\lambda t} F(t) dt \\
&\le - \int_0^s \frac{e^{-\lambda t} - e^{-\lambda s}}{\lambda} \int_\gamma \frac{\partial v_\varepsilon^2}{\partial \nu_1} \partial_t r_4 d\gamma dt \\
&\quad + \int_0^s \int_\gamma e^{-\lambda t} |w_\varepsilon^2| (|\partial_t r_5| + |r_5|) d\gamma dt + \int_0^s e^{-\lambda t} F(t) dt.
\end{aligned} \tag{4.132}$$

Now, noting (4.112) again, by Lemma 2.1 and the trace theorem, and using the first equation in (4.128), we see that

$$\begin{aligned}
&- \int_0^s \frac{e^{-\lambda t} - e^{-\lambda s}}{\lambda} \int_\gamma \frac{\partial v_\varepsilon^2}{\partial \nu_1} \partial_t r_4 d\gamma dt + \int_0^s \int_\gamma e^{-\lambda t} |w_\varepsilon^2| (|\partial_t r_5| + |r_5|) d\gamma dt \\
&\le \frac{e^{-\lambda s}}{4} \int_0^s \frac{e^{-\lambda t} - e^{-\lambda s}}{\lambda} \int_{\Omega_1} [|\partial_t v_\varepsilon^2|^2 + |\nabla v_\varepsilon^2|^2] dxdt + \frac{1}{2} \int_0^s e^{-\lambda t} F(t) dt \\
&\quad + C(\lambda) [|\partial_t r_4|^2_{H^{1/2}((0,T^*)\times\gamma)} + |r_5|^2_{H^1((0,T^*)\times\gamma)}].
\end{aligned} \tag{4.133}$$

Recalling $\lambda > 1$ and (4.131), we find

$$\begin{aligned}
&\int_0^s \frac{e^{-\lambda t} - e^{-\lambda s}}{\lambda} \int_{\Omega_1} |\partial_t v_\varepsilon^2|^2 dxdt + e^{-\lambda s} F(s) \\
&\ge \frac{e^{-\lambda s}}{4} \int_0^s \frac{e^{-\lambda t} - e^{-\lambda s}}{\lambda} \int_{\Omega_1} [|\partial_t v_\varepsilon^2|^2 + |\nabla v_\varepsilon^2|^2] dxdt + \frac{1}{2} e^{-\lambda s} F(s).
\end{aligned} \tag{4.134}$$

Combining (4.132), (4.133) and (4.134), we arrive at

$$\begin{aligned}
&\frac{1}{2} e^{-\lambda s} F(s) + \lambda \int_0^s e^{-\lambda t} F(t) dt \\
&\le \frac{3}{2} \int_0^s e^{-\lambda t} F(t) dt + C(\lambda) [|\partial_t r_4|^2_{H^{1/2}((0,T^*)\times\gamma)} + |r_5|^2_{H^1((0,T^*)\times\gamma)}], \qquad \forall\, s \in [0, T^*].
\end{aligned} \tag{4.135}$$

By choosing $\lambda = 3/2$ in (4.135), we end up with

$$F(s) \le C [|\partial_t r_4|^2_{H^{1/2}((0,T^*)\times\gamma)} + |r_5|^2_{H^1((0,T^*)\times\gamma)}], \qquad \forall\, s \in [0, T^*]. \tag{4.136}$$

On the other hand, by (4.130), (4.131) and noting (4.112) again, we get

$$\int_0^{T^*} \int_{\Omega_1} |\partial_t v_\varepsilon^2|^2 dxdt \le \int_0^{T^*} \int_\gamma \Big[w_\varepsilon^2 \partial_t r_5 - \frac{\partial v_\varepsilon^2}{\partial \nu_1} \partial_t r_4\Big] d\gamma dt + F(T^*). \tag{4.137}$$

Similar to (4.133), one deduces that

$$\begin{aligned}&\int_0^{T^*}\int_\gamma\Big[w_\varepsilon^2\partial_t r_5-\frac{\partial v_\varepsilon^2}{\partial\nu_1}\partial_t r_4\Big]d\gamma dt\\&\le\frac12\int_0^{T^*}\int_{\Omega_1}|\partial_t v_\varepsilon^2|^2dxdt+\frac12\int_0^{T^*}F(t)dt+C[|\partial_t r_4|^2_{H^{1/2}((0,T^*)\times\gamma)}+|r_5|^2_{H^1((0,T^*)\times\gamma)}].\end{aligned}\tag{4.138}$$

Now, combining (4.136), (4.137) and (4.138), and recalling (4.111), we conclude that

$$\int_0^{T^*}\int_{\Omega_1}|\partial_t v_\varepsilon^2|^2dxdt\le C[|\partial_t r_4|^2_{H^{1/2}((0,T^*)\times\gamma)}+|r_5|^2_{H^1((0,T^*)\times\gamma)}]=O(\varepsilon^{1/2}).\tag{4.139}$$

Finally, combining (4.111), (4.126), (4.129) and (4.139), it is easy to check that $(y_\varepsilon, z_\varepsilon)$ satisfy the conclusion of Lemma 4.3. □

Finally, combining Proposition 4.7 and Lemma 4.3, we end up with

Theorem 4.4 *Let Assumption 2.1 hold. Then there is a family of solutions* $\{(y_\varepsilon, z_\varepsilon)\}_{\varepsilon>0}$ *of system (1.1) in* $[0,T]$ *(the initial conditions being excepted), such that*

$$|\partial_t y_\varepsilon|^2_{L^2((0,T)\times\Omega_1)}=O(\varepsilon),\qquad E_\varepsilon(0)=E(y_\varepsilon,z_\varepsilon,\partial_t z_\varepsilon)(0)\approx 1.\tag{4.140}$$

5 Weakened observability inequality and polynomial decay rate under the GCC

This section is devoted to deriving a weakened observability inequality for equation (1.1) under the GCC, and then analyzing the polynomial decay rate of its smooth solutions.

