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Uniform Stabilization of the Higher Dimensional System of Thermoelasticity with a Nonlinear Boundary Feedback

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Abstract

Using multipler techniques and Lyapunov methods, we derive explicit decay rates for the energy in the higher-dimensional system of thermoelasticity with a nonlinear velocity feedback on part of the boundary of a thermoelastic body, which is clamped along the rest of its boundary.

Key Words: Thermoelasticity; Uniform boundary stabilization; Lyapunov methods.

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1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\Gamma = \partial\Omega$ of class C^2, and consider a n-dimensional linear, homogeneous, isotropic, and thermoelastic body occupying Ω in its non-deformed state. For a material point with configuration $x = (x_1, \cdots, x_n)$ at time t, let $u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$ and $\theta(x,t)$ denote the displacement and temperature deviation, respectively, from the natural state of the reference configuration. Then, in the absence of external forces and heat sources, u and θ satisfy the system of equations of thermoelasticity

$$\begin{cases} u'' - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \alpha\nabla\theta = 0 & \text{in } \Omega\times(0,\infty), \\ \theta' - \Delta\theta + \beta \mathrm{div} u' = 0 & \text{in } \Omega\times(0,\infty), \\ u(0) = u^0, \ u'(0) = u^1, \ \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where α, $\beta > 0$ are the coupling parameters and λ, μ are Lamé's constants satisfying

$$\mu > 0, \quad n\lambda + (n+1)\mu > 0. \tag{1.2}$$

By $'$ we denote the derivative with respect to the time variable. Δ, ∇, div denote the Laplace, gradient, and divergence operators in the space variables, respectively. $u(0)$, $u'(0)$ and $\theta(0)$ denote the functions $x \to u(x,0)$, $x \to u'(x,0)$ and $x \to \theta(x,0)$, respectively. For the derivation of (1.1), we refer to [24] and [44].

Note that condition (1.2) is weaker than the following usual condition on the Lamé coefficients (see [44] and [13, p.414])

$$n\lambda + 2\mu > 0. \tag{1.3}$$

Extensive work has been done on the problem of stabilization for system (1.1) (see [5, 9, 7, 12, 17, 20, 27, 33, 37, 40, 42, 43, 45, 46, 47, 48]). We give here a brief description about the existing literature. For a detailed survey, we refer to [37, 38].

The thermoelastic energy of (1.1) can be defined as

$$\begin{aligned} E(u,\theta,t) &= \frac{1}{2}\int_\Omega \Big[|u'(x,t)|^2 + \mu|\nabla u(x,t)|^2 \\ &\quad +(\lambda+\mu)|\mathrm{div} u(x,t)|^2 + \frac{\alpha}{\beta}|\theta(x,t)|^2\Big]dx. \end{aligned} \tag{1.4}$$

Here we have used the notation

$$|\nabla u(x,t)|^2 = \sum_{i,j=1}^{n} |\frac{\partial u_i}{\partial x_j}|^2. \tag{1.5}$$

Under the Dirichlet-Dirichlet boundary conditions

$$u = 0, \ \ \theta = 0 \quad \text{on} \quad \Gamma\times(0,\infty), \tag{1.6}$$

it is easy to verify that

$$E'(u,\theta,t) = -\frac{\alpha}{\beta}\int_\Omega |\nabla\theta(x,t)|^2 dx. \tag{1.7}$$

Therefore, the energy $E(u,\theta,t)$ decreases on $(0,\infty)$. Furthermore, in the case of one space dimension, it has been shown (see [9, 17, 20, 40]) that the energy $E(u,\theta,t)$ of system (1.1) associated with various boundary conditions decays to zero exponentially. In the case of multi-space dimension, Dafermos in his pioneering work [12] showed that, generically with respect to the domain, the energy of every solution of (1.1) and (1.6) tends to zero as $t\to\infty$. However, he also pointed out that, when Ω is a ball, non-decaying solutions do exist. More recently, Lebeau and Zuazua [33] proved that the decay rate is never uniform when Ω is convex. Thus, in order to ensure the uniform decay rate for such convex domains, additional damping mechanisms are necessary. In this aspect, the first author (see [37]) introduced a linear boundary velocity feedback acting on the elastic component of the system and established the uniform decay rate.

The purpose of this paper is to introduce a nonlinear boundary feedback which allows to test the robustness of the damping mechanisms. Under the classical polynomial growth assumption on the nonlinear boundary feedback near the origin, by using multipler techniques and Lyapunov methods, we show that the energy in the multi-dimensional system of thermoelasticity decays to zero at an exponential or polynomial rate.

Further, even if the nonlinearity does degenerate at the origin faster than any polynomial, we show that the decay rate is governed by a dissipative ordinary differential equation. This allows us to show, in particular, that if the nonlinearity degenerates at the origin exponentially then we obtain a logarithmic decay rate. To do that, we proceed as in [41] where the simpler case of the wave equation with internal damping is addressed. This type of result was obtained earlier by Lasiecka *et al* [32] for the wave equation with nonlinear boundary feedback. However, our approach, even if it uses some ingredients as in [32] (for instance, Jensen's inequality), relies essentially on the generalized Young's inequality and it is simpler. This allows us to get more explicit expressions for the nonlinearity entering in the differential inequality governing the decay of the energy.

The rest of this paper is organized as follows. In Section 2, we present our main results. In Section 3, we prove that the system of thermoelasticity with a nonlinear boundary feedback is well-posed by using the theory of nonlinear semigroups. Then, borrowing Lyapunov methods and multipler techniques, we prove our main results in Section 4. Finally, in Section 5, we briefly discuss some special case and pose an open problem.

2 Main Results

Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\Gamma=\partial\Omega$ of class C^2. Set

$$\Gamma_1 = \{x\in\Gamma \,:\, m(x)\cdot\nu(x)\le 0\}, \tag{2.1}$$
$$\Gamma_2 = \{x\in\Gamma \,:\, m(x)\cdot\nu(x)> 0\}, \tag{2.2}$$

where

$$m(x)=x-x^0=(x_1-x_1^0,\cdots,x_n-x_n^0) \tag{2.3}$$

for some $x^0 \in \mathbf{R}^n$, $\nu = (\nu_1, \cdots, \nu_n)$ denotes the unit normal on Γ directed towards the exterior of Ω and

$$m \cdot \nu = m(x) \cdot \nu(x) = \sum_{i=1}^{n} (x_i - x_i^0)\nu_i. \tag{2.4}$$

Γ_1 is assumed either to be empty or to have a nonempty interior relative to Γ.

In what follows, $H^s(\Omega)$ denotes the usual Sobolev space (see [1]) for any $s \in \mathbf{R}$. For $s \geq 0$, $H_0^s(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω with compact support in Ω. Let X be a Banach space. We denote by $C^k([0,T];X)$ the space of all k times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T];X)$ for $C^0([0,T];X)$.

We further introduce other function spaces as follows:

$$\begin{aligned} H^1_{\Gamma_1}(\Omega) &= \{u \in H^1(\Omega) \ : \ u = 0 \ \text{ on } \Gamma_1\}, && (2.5)\\ \mathcal{H} &= (H^1_{\Gamma_1}(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega). && (2.6) \end{aligned}$$

We consider the thermoelastic system with a nonlinear boundary feedback

$$\begin{cases} u'' - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\ u + \alpha\nabla\theta = 0 & \text{in } \Omega \times (0,\infty), \\ \theta' - \Delta\theta + \beta \mathrm{div}\ u' = 0 & \text{in } \Omega \times (0,\infty), \\ \theta = 0 & \text{on } \Gamma \times (0,\infty), \\ u = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \mu\dfrac{\partial u}{\partial \nu} + (\lambda+\mu)\mathrm{div}(u)\nu & \\ \quad + am \cdot \nu u + m \cdot \nu g(u') = 0 & \text{on } \Gamma_2 \times (0,\infty), \\ u(0) = u^0,\ u'(0) = u^1,\ \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \tag{2.7}$$

where $a = a(x)$ is a given nonnegative function on Γ_2 with

$$a(x) \in C^1(\Gamma_2), \tag{2.8}$$

and $g(u) = (g_1(u_1, \cdots, u_n), \cdots, g_n(u_1, \cdots, u_n)) \in (C(\mathbf{R}^n))^n$ is a given vector function. Similar nonlinear boundary feedbacks were introduced for the wave equation (see [11, 31, 32, 50, 51]) and the equations of elasticity (see [28, 29]).

The elastic Lamé operator $\mu\Delta u - (\lambda+\mu)\nabla\mathrm{div}\ u$ may be written in divergence form $\mathrm{div}[\sigma_{ij}(u)]$, where the stress tensor $\sigma_{ij}(u)$ is given by

$$\sigma_{ij}(u) = 2\mu\epsilon_{ij}(u) + \lambda\delta_{ij}\sum_{k=1}^{n}\epsilon_{kk}(u), \tag{2.9}$$

and the linearized strain tensor $\epsilon_{ij}(u)$ is given by

$$\epsilon_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big). \tag{2.10}$$

The boundary conditions describing the surface forces are given by

$$\sigma_{ij}(u)\nu_j + am \cdot \nu u_i = f_i. \tag{2.11}$$

Since

$$\mu\frac{\partial u_i}{\partial\nu}+(\lambda+\mu)\mathrm{div}(u)\nu_i=\sigma_{ij}(u)\nu_j+\mu\Big(\frac{\partial u_j}{\partial x_j}\nu_i-\frac{\partial u_j}{\partial x_i}\nu_j\Big), \tag{2.12}$$

the boundary condition in (2.7)

$$\mu\frac{\partial u}{\partial\nu}+(\lambda+\mu)\mathrm{div}(u)\nu+am\cdot\nu u+m\cdot\nu g(u')=0 \tag{2.13}$$

can be easily transferred into (2.11) with

$$f_i(u)=-m\cdot\nu g_i(u')-\mu\Big(\frac{\partial u_j}{\partial x_j}\nu_i-\frac{\partial u_j}{\partial x_i}\nu_j\Big). \tag{2.14}$$

Therefore, system (2.7) may be viewed as the system of thermoelasticity subject to a boundary feedback force of the form (2.14). This feedback mechanism is however non-optimal since, due to the presence of the first order space derivates, its regularity is not sharp (one can not guarantee that it belong to $L^2(\partial\Omega\times(0,T))$ for finite energy solutions). Therefore, the question of analyzing the stabilization under the weaker feedback forces of the form

$$f_i(u)=-m\cdot\nu g_i(u'), \tag{2.15}$$

in which the last term in (2.14) has been dropped, is an interesting open problem. This analysis has been performed in [3, 19] in the context of the system of elasticity but, to our knowledge, this issue has not been addressed for the system of thermoelasticity.

Throughout this paper, we assume that

$$\Gamma_1\neq\emptyset \;\text{ or }\; a(x)\not\equiv 0. \tag{2.16}$$

We refer to Section 5 for a brief discussion of the case where $\Gamma_1=\emptyset$ and $a(x)\equiv 0$. Under assumptions (1.2) and (2.16), one can easily show that the following norm on $(H^1_{\Gamma_1}(\Omega))^n$

$$\begin{aligned}\|u\|_{(H^1_{\Gamma_1}(\Omega))^n} &= \Big(\int_\Omega[\mu|\nabla u|^2+(\lambda+\mu)|\mathrm{div}(u)|^2]dx\\ &\quad+\int_{\Gamma_2}am\cdot\nu|u|^2d\Gamma\Big)^{1/2}\end{aligned} \tag{2.17}$$

is equivalent to the usual one induced by $(H^1(\Omega))^n$ (see [37]). Indeed, it suffices to note the following fact:

$$\mu|\nabla u|^2+(\lambda+\mu)|\mathrm{div}u|^2\geq\mu|\nabla u|^2,\;\text{ for }\lambda+\mu\geq 0, \tag{2.18}$$

and for $\lambda+\mu<0$,

$$\begin{aligned}\mu|\nabla u|^2+(\lambda+\mu)|\mathrm{div}u|^2 &= \mu\sum_{i,j=1}^n\Big|\frac{\partial u_i}{\partial x_j}\Big|^2+(\lambda+\mu)\sum_{i,j=1}^n\frac{\partial u_i}{\partial x_i}\frac{\partial u_j}{\partial x_j}\\ &\geq \mu\sum_{i,j=1}^n\Big|\frac{\partial u_i}{\partial x_j}\Big|^2+(\lambda+\mu)n\sum_i^n\Big|\frac{\partial u_i}{\partial x_i}\Big|^2\\ &= \mu\sum_{i\neq j}\Big|\frac{\partial u_i}{\partial x_j}\Big|^2+[n\lambda+(n+1)\mu]\sum_i^n\Big|\frac{\partial u_i}{\partial x_i}\Big|^2\\ &\geq \min\{\mu,\;n\lambda+(n+1)\mu\}|\nabla u|^2.\end{aligned} \tag{2.19}$$

In the sequel, we use the following energy norm on $\mathcal{H}$

$$\|(u,v,\theta)\|_{\mathcal{H}} = \Big(\int_\Omega [\mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div}(u)|^2 + |v|^2 + \frac{\alpha}{\beta}|\theta|^2]dx + \int_{\Gamma_2} am\cdot\nu|u|^2 d\Gamma\Big)^{1/2} \tag{2.20}$$

for $(u,v,\theta)\in\mathcal{H}$, which is equivalent to the usual one induced by $(H^1(\Omega))^n\times(L^2(\Omega))^n\times L^2(\Omega)$.

In Section 3, we will prove that system (2.7) generates a nonlinear continuous semigroup $S(t)$ on $\mathcal{H}$. Thus, system (2.7) is well-posed.

In order to state our main results, we first introduce some constants. In what follows, we denote by $\|\cdot\|$ the norm of $L^2(\Omega)$ or $(L^2(\Omega))^n$. Set

$$R_0 = \max_{x\in\bar\Omega}|m(x)| = \max_{x\in\bar\Omega}|\sum_{k=1}^n (x_k - x_k^0)^2|^{1/2}, \tag{2.21}$$

$$a_0 = \max_{x\in\Gamma_2} a(x), \tag{2.22}$$

$$K(a) = \frac{2a_0^2R_0^2}{\mu} + (2-n)a_0. \tag{2.23}$$

Let γ be the smallest positive constant such that

$$\int_{\Gamma_2}|u|^2 d\Gamma \le \gamma^2\|u\|^2_{(H^1_{\Gamma_1}(\Omega))^n}, \quad \forall u\in(H^1_{\Gamma_1}(\Omega))^n. \tag{2.24}$$

Let λ_0 and λ_1 be the smallest positive constants, respectively, such that

$$\|u\|\le\lambda_0\|u\|_{(H^1_{\Gamma_1}(\Omega))^n}, \quad \forall u\in(H^1_{\Gamma_1}(\Omega))^n, \tag{2.25}$$

and

$$\|u\|\le\lambda_1\|\nabla u\|, \quad \forall u\in H^1_0(\Omega). \tag{2.26}$$

As the decay rates depend also on the constant appearing in the following technical lemma, we present it before stating our main results. This lemma is helpful for dealing with the case where the potential a is large.