5.1 Weakened observability inequality

To begin with, let us introduce the following assumption:

(H) *There exist* $T_0>0$ *such that for some constant* $C>0$*, solutions of the following system*

$$\begin{cases}\Box z=0 & in\quad (0,T_0)\times\Omega,\\ z=0 & on\quad (0,T_0)\times\Gamma,\\ z(0)=z_0,\quad z_t(0)=z_1 & in\quad \Omega\end{cases}\tag{5.1}$$

satisfy

$$|z_0|^2_{H_0^1(\Omega)}+|z_1|^2_{L^2(\Omega)}\le C\int_0^{T_0}\int_{\Omega_1}|z_t|^2dxdt,\qquad\forall\,(z_0,z_1)\in H_0^1(\Omega)\times L^2(\Omega).\tag{5.2}$$

Obviously, inequality (5.2) is an internal observability estimate for system (5.1). It is well-known that there are two classes of conditions on T_0 and Ω_1 guaranteeing that (H) holds. Fix a $x_0\in\mathbf{R}^n$, and put

$$\Gamma_0\triangleq\Big\{x\in\Gamma\ \Big|\ (x-x_0)\cdot\nu(x)>0\Big\},$$

where $\nu(x)$ is the unit outward normal vector of Ω at $x \in \Gamma$. The first one is given by the *classical multiplier condition*, i.e., when $\Omega_1 = \mathcal{O}_{\varepsilon_0}(\Gamma_0) \bigcap \Omega$ for some $\varepsilon_0 > 0$ and $T_0 > 2 \max_{x \in \Omega \setminus \Omega_1} |x - x_0|$. This is the typical situation one encounters when applying the multiplier technique ([12]). The second one is when T_0 and Ω_1 satisfy the *Geometric Optics Condition* introduced in [3].

Recall that $D(\mathcal{A})$ is as in (2.3). Under assumption (H), we have the following weakened observability inequality for system (1.1):

Theorem 5.1 *Suppose T_0 and Ω_1 satisfy (H). Then there exists a constant $C > 0$ such that for any $(y_0, z_0, z_1) \in D(\mathcal{A})$ and any $T \geq T_0$, the solution (y, z, z_t) of (1.1) satisfies*

$$|(y_0, z_0, z_1)|_H^2 \leq C \int_0^T \int_{\Omega_1} (|y_t|^2 + |y_{tt}|^2) dxdt. \tag{5.3}$$

The proof of Theorem 5.1 is given in Subsection 5.3. It consists in viewing (1.1) as a perturbation of the wave equation in Ω.

Remark 5.1 *Inequality (5.3) is very likely not sharp. One can expect, under assumption (H), the following stronger inequality to hold:*

$$|(y_0, z_0, z_1)|_H^2 \leq C |y_t|^2_{H^{1/2}(0,T;L^2(\Omega_1))}. \tag{5.4}$$

This is an open problem. Note that, the WKB asymptotic expansion for the flat interface developed in Section 3 supports the possible validity of inequality (5.4). Indeed, from Subsection 3.6 we see that, in the case of non-normal incidence, by means of interpolation techniques, for small positive parameter ε, the energy $|y_t^\varepsilon|^2_{H^{1/2}(0,T;L^2(\Omega_1))}$ dissipated between $t = 0$ and $t = T$ is of the order of $1/\varepsilon^2$, the same as the total energy. We refer to [21] for the spectral analysis result for the $1 - d$ problem which might lead to sharp estimate like (5.4).

5.2 Polynomial decay rate

Our polynomial decay result in this paper reads as follows.

Theorem 5.2 *Suppose T_0 and Ω_1 satisfy (H). Then there is a constant $C > 0$ such that for any $(y_0, z_0, z_1) \in D(\mathcal{A})$, the solution of (1.1) satisfies*

$$|(y(t), z(t), z_t(t))|_H^2 \leq \frac{C}{t} |(y_0, z_0, z_1)|^2_{D(\mathcal{A})}, \qquad \forall\, t > 0. \tag{5.5}$$

The proof of Theorem 5.2 is given in Subsection 5.4.

Remark 5.2 *The result in Theorem 5.2 is not sharp in general since in [21] we have shown that the energy decay rate is $1/t^4$ for $n = 1$. Obviously in one space dimension, the condition (H) is fulfilled whenever T_0 exceeds twice the length of the unobserved interval $\Omega \setminus \Omega_1$ without any geometric restriction on Ω_1. On the other hand, the WKB asymptotic expansion for the* *flat interface developed in Section 3 shows that it is impossible to expect the same decay rate for several space dimensions. Indeed, for an incoming wave of form (3.3) with energy* $\approx \frac{1}{\varepsilon^2}$*, as shown in Subsection 3.6, the percentage of energy dissipated in the two cases of non-normal and normal incidence are* $O(\varepsilon)$ *and* $O(\sqrt{\varepsilon})$ *respectively. Note that more energy is absorbed in the case of normal incidence. This suggests that the rate of decay in the multidimensional case is slower than in the one dimensional one where only normal incidence occurs. According to Remark 5.1 and the possible sharp weakened observability inequality (5.4), it seems reasonable to expect* $1/t^2$ *to be the sharp polynomial decay rate for smooth solutions of (1.1) with initial data in* $D(\mathcal{A})$*. This is also an open problem.*

Remark 5.3 *In the absence of the GCC for the heat domain* Ω_1*, instead of (5.2), one has a weaker observability inequality of logarithmical type for the wave equation (5.1) ([8]). In view of this and using a similar argument as in this section, we may show a logarithmical decay result for smooth solutions of (1.1) without any geometric restriction on* Ω_1*. The details will be presented elsewhere.*

5.3 Proof of Theorem 5.1

We set

$$w = y\chi_{\Omega_1} + z\chi_{\Omega_2}. \tag{5.6}$$

Then, by equation (1.1), it is easy to see that $w \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ satisfies

$$\begin{cases} \Box w = (y_{tt} - y_t)\chi_{\Omega_1} & \text{in} \quad (0,T) \times \Omega, \\ w = 0 & \text{on} \quad (0,T) \times \Gamma, \\ w(0) = y_0\chi_{\Omega_1} + z_0\chi_{\Omega_2}, \quad w_t(0) = (\Delta y_0)\chi_{\Omega_1} + z_1\chi_{\Omega_2} & \text{in} \quad \Omega. \end{cases} \tag{5.7}$$

We decompose w as $w = p + q$, where p and q are respectively solutions of

$$\begin{cases} \Box p = 0 & \text{in} \quad (0,T) \times \Omega, \\ p = 0 & \text{on} \quad (0,T) \times \Gamma, \\ p(0) = y_0\chi_{\Omega_1} + z_0\chi_{\Omega_2}, \quad p_t(0) = (\Delta y_0)\chi_{\Omega_1} + z_1\chi_{\Omega_2} & \text{in} \quad \Omega \end{cases} \tag{5.8}$$

and

$$\begin{cases} \Box q = (y_{tt} - y_t)\chi_{\Omega_1} & \text{in} \quad (0,T) \times \Omega, \\ q = 0 & \text{on} \quad (0,T) \times \Gamma, \\ q(0) = q_t(0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{5.9}$$

Recall that $T > T_0$. Hence, by assumption (H), we conclude that there is a constant $C > 0$ such that solutions of system (5.8) satisfy

$$|y_0\chi_{\Omega_1} + z_0\chi_{\Omega_2}|^2_{H_0^1(\Omega)} + |(\Delta y_0)\chi_{\Omega_1} + z_1\chi_{\Omega_2}|^2_{L^2(\Omega)} \le C\int_0^T\int_{\Omega_1} |p_t|^2 dxdt. \tag{5.10}$$

Now, by (5.6) and (5.10), noting the definition of H in (1.2) and (1.3), it is easy to see that

$$|(y_0, z_0, z_1)|_H^2 \le C\left[\int_0^T\int_{\Omega_1} |q_t|^2 dxdt + \int_{\Omega_1} |y_t|^2 dxdt\right]. \tag{5.11}$$

However, applying the usual energy method to system (5.9), one finds that

$$\int_0^T \int_{\Omega_1} |q_t|^2 dxdt \le C \int_0^T \int_{\Omega_1} |y_{tt} - y_t|^2 dxdt \le C \int_0^T \int_{\Omega_1} (|y_t|^2 + |y_{tt}|^2) dxdt. \tag{5.12}$$

Consequently, combining (5.11) and (5.12), one obtains (5.3) immediately. This completes the proof of Theorem 5.1. □

5.4 Proof of Theorem 5.2

Before giving the proof of Theorem 5.2, we recall two known technical results.

The first one is the following (*see* for example, Lemma 5.2 in [1]):

Proposition 5.1 *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive numbers satisfying*

$$a_{k+1} \le a_k - C a_{k+1}^{2+\alpha}, \qquad \forall\, k \ge 1$$

for some constants $C > 0$ and $\alpha > -1$. Then there is a constant $M = M(C, \alpha) > 0$ such that

$$a_k \le \frac{M}{(k+1)^{\frac{1}{1+\alpha}}}, \qquad \forall\, k \ge 1.$$

The next one can be found in [19, pp. 42, Proposition 3.3].

Proposition 5.2 *Let the closed linear operator A generate a contraction semigroup on a Banach space V. Then, for $v \in D(A^2)$, one has*

$$|Av|_V^2 \le 4|v|_V |A^2 v|_V.$$

A direct consequence of Proposition 5.2 is as follows.

Corollary 5.1 *Let the closed linear operator A generate a contraction semigroup on a Banach space V, and $0 \in \rho(A)$. Then, for $v \in D(A)$, one has*

$$|v|_V^2 \le 4|A^{-1}v|_V |Av|_V. \tag{5.13}$$

Proof of Theorem 5.2. The proof is divided into two steps.

Step 1. Let us derive some dissipation laws for equations (1.1). From (1.5), for any $T \ge S \ge 0$, we get

$$E(T) - E(S) = -\int_S^T \int_{\Omega_1} |y_t|^2 dxdt, \tag{5.14}$$

where E is defined by (1.4).

Denote by ψ and ϕ the heat and wave components of $\mathcal{A}^{-1}(y_0, z_0, z_1)$, respectively. Set

$$Y = Y(t,x) \triangleq \int_0^t y(\tau, x) d\tau + \psi(x), \qquad Z = Z(t,x) \triangleq \int_0^t z(\tau, x) d\tau + \phi(x). \tag{5.15}$$

Then, from (2.5) and (2.6), we see that $(Y,Z,Z_t)=\widetilde{X}$ solves (1.1) with initial data (ψ,ϕ,z_0).

We define

$$E_{-1}(t)\stackrel{\triangle}{=}E(Y,Z,Z_t)(t), \tag{5.16}$$

where (Y,Z,Z_t) is given by (5.15). Recall that (Y,Z,Z_t) satisfies (1.1). Hence, by (5.14), we get

$$E_{-1}(T)-E_{-1}(S)=-\int_S^T\int_{\Omega_1}|Y_t|^2dxdt=-\int_S^T\int_{\Omega_1}|y|^2dxdt. \tag{5.17}$$

Similarly, denoting by ξ and η the heat and wave components of $\mathcal{A}^{-2}(y_0,z_0,z_1)$, respectively, put

$$M=M(t,x)\stackrel{\triangle}{=}\int_0^t Y(\tau,x)d\tau+\xi(x),\qquad N=N(t,x)\stackrel{\triangle}{=}\int_0^t Z(\tau,x)d\tau+\eta(x). \tag{5.18}$$

Then, (M,N,N_t) solves (1.1) with initial data (ξ,η,ϕ).