Lemma 2.1 *There exists a constant $k_1>0$, independent of u, such that the solution φ of*

$$\begin{cases} \mu\Delta\varphi + (\lambda+\mu)\nabla\mathrm{div}\varphi = 0 & \text{in } \Omega,\\ \varphi = u & \text{on } \Gamma, \end{cases} \tag{2.27}$$

satisfies

$$\|\varphi\|^2 \le k_1^2\int_\Gamma |u|^2 d\Gamma, \tag{2.28}$$

and

$$\int_\Omega [\mu\nabla u\cdot\nabla\varphi + (\lambda+\mu)\mathrm{div}(u)\mathrm{div}(\varphi)]dx \ge 0, \tag{2.29}$$

for all $u\in(H^1(\Omega))^n$, where

$$\nabla u\cdot\nabla\varphi = \sum_{i=1}^n \nabla u_i\cdot\nabla\varphi_i. \tag{2.30}$$

Proof. For any $f \in (L^2(\Omega))^n$, we consider

$$\begin{cases} \mu\Delta\psi + (\lambda+\mu)\nabla \mathrm{div}\psi = f & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma. \end{cases} \tag{2.31}$$

Multiplying (2.31) by φ and integrating over Ω by parts, we obtain

$$\int_\Omega \varphi \cdot f dx = \int_\Gamma \Big[\mu\frac{\partial\psi}{\partial\nu} + (\lambda+\mu)\mathrm{div}(\psi)\nu\Big] \cdot u d\Gamma. \tag{2.32}$$

On the other hand, there exists a constant $c > 0$ such that

$$\|\mu\frac{\partial\psi}{\partial\nu} + (\lambda+\mu)\mathrm{div}(\psi)\nu\|_{(L^2(\Gamma))^n} \leq c\|f\|, \ \ \forall\, f \in (L^2(\Omega))^n. \tag{2.33}$$

To prove this inequality, we let $h = (h_1, \cdots, h_n)$ be a vector field in $(C^1(\bar{\Omega}))^n$ satisfying

$$h = \nu \quad \text{on } \Gamma. \tag{2.34}$$

Multiplying (2.31) by $h \cdot \nabla\psi$ and integrating over Ω by parts, we obtain

$$\begin{aligned} \int_\Omega \Delta\psi_i h_k \frac{\partial\psi_i}{\partial x_k} &= \frac{1}{2}\int_\Gamma h\cdot\nu\Big|\frac{\partial\psi_i}{\partial\nu}\Big|^2 d\Gamma + \frac{1}{2}\int_\Omega \mathrm{div}(h)|\nabla\psi_i|^2 dx \\ &\quad - \int_\Omega \frac{\partial\psi_i}{\partial x_j}\frac{\partial\psi_i}{\partial x_k}\frac{\partial h_k}{\partial x_j}, \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} \int_\Omega \frac{\partial \mathrm{div}(\psi)}{\partial x_i} h_k \frac{\partial\psi_i}{\partial x_k} &= \frac{1}{2}\int_\Gamma h\cdot\nu|\mathrm{div}(\psi)|^2 d\Gamma + \frac{1}{2}\int_\Omega \mathrm{div}(h)|\mathrm{div}(\psi)|^2 dx \\ &\quad - \int_\Omega \mathrm{div}(\psi)\frac{\partial\psi_i}{\partial x_k}\frac{\partial h_k}{\partial x_i}. \end{aligned} \tag{2.36}$$

Here we have used the summation convention for repeated indices. It therefore follows that there exists a constant $c > 0$ such that

$$\begin{aligned} &\int_\Gamma \Big[\mu\Big|\frac{\partial\psi}{\partial\nu}\Big|^2 + (\lambda+\mu)|\mathrm{div}(\psi)|^2\Big] d\Gamma \\ \leq\ & c\Big[\int_\Omega \Big(\mu|\nabla\psi_i|^2 + (\lambda+\mu)|\mathrm{div}(\psi)|^2\Big)dx + \|f\|^2\Big]. \end{aligned} \tag{2.37}$$

In addition, multiplying (2.31) by ψ and integrating over Ω by parts, we deduce that there exists a constant $c > 0$ such that

$$\int_\Omega \Big(\mu|\nabla\psi_i|^2 + (\lambda+\mu)|\mathrm{div}(\psi)|^2\Big)dx \leq c\|f\|^2. \tag{2.38}$$

Hence, we have

$$\int_\Gamma \Big[\mu\Big|\frac{\partial\psi}{\partial\nu}\Big|^2 + (\lambda+\mu)|\mathrm{div}(\psi)|^2\Big] d\Gamma \leq c\|f\|^2, \tag{2.39}$$

which implies (2.33). Consequently, (2.28) follows from (2.32) and (2.33).

To prove inequality (2.29), we multiply (2.27) by u and φ, respectively, and integrate over Ω by parts. This gives

$$\begin{aligned}\int_\Omega [\mu\nabla u\cdot\nabla\varphi+(\lambda+\mu)\mathrm{div}(u)\mathrm{div}(\varphi)]dx &= \int_\Gamma [\mu\frac{\partial\varphi}{\partial\nu}\cdot u+(\lambda+\mu)\nu\cdot u\mathrm{div}(\varphi)]d\Gamma \\ &= \int_\Omega [\mu|\nabla\varphi|^2+(\lambda+\mu)|\mathrm{div}(\varphi)|^2]dx \\ &\geq 0. \end{aligned} \tag{2.40}$$

□

The thermoelastic energy of (2.7) is defined by

$$E(t)=E(u,\theta,t)=\frac{1}{2}\|(u(t),u'(t),\theta(t))\|^2_{\mathcal{H}}. \tag{2.41}$$

By a straightforward calculation, we obtain

$$E'(t)=-\int_{\Gamma_2} m\cdot\nu g(u'(t))\cdot u'(t)d\Gamma-\frac{\alpha}{\beta}\|\nabla\theta(t)\|^2. \tag{2.42}$$

If g satisfies that $g(u)\cdot u\geq 0$ for all $u\in\mathbf{R}^n$, then the energy $E(t)$ decreases in $(0,\infty)$.

What is more, we have the following decay rates. These are our main results of this paper.

Theorem 2.2 *Let Γ_1 and Γ_2 be given by (2.1) and (2.2), respectively, satisfying*

$$\Gamma_1\cap\bar{\Gamma}_2=\emptyset. \tag{2.43}$$

Suppose that (2.16) holds. Let the function $g\in(C(\mathbf{R}^n))^n$ satisfy the following conditions:

$$\begin{aligned} g(0) &= 0, && (2.44)\\ |g(u^1)-g(u^2)| &\leq k_2|u^1-u^2|^q,\ \ \forall u^1,u^2\in\mathbf{R}^n \text{ with } |u^1-u^2|\leq 1, && (2.45)\\ |g(u^1)-g(u^2)| &\leq k_2|u^1-u^2|,\ \ \forall u^1,u^2\in\mathbf{R}^n \text{ with } |u^1-u^2|\geq 1, && (2.46)\\ g(u)\cdot u &\geq k_3|u|^{p+1},\ \ \forall u\in\mathbf{R}^n \text{ with } |u|\leq 1, && (2.47)\\ g(u)\cdot u &\geq k_3|u|^2,\ \ \forall u\in\mathbf{R}^n \text{ with } |u|\geq 1, && (2.48)\\ 0 &\leq [g(u^1)-g(u^2)]\cdot(u^1-u^2),\ \ \forall u^1,\ u^2\in\mathbf{R}^n, && (2.49)\end{aligned}$$

for some constants $k_2,\ k_3>0$ and $p,\ q$ with $0<q\leq 1$. Suppose the function $a(x)$ satisfies

$$2K(a)R_0\gamma^2<1,\quad \text{for } n\leq 2, \tag{2.50}$$

or

$$a_0\leq\frac{(n-2)\mu}{2R_0^2},\quad \text{for } n\geq 3. \tag{2.51}$$

Then we have

(1) If $p=q=1$, there exist some constants $M\geq 1$ and $\omega>0$, independent of (u^0,u^1,θ^0), such that

$$E(t)\leq ME(0)e^{-\omega t},\ \ \forall t\geq 0, \tag{2.52}$$

for all solutions of (2.7) with $(u^0, u^1, \theta^0) \in \mathcal{H}$.

(2) If $p+1 > 2q$, *there exists* $\tau > 0$, *depending on* (u^0, u^1, θ^0), *such that*

$$E(t) \leq 4E(0)\Big(1 + \frac{p+1-2q}{2q}\tau t\Big)^{-\frac{2q}{p+1-2q}}, \quad \forall t \geq 0, \tag{2.53}$$

for all solutions of (2.7) with $(u^0, u^1, \theta^0) \in \mathcal{H}$. *Further, the constants* M, ω *and* τ *can be explicitly given by*

$$\sigma_0 = \frac{p+1-2q}{2q}, \tag{2.54}$$

$$\varepsilon = \begin{cases} (1-2K(a)R_0\gamma^2)/8, & n \leq 2, \\ 1/8, & n \geq 3, \end{cases} \tag{2.55}$$

$$C_1 = \frac{2R_0}{\sqrt{\mu}} + (n-1)\lambda_0, \tag{2.56}$$

$$C_2 = \frac{\alpha^2 R_0^2}{\mu\varepsilon} + \frac{\alpha^2(n-1)^2\lambda_0^2}{4\varepsilon} + \frac{\alpha\lambda_1^2}{\beta}, \tag{2.57}$$

$$C_3 = \frac{2R_0^2}{\mu} + \frac{(n-1)^2 R_0\gamma^2}{4\varepsilon}, \tag{2.58}$$

$$C_4 = 1 + C_3 k_2^2, \tag{2.59}$$

$$b_1 = \frac{1}{2}C_1^{-1}E^{-\sigma_0}(0), \tag{2.60}$$

$$b_2 = \alpha/[E^{\sigma_0}(0)(\sigma_0\alpha C_1 + \beta C_2)], \tag{2.61}$$

$$b_3 = k_3/[E^{\sigma_0}(0)(C_4 + \sigma_0 C_1 k_3)], \tag{2.62}$$

$$b_4 = k_3(p+1)^{(p+1)/(2q)}/[C_1\sigma_0 E^{\sigma_0}(0)k_3(p+1)^{(p+1)/(2q)} + q(2C_4)^{(p+1)/(2q)}[R_0 mes(\Gamma_2)(p+1-2q)]^{\sigma_0}], \tag{2.63}$$

$$\delta_1 = \min\{1/2C_1,\ \alpha/(\beta C_2),\ k_3/C_4\}, \tag{2.64}$$

$$\delta_2 = \min\{b_1,\ b_2,\ b_3,\ b_4\}, \tag{2.65}$$

$$M = (1+\delta_1 C_1)/(1-\delta_1 C_1) \leq 3, \tag{2.66}$$

$$\omega = \delta_1/(1+\delta_1 C_1), \tag{2.67}$$

$$\tau = \delta_2/\Big[2(2E(0))^{-\sigma_0}[1+\delta_2 C_1 E^{\sigma_0}(0)]^{\sigma_0+1}\Big]. \tag{2.68}$$

If the potential a does not satisfy the smallness conditions (2.50) and (2.51), then we do not know whether or not Theorem 2.2 still holds. However, we can deal with large potentials a under an additional condition on Γ_2. Namely, we have

Theorem 2.3 *Assume that* Γ_1, Γ_2 *and* g *satisfy the conditions of Theorem 2.2. We further assume that*

$$m \cdot \nu \geq \eta > 0 \quad on\ \Gamma_2. \tag{2.69}$$

Suppose (2.16) holds. Then the decay properties (2.52) and (2.53) hold.

Furthermore, the explicit values of the constants M, ω and τ are as follows:

$$\begin{aligned}
\sigma_0 &= \frac{p+1-2q}{2q}, &(2.70)\\
\varepsilon &= 1/8, &(2.71)\\
C_0 &= \max\{0, \frac{2a_0R_0^2}{\mu}+2-n\}, &(2.72)\\
C_1' &= \frac{2R_0}{\sqrt{\mu}}+(n-1)\lambda_0+C_0k_1\gamma, &(2.73)\\
C_2' &= \frac{\alpha^2R_0^2}{\mu\varepsilon}+\frac{\alpha^2(n-1)^2\lambda_0^2}{4\varepsilon}+\frac{C_0^2\alpha^2k_1^2\gamma^2}{4\varepsilon}+\frac{\alpha\lambda_1^2}{\beta}, &(2.74)\\
C_3' &= \frac{2R_0^2}{\mu}+\frac{(n-1+C_0)^2R_0\gamma^2}{4\varepsilon}, &(2.75)\\
C_4' &= R_0+\frac{C_0^2k_1^2}{4\varepsilon}+C_3'k_2^2R_0, &(2.76)\\
b_1' &= \frac{1}{2}C_1'^{-1}E^{-\sigma_0}(0), &(2.77)\\
b_2' &= \alpha/[E^{\sigma_0}(0)(\sigma_0\alpha C_1'+\beta C_2')], &(2.78)\\
b_3' &= k_3\eta/[E^{\sigma_0}(0)(C_4'+\sigma_0C_1'k_3R_0)], &(2.79)\\
b_4' &= \eta k_3(p+1)^{(p+1)/(2q)}/[C_1'\sigma_0E^{\sigma_0}(0)R_0k_3(p+1)^{(p+1)/(2q)}\\
&\qquad +q[2C_4']^{(p+1)/(2q)}[mes(\Gamma_2)(p+1-2q)]^{\sigma_0}], &(2.80)\\
\delta_1' &= \min\{1/2C_1',\ \alpha/(\beta C_2'),\ \eta k_3/C_4'\}, &(2.81)\\
\delta_2' &= \min\{b_1',\ b_2',\ b_3',\ b_4'\}, &(2.82)\\
M &= (1+\delta_1'C_1')/(1-\delta_1'C_1')\le 3, &(2.83)\\
\omega &= \delta_1'/(1+\delta_1'C_1'), &(2.84)\\
\tau &= \delta_2'/\Big[2(2E(0))^{-\sigma_0}[1+\delta_2'C_1'E^{\sigma_0}(0)]^{\sigma_0+1}\Big]. &(2.85)
\end{aligned}$$

Example 2.4 It is easy to see that the following function

$$g(u)=g(u_1,u_2,\cdots u_n)=\begin{cases}(u_1^2+u_2^2+\cdots+u_n^2)u, & \text{if } |u|\le 1,\\ u, & \text{if } |u|\ge 1,\end{cases} \tag{2.86}$$

satisfies the conditions of Theorem 2.2 with $p=3$ and any $0<q\le 1$. In fact, (2.44), (2.46), (2.47) and (2.48) are obvious. In addition, it is easy to verify that g is globally Lipschitz. Thus, for any $u^1, u^2\in \mathbf{R}^n$ with $|u^1-u^2|\le 1$, we have

$$|g(u^1)-g(u^2)|\le c|u^1-u^2|\le c|u^1-u^2|^q. \tag{2.87}$$

To show (2.49), it suffices to show that the Jacobian matrix

$$\Big(\frac{\partial g_i}{\partial u_j}\Big)=\begin{pmatrix}\frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_n}\\ \vdots & & \vdots\\ \frac{\partial g_n}{\partial u_1} & \cdots & \frac{\partial g_n}{\partial u_n}\end{pmatrix} \tag{2.88}$$

is positive semi-definite. If $|u| \geq 1$, this is obvious. If $|u| \leq 1$, then

$$\left(\frac{\partial g_i}{\partial u_j}\right) = \begin{pmatrix} 3u_1^2 & 2u_1u_2 & \cdots & 2u_1u_n \\ 2u_1u_2 & 3u_2^2 & \cdots & 2u_2u_n \\ \cdots & \cdots & \cdots & \cdots \\ 2u_1u_n & 2u_2u_n & \cdots & 3u_n^2 \end{pmatrix} \tag{2.89}$$

is also positive semi-definite as we have

$$\begin{aligned} \begin{vmatrix} 3u_1^2 & 2u_1u_2 & \cdots & 2u_1u_k \\ 2u_1u_2 & 3u_2^2 & \cdots & 2u_2u_k \\ \cdots & \cdots & \cdots & \cdots \\ 2u_1u_k & 2u_2u_k & \cdots & 3u_k^2 \end{vmatrix} &= u_1^2u_2^2\cdots u_k^2 \begin{vmatrix} 3 & 2 & \cdots & 2 \\ 2 & 3 & \cdots & 2 \\ \cdots & \cdots & \cdots & \cdots \\ 2 & 2 & \cdots & 3 \end{vmatrix} \\ &= u_1^2u_2^2\cdots u_k^2[3+2(k-1)] \\ &\geq 0. \end{aligned} \tag{2.90}$$

On the other hand, if we take $g(u) = (h(u_1), \cdots, h(u_n))$, where

$$h(s) = \begin{cases} |s|^{p-1}s & \text{if } |s| \leq 1, \\ s, & \text{if } |s| \geq 1, \end{cases} \tag{2.91}$$

is a function on $\mathbf{R}$, then g does not satisfy (2.47) and (2.48). But one can expect that this kind of functions should produce good decay rates. Therefore, we amend the conditions on g in Theorems 2.2 and 2.3 and obtain the following theorem. □

Theorem 2.5 *Let $h(s)$ be a continuous function on $\mathbf{R}$ satisfying the following conditions:*

$$\begin{aligned} h(0) &= 0, & (2.92) \\ |h(s_1) - h(s_2)| &\leq k_2|s_1 - s_2|^q, \ \ \forall s_1, s_2 \in \mathbf{R} \ with \ |s_1 - s_2| \leq 1, & (2.93) \\ |h(s_1) - h(s_2)| &\leq k_2|s_1 - s_2|, \ \ \forall s_1, s_2 \in \mathbf{R} \ with \ |s_1 - s_2| \geq 1, & (2.94) \\ h(s)s &\geq k_3|s|^{p+1}, \ \ \forall s \in \mathbf{R} \ with \ |s| \leq 1, & (2.95) \\ h(s)s &\geq k_3|s|^2, \ \ \forall s \in \mathbf{R} \ with \ |s| \geq 1, & (2.96) \\ 0 &\leq [h(s^1) - h(s^2)](s^1 - s^2), \ \ \forall s^1, \ s^2 \in \mathbf{R}, & (2.97) \end{aligned}$$

for some constants $k_2, k_3 > 0$ and p, q with $0 < q \leq 1$. Suppose that

$$g(u) = (h(u_1), \cdots, h(u_n)). \tag{2.98}$$

Assume that the conditions on g in Theorem 2.2 are replaced by the above conditions and the other conditions in Theorems 2.2 and 2.3 are kept unchanged. Then Theorems 2.2 and 2.3 still hold with the same constants except that b_4 is replaced by

$$\begin{aligned} &k_3(p+1)^{(p+1)/(2q)}/[C_1\sigma_0 E^{\sigma_0}(0)k_3(p+1)^{(p+1)/(2q)} \\ &\qquad +q(2C_4)^{(p+1)/(2q)}[nR_0 mes(\Gamma_2)(p+1-2q)]^{\sigma_0}], \end{aligned} \tag{2.99}$$

and b_4' replaced by

$$\eta k_3(p+1)^{(p+1)/(2q)}/[C_1'\sigma_0 E^{\sigma_0}(0)R_0k_3(p+1)^{(p+1)/(2q)} \\ +q[2C_4']^{(p+1)/(2q)}[nmes(\Gamma_2)(p+1-2q)]^{\sigma_0}]. \tag{2.100}$$

Furthermore, if $p>1$ and $q=1/p$, then the decay rate (2.53) can be refined to

$$E(t) \le 4E(0)\Big(1+\frac{p-1}{2}\tau t\Big)^{-\frac{2}{p-1}}, \quad \forall t \ge 0, \tag{2.101}$$

with the same constants except the following changes:

$$\begin{aligned} \sigma_0 &= \frac{p-1}{2}, & (2.102)\\ b_4 &= (p+1)^{(p+1)/2}/[C_1\sigma_0E^{\sigma_0}(0)(p+1)^{(p+1)/2} \\ &\qquad +(2C_5)^{(p+1)/2}[R_0mes(\Gamma_2)(p-1)]^{\sigma_0}], & (2.103)\\ b_4' &= \eta(p+1)^{(p+1)/2}/[C_1'\sigma_0E^{\sigma_0}(0)R_0(p+1)^{(p+1)/2} \\ &\qquad +[2C_5']^{(p+1)/2}[mes(\Gamma_2)(p-1)]^{\sigma_0}], & (2.104)\\ C_5 &= \frac{1+C_3k_2k_3}{k_3}, & (2.105)\\ C_5' &= \frac{4\varepsilon R_0 + C_0k_1^2 + 4\varepsilon C_3' k_2k_3R_0}{k_3}. & (2.106) \end{aligned}$$

Remark 2.6 Note that condition (2.95) on h implies that

$$|h(s)| \ge k_3|s|^p, \quad \forall s \in \mathbf{R} \text{ with } |s| \le 1. \tag{2.107}$$

This means that $h(s)$ can not degenerate at the origin faster than $|s|^p$. The following theorem provides a decay rate when the nonlinearity h degenerates faster than any polynomial. □

Theorem 2.7 *Let $h(s)$ be a continuous function on $\mathbf{R}$ satisfying the following conditions:*

$$\begin{aligned} h(0) &= 0, & (2.108)\\ |h(s_1)-h(s_2)| &\le k_2|s_1-s_2|^q, \quad \forall s_1,s_2 \in \mathbf{R} \textit{ with } |s_1-s_2| \le 1, & (2.109)\\ |h(s_1)-h(s_2)| &\le k_2|s_1-s_2|, \quad \forall s_1,s_2 \in \mathbf{R} \textit{ with } |s_1-s_2| \ge 1, & (2.110)\\ h(s)s &\ge k_3|s|^2, \quad \forall s \in \mathbf{R} \textit{ with } |s| \ge 1, & (2.111)\\ 0 &\le [h(s^1)-h(s^2)](s^1-s^2), \quad \forall s^1,\ s^2 \in \mathbf{R}, & (2.112) \end{aligned}$$

for some constants $k_2, k_3 > 0$ and $0 < q \le 1$. Suppose that

$$g(u) = (h(u_1), \cdots, h(u_n)). \tag{2.113}$$

Assume that (2.16), (2.43) and (2.69) hold. Let $\varphi(s)$ denote a increasing and convex function defined on $[0,\infty)$ and twice differentiable outside $s=0$ such that $\varphi(s^{2q}) \le h(s)s$ on $[-1,1]$.

Then the energy $E(t)$ of solutions of (2.7) with $(u^0, u^1, \theta^0) \in \mathcal{H}$ satisfies the following decay rate:

$$E(t) \le 2V(t), \quad \textit{for } t \ge 0, \tag{2.114}$$

where $V(t)$ is the solution of the following differential equation:

$$V'(t) = -\frac{\delta V(t)}{2b}\varphi'\Big(\frac{aV(t)}{b}\Big) - n\delta(R_0 + 2C_0^2k_1^2 + C_3'k_2^2R_0)\mathrm{mes}(\Gamma_2)\varphi\Big(\frac{aV(t)}{b}\Big), \tag{2.115}$$

where δ is a sufficiently small positive constant and

$$a = \frac{1}{2n(R_0 + 2C_0^2k_1^2 + C_3'k_2^2R_0)\mathrm{mes}(\Gamma_2)}, \tag{2.116}$$

$$b = 1 + \delta C_1'\varphi'(aE(0)). \tag{2.117}$$

Furthermore, we have

$$\lim_{t\to\infty} E(t) = \lim_{t\to\infty} V(t) = 0. \tag{2.118}$$

Remark 2.8 The function φ which satisfies the conditions of Theorem 2.7 always exists. For example, we set

$$\bar{\varphi}(s) = \mathrm{conv}[\min\{s^{1/(2q)}h(s^{1/(2q)}),\ -s^{1/(2q)}h(-s^{1/(2q)})\}], \quad \text{for } 0 \le s \le 1, \tag{2.119}$$

and extend it to $[0, \infty)$. Here conv denotes the convex envelope of a function. Then we can take an increasing, convex and twice differentiable function $\varphi(s)$ such that $\varphi(s) \le \bar{\varphi}(s)$. □

In the special case where $s^{1/(2q)}h(s^{1/(2q)})$ is convex and $h(s)$ is odd, we have:

Corollary 2.9 *Assume that all the conditions of Theorem 2.7 hold. If, further, h is odd in $[-1, 1]$ and $s^{1/(2q)}h(s^{1/(2q)})$ is convex on $[0, 1]$, then the energy $E(t)$ of (2.7) satisfies the following decay rate:*

$$E(t) \le 2V(t), \quad \textit{for } t \ge 0, \tag{2.120}$$

where, for t large enough, $V(t)$ satisfies the following differential equation:

$$V'(t) = -K_1V^{1/(2q)}h((\frac{aV}{b})^{1/(2q)}) - K_2V^{1/q}h'((\frac{aV}{b})^{1/(2q)}), \tag{2.121}$$

where K_1, K_2 are positive constants independent of V.

Corollary 2.10 *Assume that all the conditions of Theorem 2.7 hold. If, further, h satisfies (2.95), then the decay properties (2.52) and (2.53) hold.*

Corollary 2.10 shows that Theorem 2.5 is covered by Theorem 2.7 when (2.69) holds. Since Theorem 2.7 does not include the case where (2.69) does not hold, we separate it from Theorem 2.7.

We now give an example of logarithmic decay rate which complements the example of polynomial decay rate existing in the literature.

Example 2.11 *Logarithmic Decay Rate.* Let $h(s)$ satisfy

$$h(s) = s^3 e^{-\frac{1}{s^2}}, \quad |s| \le 1, \tag{2.122}$$

$$c_1|s| \le h(s) \le |s|, \quad |s| \ge 1. \tag{2.123}$$

It is easy to see that h satisfies all the conditions of Corollary 2.9 with $q = 1$. Consequently, for t large enough, by (2.121), V satisfies

$$V'(t) \le -\omega V^2 e^{-\frac{b}{aV}}, \tag{2.124}$$

which is the same as

$$\left(e^{\frac{b}{aV}}\right)' \ge \frac{b\omega}{a}, \tag{2.125}$$

where ω is a positive constant independent of V. Solving the inequality, we obtain the logarithmic decay rate

$$V(t) \le \frac{b}{a}\Big[\log\Big(\frac{b\omega}{a}t + e^{\frac{b}{aV(0)}}\Big)\Big]^{-1}. \tag{2.126}$$

□

Remark 2.12 The decay rate of the form (2.101) has been established for the wave equation [21, p.127] and the compactly coupled wave equations [22]. It can be seen from the proof of Theorem 2.5 at the end of section 4 that the key point to obtain this decay rate $t^{-2/(p-1)}$ is to enlarge the following inequality

$$s^2 + |h(s)|^2 \le c(h(s)s)^{2/(p+1)}. \tag{2.127}$$

This can be done as follows. We first deduce from (2.95) that

$$s^2 \le c(h(s)s)^{2/(p+1)}. \tag{2.128}$$

Next, we have

$$|h(s)| \le c|s|^{1/p}, \tag{2.129}$$

$$|h(s)|^{2p/(p+1)} \le c|s|^{2/(p+1)}, \tag{2.130}$$

$$|h(s)|^{(2p+2-2)/(p+1)} \le c|s|^{2/(p+1)}, \tag{2.131}$$

$$|h(s)|^2 \le c|s|^{2/(p+1)}|h(s)|^{2/(p+1)} = c|h(s)s|^{2/(p+1)}. \tag{2.132}$$

In the situation of Theorems 2.2 and 2.3, since, for the vector function $g(u)$ and vector u, we have

$$|u|^{2/(p+1)}|g(u)|^{2/(p+1)} \ge |u \cdot g(u)|^{2/(p+1)} \tag{2.133}$$

the analogous inequality of (2.127) for g no longer holds. Thus, the decay rate (2.53) of Theorems 2.2 and 2.3 can not be refined to (2.101) in the case where $p > 1$ and $q = 1/p$. □

Remark 2.13 Observe that (2.44), (2.45) and (2.47) imply that $q \le p$. Therefore, under the assumption $0 < q \le 1$, we have $2q \le p + 1$. If $2q = p + 1$, then $q - 1 = p - q \ge 0$. Hence we have $q = p = 1$. Therefore, we have only two cases: $q = p = 1$ and $2q < p + 1$, as stated in Theorem 2.2. □

Remark 2.14 Under condition (2.43), assumptions (2.1) and (2.2) imply that the domain Ω is simply connected and star-shaped with respect to $x^0 \in \Omega$ (in which case $\Gamma_1 = \emptyset$) or $\Omega = \Omega_1 - \overline{\Omega}_2$, both Ω_1 and Ω_2 being star-shaped with respect to x^0. If $\Gamma_1 \cap \bar{\Gamma}_2 \neq \emptyset$, it is well known that the solution of (2.7) is not regular enough (see [14]) to perform the integrations by parts we will do later. Thus, the obtention of decay rates in this case is an open problem. The extension of the results of this paper to the case $\Gamma_1 \cap \bar{\Gamma}_2 \neq \emptyset$ requires a careful analysis of the singularities that the solution may develop on $\Gamma_1 \cap \bar{\Gamma}_2$ as in [16]. □

Remark 2.15 The expressions of ω and τ in the decay rates may look complex. But they are useful since they provide explicit estimates of the dependence of the decay rates ω and τ on the various parameters $\alpha, \beta, \Omega, k_2, k_3 \cdots$, etc. For example, if we take a linear boundary feedback $g(s) = k_3 s$ and let $k_3 \to 0$, then it follows from (2.64) and (2.67) that $\omega \to 0$. In addition, by these expressions, we can analyze the limit of the polynomial decay rate $\left(1 + \frac{p+1-2q}{2q}\tau t\right)^{-\frac{2q}{p+1-2q}}$ as p, q tend to 1 and recover the exponential decay of the case $p = q = 1$. Indeed, it is easy to see that

$$\lim_{p,q\to 1} \delta_2(p,q) = \min\{1/2C_1,\ \alpha/(\beta C_2),\ k_3/C_4\} = \delta_1, \tag{2.134}$$

$$\lim_{p,q\to 1} \tau(p,q) = \delta_1/\Big[2(1+\delta_1 C_1)\Big] = \omega/2, \tag{2.135}$$

$$\lim_{p,q\to 1} \left(1 + \frac{p+1-2q}{2q}\tau t\right)^{-\frac{2q}{p+1-2q}} = e^{-\omega t/2}. \tag{2.136}$$

□

Remark 2.16 For the linear elastodynamic system, the uniform stabilization with the boundary feedback of the form

$$\sigma_{ij}(u)\nu_j = u_i' \tag{2.137}$$

was established by Horn [18] by developing microlocal estimates (see [19]) for tangential derivatives of the solutions of the elastodynamic system. However, this remains to be done for the system of thermoelasticity.

3 Well-posedness

In this section, we use the theory of nonlinear semigroups to treat the problem of well-posedness of (2.7). Therefore, we formulate (2.7) as an abstract Cauchy problem.

Let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $(H^1_{\Gamma_1}(\Omega))^n$ and $[(H^1_{\Gamma_1}(\Omega))^n]'$ or $H^1_0(\Omega)$ and $H^{-1}(\Omega)$. We define the duality operator $A : (H^1_{\Gamma_1}(\Omega))^n \to [(H^1_{\Gamma_1}(\Omega))^n]'$ by

$$\langle Au, v\rangle = (u,v)_{(H^1_{\Gamma_1}(\Omega))^n}, \ \ \forall\, u,\ v \in (H^1_{\Gamma_1}(\Omega))^n, \tag{3.1}$$

and the duality operator $A_0 : H^1_0(\Omega) \to H^{-1}(\Omega)$ by

$$\langle A_0 u, v\rangle = (u,v)_{H^1_0(\Omega)}, \ \ \forall\, u,\ v \in H^1_0(\Omega). \tag{3.2}$$

The Riesz representation theorem ensures that A and A_0 are isometric isomorphisms of $(H^1_{\Gamma_1}(\Omega))^n$ onto $[(H^1_{\Gamma_1}(\Omega))^n]'$ and $H^1_0(\Omega)$ onto $H^{-1}(\Omega)$, respectively.

Also, we define a nonlinear operator B by

$$\langle Bu, v\rangle = \int_{\Gamma_2} m\cdot \nu g(u)\cdot v d\Gamma, \ \ \forall\, u,\ v \in (H^1_{\Gamma_1}(\Omega))^n. \tag{3.3}$$

Lemma 3.1 *Suppose that the function $g \in (C(\mathbf{R}^n))^n$. If there exist constants $c > 0$ and $\sigma \geq 0$ such that for $u \in \mathbf{R^n}$*

$$|g(u)| \leq \begin{cases} c[1+|u|^{n/(n-2)}], & n\geq 3,\\ c(1+|u|^{\sigma}), & n=2, \end{cases} \tag{3.4}$$

then B maps $(H^1_{\Gamma_1}(\Omega))^n$ into $[(H^1_{\Gamma_1}(\Omega))^n]'$. Furthermore, B is hemicontinuous, that is, we have

$$\lim_{t\to 0}\langle B(u^1+tu^2), v\rangle = \langle Bu^1, v\rangle \tag{3.5}$$

for any $u^1,\ u^2,\ v\in (H^1_{\Gamma_1}(\Omega))^n$.

Proof. If $n \geq 3$, then, by the trace theorem (see [36, Chap.1]) and the imbedding theorem (see [1, p.217]), we have

$$H^1_{\Gamma_1}(\Omega) \hookrightarrow H^{1/2}(\Gamma) \subset L^{(2n-2)/(n-2)}(\Gamma). \tag{3.6}$$

Thus, we have

$$\begin{aligned} |\langle Bu, v\rangle| &= |\int_{\Gamma_2} m\cdot\nu g(u)\cdot v d\Gamma| \\ &\leq c\Big(\int_{\Gamma_2}|g(u)|^{(2n-2)/n}d\Gamma\Big)^{n/(2n-2)}\Big(\int_{\Gamma_2}|v|^{(2n-2)/(n-2)}d\Gamma\Big)^{(n-2)/(2n-2)} \\ &\leq c\Big[1+\Big(\int_{\Gamma_2}|u|^{(2n-2)/(n-2)}d\Gamma\Big)^{n/(2n-2)}\Big]\|v\|_{(H^1_{\Gamma_1}(\Omega))^n} \\ &\leq c\Big[1+\|u\|^{n/(n-2)}_{(H^1_{\Gamma_1}(\Omega))^n}\Big]\|v\|_{(H^1_{\Gamma_1}(\Omega))^n}. \end{aligned} \tag{3.7}$$

This shows that $Bu \in [(H^1_{\Gamma_1}(\Omega))^n]'$.

If $n = 2$, then, by the trace theorem (see [36, Chap.1]) and the imbedding theorem (see [1, p.217]), we have

$$H^1_{\Gamma_1}(\Omega) \hookrightarrow H^{1/2}(\Gamma) \subset L^r(\Gamma) \tag{3.8}$$

for any $2 \leq r < \infty$. In addition, we may as well assume that $\sigma \geq 1$ since, by condition (3.4), we have

$$|g(u)| \leq 2c \leq 2c(1+|u|^{\sigma+1}), \ \text{ for } |u| \leq 1, \tag{3.9}$$

and

$$|g(u)| \leq c(1+|u|^{\sigma+1}) \leq 2c(1+|u|^{\sigma+1}), \ \text{ for } |u| \geq 1. \tag{3.10}$$

It therefore follows that

$$\begin{aligned} |\langle Bu, v\rangle| &= |\int_{\Gamma_2} m\cdot\nu g(u)\cdot v d\Gamma| \\ &\leq c(\int_{\Gamma_2}|g(u)|^2 d\Gamma)^{1/2}(\int_{\Gamma_2}|v|^2 d\Gamma)^{1/2} \\ &\leq c[1+(\int_{\Gamma_2}|u|^{2\sigma}d\Gamma)^{1/2}]\|v\|_{(H^1_{\Gamma_1}(\Omega))^n} \\ &\leq c[1+\|u\|^{\sigma}_{(H^1_{\Gamma_1}(\Omega))^n}]\|v\|_{(H^1_{\Gamma_1}(\Omega))^n}. \end{aligned} \tag{3.11}$$

This shows that $Bu \in [(H^1_{\Gamma_1}(\Omega))^n]'$.