We define

$$E_{-2}(t)\stackrel{\triangle}{=}E(M,N,N_t)(t). \tag{5.19}$$

Then, by (5.14), we get

$$E_{-2}(T)-E_{-2}(S)=-\int_S^T\int_{\Omega_1}|M_t|^2dxdt. \tag{5.20}$$

It is obvious that

$$\begin{array}{llll}\Delta Y=Y_t=y, & \Delta M=M_t=Y & & \text{in } \Omega_1,\\ \Delta Z=Z_{tt}=z_t, & N_t=Z, & \Delta N=N_{tt}=Z_t=z & \text{in } \Omega_2.\end{array} \tag{5.21}$$

Finally, we put

$$\mathcal{E}(t)\stackrel{\triangle}{=}E_{-2}(t)+E_{-1}(t)+E(t). \tag{5.22}$$

Then, by (5.14), (5.17) and (5.20), and noting (5.21), we see that

$$\begin{aligned}\mathcal{E}(S)-\mathcal{E}(T)&=\int_S^T\int_{\Omega_1}(|M_t|^2+|Y_t|^2+|y_t|^2)dxdt\\ &=\int_S^T\int_{\Omega_1}(|Y|^2+|Y_t|^2+|Y_{tt}|^2)dxdt.\end{aligned} \tag{5.23}$$

Step 2. Applying Theorem 5.1 to (Y,Z,Z_t) , we get

$$E_{-1}(0)\le C\int_0^T\int_{\Omega_1}(|Y_t|^2+|Y_{tt}|^2)dxdt. \tag{5.24}$$

Combining (5.23) and (5.24), we see that

$$E_{-1}(S)\le C\Big(\mathcal{E}(S)-\mathcal{E}(T)\Big),\qquad \forall\, 0\le S\le T<\infty. \tag{5.25}$$

Now, let us fix $T>T_0$. We set

$$\alpha_m=E(mT),\qquad m=0,1,2,\cdots.$$

Also, put

$$E_1(t) = E(y_t, z_t, z_{tt})(t).$$

Applying (5.13) in Corollary 5.1 to $\mathcal{A}$ defined by (2.2)–(2.3), one deduces that

$$\mathcal{E}(t) \leq C\sqrt{E_{-1}(t)E_1(t)}, \tag{5.26}$$

Then, by (5.25) and (5.26), we find

$$C(\alpha_m - \alpha_{m+1}) \geq \frac{\alpha_m^2}{E_1(0)}, \tag{5.27}$$

where $C > 0$ is a constant, independent of m or (y_0, z_0, z_1).

Finally, applying Proposition 5.1 to (5.27), one gets (5.5) easily. □

6 Appendix A: Proof of Theorem 2.1

First, let us show that $D(\mathcal{A})$ is dense in H. For this purpose, we fix any $(f, g, h) \in H$. Obviously, $(f, g) \in H_0^1(\Omega)$. Thus, for any given $\delta > 0$, there exists a function $w^\delta \in C_0^\infty(\Omega)$ such that

$$|w^\delta - (f, g)|_{H_0^1(\Omega)} < \delta/2. \tag{6.1}$$

It is easy to see that there is a sufficiently small $\eta > 0$ such that $\mathcal{O}_\eta(\gamma \cap \mathrm{supp}\,(w^\delta)) \subset \Omega$. Obviously, one can find a function $\Phi \in C_0^\infty(\Omega)$ such that $\Phi(x) = 1$ for all $x \in \mathcal{O}_\eta(\gamma \cap \mathrm{supp}\,(w^\delta))$. Put $f^\delta \triangleq w^\delta|_{\Omega_1}$, $g^\delta \triangleq w^\delta|_{\Omega_2}$ and $\phi^\delta \triangleq (h - \Phi\Delta w^\delta)|_{\Omega_2}$. Note that $\phi^\delta \in L^2(\Omega_2)$. Thus there is a function $\psi^\delta \in C_0^\infty(\Omega_2)$ such that $|\psi^\delta - \phi^\delta|_{L^2(\Omega_2)} < \delta/2$. Now, put $h^\delta \triangleq (\Phi\Delta w^\delta + \psi^\delta)|_{\Omega_2}$. Then

$$|h - h^\delta|_{L^2(\Omega_2)} = |\psi^\delta - \phi^\delta|_{L^2(\Omega_2)} < \delta/2. \tag{6.2}$$

It is easy to check that $(f^\delta, g^\delta, h^\delta) \in D(\mathcal{A})$. Furthermore, in view of (6.1) and (6.2), we get

$$|(f^\delta, g^\delta, h^\delta) - (f, g, h)|_H < \delta.$$

Thus, we conclude that $D(\mathcal{A})$ is dense in H.

Next, for any $Y = (Y_1, Y_2, Y_3) \in D(\mathcal{A})$, we have

$$\begin{aligned}
&\mathrm{Re}\,(\mathcal{A}Y, Y)_H = \mathrm{Re}\,((\Delta Y_1, Y_3, \Delta Y_2), (Y_1, Y_2, Y_3))_H \\
&= \mathrm{Re}\Big[(\nabla\Delta Y_1, \nabla Y_1)_{(L^2(\Omega_1))^n} + (\nabla Y_3, \nabla Y_2)_{(L^2(\Omega_2))^n} + (\Delta Y_2, Y_3)_{L^2(\Omega_2)}\Big] \\
&= \mathrm{Re}\Big[- |\Delta Y_1|^2_{L^2(\Omega_1)} + \Big(\Delta Y_1, \frac{\partial Y_1}{\partial \nu_1}\Big)_{L^2(\gamma)} + \Big(Y_3, \frac{\partial Y_2}{\partial \nu_2}\Big)_{L^2(\gamma)} \\
&\quad -(Y_3, \Delta Y_2)_{L^2(\Omega_2)} + (\Delta Y_2, Y_3)_{L^2(\Omega_2)}\Big] \\
&= -|\Delta Y_1|^2_{L^2(\Omega_1)} \leq 0.
\end{aligned} \tag{6.3}$$

Finally, let us show that $0 \in \rho(\mathcal{A})$. For this purpose, we fix any $F = (F_1, F_2, F_3) \in H$ and we look for $Y = (Y_1, Y_2, Y_3) \in D(\mathcal{A})$ such that

$$\mathcal{A}Y = F. \tag{6.4}$$

By (2.2), (2.3) and (1.2), it follows from (6.4) that

$$Y_3 = F_2 \in H^1_{\Gamma_2}(\Omega_2). \tag{6.5}$$

Put $f = (F_1, F_3)(\in L^2(\Omega))$. We solve the following equation

$$\begin{cases} \Delta\phi = f & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma. \end{cases} \tag{6.6}$$