If $n = 1$, this is just the consequence of the following embedding

$$H^1_{\Gamma_1}(\Omega) \hookrightarrow C(\bar{\Omega}). \tag{3.12}$$

It remains to prove that B is hemicontinuous. By the continuity of g, we deduce that for any $u^1,\ u^2,\ v \in (H^1_{\Gamma_1}(\Omega))^n$

$$g(u^1 + tu^2) \to g(u^1) \text{ as } t \to 0 \text{ a.e. on } \Gamma. \tag{3.13}$$

It therefore follows from Lebesgue's dominated convergence theorem and (3.4) that

$$\begin{aligned}
\lim_{t\to 0}\langle B(u^1 + tu^2), v\rangle &= \int_{\Gamma_2} m \cdot \nu g(u^1 + tu^2) \cdot v d\Gamma \\
&= \int_{\Gamma_2} m \cdot \nu g(u^1) \cdot v d\Gamma \\
&= \langle Bu^1, v\rangle.
\end{aligned} \tag{3.14}$$

This shows that B is hemicontinuous. □

Note that if g satisfies (2.45) and (2.46) of Theorem 2.2 (or (2.93) and (2.94) of Theorem 2.5), then g satisfies (3.4). Thus B maps $(H^1_{\Gamma_1}(\Omega))^n$ into $[(H^1_{\Gamma_1}(\Omega))^n]'$.

Using the operators $A,\ A_0$ and B, we can formally transform problem (2.7) into an abstract Cauchy problem. In doing so, we multiply the first equation of (2.7) by $v \in (H^1_{\Gamma_1}(\Omega))^n$ and integrate over Ω by parts. This gives

$$\begin{aligned}
0 &= \int_\Omega (u'' - \mu\Delta u - (\lambda+\mu)\nabla \text{div } u + \alpha\nabla\theta) \cdot v dx \\
&= \int_\Omega u'' \cdot v dx - \mu \int_\Gamma \frac{\partial u}{\partial \nu} \cdot v d\Gamma - (\lambda+\mu)\int_\Gamma v \cdot \nu \text{div}(u) d\Gamma \\
&\quad + \int_\Omega (\mu\nabla u \cdot \nabla v + (\lambda+\mu)\text{div } u \text{div } v) dx + \int_\Omega \alpha\nabla\theta \cdot v dx \\
&= \int_\Omega u'' \cdot v dx + \int_\Gamma am \cdot \nu u \cdot v d\Gamma + \int_\Gamma m \cdot \nu g(u') \cdot v d\Gamma \\
&\quad + \int_\Omega (\mu\nabla u \cdot \nabla v + (\lambda+\mu)\text{div } u \text{div } v) dx + \int_\Omega \alpha\nabla\theta \cdot v dx \\
&= \langle u'', v\rangle + \langle Au, v\rangle + \langle Bu', v\rangle + \langle \alpha\nabla\theta, v\rangle,
\end{aligned} \tag{3.15}$$

and therefore,

$$u'' + Au + Bu' + \alpha\nabla\theta = 0. \tag{3.16}$$

Similarly, multiplying the second equation of (2.7) by $v \in H^1_0(\Omega)$ and integrating over Ω by parts, we obtain

$$\theta' + A_0\theta + \beta \text{div } u' = 0. \tag{3.17}$$

Setting

$$\Phi = (u, u', \theta), \tag{3.18}$$

$$\Lambda\Phi = (-u',\ Au + Bu' + \alpha\nabla\theta,\ A_0\theta + \beta\text{div } u'), \tag{3.19}$$

we transform (2.7) into

$$\begin{cases} \Phi' + \Lambda\Phi = 0, & t > 0, \\ \Phi(0) = (u^0, u^1, \theta^0). \end{cases} \tag{3.20}$$

This leads us to define the solution of (2.7) as that of (3.20).

Consider the nonlinear operator Λ on $\mathcal{H} = (H^1_{\Gamma_1}(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega)$ with the domain

$$D(\Lambda) = \{(u, v, \theta) \in \mathcal{H} \;:\; \quad v \in (H^1_{\Gamma_1}(\Omega))^n,\; Au + Bv \in (L^2(\Omega))^n \\ \theta \in H^2(\Omega) \cap H^1_0(\Omega)\}. \tag{3.21}$$

We are going to prove that $-\Lambda$ is m-dissipative. For the definition of m-dissipativeness, we refer to [6, p.71].

In what follows, we denote by $(\cdot, \cdot)$ the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^n$.

Lemma 3.2 *Suppose that g satisfies (2.44), (2.49) (or (2.92), (2.97)) and (3.4). Then the operator $-\Lambda$ is m-dissipative on $\mathcal{H}$.*

Proof. By (2.49) (or (2.97)), we obtain that for any $(u^1, v^1, \theta^1),\ (u^2, v^2, \theta^2) \in D(\Lambda)$

$$\begin{aligned} & (\Lambda(u^1, v^1, \theta^1) - \Lambda(u^2, v^2, \theta^2), (u^1, v^1, \theta^1) - (u^2, v^2, \theta^2))_{\mathcal{H}} \\ = \;& (v^2 - v^1, u^1 - u^2)_{(H^1_{\Gamma_1}(\Omega))^n} + \frac{\alpha}{\beta}(A_0(\theta^1 - \theta^2) + \beta \mathrm{div}(v^1 - v^2), \theta^1 - \theta^2) \\ & + (A(u^1 - u^2) + Bv^1 - Bv^2 + \alpha\nabla(\theta^1 - \theta^2), v^1 - v^2) \\ = \;& \int_{\Gamma_2} m \cdot \nu [g(v^1) - g(v^2)] \cdot (v^1 - v^2) d\Gamma + \frac{\alpha}{\beta} \|\nabla(\theta^1 - \theta^2)\|^2 \\ \geq \;& 0. \end{aligned} \tag{3.22}$$

Thus, $-\Lambda$ is dissipative.

It remains to show that

$$(I + \Lambda)(D(\Lambda)) = \mathcal{H}. \tag{3.23}$$

Namely, we want to prove that

$$\begin{cases} u - v = \varphi, \\ v + Au + Bv + \alpha\nabla\theta = \psi, \\ \theta + A_0\theta + \beta \mathrm{div} v = \xi, \end{cases} \tag{3.24}$$

has a solution $(u, v, \theta) \in D(\Lambda)$ for every $(\varphi, \psi, \xi) \in \mathcal{H}$. For this, it suffices to prove that the following problem

$$\begin{cases} v + Av + Bv + \alpha\nabla\theta = \psi - A\varphi, \\ \theta + A_0\theta + \beta \mathrm{div} v = \xi, \end{cases} \tag{3.25}$$

has a solution $v \in (H^1_{\Gamma_1}(\Omega))^n$ and $\theta \in H^2(\Omega) \cap H^1_0(\Omega)$ for every $(\varphi, \psi, \xi) \in \mathcal{H}$. In fact, if this has been done, then, by setting $u = v + \varphi$, it is easy to see that (u, v, θ) satifies (3.24). Further, we have

$$Au + Bv = \psi - v - \alpha\nabla\theta \in (L^2(\Omega))^n. \tag{3.26}$$

Thus, we have $(u, v, \theta) \in D(\Lambda)$.

To prove that (3.25) has a required solution, it suffices to show that the nonlinear operator $\mathcal{A}$ defined by

$$\mathcal{A}(v,\theta) = (v + Av + Bv + \alpha\nabla\theta, \ \frac{\alpha}{\beta}\theta + \frac{\alpha}{\beta}A_0\theta + \alpha \text{div} v) \tag{3.27}$$

maps $(H^1_{\Gamma_1}(\Omega))^n \times H^1_0(\Omega)$ onto $[(H^1_{\Gamma_1}(\Omega))^n]' \times H^{-1}(\Omega)$, that is

$$\mathcal{A}((H^1_{\Gamma_1}(\Omega))^n \times H^1_0(\Omega)) = [(H^1_{\Gamma_1}(\Omega))^n]' \times H^{-1}(\Omega). \tag{3.28}$$

In fact, if this has been done, then, since $\psi - A\varphi \in [(H^1_{\Gamma_1}(\Omega))^n]'$ and $\frac{\alpha}{\beta}\xi \in L^2(\Omega)$, there exist $(v,\theta) \in (H^1_{\Gamma_1}(\Omega))^n \times H^1_0(\Omega)$ such that (3.25) holds. Further, since $A_0\theta = \xi - \theta - \beta \text{div} v \in L^2(\Omega)$, we have $\theta \in H^2(\Omega) \cap H^1_0(\Omega)$.

To prove (3.28), by Theorem 1.3 of [6, p.40], it suffices to show $\mathcal{A}$ is monotone, coercive and hemicontinuous. For any (v^1,θ^1), $(v^2,\theta^2) \in (H^1_{\Gamma_1}(\Omega))^n \times H^1_0(\Omega)$, we have

$$\begin{aligned} & \langle \mathcal{A}(v^1,\theta^1) - \mathcal{A}(v^2,\theta^2), (v^1,\theta^1) - (v^2,\theta^2)\rangle \\ = \ & \langle v^1 - v^2, v^1 - v^2\rangle + \langle A(v^1 - v^2), v^1 - v^2\rangle + \langle Bv^1 - Bv^2, v^1 - v^2\rangle \\ & + \langle \alpha\nabla(\theta^1 - \theta^2), v^1 - v^2\rangle + \frac{\alpha}{\beta}\langle \theta^1 - \theta^2, \theta^1 - \theta^2\rangle \\ & + \frac{\alpha}{\beta}\langle A_0(\theta^1 - \theta^2), \theta^1 - \theta^2\rangle + \alpha\langle \text{div}(v^1 - v^2), \theta^1 - \theta^2\rangle \\ \geq \ & 0. \end{aligned} \tag{3.29}$$

So $\mathcal{A}$ is monotone. Similarly, we have

$$\begin{aligned} \langle \mathcal{A}(v,\theta), (v,\theta)\rangle &= \langle v, v\rangle + \langle Av, v\rangle + \langle Bv, v\rangle \\ & \quad + \frac{\alpha}{\beta}\langle \theta,\theta\rangle + \frac{\alpha}{\beta}\langle A_0\theta,\theta\rangle \\ &\geq \|v\|^2_{(H^1_{\Gamma_1}(\Omega))^n} + \frac{\alpha}{\beta}\|\theta\|^2_{H^1_0(\Omega)}. \end{aligned} \tag{3.30}$$

Thus, $\mathcal{A}$ is coercive. On the other hand, by Lemma 3.1, we can readily deduce that $\mathcal{A}$ is hemicontinuous. That is, we have

$$\lim_{t\to 0}\langle \mathcal{A}[(v^1,\theta^1) + t(v^2,\theta^2)], (v,\theta)\rangle = \langle \mathcal{A}(v^1,\theta^1), (v,\theta)\rangle \tag{3.31}$$

for any (v^1,θ^1), (v^2,θ^2), $(v,\theta) \in (H^1_{\Gamma_1}(\Omega))^n \times H^1_0(\Omega)$. This completes the proof. □

Lemma 3.3 *Suppose that (2.43) holds. If g satisfies (2.44) and (3.4), then $D(\Lambda)$ is dense in $\mathcal{H}$. Further, if g satisfies (2.45) and (2.46) (or (2.93) and (2.94)), we have*

$$\begin{aligned} D(\Lambda) \subset D_0 = \{(u,v,\theta) \in \mathcal{H} \ : \ & u \in (H^s(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n, \ v \in (H^1_{\Gamma_1}(\Omega))^n, \\ & \theta \in H^2(\Omega) \cap H^1_0(\Omega), \\ & \mu\frac{\partial u}{\partial \nu} + (\lambda+\mu)\nu \text{div} u + am\cdot\nu u + m\cdot\nu g(v) = 0 \ \text{ on } \Gamma_2\}, \end{aligned} \tag{3.32}$$

for some $s > 3/2$.

Proof. Set

$$D = \{(u,v,\theta) \in \mathcal{H} \ : \ u \in (H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n, \ v \in (H^1_0(\Omega))^n, \\ \theta \in H^2(\Omega) \cap H^1_0(\Omega), \\ \mu \frac{\partial u}{\partial \nu} + (\lambda+\mu)\nu \mathrm{div} u + am\cdot\nu u = 0 \ \text{ on } \Gamma_2\}. \tag{3.33}$$

To prove that $D(\Lambda)$ is dense in $\mathcal{H}$, it suffices to prove that D is dense in $\mathcal{H}$ and $D \subset D(\Lambda)$. Firstly, we prove that D is dense in $\mathcal{H}$. For this, it suffices to prove that

$$W = \{w \in (H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n \ : \ \mu \frac{\partial w}{\partial \nu} + (\lambda+\mu)\nu \mathrm{div} w + am\cdot\nu w = 0 \ \text{ on } \Gamma_2\} \tag{3.34}$$

is dense in $(H^1_{\Gamma_1}(\Omega))^n$. Let $v \in (H^1_{\Gamma_1}(\Omega))^n$ be such that

$$(v,w)_{(H^1_{\Gamma_1}(\Omega))^n} = 0, \quad w \in W. \tag{3.35}$$

For any fixed $f \in (L^2(\Omega))^n$, we consider the following elliptic problem

$$\begin{cases} -\mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u = f & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_1, \\ \mu \frac{\partial u}{\partial \nu} + (\lambda+\mu)\nu \mathrm{div} u + am\cdot\nu u = 0 & \text{on } \Gamma_2. \end{cases} \tag{3.36}$$

By the elliptic regularity theory, problem (3.36) has a solution $u \in W$. Thus, by (3.35), we have

$$(v,f) = (v,u)_{(H^1_{\Gamma_1}(\Omega))^n} = 0. \tag{3.37}$$

Hence, we deduce that $v = 0$. It therefore follows from the Hahn-Banach theorem that W is dense in $(H^1_{\Gamma_1}(\Omega))^n$.

Next, we prove that $D \subset D(\Lambda)$. Let $(u,v,\theta) \in D$. To prove that $(u,v,\theta) \in D(\Lambda)$, it suffices to prove that $Au + Bv \in (L^2(\Omega))^n$. For this, let $w \in (H^1_{\Gamma_1}(\Omega))^n$. By the definition of A and B, we have

$$\begin{aligned} \langle Au + Bv, w\rangle &= (u,w)_{(H^1_{\Gamma_1}(\Omega))^n} + \int_{\Gamma_2} m\cdot\nu g(v)\cdot w d\Gamma \\ &= \int_\Omega (\mu\nabla u\cdot\nabla w + (\lambda+\mu)\mathrm{div}\ u \mathrm{div}\ w) dx \\ &\quad + \int_{\Gamma_2} am\cdot\nu u\cdot w d\Gamma \ \ (\text{use } \ g(0)=0) \\ &= -\int_\Omega (\mu\Delta u + (\lambda+\mu)\nabla \mathrm{div}\ u)\cdot w dx, \end{aligned} \tag{3.38}$$

since $u \in (H^2(\Omega))^n$. It therefore follows that there exists a constant c, depending on u, such that

$$|\langle Au + Bv, w\rangle| \le c\|w\|, \tag{3.39}$$

which implies that $Au + Bv \in (L^2(\Omega))^n$. Thus $(u,v,\theta) \in D(\Lambda)$.

It remains to prove that $D(\Lambda) \subset D_0$ if g satisfies (2.45) and (2.46) (or (2.93) and (2.94)). Then it is sufficient to prove that $u \in (H^s(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n$ for some $s > 3/2$ if $(u, v, \theta) \in D(\Lambda)$. For this, let $f = Au + Bv$ and $h = -m \cdot \nu g(v)$. Then, for any $w \in (H^1_{\Gamma_1}(\Omega))^n$, we have

$$\begin{aligned} \langle f, w \rangle &= \int_\Omega (\mu \nabla u \cdot \nabla w + (\lambda + \mu) \text{div } u \text{div } w) dx \\ &\quad + \int_{\Gamma_2} am \cdot \nu u \cdot w d\Gamma + \int_{\Gamma_2} m \cdot \nu g(v) \cdot w d\Gamma. \end{aligned} \tag{3.40}$$

This shows that u is a weak solution of the following elliptic problem

$$\begin{cases} -\mu \Delta u - (\lambda + \mu) \nabla \text{div } u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \mu \dfrac{\partial u}{\partial \nu} + (\lambda + \mu) \nu \text{div} u + am \cdot \nu u = h & \text{on } \Gamma_2. \end{cases} \tag{3.41}$$

By assumption, we have

$$f = Au + Bv \in (L^2(\Omega))^n. \tag{3.42}$$

We further prove that

$$h \in (H^\sigma(\Gamma))^n \tag{3.43}$$

for some $0 < \sigma < q/2$. For this, we let

$$S_1 = \{(x, y) \in \Gamma \times \Gamma \ : \ |x - y| \leq 1\}, \tag{3.44}$$

$$S_{11} = \{(x, y) \in S_1 \ : \ |v(x) - v(y)| \leq 1\}, \tag{3.45}$$

$$S_{12} = \{(x, y) \in S_1 \ : \ |v(x) - v(y)| \geq 1\}. \tag{3.46}$$

To prove (3.43), by definition (see [15, p.20]), it suffices to prove

$$\iint_{\Gamma \times \Gamma} \frac{|g(v(x)) - g(v(y))|^2}{|x - y|^{n-1+2\sigma}} d\Gamma(x) d\Gamma(y) < \infty. \tag{3.47}$$

The integral over $\Gamma \times \Gamma - S_1$ is bounded above by

$$c \iint_{\Gamma \times \Gamma} |g(v(x)) - g(v(y))|^2 d\Gamma(x) d\Gamma(y) \leq c \int_\Gamma |g(v(x))|^2 d\Gamma(x) < \infty, \tag{3.48}$$

since $g(v(x)) \in L^2(\Gamma)$ which is true because $v \in (H^{1/2}(\Gamma))^n$ and g satisfies (2.45) and (2.46) (or (2.93) and (2.94)). Therefore, it suffices to prove

$$\begin{aligned} I &= \iint_{S_1} \frac{|g(v(x)) - g(v(y))|^2}{|x - y|^{n-1+2\sigma}} d\Gamma(x) d\Gamma(y) \\ &= \iint_{S_{11}} \frac{|g(v(x)) - g(v(y))|^2}{|x - y|^{n-1+2\sigma}} d\Gamma(x) d\Gamma(y) \\ &\quad + \iint_{S_{12}} \frac{|g(v(x)) - g(v(y))|^2}{|x - y|^{n-1+2\sigma}} d\Gamma(x) d\Gamma(y) \\ &= I_1 + I_2 \\ &< \infty. \end{aligned} \tag{3.49}$$

By (2.45) and Hölder's inequality, we have

$$\begin{aligned} I_1 &\le \iint_{S_{11}} \frac{|v(x)-v(y)|^{2q}}{|x-y|^{n-1+2\sigma}} d\Gamma(x)d\Gamma(y) \\ &\le \Big(\iint_{S_{11}} \frac{|v(x)-v(y)|^2}{|x-y|^n} d\Gamma(x)d\Gamma(y)\Big)^q \\ &\quad \times\Big(\iint_{S_1} \frac{1}{|x-y|^{(n-1+2\sigma-nq)/(1-q)}} d\Gamma(x)d\Gamma(y)\Big)^{1-q}. \end{aligned} \tag{3.50}$$

Since $v \in (H^1_{\Gamma_1}(\Omega))^n$, by the trace theorem, we have $v \in (H^{1/2}(\Gamma))^n$, and then we have

$$\iint_{S_1} \frac{|v(x)-v(y)|^2}{|x-y|^n} d\Gamma(x)d\Gamma(y) < \infty. \tag{3.51}$$

In addition, since for $0 < 2\sigma < q$

$$\frac{n-1+2\sigma-nq}{1-q} = n + \frac{2\sigma-1}{1-q} < n-1, \tag{3.52}$$

we have

$$\iint_{S_1} \frac{1}{|x-y|^{(n-1+2\sigma-nq)/(1-q)}} d\Gamma(x)d\Gamma(y) < \infty. \tag{3.53}$$

It therefore follows that $I_1 < \infty$. On the other hand, by (2.46) and (3.51), we have

$$\begin{aligned} I_2 &\le \iint_{S_{12}} \frac{|v(x)-v(y)|^2}{|x-y|^{n-1+2\sigma}} d\Gamma(x)d\Gamma(y) \\ &\le \iint_{S_{12}} \frac{|v(x)-v(y)|^2}{|x-y|^n} d\Gamma(x)d\Gamma(y) \\ &< \infty. \end{aligned} \tag{3.54}$$

Thus we have proved (3.49). In a similar way, we can show that $I < \infty$ if (2.93) holds.

Finally, we prove that

$$u \in (H^s(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n \tag{3.55}$$

with

$$s = \frac{3}{2} + \sigma > \frac{3}{2}. \tag{3.56}$$

Let $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ denote the unit normal on Γ directed to the exterior of Ω and consider the following system

$$\sum_{j=1}^n [(\lambda+\mu)\nu_i\nu_j + \delta_{ij}\mu]\xi_j = h_i, \quad i = 1, \cdots, n, \tag{3.57}$$

where δ_{ij} denote the Kronecker symbol, i.e.,

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases} \tag{3.58}$$

It is easy to see that the system has a solution $\xi = (\xi_1, \cdots, \xi_n) \in (H^\sigma(\Gamma))^n)$ for $h \in (H^\sigma(\Gamma))^n$. By the trace theorem (see, e.g., [36, p.39, Theorem 8.3]), there exists $\varphi \in (H^s(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n$ such that

$$\frac{\partial \varphi}{\partial \nu} = \xi, \quad \varphi = 0 \quad \text{on } \Gamma. \tag{3.59}$$

Let $\{\tau^k(x)\}_{k=1}^{n-1}$ be a tangential vector field such that$\{\nu(x), \tau^1(x), \cdots, \tau^{n-1}(x)\}$ forms an orthonormal basis in $\mathbf{R}^n$ for almost all $x \in \Gamma$. Hence, there exist $\gamma^{k,j}$ $(j = 1, 2, \cdots, n;\ k = 1, 2, \cdots, n-1)$ depending on $\{\nu(x), \tau^1(x), \cdots, \tau^{n-1}(x)\}$ such that

$$\begin{aligned} \frac{\partial \varphi_j}{\partial x_j} &= \nu_j \frac{\partial \varphi_j}{\partial \nu} + \sum_{k=1}^{n-1} \gamma^{k,j} \frac{\partial \varphi_j}{\partial \tau^k} \\ &= \nu_j \frac{\partial \varphi_j}{\partial \nu} \\ &= \nu_j \xi_j \qquad \text{on } \Gamma, \qquad j = 1, 2, \cdots, n. \end{aligned} \tag{3.60}$$

It therefore follows from (3.57) that

$$\mu \frac{\partial \varphi}{\partial \nu} + (\lambda + \mu) \nu \operatorname{div} \varphi + am \cdot \nu \varphi = h \quad \text{on } \Gamma_2. \tag{3.61}$$

Setting

$$\psi = u - \varphi, \tag{3.62}$$

then ψ satisfies

$$\begin{cases} -\mu \Delta \psi - (\lambda + \mu) \nabla \operatorname{div}\ \psi = F & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma_1, \\ \mu \dfrac{\partial \psi}{\partial \nu} + (\lambda + \mu) \nu \operatorname{div} \psi + am \cdot \nu \psi = 0 & \text{on } \Gamma_2, \end{cases} \tag{3.63}$$

where

$$F = f - \mu \Delta \varphi - (\lambda + \mu) \nabla \operatorname{div}\ \varphi \in (H^{s-2}(\Omega))^n. \tag{3.64}$$

Thus, problem (3.41) is equivalent to (3.63). By the classical variational methods (see, e.g., [13]), for every $F \in ((H^1_{\Gamma_1}(\Omega)')^n$, problem (3.63) has a unique weak solution $\psi \in (H^1_{\Gamma_1}(\Omega))^n$ in the sense of distribution

$$\int_\Omega (\mu \nabla \psi \cdot \nabla \phi + (\lambda + \mu) \operatorname{div} \psi \operatorname{div} \phi) dx + \int_{\Gamma_2} am \cdot \nu \psi \cdot \phi d\Gamma = \int_\Omega F \cdot \phi dx \quad \forall \phi \in (H^1_{\Gamma_1}(\Omega))^n. \tag{3.65}$$

Moreover, by the classical Nirenberg's translation method (see, e.g., [2, p.107, Lemma 9.2] or [36, p.124]), we prove that, if $F \in (L^2(\Omega))^n$, then $\psi \in (H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n$. Our proof is essentially the same as that of Lemma 9.2 of [2, p.107]. Since the regularity is local property, it suffices to prove that, for any $x \in \bar{\Omega}$, there exists a neighborhoood $O(x)$ such that $\psi \in (H^2(O(x) \cap \Omega))^n$. We consider the case $x \in \Gamma$ as the case $x \in \Omega$ is much easier. For simplicity, we may as well assume that $x = 0 \in \Gamma_2$ and the boudary is flat with the normal oriented in the direction x_n since the general case can be transformed into the special case by a mapping of class C^2 (see Remark 3.4 below). Therefore, there exists a hemisphere $G_R = \{x \ :\ |x| < R,\ x_n > 0\}$ such that $G_R \subset \Omega$ and $\Gamma_{2G} = \{x \in \bar{G}_R \ :\ x_n = 0\} \subset \Gamma_2$.

Let $0 < R' < R$ and $R'' = \frac{1}{2}(R' + R)$ and let ζ denote a real function which is infinitely differentiale on $\mathbf{R}^n$ and $\zeta = 1$ on $G_{R'}$ and $\zeta = 0$ outside $G_{R''}$. Note that ζ need not vanish on the flat part Γ_{2G} of the boundary of G_R. Let $C_0^\infty(G_R \cup \Gamma_{2G})$ denote the function space of all infinitely differentiable functions with support in $G_R \cup \Gamma_{2G}$. By (3.65), we have for any $\phi \in (C_0^\infty(G_R \cup \Gamma_{2G}))^n$

$$\int_{G_R} (\mu \nabla \psi \cdot \nabla \phi + (\lambda + \mu)\text{div}\psi \text{div}\phi)dx + \int_{\Gamma_{2G}} am \cdot \nu \psi \cdot \phi d\Gamma = \int_{G_R} F \cdot \phi dx. \qquad (3.66)$$

Define the bilinear form $B(\psi, \phi)$ by

$$B(\psi, \phi) = \int_{G_R} (\mu \nabla \psi \cdot \nabla \phi + (\lambda + \mu)\text{div}\psi \text{div}\phi)dx + \int_{\Gamma_{2G}} am \cdot \nu \psi \cdot \phi d\Gamma. \qquad (3.67)$$

Then we have

$$|B(\psi, \phi)| \leq \|F\|\|\phi\| \quad \forall \phi \in (C_0^\infty(G_R \cup \Gamma_{2G}))^n. \qquad (3.68)$$

For a real number, we define the difference operator δ_h^i by

$$\delta_h^i u = h^{-1}[u(x_1, \cdots, x_i + h, \cdots, x_n) - u(x_1, \cdots, x_i, \cdots, x_n)] \quad i = 1, 2, \cdots, n.$$

We now want to estimate the difference quotients $\delta_h^i(\zeta\psi)$ for $i = 1, 2, \cdots, n-1$. Since

$$\begin{aligned}
B(\delta_h^i(\zeta\psi), \phi) &= \int_{G_R} (\mu \nabla(\delta_h^i(\zeta\psi)) \cdot \nabla \phi + (\lambda + \mu)\text{div}(\delta_h^i(\zeta\psi))\text{div}\phi)dx \\
&\quad + \int_{\Gamma_{2G}} am \cdot \nu(\delta_h^i(\zeta\psi)) \cdot \phi d\Gamma \\
&= \int_{G_R} (\mu \delta_h^i(\nabla(\zeta\psi)) \cdot \nabla \phi + (\lambda + \mu)\delta_h^i(\text{div}(\zeta\psi))\text{div}\phi)dx \\
&\quad + \int_{\Gamma_{2G}} am \cdot \nu(\delta_h^i(\zeta\psi)) \cdot \phi d\Gamma \\
&= \int_{G_R} (\mu \delta_h^i(\psi_j \nabla \zeta + \zeta \nabla \psi_j) \cdot \nabla \phi_j + (\lambda + \mu)\delta_h^i(\nabla \zeta \cdot \psi + \zeta \text{div}\psi)\text{div}\phi)dx \\
&\quad + \int_{\Gamma_{2G}} am \cdot \nu(\delta_h^i(\zeta\psi)) \cdot \phi d\Gamma \\
&= \mu \int_{G_R} (\delta_h^i(\psi_j \nabla \zeta) \cdot \nabla \phi_j + \zeta \nabla \psi_j \cdot \delta_{-h}^i(\nabla \phi_j))dx \\
&\quad + (\lambda + \mu) \int_{G_R} (\delta_h^i(\nabla \zeta \cdot \psi)\text{div}\phi + \zeta \text{div}\psi)\delta_{-h}^i(\text{div}\phi))dx \\
&\quad + \int_{\Gamma_{2G}} am \cdot \nu \zeta \psi \cdot \delta_{-h}^i(\phi) d\Gamma \\
&= \mu \int_{G_R} (\delta_h^i(\psi_j \nabla \zeta) \cdot \nabla \phi_j + \nabla \psi_j \cdot \nabla(\zeta \delta_{-h}^i \phi_j) - \nabla \psi_j \cdot \nabla \zeta \delta_{-h}^i \phi_j)dx \\
&\quad + (\lambda + \mu) \int_{G_R} (\delta_h^i(\nabla \zeta \cdot \psi)\text{div}\phi + \text{div}\psi \text{div}(\zeta \delta_{-h}^i \phi) - \text{div}\psi \nabla \zeta \cdot \delta_{-h}^i \phi)dx \\
&\quad + \int_{\Gamma_{2G}} am \cdot \nu \psi \cdot (\zeta \delta_{-h}^i \phi) d\Gamma \\
&= B(\psi, \zeta \delta_{-h}^i \phi) + \mu \int_{G_R} (\delta_h^i(\psi_j \nabla \zeta) \cdot \nabla \phi_j - \nabla \psi_j \cdot \nabla \zeta \delta_{-h}^i \phi_j)dx \\
&\quad + (\lambda + \mu) \int_{G_R} (\delta_h^i(\nabla \zeta \cdot \psi)\text{div}\phi - \text{div}\psi \nabla \zeta \cdot \delta_{-h}^i \phi)dx. \qquad (3.69)
\end{aligned}$$

It therefore follows from (3.68) that

$$\begin{aligned} |B(\delta_h^i(\zeta\psi),\phi)| &\leq \|F\|\|\zeta\delta_{-h}^i\phi\| + C\|\phi\|_{(H^1(G_R))^n}\|\psi\|_{(H^1(G_R))^n} \\ &\leq C(\|F\| + \|\psi\|_{(H^1(G_R))^n})\|\phi\|_{(H^1(G_R))^n}. \end{aligned} \tag{3.70}$$

Let $(H^1_{\Gamma_{2G}}(G_R))^n$ be the completion of $(C_0^\infty(G_R \cup \Gamma_{2G}))^n$ in $(H^1(G_R))^n$. Then by a density argument, we obtain for any $\phi \in (H^1_{\Gamma_{2G}}(G_R))^n$

$$|B(\delta_h^i(\zeta\psi),\phi)| \leq C(\|F\| + \|\psi\|_{(H^1(G_R))^n})\|\phi\|_{(H^1(G_R))^n}. \tag{3.71}$$

Since $\delta_h^i(\zeta\psi) \in (H^1_{\Gamma_{2G}}(G_R))^n$ if h is small enough, we deduce

$$|B(\delta_h^i(\zeta\psi),\delta_h^i(\zeta\psi))| \leq C(\|F\| + \|\psi\|_{(H^1(G_R))^n})\|\delta_h^i(\zeta\psi)\|_{(H^1(G_R))^n}. \tag{3.72}$$

On the other hand, it is clear that

$$|B(\delta_h^i(\zeta\psi),\delta_h^i(\zeta\psi))| \geq C\|\delta_h^i(\zeta\psi)\|^2_{(H^1(G_R))^n}.$$

Hence it follows from (3.72) that

$$\|\delta_h^i(\zeta\psi)\|_{(H^1(G_R))^n} \leq C(\|F\| + \|\psi\|_{(H^1(G_R))^n}). \tag{3.73}$$

Since $\zeta = 1$ on $G_{R'}$, by Theorem 3.16 of [2, p.45], we deduce that $\frac{\partial\psi_i}{\partial x_j} \in (H^1(G_{R'}))^n$ for all $i = 1, ..., n,\ j = 1, ..., n-1$. It remains to show that $\frac{\partial^2\psi}{\partial x_n^2} \in (L^2(G_{R'}))^n$. To do this we have to distinguish the components ψ_i for $i = 1, ..., n-1$ and for $i = n$. In what concerns $i = 1, ..., n-1$, we have

$$-\mu\frac{\partial^2\psi_i}{\partial x_n^2} = \mu\Delta'\psi_i + (\lambda+\mu)\frac{\partial}{\partial x_i}(\mathrm{div}\psi) + F_i \in L^2(G_{R'}),$$

while

$$-(\lambda+2\mu)\frac{\partial^2\psi_n}{\partial x_n^2} = \mu\Delta'\psi_n + (\lambda+\mu)\frac{\partial}{\partial x_n}\Big(\frac{\partial\psi_1}{\partial x_1} + ... + \frac{\partial\psi_{n-1}}{\partial x_{n-1}}\Big) + F_n \in L^2(G_{R'}),$$

where $\Delta' = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_{n-1}^2}$.