Obviously, $\phi \in H^2(\Omega) \cap H^1_0(\Omega)$. Denote

$$Y_1 = \phi|_{\Omega_1}, \qquad Y_2 = \phi|_{\Omega_2}. \tag{6.7}$$

Note that $\chi_{\Omega_1} f = F_1 \in H^1_{\Gamma_1}(\Omega_1)$. We see that

$$\Delta Y_1 \in H^1(\Omega_1),\ (Y_1, Y_2) \in H^1(\Omega),\ Y_1|_{\Gamma_1} = \Delta Y_1|_{\Gamma_1} = Y_2|_{\Gamma_2} = 0 \ \text{ and } \ \Delta Y_1|_\gamma = Y_3|_\gamma. \tag{6.8}$$

From (6.5), (6.8), we see that $(Y_1, Y_2, Y_3) \in D(\mathcal{A})$. Therefore $\mathcal{A}^{-1}$ exists and it is a bounded operator in H. Also, it is easy to check that $\mathcal{A}^{-1}$ is a compact operator in H.

Now, we see that $\mathcal{A}$ generates a C_0−semigroup in H. □

7 Appendix B: Proof of Propositions 4.2 and 4.6

This Appendix is devoted to the proof of Propositions 4.2 and 4.6. It is easy to see that the last term in (4.23) is equal to $O(|\tilde{x} - \tilde{x}^\pm(t)|^3)$, which may be ignored in the proof of Propositions 4.2 and 4.6. Hence, without loss of generality, in the rest of this Appendix we assume $a^\pm_{ijk} = 0$ for all $i, j, k = 1, \cdots, n$.

Proof of Proposition 4.2. The computations are as follows.

Verification of (4.36): Recalling that $\tilde{x}^\pm(t_0) = \tilde{x}(x_0) = \tilde{x}_0$, from (4.12), (4.13) and (4.24), one finds (recall (4.10) for $J(\tilde{x})$)

$$\begin{aligned} \tilde{x}^\pm_t(t) &= (J(\tilde{x}^\pm(t)))^{-1}\dot{x}^\pm(t) = 2(J(\tilde{x}^\pm(t)))^{-1}\xi^\pm(t) \\ &= 2(J(\tilde{x}_0))^{-1}\xi^\pm(t_0) + O(|t - t_0|) = 2(\eta^\pm_1, \eta'_\pm)^T + O(|t - t_0|) \end{aligned} \tag{7.1}$$

as $t \to t_0$. Hence, recalling that $\tilde{x}_0 = (0, \tilde{x}'_0)$, we see that when $(t, \tilde{x})$ tends to $(t_0, \tilde{x}_0)$, it holds

$$\begin{aligned} &\tilde{x} - \tilde{x}^\pm(t) \\ &= \tilde{x} - \tilde{x}^\pm(t_0) - \tilde{x}^\pm_t(t_0)(t - t_0) + O(|t - t_0|^2) \\ &= \tilde{x} - \tilde{x}_0 - \tilde{x}^\pm_t(t_0)(t - t_0) + O(|t - t_0|^2 + |\tilde{x} - \tilde{x}_0|^2) \\ &= \tilde{x} - \tilde{x}_0 - 2(\eta^\pm_1, \eta'_\pm)(t - t_0) + O(|t - t_0|^2 + |\tilde{x} - \tilde{x}_0|^2) \\ &= (\tilde{x}_1 - 2\eta^\pm_1(t - t_0), \tilde{x}' - \tilde{x}'_0 - 2\eta'_\pm(t - t_0)) + O(|t - t_0|^2 + |\tilde{x} - \tilde{x}_0|^2), \end{aligned} \tag{7.2}$$

which gives (4.36).

Verification of (4.37): By (4.23) and (4.35), we get

$$\tilde{\phi}^{\pm}(t,\tilde{x}) = O(|\tilde{x} - \tilde{x}^{\pm}(t)|).$$

This fact, combined with (4.36), yields (4.37) immediately.

Verification of (4.38) and (4.39): From (4.23), we see that

$$\begin{aligned}
&\partial_t\tilde{\phi}^{\pm}(t,\tilde{x}) \\
&= -\xi^{\pm}(t)^T\partial_t x^{\pm}(t) + \partial_t\xi^{\pm}(t)^T\Big(x(\tilde{x}) - x^{\pm}(t)\Big) - \Big(x(\tilde{x}) - x^{\pm}(t)\Big)^T M^{\pm}(t)\partial_t x^{\pm}(t) \\
&\quad + \frac{1}{2}\Big(x(\tilde{x}) - x^{\pm}(t)\Big)^T \partial_t M^{\pm}(t)\Big(x(\tilde{x}) - x^{\pm}(t)\Big) \\
&= -\xi^{\pm}(t)^T\partial_t x^{\pm}(t) + O(|\tilde{x} - \tilde{x}^{\pm}(t)|).
\end{aligned} \tag{7.3}$$

However, by (4.12) and (4.13), and noting (4.14), we have

$$\xi^{\pm}(t)^T\partial_t x^{\pm}(t) = 2|\xi^{\pm}(t)|^2 = \frac{1}{2}. \tag{7.4}$$

Combining (7.3) and (7.4), we arrive at (4.38).

On the other hand, from (4.23) and (4.24), we see that

$$\begin{aligned}
\nabla_{\tilde{x}}\tilde{\phi}^{\pm}(t,\tilde{x}) &= (J(\tilde{x}))^T\xi^{\pm}(t) + (J(\tilde{x}))^T M^{\pm}(t)\Big(x(\tilde{x}) - x^{\pm}(t)\Big) \\
&= (J(\tilde{x}))^T\xi^{\pm}(t) + O(|\tilde{x} - \tilde{x}^{\pm}(t)|) \\
&= (J(\tilde{x}_0))^T\xi^{\pm}(t_0) + O(|t - t_0| + |\tilde{x} - \tilde{x}_0| + |\tilde{x} - \tilde{x}^{\pm}(t)|) \\
&= \sigma^{\pm} + O(|t - t_0| + |\tilde{x} - \tilde{x}_0| + |\tilde{x} - \tilde{x}^{\pm}(t)|).
\end{aligned} \tag{7.5}$$

Now combining (4.36) and (7.5), and noting (4.25), we conclude (4.39).