By interpolation (see, e.g., [36, p.29, Theorem 6.2]), for $F \in (H^{s-2}(\Omega))^n$ we have $\psi \in (H^s(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n$, and then $u \in (H^s(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n$. This completes the proof. □

Remark 3.4 Let us briefly explain how to deal with the case where the boundary is not flat by a flattening procedure. Suppose that $x^0 \in \Gamma_2$ and the boundary of Ω near x^0 can be expressed by

$$\Phi(x_1, x_2, \ldots, x_n) = 0.$$

Let

$$y_i = y_i(x_1, x_2, \ldots, x_n) = x_i, \quad i = 1, \ldots, n-1, \tag{3.74}$$
$$y_n = y_n(x_1, x_2, \ldots, x_n) = \Phi(x_1, x_2, \ldots, x_n) \tag{3.75}$$

be the invertible mappling of class C^2 which transforms Ω into $\tilde{\Omega}$. We denote by $x = x(y)$ the inverted mapping of the above mapping $y = y(x)$ and set

$$\tilde{\psi}(y) = \psi(x(y)).$$

By calculations, we obtain

$$\frac{\partial \psi_i}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial \tilde{\psi}_i}{\partial y_k} \frac{\partial y_k}{\partial x_j}, \tag{3.76}$$

$$\frac{\partial^2 \psi_i}{\partial x_j^2} = \sum_{k,l=1}^{n} \frac{\partial^2 \tilde{\psi}_i}{\partial y_k \partial y_l} \frac{\partial y_k}{\partial x_j} \frac{\partial y_l}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial \tilde{\psi}_i}{\partial y_k} \frac{\partial^2 y_k}{\partial x_j^2}. \tag{3.77}$$

Therefore, the first equation of (3.63) is transformed into

$$\begin{aligned} \tilde{F}_i(y) = & -\mu\Big(\sum_{j,k,l=1}^{n} \frac{\partial^2 \tilde{\psi}_i}{\partial y_k \partial y_l} \frac{\partial y_k}{\partial x_j} \frac{\partial y_l}{\partial x_j} + \sum_{j,k=1}^{n} \frac{\partial \tilde{\psi}_i}{\partial y_k} \frac{\partial^2 y_k}{\partial x_j^2} \Big) \\ & -(\lambda+\mu)\Big(\sum_{j,k,l=1}^{n} \frac{\partial^2 \tilde{\psi}_j}{\partial y_k \partial y_l} \frac{\partial y_k}{\partial x_j} \frac{\partial y_l}{\partial x_i} + \sum_{j,k=1}^{n} \frac{\partial \tilde{\psi}_j}{\partial y_k} \frac{\partial^2 y_k}{\partial x_j \partial x_i} \Big), \qquad i = 1, \dots, n, \end{aligned} \tag{3.78}$$

where $\tilde{F}_i(y) = F_i(x(y))$. Moreover, since the unit normal $\nu(x)$ on the boundary near x^0 is equal to

$$\nu(x) = \frac{\nabla y_n}{|\nabla y_n|},$$

we have

$$\frac{\partial \psi_i}{\partial \nu} = \sum_{j=1}^{n} \frac{\partial \psi_i}{\partial x_j} \nu_j = \sum_{j,k=1}^{n} \frac{\partial \tilde{\psi}_i}{\partial y_k} \frac{\partial y_k}{\partial x_j} \frac{\partial y_n}{\partial x_j} |\nabla y_n|^{-1}, \tag{3.79}$$

$$\nu_i \mathrm{div}\psi = \sum_{j=1}^{n} \frac{\partial \psi_j}{\partial x_j} \nu_i = \sum_{j,k=1}^{n} \frac{\partial \tilde{\psi}_j}{\partial y_k} \frac{\partial y_k}{\partial x_j} \frac{\partial y_n}{\partial x_i} |\nabla y_n|^{-1}. \tag{3.80}$$

Hence, the boundary condition of (3.63) is transformed into the following form

$$\mu \sum_{j,k=1}^{n} \frac{\partial \tilde{\psi}_i}{\partial y_k} \frac{\partial y_k}{\partial x_j} \frac{\partial y_n}{\partial x_j} + (\lambda+\mu) \sum_{j,k=1}^{n} \frac{\partial \tilde{\psi}_j}{\partial y_k} \frac{\partial y_k}{\partial x_j} \frac{\partial y_n}{\partial x_i} + \tilde{a}\tilde{\psi}_i = 0, \quad i = 1, \dots, n, \tag{3.81}$$

where $\tilde{a} = \tilde{a}(y) \geq 0$. Since the already known H^1-regularity guarantees the first order derivative terms of the unkown $\tilde{\psi}$ in (3.78) are in L^2, they may be put on the right hand side and it is then sufficient to keep the second order terms on the left hand side. Therefore, since $\nu(y) = (0, \dots, 0, 1)$ on the boundary of $\tilde{\Omega}$ near $y^0 = y(x^0)$, it follows that the bilinear form corresponding to equation (3.78) and boundary condition (3.81) is

$$\begin{aligned} \tilde{B}(\tilde{\psi}, \tilde{\phi}) = & \ \mu \int_{\tilde{G}_R} \sum_{i,j,k,l=1}^{n} \frac{\partial \tilde{\psi}_i}{\partial y_k} \frac{\partial \tilde{\phi}_i}{\partial y_l} \frac{\partial y_k}{\partial x_j} \frac{\partial y_l}{\partial x_j} dy + (\lambda+\mu) \int_{\tilde{G}_R} \sum_{i,j,k,l=1}^{n} \frac{\partial \tilde{\psi}_j}{\partial y_k} \frac{\partial \tilde{\phi}_i}{\partial y_l} \frac{\partial y_k}{\partial x_j} \frac{\partial y_l}{\partial x_i} dy \\ & + \int_{\tilde{\Gamma}_{2G}} \tilde{a}\tilde{\psi} \cdot \tilde{\phi} d\Gamma, \end{aligned} \tag{3.82}$$

where $\tilde{G}_R$ and $\tilde{\Gamma}_{2G}$ denote the same subsets in the $y-$space as G_R and Γ_{2G}. It is clear that the bilinear $\tilde{B}(\tilde{\psi}, \tilde{\phi})$ has the same properties as $B(\psi, \phi)$. Therefore, the above proof of regularity is valid for the general case (see, e.g., [8]).

By the classical theory of nonlinear semigroups (see [6, Chap. 3]), we have

Theorem 3.5 *Let Γ_1 and Γ_2 be given by (2.1) and (2.2). Let (2.16) and (2.43) hold. Suppose that the function g satisfies (2.44), (2.49) (or (2.92), (2.97)) and (3.4). Then we have*

(i) For every initial condition $(u^0, u^1, \theta^0) \in \mathcal{H}$, problem (2.7) has a unique mild solution satisfying

$$(u, u', \theta) \in C([0, \infty); \mathcal{H}). \tag{3.83}$$

Moreover, if (u, θ) and (v, ξ) are two solutions corresponding to initial states (u^0, u^1, θ^0) and (v^0, v^1, ξ^0), respectively, then we have for every $t \in [0, \infty)$

$$\|(u(t), u'(t), \theta(t)) - (v(t), v'(t), \xi(t))\|_{\mathcal{H}} \leq \|(u^0, u^1, \theta^0) - (v^0, v^1, \xi^0)\|_{\mathcal{H}}. \tag{3.84}$$

(ii) Further, if g satisfies (2.45) and (2.46) (or (2.93) and (2.94)), then, for every initial condition $(u^0, u^1, \theta^0) \in D(\Lambda)$, problem (2.7) has a unique classical solution satisfying

$$\begin{aligned} u &\in L^\infty([0, \infty); (H^s(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n), && (3.85)\\ u' &\in L^\infty([0, \infty); (H^1_{\Gamma_1}(\Omega))^n), && (3.86)\\ u'' &\in L^\infty([0, \infty); (L^2(\Omega))^n), && (3.87)\\ \theta &\in L^\infty([0, \infty); H^2(\Omega) \cap H^1_0(\Omega)), && (3.88)\\ \theta' &\in L^\infty([0, \infty); L^2(\Omega)), && (3.89) \end{aligned}$$

for some $s > 3/2$.

4 Proofs of the Main Results

In this section, we borrow the Lyapunov method to prove our main results.

Let u, θ be the solution of (2.7) and φ the solution of (2.27) corresponding to the present solution u of (2.7) (Note that φ depends on t as u does). Let δ be any positive number. We define a functional V on $\mathcal{H}$ by

$$F(t) = F(u, \theta, t) = \int_\Omega [2u'_i m \cdot \nabla u_i + (n-1)u'_i u_i + C u'_i \varphi_i] dx, \tag{4.1}$$

$$V(t) = V(u, \theta, t) = E(t) + \delta [E(t)]^\sigma F(t), \tag{4.2}$$

where C and σ are nonnegative constants that will be determined later. Here we have used the summation convention for repeated indices. Thus, equality (4.1) means that

$$F(t) = \sum_{i=1}^n \int_\Omega [2u'_i m \cdot \nabla u_i + (n-1)u'_i u_i + C\varphi_i u'_i] dx. \tag{4.3}$$

Evidently, the functional V is actually a generalized energy functional which is closely related to the energy functional $E(t)$. Such similar functionals were constructed for the wave equation (see [10, 25, 51]), thermoelastic plate models (see [42, 43]), viscoelasticity (see [34, 35]) and thermoviscoelasticity (see [39]).

We will see below that the term $Cu_i'\varphi_i$ plays a key role in dealing with the case where the potential a is large.

We are going to show that V satisfies

$$V'(t) \leq -cV^{\sigma+1}(t), \tag{4.4}$$

where c is a positive constant. Then, by solving differential inequality (4.4), Theorems 2.2, 2.3 and 2.5 are proved.

We first show that, by choosing δ sufficiently small, V and E are equivalent.

Lemma 4.1 *Let Γ_1 and Γ_2 be given by (2.1) and (2.2), respectively. Suppose that (2.16) holds. Let the function $g \in (C(\mathbf{R}^n))^n$ satisfy (2.44), (2.49) (or (2.92), (2.97)) and (3.4). Then we have*

$$(1-\delta C_1''E^\sigma(t)))E(t) \leq V(t) \leq (1+\delta C_1''E^\sigma(t)))E(t), \tag{4.5}$$

for all solutions u, θ of (2.7), where the positive constant C_1'' is given by

$$C_1'' = \frac{2R_0}{\sqrt{\mu}} + (n-1)\lambda_0 + Ck_1\gamma. \tag{4.6}$$

Proof. It is easy to see that

$$|2\int_\Omega u_i' m \cdot \nabla u_i dx| \leq \frac{R_0}{\sqrt{\mu}}[\|u_i'(t)\|^2 + \mu\|\nabla u_i(t)\|^2] \leq \frac{2R_0}{\sqrt{\mu}}E(t). \tag{4.7}$$

Using (2.25), we obtain

$$|\int_\Omega u_i' u_i dx| \leq \frac{\lambda_0}{2}[\|u'(t)\|^2 + \|u(t)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}] \leq \lambda_0 E(t). \tag{4.8}$$

Using (2.28) and (2.24), we have

$$\begin{aligned} |\int_\Omega u_i'\varphi_i dx| &= \frac{k_1\gamma}{2}|\int_\Omega 2u_i'\frac{\varphi_i}{k_1\gamma}dx| \\ &\leq \frac{k_1\gamma}{2}[\|u_i'(t)\|^2 + \frac{1}{k_1^2\gamma^2}\|\varphi_i(t)\|^2] \\ &\leq \frac{k_1\gamma}{2}[\|u_i'(t)\|^2 + \frac{1}{\gamma^2}\int_{\Gamma_2}|u_i|^2 d\Gamma] \\ &\leq \frac{k_1\gamma}{2}[\|u'(t)\|^2 + \|u(t)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}] \\ &\leq k_1\gamma E(t). \end{aligned} \tag{4.9}$$

Noting (4.6), we deduce from (4.7), (4.8) and (4.9) that

$$|F(t)| \leq C_1''E(t), \tag{4.10}$$

which implies (4.5). □

We then estimate $F'(t)$.

Lemma 4.2 *Let Γ_1 and Γ_2 be given by (2.1) and (2.2), respectively, satisfying (2.43). Suppose that (2.16) holds and the function $g \in (C(\mathbf{R}^n))^n$ satisfies (2.44), (2.45), (2.46) and (2.49) (or (2.92), (2.93), (2.94) and (2.97)). Let u, θ be a classical solution of (2.7) with $(u^0, u^1, \theta^0) \in D(\Lambda)$. Then we have*

$$\begin{aligned} F'(t) &\leq -E(t) + C_2''\|\nabla\theta(t)\|^2 \\ &\quad + \int_{\Gamma_2} \Big[\Big(m\cdot\nu + \frac{C^2k_1^2}{4\varepsilon}\Big)|u'|^2 + C_3'' m\cdot\nu|g(u')|^2\Big]d\Gamma, \end{aligned} \tag{4.11}$$

where

$$C_2'' = \frac{\alpha^2R_0^2}{\mu\varepsilon} + \frac{\alpha^2(n-1)^2\lambda_0^2}{4\varepsilon} + \frac{C^2\alpha^2k_1^2\gamma^2}{4\varepsilon} + \frac{\alpha\lambda_1^2}{\beta}, \tag{4.12}$$

$$C_3'' = \frac{2R_0^2}{\mu} + \frac{(n-1+C)^2R_0\gamma^2}{4\varepsilon}, \tag{4.13}$$

and the constant C is given by

$$C = C_0 = \max\{0, \frac{2a_0R_0^2}{\mu} + 2 - n\}. \tag{4.14}$$

If, further, the function $a(x)$ satisfies (2.50) or (2.51), then we can take

$$C = 0. \tag{4.15}$$

The constant ε is given by

$$\varepsilon = \begin{cases} 1/8, & \text{if } C = C_0, \\ 1/8, & \text{if } C = 0 \text{ and } a(x) \text{ satisfies (2.51)}, \\ (1 - 2K(a)R_0\gamma^2)/8, & \text{if } C = 0 \text{ and } a(x) \text{ satisfies (2.50)}. \end{cases} \tag{4.16}$$

Proof. Since $(u^0, u^1, \theta^0) \in D(\Lambda)$, we have $u \in L^\infty([0,\infty); (H^s(\Omega)\cap H^1_{\Gamma_1}(\Omega))^n)$ for some $s > 3/2$ and $\theta \in L^\infty([0,\infty); H^2(\Omega)\cap H^1_0(\Omega))$. Hence, the following integrations by parts are valid.