Verification of (4.40): From the first equality in (7.3) and noting (7.4), we find

$$\partial_{tt}\tilde{\phi}^{\pm}(t,\tilde{x}) = -\partial_t\xi^{\pm}(t)^T\partial_t x^{\pm}(t) + \partial_t x^{\pm}(t)^T M^{\pm}(t)\partial_t x^{\pm}(t) + O(|\tilde{x} - \tilde{x}^{\pm}(t)|). \tag{7.6}$$

From (4.12) and (4.13), we have

$$\partial_t\xi^{\pm}(t) = 0. \tag{7.7}$$

Combining (7.6), (7.7), and noting (4.36), (4.12) and (4.13), we arrive at

$$\begin{aligned}
\partial_{tt}\tilde{\phi}^{\pm}(t,0,\tilde{x}') &= \dot{x}^{\pm}(t_0)^T M^{\pm}(t_0)\dot{x}^{\pm}(t_0) + O(|t - t_0| + |\tilde{x}' - \tilde{x}_0'|) \\
&= 4\xi^{\pm}(t_0)^T M^{\pm}(t_0)\xi^{\pm}(t_0) + O(|t - t_0| + |\tilde{x}' - \tilde{x}_0'|).
\end{aligned} \tag{7.8}$$

Noting (4.24), this fact yields (4.40).

Verification of (4.41): From the first equality in (7.3), and noting (7.7), we see that

$$\begin{aligned}\partial_t \nabla_{\tilde{x}} \tilde{\phi}^{\pm}(t, \tilde{x}) &= \partial_t \xi^{\pm}(t)^T J(\tilde{x}) - (J(\tilde{x}))^T M^{\pm}(t) \partial_t x^{\pm}(t) + O(|\tilde{x} - \tilde{x}^{\pm}(t)|) \\ &= -(J(\tilde{x}))^T M^{\pm}(t) \partial_t x^{\pm}(t) + O(|\tilde{x} - \tilde{x}^{\pm}(t)|).\end{aligned} \tag{7.9}$$

Hence, from (4.36), (7.9), (4.12) and (4.13), we get

$$\begin{aligned}\partial_t \nabla_{\tilde{x}} \tilde{\phi}^{\pm}(t, 0, \tilde{x}') &= -(J(\tilde{x}_0))^T M^{\pm}(t_0) \dot{x}^{\pm}(t_0) + O(|t - t_0| + |\tilde{x}' - \tilde{x}'_0|) \\ &= -2(J(\tilde{x}_0))^T M^{\pm}(t_0) \xi^{\pm}(t_0) + O(|t - t_0| + |\tilde{x}' - \tilde{x}'_0|).\end{aligned} \tag{7.10}$$

Noting (4.24), this fact yields (4.41).

Verification of (4.42): From the first equality in (7.5), we obtain that

$$\begin{aligned}\nabla^2_{\tilde{x}} \tilde{\phi}^{\pm}(t, \tilde{x}) &= \nabla_{\tilde{x}}\Big((J(\tilde{x}))^T \xi^{\pm}(t)\Big) + (J(\tilde{x}))^T M^{\pm}(t) J(\tilde{x}) \\ &\quad + \nabla_{\tilde{x}} (J(\tilde{x}))^T M^{\pm}(t) \Big(x(\tilde{x}) - x^{\pm}(t)\Big) \\ &= \nabla_{\tilde{x}}\Big((J(\tilde{x}))^T \xi^{\pm}(t)\Big) + (J(\tilde{x}))^T M^{\pm}(t) J(\tilde{x}) + O(|\tilde{x} - \tilde{x}^{\pm}(t)|) \\ &= \nabla_{\tilde{x}}\Big((J(\tilde{x}_0))^T \xi^{\pm}(t_0)\Big) + (J(\tilde{x}_0))^T M^{\pm}(t_0) J(\tilde{x}_0) \\ &\quad + O(|t - t_0| + |\tilde{x} - \tilde{x}_0| + |\tilde{x} - \tilde{x}^{\pm}(t)|).\end{aligned} \tag{7.11}$$

Hence, combining (7.11) and (4.36), and noting (4.24), we end up with (4.42). This completes the proof of Proposition 4.2. □

Proof of Proposition 4.6. First, from (4.23) and (4.35), we see that when $\tilde{x}$ is close to $\tilde{x}^{\pm}(t)$, it holds

$$\operatorname{Im} \tilde{\phi}^{\pm}(t, \tilde{x}) = \frac{1}{2}\Big(\tilde{x} - \tilde{x}^{\pm}(t)\Big)^T \operatorname{Im} \widetilde{M}^{\pm}(t) \Big(\tilde{x} - \tilde{x}^{\pm}(t)\Big) + O(|\tilde{x} - \tilde{x}^{\pm}(t)|^3), \tag{7.12}$$

where

$$\widetilde{M}^{\pm}(t) = \Big(J(\tilde{x}^{\pm}(t))\Big)^T M^{\pm}(t) J(\tilde{x}^{\pm}(t)).$$

Recall that, in view of Remark 4.1 and Proposition 4.5, $\operatorname{Im} M^{\pm}(t)$ is positive define. Hence so is $\operatorname{Im} \widetilde{M}^{\pm}(t)$. Therefore, for some constant $c > 0$, it holds

$$\operatorname{Im} \tilde{\phi}^{\pm}(t, \tilde{x}) \geq c|\tilde{x} - \tilde{x}^{\pm}(t)|^2 \tag{7.13}$$

whenever $\tilde{x}$ is close to $\tilde{x}^{\pm}(t)$. Combining (7.13) and (4.36), and noting $\eta_1^{\pm} \neq 0$ (*see* (4.25)), we conclude (4.60). This completes the proof of Proposition 4.6. □

8 Appendix C: Proof of Proposition 4.12

This section is devoted to the proof of Proposition 4.12.

Put

$$R^{\pm}(t,\tilde{x}) \triangleq \xi^{\pm}(t)^T\Big(x(\tilde{x})-x^{\pm}(t)\Big)+\frac{1}{2}\Big(x(\tilde{x})-x^{\pm}(t)\Big)^T M^{\pm}(t)\Big(x(\tilde{x})-x^{\pm}(t)\Big).$$

Then, by (4.23), we have

$$\tilde{\phi}^{\pm}(t,\tilde{x}) = R^{\pm}(t,\tilde{x}) + \sum_{i,j,k=1}^{n} a_{ijk}^{\pm}\Big(x_i(\tilde{x})-x_i^{\pm}(t)\Big)\Big(x_j(\tilde{x})-x_j^{\pm}(t)\Big)\Big(x_k(\tilde{x})-x_k^{\pm}(t)\Big). \tag{8.1}$$

From (8.1), and noting (4.10), (4.12) and (4.13), we see that for $i',j',k'=1,2,\cdots,n$, it holds (recall (4.10) for $g_{ij}(\tilde{x})$)

$$\begin{aligned}
&\partial_{ttt}\tilde{\phi}^{\pm}(t_0,\tilde{x}_0) = \partial_{ttt}R^{\pm}(t_0,\tilde{x}_0) - 8\sum_{i,j,k=1}^{n} a_{ijk}^{\pm}\xi_i^{\pm}(t_0)\xi_j^{\pm}(t_0)\xi_k^{\pm}(t_0),\\
&\partial_{tt\tilde{x}_{k'}}\tilde{\phi}^{\pm}(t_0,\tilde{x}_0) = \partial_{tt\tilde{x}_{k'}}R^{\pm}(t_0,\tilde{x}_0) + 24\sum_{i,j,k=1}^{n} a_{ijk}^{\pm}\xi_i^{\pm}(t_0)\xi_j^{\pm}(t_0)g_{kk'}^{\pm}(\tilde{x}_0),\\
&\partial_{t\tilde{x}_{j'}\tilde{x}_{k'}}\tilde{\phi}^{\pm}(t_0,\tilde{x}_0) = \partial_{t\tilde{x}_{j'}\tilde{x}_{k'}}R^{\pm}(t_0,\tilde{x}_0) - 12\sum_{i,j,k=1}^{n} a_{ijk}^{\pm}\xi_i^{\pm}(t_0)g_{jj'}(\tilde{x}_0)g_{kk'}(\tilde{x}_0),\\
&\partial_{\tilde{x}_{i'}\tilde{x}_{j'}\tilde{x}_{k'}}\tilde{\phi}^{\pm}(t_0,\tilde{x}_0) = \partial_{\tilde{x}_{i'}\tilde{x}_{j'}\tilde{x}_{k'}}R^{\pm}(t_0,\tilde{x}_0) + 6\sum_{i,j,k=1}^{n} a_{ijk}^{\pm}g_{ii'}(\tilde{x}_0)g_{jj'}(\tilde{x}_0)g_{kk'}(\tilde{x}_0).
\end{aligned} \tag{8.2}$$

According to Proposition 4.4, it suffices to choose a_{ijk}^{+} such that

$$\begin{aligned}
&\partial_{ttt}\tilde{\phi}^{+}(t_0,\tilde{x}_0) = \partial_{ttt}\tilde{\phi}^{-}(t_0,\tilde{x}_0), \quad \partial_{tt\tilde{x}_{k'}}\tilde{\phi}^{+}(t_0,\tilde{x}_0) = \partial_{tt\tilde{x}_{k'}}\tilde{\phi}^{-}(t_0,\tilde{x}_0),\\
&\partial_{t\tilde{x}_{j'}\tilde{x}_{k'}}\tilde{\phi}^{+}(t_0,\tilde{x}_0) = \partial_{t\tilde{x}_{j'}\tilde{x}_{k'}}\tilde{\phi}^{-}(t_0,\tilde{x}_0), \quad \partial_{\tilde{x}_{i'}\tilde{x}_{j'}\tilde{x}_{k'}}\tilde{\phi}^{+}(t_0,\tilde{x}_0) = \partial_{\tilde{x}_{i'}\tilde{x}_{j'}\tilde{x}_{k'}}\tilde{\phi}^{-}(t_0,\tilde{x}_0),
\end{aligned} \tag{8.3}$$

where $i',j',k'=2,\cdots,n$. For this purpose, put

$$\tilde{a}_{pqr}^{\pm} = \sum_{i,j,k=1}^{n} a_{ijk}^{\pm}g_{ip}(\tilde{x}_0)g_{jq}(\tilde{x}_0)g_{kr}(\tilde{x}_0), \qquad p,q,r=1,2,\cdots,n. \tag{8.4}$$

First, from the fourth equality in (8.3) and (8.2), we get

$$\tilde{a}_{i'j'k'}^{+} = \frac{\partial_{\tilde{x}_{i'}\tilde{x}_{j'}\tilde{x}_{k'}}R^{-}(t_0,\tilde{x}_0) - \partial_{\tilde{x}_{i'}\tilde{x}_{j'}\tilde{x}_{k'}}R^{+}(t_0,\tilde{x}_0)}{6} + \tilde{a}_{i'j'k'}^{-}, \qquad i',j',k'=2,\cdots,n.$$

Next, from (4.24) and (4.10), one has

$$\xi_i^{\pm}(t_0) = \sum_{\ell=1}^{n} g_{i\ell}(\tilde{x}_0)\eta_\ell^{\pm}. \tag{8.5}$$