By (4.1), we have

$$\begin{aligned} F'(t) &= \int_\Omega 2u_i'' m\cdot\nabla u_i dx + \int_\Omega 2u_i' m\cdot\nabla u_i' dx \\ &\quad + \int_\Omega (n-1)u_iu_i''dx + (n-1)\|u_i'\|^2 \\ &\quad + C\int_\Omega \varphi_i u_i'' dx + C\int_\Omega \varphi_i' u_i' dx. \end{aligned} \tag{4.17}$$

We now estimate every integral in (4.17) as follows. Since $u = 0$ on Γ_1, we have

$$\frac{\partial u_i}{\partial x_k} = \frac{\partial u_i}{\partial \nu}\nu_k \quad \text{on} \quad \Gamma_1. \tag{4.18}$$

Thus, we obtain

$$
\begin{aligned}
& 2\int_\Omega u_i'' m\cdot\nabla u_i dx \\
= \ & 2\int_\Omega \Big[\mu\Delta u_i + (\lambda+\mu)\frac{\partial}{\partial x_i}(\mathrm{div} u) - \alpha\frac{\partial\theta}{\partial x_i}\Big] m\cdot\nabla u_i dx \\
= \ & \mu\int_\Gamma \Big[2\frac{\partial u_i}{\partial\nu} m\cdot\nabla u_i - m\cdot\nu|\nabla u_i|^2\Big] d\Gamma + (n-2)\mu\|\nabla u_i\|^2 \\
& +(\lambda+\mu)\int_\Gamma \Big[2\mathrm{div}(u) m_k\nu_i\frac{\partial u_i}{\partial x_k} - m\cdot\nu|\mathrm{div} u|^2\Big] d\Gamma \\
& +(n-2)(\lambda+\mu)\|\mathrm{div} u\|^2 - 2\alpha\int_\Omega (m\cdot\nabla u_i)\frac{\partial\theta}{\partial x_i} dx \\
= \ & \int_{\Gamma_1} m\cdot\nu\Big[\mu|\frac{\partial u_i}{\partial\nu}|^2 + (\lambda+\mu)|\mathrm{div} u|^2\Big] d\Gamma \\
& +2\int_{\Gamma_2} \Big[\mu\frac{\partial u_i}{\partial\nu} + (\lambda+\mu)\nu_i\mathrm{div} u\Big] m\cdot\nabla u_i d\Gamma \\
& -\int_{\Gamma_2} m\cdot\nu[\mu|\nabla u_i|^2 + (\lambda+\mu)|\mathrm{div} u|^2] d\Gamma \\
& +(n-2)[\mu\|\nabla u_i\|^2 + (\lambda+\mu)\|\mathrm{div} u\|^2] \\
& -2\alpha\int_\Omega (m\cdot\nabla u_i)\frac{\partial\theta}{\partial x_i} dx.
\end{aligned}
\tag{4.19}
$$

Since $m\cdot\nu\le 0$ on Γ_1, we have

$$\int_{\Gamma_1} m\cdot\nu\Big[\mu|\frac{\partial u_i}{\partial\nu}|^2 + (\lambda+\mu)|\mathrm{div} u|^2\Big] d\Gamma \le 0. \tag{4.20}$$

In addition, we have

$$-2\alpha\int_\Omega (m\cdot\nabla u_i)\frac{\partial\theta}{\partial x_i} dx \le \varepsilon\mu\|\nabla u_i(t)\|^2 + \frac{\alpha^2 R_0^2}{\mu\varepsilon}\|\nabla\theta(t)\|^2. \tag{4.21}$$

It therefore follows from (4.19)-(4.21) that

$$
\begin{aligned}
& 2\int_\Omega u_i'' m\cdot\nabla u_i dx \\
\le \ & 2\int_{\Gamma_2} \Big[\mu\frac{\partial u_i}{\partial\nu} + (\lambda+\mu)\nu_i\mathrm{div} u\Big] m\cdot\nabla u_i d\Gamma \\
& -\int_{\Gamma_2} m\cdot\nu[\mu|\nabla u_i|^2 + (\lambda+\mu)|\mathrm{div} u|^2] d\Gamma \\
& +(n-2)[\mu\|\nabla u_i\|^2 + (\lambda+\mu)\|\mathrm{div} u\|^2] \\
& +\varepsilon\mu\|\nabla u_i(t)\|^2 + \frac{\alpha^2 R_0^2}{\mu\varepsilon}\|\nabla\theta(t)\|^2.
\end{aligned}
\tag{4.22}
$$

For the second integral in (4.17), we have

$$2\int_\Omega u_i' m\cdot\nabla u_i' dx = -n\|u_i'(t)\|^2 + \int_{\Gamma_2} m\cdot\nu|u_i'|^2 d\Gamma. \tag{4.23}$$

Using (2.25), we deduce

$$
\begin{aligned}
& (n-1)\int_\Omega u_i u_i'' dx \\
= \;& (n-1)\int_\Omega u_i\Big[\mu\Delta u_i + (\lambda+\mu)\frac{\partial}{\partial x_i}(\mathrm{div}u) - \alpha\frac{\partial\theta}{\partial x_i}\Big]dx \\
= \;& (n-1)\int_{\Gamma_2}\Big[\mu\frac{\partial u_i}{\partial\nu} + (\lambda+\mu)\nu_i\mathrm{div}u\Big]u_i d\Gamma \\
& -(n-1)[\mu\|\nabla u_i(t)\|^2 + (\lambda+\mu)\|\mathrm{div}u(t)\|^2] \\
& -(n-1)\alpha\int_\Omega u_i\frac{\partial\theta}{\partial x_i}dx \\
\le \;& (n-1)\int_{\Gamma_2}\Big[\mu\frac{\partial u_i}{\partial\nu} + (\lambda+\mu)\nu_i\mathrm{div}u\Big]u_i d\Gamma \\
& -(n-1)[\mu\|\nabla u_i(t)\|^2 + (\lambda+\mu)\|\mathrm{div}u(t)\|^2] \\
& +(n-1)^2\frac{\alpha^2\lambda_0^2}{4\varepsilon}\|\nabla\theta(t)\|^2 + \varepsilon\|u(t)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}.
\end{aligned} \tag{4.24}
$$

By (2.28) and (2.29), we have

$$
\begin{aligned}
& C\int_\Omega \varphi_i u_i'' dx \\
= \;& C\int_\Omega \varphi_i\Big[\mu\Delta u_i + (\lambda+\mu)\frac{\partial}{\partial x_i}(\mathrm{div}u) - \alpha\frac{\partial\theta}{\partial x_i}\Big]dx \\
= \;& C\int_{\Gamma_2}\Big[\mu\frac{\partial u_i}{\partial\nu} + (\lambda+\mu)\nu_i\mathrm{div}u\Big]\varphi_i d\Gamma \\
& -C\int_\Omega[\mu\nabla u_i(t)\nabla\varphi_i(t) + (\lambda+\mu)\mathrm{div}u(t)\mathrm{div}\varphi(t)]dx \\
& -C\alpha\int_\Omega \varphi_i\frac{\partial\theta}{\partial x_i}dx \\
\le \;& C\int_{\Gamma_2}\Big[\mu\frac{\partial u_i}{\partial\nu} + (\lambda+\mu)\nu_i\mathrm{div}u\Big]u_i d\Gamma \\
& +\frac{C^2\alpha^2k_1^2\gamma^2}{4\varepsilon}\|\nabla\theta(t)\|^2 + \frac{\varepsilon}{k_1^2\gamma^2}\|\varphi_i(t)\|^2 \\
\le \;& C\int_{\Gamma_2}[\mu\frac{\partial u_i}{\partial\nu} + (\lambda+\mu)\nu_i\mathrm{div}u]u_i d\Gamma \\
& +\frac{C^2\alpha^2k_1^2\gamma^2}{4\varepsilon}\|\nabla\theta(t)\|^2 + \varepsilon\|u(t)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}.
\end{aligned} \tag{4.25}
$$

Using (2.28), we have

$$
\begin{aligned}
C\int_\Omega \varphi_i' u_i' dx \;\le\;& \frac{C^2}{4\varepsilon}\|\varphi_i'(t)\|^2 + \varepsilon\|u_i'(t)\|^2 \\
\le\;& \frac{C^2k_1^2}{4\varepsilon}\int_{\Gamma_2}|u_i'(t)|^2 d\Gamma + \varepsilon\|u_i'(t)\|^2.
\end{aligned} \tag{4.26}
$$

It therefore follows from (4.17) and (4.22)-(4.26) that

$$\begin{aligned}
F'(t) \leq\ & -[\mu\|\nabla u_i\|^2 + (\lambda+\mu)\|\mathrm{div}u\|^2 + \|u_i'(t)\|^2] \\
& +3\varepsilon\|u(t)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} + \varepsilon\|u_i'(t)\|^2 \\
& +\Big[\frac{\alpha^2R_0^2}{\mu\varepsilon} + \frac{(n-1)^2\alpha^2\lambda_0^2}{4\varepsilon} + \frac{C^2\alpha^2k_1^2\gamma^2}{4\varepsilon}\Big]\|\nabla\theta(t)\|^2 \\
& +2\int_{\Gamma_2}\Big[\mu\frac{\partial u_i}{\partial\nu} + (\lambda+\mu)\nu_i\mathrm{div}u\Big]m\cdot\nabla u_i d\Gamma \quad (=I_1) \\
& -\int_{\Gamma_2} m\cdot\nu[\mu|\nabla u_i|^2 + (\lambda+\mu)|\mathrm{div}u|^2]d\Gamma \quad (=I_2) \\
& +\int_{\Gamma_2}\Big[m\cdot\nu + \frac{C^2k_1^2}{4\varepsilon}\Big]|u_i'(t)|^2 d\Gamma \quad (=I_3) \\
& +(n-1+C)\int_{\Gamma_2}[\mu\frac{\partial u_i}{\partial\nu} + (\lambda+\mu)\nu_i\mathrm{div}u]u_i d\Gamma \quad (=I_4) \\
\leq\ & -2E(t) + 6\varepsilon E(t) \\
& +\Big[\frac{\alpha^2R_0^2}{\mu\varepsilon} + \frac{(n-1)^2\alpha^2\lambda_0^2}{4\varepsilon} + \frac{C^2\alpha^2k_1^2\gamma^2}{4\varepsilon}\Big]\|\nabla\theta(t)\|^2 + \frac{\alpha}{\beta}\|\theta(t)\|^2 \\
& +I_1+I_2+I_3+I_4 \\
& +\int_{\Gamma_2} am\cdot\nu|u_i|^2 d\Gamma \quad (=I_5).
\end{aligned} \tag{4.27}$$

Note that we add here two terms

$$\frac{\alpha}{\beta}\|\theta(t)\|^2, \quad \int_{\Gamma_2} am\cdot\nu|u_i|^2 d\Gamma \tag{4.28}$$

to make up $[\mu\|\nabla u_i\|^2 + (\lambda+\mu)\|\mathrm{div}u\|^2 + \|u_i'(t)\|^2]$ into $E(t)$. Since

$$\begin{aligned}
I_1 =\ & -2\int_{\Gamma_2}[am\cdot\nu u_i + m\cdot\nu g_i(u')]m\cdot\nabla u_i d\Gamma \\
\leq\ & \int_{\Gamma_2} m\cdot\nu\Big[\frac{2a^2R_0^2}{\mu}|u_i|^2 + \frac{2R_0^2}{\mu}|g(u')|^2 + \mu|\nabla u_i|^2\Big]d\Gamma,
\end{aligned} \tag{4.29}$$

we have

$$I_1 + I_2 \leq \int_{\Gamma_2} m\cdot\nu\Big[\frac{2a^2R_0^2}{\mu}|u_i|^2 + \frac{2R_0^2}{\mu}|g(u')|^2\Big]d\Gamma. \tag{4.30}$$

In addition,

$$\begin{aligned}
I_4 + I_5 =\ & -(n-1+C)\int_{\Gamma_2}[am\cdot\nu u_i + m\cdot\nu g_i(u')]u_i d\Gamma + \int_{\Gamma_2} am\cdot\nu|u_i|^2 d\Gamma \\
\leq\ & -(n-2+C)\int_{\Gamma_2} am\cdot\nu|u_i|^2 d\Gamma \\
& +\int_{\Gamma_2} m\cdot\nu\Big[\frac{(n-1+C)^2R_0\gamma^2}{4\varepsilon}|g_i(u')|^2 + \frac{\varepsilon}{R_0\gamma^2}|u_i|^2\Big]d\Gamma \\
\leq\ & -(n-2+C)\int_{\Gamma_2} am\cdot\nu|u_i|^2 d\Gamma \\
& +\frac{(n-1+C)^2R_0\gamma^2}{4\varepsilon}\int_{\Gamma_2} m\cdot\nu|g(u')|^2 d\Gamma + \varepsilon\|u(t)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}.
\end{aligned} \tag{4.31}$$

Noting definitions (4.12) and (4.13) of C_2'' and C_3'', it therefore follows from (4.27)-(4.31) that

$$\begin{aligned} F'(t) &\le -2E(t) + 8\varepsilon E(t) + C_2''\|\nabla\theta(t)\|^2 \\ &\quad + \int_{\Gamma_2} \Big[m\cdot\nu + \frac{C^2k_1^2}{4\varepsilon}\Big]|u'|^2 d\Gamma + C_3''\int_{\Gamma_2} m\cdot\nu|g(u')|^2 d\Gamma \\ &\quad + \int_{\Gamma_2} am\cdot\nu\Big[\frac{2aR_0^2}{\mu} - n + 2 - C)\Big]|u_i|^2 d\Gamma \ \ (= I). \end{aligned} \tag{4.32}$$

Thus, by taking $C = C_0$ and $\varepsilon = 1/8$, we have $I \le 0$, and then we deduce (4.11).

If, further, the function $a(x)$ satisfies (2.51), we can take $C = 0$ and we still have $I \le 0$. Then (4.11) follows.

Likewise, if the function $a(x)$ satisfies (2.50), then we can also take $C = 0$. However, in this case, since I is no longer negative, we estimate I as follows. Let $K(a)$ be given by (2.23). Using (2.24), we deduce

$$\begin{aligned} I &\le K(a)\int_{\Gamma_2} m\cdot\nu|u|^2 d\Gamma \\ &\le K(a)R_0\gamma^2\|u(t)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} \\ &\le 2K(a)R_0\gamma^2E(t). \end{aligned} \tag{4.33}$$

Thus, by taking $\varepsilon = (1 - 2K(a)R_0\gamma^2)/8$, we also deduce (4.11) . This completes the proof of Lemma 4.2. □

We are now ready to prove our main results. The method of the proof is analogous to the one developed by the second author in [51].

Proof of Theorem 2.2. We first assume that $(u^0, u^1, \theta^0) \in D(\Lambda)$. Then Lemma 4.2 is valid. We note that $C = 0$ because the function $a(x)$ satisfies (2.50) or (2.51). Then the constant C_1'' in Lemma 4.1 and the constants C_2'' and C_3'' in Lemma 4.2 become C_1, C_2, C_3 in Theorem 2.2, respectively. By (2.42), (4.10) and Lemmas 4.1 and 4.2, we have

$$\begin{aligned} V'(t) &= E'(t) + \delta\frac{d}{dt}(E^\sigma(t)F(t)) \\ &= E'(t) + \delta\sigma E^{\sigma-1}(t)E'(t)F(t) + \delta E^\sigma(t)F'(t) \\ &\le \Big[1 - \delta C_1\sigma E^\sigma(t)\Big]E'(t) - \delta E^{\sigma+1}(t) \\ &\quad + \delta C_2E^\sigma(t)\|\nabla\theta(t)\|^2 + \delta E^\sigma(t)\int_{\Gamma_2} m\cdot\nu[|u'|^2 + C_3|g(u')|^2]d\Gamma \\ &= -\delta E^{\sigma+1}(t) + \Big[\delta E^\sigma(t)\Big(\frac{\sigma\alpha C_1}{\beta} + C_2\Big) - \frac{\alpha}{\beta}\Big]\|\nabla\theta(t)\|^2 \\ &\quad - \Big[1 - \delta C_1\sigma E^\sigma(t)\Big]\int_{\Gamma_2} m\cdot\nu g(u')\cdot u' d\Gamma \\ &\quad + \delta E^\sigma(t)\int_{\Gamma_2} m\cdot\nu[|u'|^2 + C_3|g(u')|^2]d\Gamma. \end{aligned} \tag{4.34}$$

We now distinguish the cases $p = q = 1$ and $p + 1 > 2q$.

Case I: $p = q = 1$. In this case, we take $\sigma = 0$ and $\delta = \delta_1$ (see (2.64)) in (4.34). By definitions (2.64) and (2.59) of δ_1 and C_4, it follows from (4.34) and (2.49) that

$$\begin{aligned} V'(t) &= -\delta_1 E(t) - \int_{\Gamma_2} m\cdot\nu g(u')\cdot u' d\Gamma \\ &\quad +\delta_1 C_4 \int_{\Gamma_2} m\cdot\nu |u'|^2 d\Gamma \\ &\leq -\delta_1 E(t) + [\delta_1 C_4 - k_3]\int_{\Gamma_2} m\cdot\nu|u'|^2 d\Gamma \\ &\leq -\delta_1 E(t) \quad (\text{ use } \quad (4.5)) \\ &\leq -\frac{\delta_1}{1+\delta_1 C_1} V(t). \end{aligned} \tag{4.35}$$

Solving this differential inequality and using (4.5), we obtain

$$\begin{aligned} E(t) &\leq \frac{1}{1-\delta_1 C_1} V(t) \\ &\leq \frac{1}{1-\delta_1 C_1} V(0) e^{-\delta_1 t/(1+\delta_1 C_1)} \\ &\leq \frac{1+\delta_1 C_1}{1-\delta_1 C_1} E(0) e^{-\delta_1 t/(1+\delta_1 C_1)}. \end{aligned} \tag{4.36}$$

This is (2.52).