Therefore, by the third equality in (8.2), and noting (8.5) and (8.4), we get

$$\begin{aligned}
\partial_{t\tilde{x}_{j'}\tilde{x}_{k'}}\tilde{\phi}^{\pm}(t_0,\tilde{x}_0) &= \partial_{t\tilde{x}_{j'}\tilde{x}_{k'}}R^{\pm}(t_0,\tilde{x}_0) - 12\sum_{i,j,k,\ell=1}^{n} a_{ijk}^{\pm}g_{i\ell}(\tilde{x}_0)g_{jj'}(\tilde{x}_0)g_{kk'}(\tilde{x}_0)\eta_\ell^{\pm}\\
&= \partial_{t\tilde{x}_{j'}\tilde{x}_{k'}}R^{\pm}(t_0,\tilde{x}_0) - 12\sum_{\ell=1}^{n}\tilde{a}_{\ell j'k'}^{\pm}\eta_\ell^{\pm}\\
&= \partial_{t\tilde{x}_{j'}\tilde{x}_{k'}}R^{\pm}(t_0,\tilde{x}_0) - 12\sum_{\ell=2}^{n}\tilde{a}_{\ell j'k'}^{\pm}\eta_\ell^{\pm} - 12\tilde{a}_{1j'k'}^{\pm}\eta_1^{\pm}.
\end{aligned} \tag{8.6}$$

Recall that $\eta_1^+ \neq 0$ (*see* Proposition 4.1). Hence, by (8.6) and the third equality in (8.3), we find

$$\tilde{a}_{1j'k'}^+ = \frac{1}{\eta_1^+}\left[\frac{\partial_{t\tilde{x}_{j'}\tilde{x}_{k'}}R^+(t_0,\tilde{x}_0) - \partial_{t\tilde{x}_{j'}\tilde{x}_{k'}}R^-(t_0,\tilde{x}_0)}{12} - \sum_{\ell=2}^{n}(\tilde{a}_{\ell j'k'}^+\eta_\ell^+ - \tilde{a}_{\ell j'k'}^-\eta_\ell^-) + \tilde{a}_{1j'k'}^-\eta_1^-\right],$$

where $j', k' = 2, \cdots, n$.

Similarly, one may determine $\tilde{a}_{11k'}^+$ ($k' = 2, \cdots, n$) and $\tilde{a}_{111}^+$ from the second and first equalities in (8.3) and (8.2), respectively. Therefore, we have determined $\tilde{a}_{i'j'k'}^+$ for all $i', j', k' = 1, 2, \cdots, n$.

Finally, from (8.4) and noting that the matrix $J(\tilde{x}_0) = \Big(g_{ij}(\tilde{x}_0)\Big)_{1\leq i,j\leq n}$ is invertible, one may determine the desired a_{ijk}^+ for all $i, j, k = 1, 2, \cdots, n$. This completes the proof of Proposition 4.12. □

References

[1] K. Ammari and M. Tucsnak, *Stabilization of Bernoulli-Euler beams by means of a pointwise feedback force, SIAM J. Control Optim.,* 39 (2000), 1160–1181.

[2] V. M. Babich, *The higher-dimensional WKB method or ray method, Encyclopedia of Mathematical Sciences,* **34**, Springer-Verlag, 1997, 91–131.

[3] C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, SIAM J. Control Optim.,* **30** (1992), 1024–1065.

[4] E. Fernández-Cara and E. Zuazua, *The cost of approximate controllability for heat equations: the linear case, Adv. Differential Equations,* **5** (2000), 465–514.

[5] A. V. Fursikov and O. Yu. Imanuvilov, *Controllability of Evolution Equations,* Lecture Notes Series 34, Research Institute of Mathematics, Seoul National University, Seoul, Korea, 1994.

[6] A. Haraux, *Systèmes dynamiques dissipatifs et applications,* Masson, Paris, 1991.

[7] L. Hörmander, *The Analysis of Linear Partial Differential Operators (I, III),* Springer-Verlag, Berlin, 1983, 1985.

[8] G. Lebeau and L. Robbiano, *Stabilisation de l'équation des ondes par le bord, Duke Math. J.,* **86** (1997), 465–491.

[9] W. Li, *Observability estimate for the ultra-parabolic equations,* 2003, in submission.

[10] W. Li and X. Zhang, *Controllability of parabolic and hyperbolic equations: towards a unified theory,* 2003, in submission.

[11] J. L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications (Vol. I),* Springer-Verlag, New York, 1972.

[12] J. L. Lions, *Contrôlabilité exacte, stabilisation et perturbations de systèmes distribués. Tome 1: Contrôlabilité exacte*, RMA 8, Masson, Paris, 1988.

[13] F. Macià and E. Zuazua, *On the lack of observability for wave equations: a Gaussian beam approach, Asymptot. Anal.,* **32** (2002), 1–26.

[14] L. Miller, *Refraction of high-frequency waves density by sharp interfaces and semiclassical measures at the boundary, J. Math. Pures Appl.,* **79** (2000), 227–269.

[15] J. Ralston, *Gaussian beams and the propagation of singularities,* in *Studies in Partial Differential Equations,* Edited by W. Littman, MAA Studies in Mathematics **23**, Washington, 1982.

[16] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: recent progress and open problems, SIAM Rev.,* **20** (1978), pp. 639–739.

[17] L. Y. Sung, *On the perfectly reflecting boundary conditions, Comm. Partial Differential Equations,* **9** (1984), 943–953.

[18] S. Tabachnikov, *Billiards,* Panor. Synth., No. 1, 1995.

[19] M. E. Taylor, *Pseudodifferential operators,* Princeton Mathematical Series, **34**, Princeton University Press, Princeton, N.J., 1981.

[20] X. Zhang, *Explicit observability estimate for the wave equation with potential and its application, Royal Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.,* **456** (2000), 1101–1115.

[21] X. Zhang and E. Zuazua, *Polynomial decay and control of a* $1-d$ *hyperbolic-parabolic coupled system, J. Differential Equations,* to appear.

[22] E. Zuazua, *Exponential decay for semilinear wave equations with localized damping, Comm. Partial Differential Equations,* **15** (1990), 205–235.

[23] E. Zuazua, *Controllability of partial differential equations and its semi-discrete approximations, Discrete Contin. Dyn. Syst.,* **8** (2002), 469–513.