Case II: $p+1 > 2q$. We first estimate

$$E^\sigma(t)\int_{\Gamma_2\cap[|u'|\leq 1]} m\cdot\nu[|u'|^2 + C_3|g(u')|^2] d\Gamma. \tag{4.37}$$

Applying Young's inequality

$$ab \leq \frac{a^s}{s} + \frac{b^\tau}{\tau}, \quad \forall a, b \geq 0, \text{ and } s,\tau > 0 \text{ with } \frac{1}{s} + \frac{1}{\tau} = 1, \tag{4.38}$$

we have for any $b > 0$

$$\begin{aligned} & E^\sigma \int_{\Gamma_2\cap[|u'|\leq 1]} m\cdot\nu|u'|^{2q} d\Gamma \\ \leq\ & \frac{p+1-2q}{p+1} b^{(p+1)/(p+1-2q)} E^{\sigma(p+1)/(p+1-2q)} \\ & + \frac{2q}{(p+1)b^{(p+1)/2q}}\Big(\int_{\Gamma_2\cap[|u'|\leq 1]} m\cdot\nu|u'|^{2q}\Gamma\Big)^{(p+1)/(2q)} \\ \leq\ & \frac{p+1-2q}{p+1} b^{(p+1)/(p+1-2q)} E^{\sigma(p+1)/(p+1-2q)} \\ & + \frac{2q}{(p+1)b^{(p+1)/(2q)}}\Big(\int_{\Gamma_2\cap[|u'|\leq 1]} m\cdot\nu d\Gamma\Big)^{(p+1-2q)/(2q)} \\ & \qquad \times \int_{\Gamma_2\cap[|u'|\leq 1]} m\cdot\nu|u'|^{p+1} d\Gamma \\ \leq\ & \frac{p+1-2q}{p+1} b^{(p+1)/(p+1-2q)} E^{\sigma(p+1)/(p+1-2q)} \\ & + \frac{2q[R_0 mes(\Gamma_2)]^{(p+1-2q)/(2q)}}{k_3(p+1)b^{(p+1)/(2q)}} \int_{\Gamma_2\cap[|u'|\leq 1]} m\cdot\nu g(u')\cdot u' d\Gamma. \end{aligned} \tag{4.39}$$

Thus, by taking

$$b = \Big(\frac{p+1}{2C_4(p+1-2q)}\Big)^{(p+1-2q)/(p+1)}, \tag{4.40}$$

we obtain

$$\begin{aligned}
& E^\sigma(t)\int_{\Gamma_2\cap[|u'|\le 1]} m\cdot\nu[|u'|^2 + C_3|g(u')|^2]d\Gamma \\
& \text{(note that } |u'|^2 \le |u'|^{2q} \text{ since } q\le 1) \\
\le\ & C_4 E^\sigma(t)\int_{\Gamma_2\cap[|u'|\le 1]} m\cdot\nu|u'|^{2q}\Gamma \\
\le\ & \frac{1}{2}E^{\sigma(p+1)/(p+1-2q)} \\
& +\frac{q(2C_4)^{(p+1)/(2q)}[R_0 mes(\Gamma_2)(p+1-2q)]^{(p+1-2q)/(2q)}}{k_3(p+1)^{(p+1)/(2q)}} \\
& \qquad \times\int_{\Gamma_2\cap[|u'|\le 1]} m\cdot\nu g(u')\cdot u' d\Gamma.
\end{aligned} \tag{4.41}$$

It therefore follows from (4.34) and (4.41) that

$$\begin{aligned}
V'(t) \le\ & -\delta E^{\sigma+1}(t) + \Big[\delta E^\sigma(t)\Big(\frac{\sigma\alpha C_1}{\beta}+C_2\Big) - \frac{\alpha}{\beta}\Big]\|\nabla\theta(t)\|^2 \\
& +\Big[\delta C_1\sigma E^\sigma(0) + \frac{\delta C_4 E^\sigma(0)}{k_3} - 1\Big]\int_{\Gamma_2\cap[|u'|\ge 1]} m\cdot\nu g(u')\cdot u' d\Gamma \\
& +\Big[\delta C_1\sigma E^\sigma(0) - 1\Big]\int_{\Gamma_2\cap[|u'|\le 1]} m\cdot\nu g(u')\cdot u' d\Gamma \\
& +\delta E^\sigma(t)\int_{\Gamma_2\cap[|u'|\le 1]} m\cdot\nu[|u'|^2 + C_3|g(u')|^2]d\Gamma \\
\le\ & -\delta E^{\sigma+1}(t) + \frac{\delta}{2}E^{\sigma(p+1)/(p+1-2q)} \\
& +\Big[\delta E^\sigma(t)\Big(\frac{\sigma\alpha C_1}{\beta}+C_2\Big) - \frac{\alpha}{\beta}\Big]\|\nabla\theta(t)\|^2 \\
& +\Big[\delta C_1\sigma E^\sigma(0) + \frac{\delta C_4 E^\sigma(0)}{k_3} - 1\Big]\int_{\Gamma_2\cap[|u'|\ge 1]} m\cdot\nu g(u')\cdot u' d\Gamma \\
& +\Big[\delta C_1\sigma E^\sigma(0) - 1\Big]\int_{\Gamma_2\cap[|u'|\le 1]} m\cdot\nu g(u')\cdot u' d\Gamma \\
& +\delta\frac{q(2C_4)^{(p+1)/(2q)}[R_0 mes(\Gamma_2)(p+1-2q)]^{(p+1-2q)/(2q)}}{k_3(p+1)^{(p+1)/(2q)}} \\
& \qquad \times\int_{\Gamma_2\cap[|u'|\le 1]} m\cdot\nu g(u')\cdot u' d\Gamma.
\end{aligned} \tag{4.42}$$

We now choose σ so that

$$\sigma + 1 = \frac{\sigma(p+1)}{p+1-2q}. \tag{4.43}$$

Then

$$\sigma = \frac{p+1-2q}{2q} = \sigma_0. \tag{4.44}$$

By taking $\delta = \delta_2$ (see (2.65)) and $\sigma = \sigma_0$ in (4.42), we obtain

$$V'(t) \leq -\frac{\delta_2}{2} E^{\sigma_0+1}(t). \tag{4.45}$$

By (4.5), we have

$$V(t) \leq [1 + \delta_2 C_1 E^{\sigma_0}(0)] E(t). \tag{4.46}$$

It therefore follows from (4.45) that

$$V'(t) \leq -\frac{\delta_2}{2[1 + \delta_2 C_1 E^{\sigma_0}(0)]^{\sigma_0}} V^{\sigma_0+1}(t). \tag{4.47}$$

Solving this differential inequality and noting definition (2.65) of δ_2, we obtain

$$\begin{aligned} V(t) &\leq \Big[(V(0))^{-\sigma_0} + \frac{\sigma_0 \delta_2 t}{2[1 + \delta_2 C_1 E^{\sigma_0}(0)]^{\sigma_0+1}}\Big]^{-1/\sigma_0} \\ &\leq \Big[(E(0) + \delta_2 C_1 E^{\sigma_0+1}(0))^{-\sigma_0} + \frac{\sigma_0 \delta_2 t}{2[1 + \delta_2 C_1 E^{\sigma_0}(0)]^{\sigma_0+1}}\Big]^{-1/\sigma_0} \\ &\leq 2E(0)\Big[1 + \frac{\sigma_0 \delta_2 t}{2(2E(0))^{-\sigma_0}[1 + \delta_2 C_1 E^{\sigma_0}(0)]^{\sigma_0+1}}\Big]^{-1/\sigma_0}, \end{aligned} \tag{4.48}$$

which, combined with (4.5), implies (2.53).

Finally, if $(u^0, u^1, \theta^0) \in \mathcal{H}$, in view of (3.84), we can show that Theorem 2.2 still holds by a density argument. □

Proof of Theorem 2.3. We first note that $C = C_0$ and the constant C_1'' in Lemma 4.1 and the constants C_2'' and C_3'' in Lemma 4.2 become C_1', C_2', C_3' in Theorem 2.3, respectively. In this case, (4.34) becomes

$$\begin{aligned} V'(t) \leq\ & -\delta E^{\sigma+1}(t) + \Big[\delta E^{\sigma}(t)\Big(\frac{\sigma \alpha C_1'}{\beta} + C_2'\Big) - \frac{\alpha}{\beta}\Big] \|\nabla \theta(t)\|^2 \\ & -\Big[1 - \delta C_1' \sigma E^{\sigma}(t)\Big] \int_{\Gamma_2} m \cdot \nu g(u') \cdot u' d\Gamma \\ & +\delta E^{\sigma}(t) \int_{\Gamma_2} \Big[(m \cdot \nu + \frac{C_0^2 k_1^2}{4\varepsilon}) |u'|^2 + C_3' m \cdot \nu |g(u')|^2\Big] d\Gamma. \end{aligned} \tag{4.49}$$

Because the additional term $\frac{C_0^2 k_1^2}{4\varepsilon}$ does not contain the factor $m \cdot \nu$, we have to impose condition (2.69) on Γ_2 so that the sum of the integrals on Γ_2 becomes negative if δ is sufficiently small. The rest of the proof is the same as that of Theorem 2.2 except that C_4 is replaced by C_4' and the integral $\int_{\Gamma_2} m \cdot \nu g(u') \cdot u' d\Gamma$ is enlarged to $R_0 \int_{\Gamma_2} g(u') \cdot u' d\Gamma$ or reduced to $\eta \int_{\Gamma_2} g(u') \cdot u' d\Gamma$. □

Proof of Theorem 2.5. The proof of the first part of Theorem 2.5 is the same as that of Theorems 2.2 and 2.3 except that all u' and $g(u')$ are replaced by u_i' and $h(u_i')$, respectively. Also, in this case, the constant b in (4.40) is taken as

$$b = \Big(\frac{p+1}{2C_4 n(p+1-2q)}\Big)^{(p+1-2q)/(p+1)}, \tag{4.50}$$

and (4.41) becomes

$$\begin{aligned}
& E^\sigma(t)\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu[|u_i'|^2+C_3|h(u_i')|^2]d\Gamma \\
\le\ & \frac{1}{2n}E^{\sigma(p+1)/(p+1-2q)} \\
& +\frac{q(2C_4)^{(p+1)/(2q)}[nR_0mes(\Gamma_2)(p+1-2q)]^{(p+1-2q)/(2q)}}{k_3(p+1)^{(p+1)/(2q)}} \\
& \qquad\times\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu h(u_i')u_i'd\Gamma. \qquad (4.51)
\end{aligned}$$

Note that the space dimension n appears here.

However, the proof of the decay rate (2.101) is a bit different. Hence we present it as follows.

We first look at the case corresponding to Theorem 2.2. Since $p>1$ and $q=1/p$, as in Remark 2.12, we have

$$|s|^2+C_3|h(s)|^2\le C_5(h(s)s)^{2/(p+1)},\quad |s|\le 1, \qquad (4.52)$$

where

$$C_5=\frac{1+C_3k_2k_3}{k_3}. \qquad (4.53)$$

It therefore follows that for any $b>0$

$$\begin{aligned}
& E^\sigma(t)\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu[|u_i'|^2+C_3|h(u_i')|^2]d\Gamma \\
\le\ & C_5\frac{p-1}{p+1}b^{(p+1)/(p-1)}E^{\sigma(p+1)/(p-1)} \\
& +\frac{2C_5}{(p+1)b^{(p+1)/2}}\Big(\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu(u_i'h(u_i'))^{2/(p+1)}\Gamma\Big)^{(p+1)/2} \\
\le\ & C_5\frac{p-1}{p+1}b^{(p+1)/(p-1)}E^{\sigma(p+1)/(p-1)} \\
& +\frac{2C_5}{(p+1)b^{(p+1)/2}}\Big(\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu d\Gamma\Big)^{(p-1)/2} \\
& \qquad\times\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu u_i'h(u_i')d\Gamma \\
\le\ & C_5\frac{p-1}{p+1}b^{(p+1)/(p-1)}E^{\sigma(p+1)/(p-1)} \\
& +\frac{2C_5[R_0mes(\Gamma_2)]^{(p-1)/2}}{(p+1)b^{(p+1)/2}}\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu u_i'h(u_i')d\Gamma. \qquad (4.54)
\end{aligned}$$

Thus, by taking

$$b=\Big(\frac{p+1}{2C_5(p-1)}\Big)^{(p-1)/(p+1)}, \qquad (4.55)$$

we obtain

$$
\begin{aligned}
& E^\sigma(t)\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu[|u_i'|^2+C_3|h(u_i')|^2]d\Gamma \\
\le\ & \frac{1}{2}E^{\sigma(p+1)/(p-1)}+\frac{(2C_5)^{(p+1)/2}[R_0mes(\Gamma_2)(p-1)]^{(p-1)/2}}{(p+1)^{(p+1)/2}} \\
& \times\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu u_i'h(u_i')d\Gamma. \qquad (4.56)
\end{aligned}
$$

It therefore follows from (4.34) that

$$
\begin{aligned}
V'(t)\ \le\ & -\delta E^{\sigma+1}(t)+\Big[\delta E^\sigma(t)\Big(\frac{\sigma\alpha C_1}{\beta}+C_2\Big)-\frac{\alpha}{\beta}\Big]\|\nabla\theta(t)\|^2 \\
& +\Big[\delta C_1\sigma E^\sigma(0)+\frac{\delta C_4E^\sigma(0)}{k_3}-1\Big]\int_{\Gamma_2\cap[|u_i'|\ge 1]} m\cdot\nu u_i'h(u_i')d\Gamma \\
& +\Big[\delta C_1\sigma E^\sigma(0)-1\Big]\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu u_i'h(u_i')d\Gamma \\
& +\delta E^\sigma(t)\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu[|u_i'|^2+C_3|h(u_i')|^2]d\Gamma \\
\le\ & -\delta E^{\sigma+1}(t)+\frac{\delta}{2}E^{\sigma(p+1)/(p-1)} \\
& +\Big[\delta E^\sigma(t)\Big(\frac{\sigma\alpha C_1}{\beta}+C_2\Big)-\frac{\alpha}{\beta}\Big]\|\nabla\theta(t)\|^2 \\
& +\Big[\delta C_1\sigma E^\sigma(0)+\frac{\delta C_4E^\sigma(0)}{k_3}-1\Big]\int_{\Gamma_2\cap[|u_i'|\ge 1]} m\cdot\nu u_i'h(u_i')d\Gamma \\
& +\Big[\delta C_1\sigma E^\sigma(0)-1\Big]\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu u_i'h(u_i')d\Gamma \\
& +\delta\frac{(2C_5)^{(p+1)/2}[R_0mes(\Gamma_2)(p-1)]^{(p-1)/2}}{(p+1)^{(p+1)/2}} \\
& \times\int_{\Gamma_2\cap[|u_i'|\le 1]} m\cdot\nu u_i'h(u_i')d\Gamma. \qquad (4.57)
\end{aligned}
$$

By taking $\delta=\delta_2$ (see (2.65)) and $\sigma=(p-1)/2$, we obtain

$$
V'(t)\le-\frac{\delta_2}{2}E^{(p+1)/2}(t). \qquad (4.58)
$$

The remainder of the proof is the same as that of Theorem 2.2.

The proof of the case corresponding to Theorem 2.3 is similar. □

We now use the method developed in [41] to prove Theorem 2.7.

Proof of Theorem 2.7. If $E(t_0)=0$ for some $t_0\ge 0$, then, by (2.42), we have $E(t)\equiv 0$ for $t\ge t_0$ and then the theorem holds. Therefore, we may assume that $E(t)>0$ for $t\ge 0$. This assumption ensures that, in the following proof, $\varphi''(aE(t))$ makes sense as we have assumed that $\varphi(s)$ is twice differentiable outside $s=0$.

We first note that $C = C_0$ and the constant C_1'' in Lemma 4.1 and the constants C_2'' and C_3'' in Lemma 4.2 become C_1', C_2', C_3' in Theorem 2.3, respectively. Set

$$V(t) = E(t) + \delta\varphi'(aE(t))F(t), \tag{4.59}$$

where the constant a is given by (2.116). By (2.42), (4.10) and Lemmas 4.1 and 4.2, we have

$$\begin{aligned}
V'(t) &= E'(t) + \delta a\varphi''(aE(t))E'(t)F(t) + \delta\varphi'(aE(t))F'(t) \\
&\le \Big[1 - \delta a C_1'\varphi''(aE(0))E(0)\Big]E'(t) - \delta\varphi'(aE(t))E(t) \\
&\quad + \delta C_2'\varphi'(aE(0))\|\nabla\theta(t)\|^2 \\
&\quad + \delta\varphi'(aE(t))\int_{\Gamma_2}\Big[(m\cdot\nu + 2C_0^2k_1^2|u'|^2 + C_3' m\cdot\nu\sum_{i=1}^n |h(u_i')|^2\Big]d\Gamma \\
&\le -\delta\varphi'(aE(t))E(t) + \Big[\delta C_2'\varphi'(aE(0)) + \alpha\delta a C_1'\varphi''(aE(0))E(0)/\beta - \alpha/\beta\Big]\|\nabla\theta(t)\|^2 \\
&\quad + \Big[\eta\delta a C_1'\varphi''(aE(0))E(0) + \delta\varphi'(aE(0))[R_0 + 2C_0^2k_1^2 + C_3'k_2^2R_0]/k_3 - \eta\Big] \\
&\qquad \times \sum_{i=1}^n \int_{\Gamma_2\cap[|u_i'|\ge 1]} h(u_i')u_i' d\Gamma \\
&\quad + \Big[\eta\delta a C_1'\varphi''(aE(0))E(0) - \eta\Big]\sum_{i=1}^n \int_{\Gamma_2\cap[|u_i'|\le 1]} h(u_i')u_i' d\Gamma \\
&\quad \text{(note that } |u'|^2 \le |u'|^{2q} \text{ since } q \le 1 \text{ and } |u_i'| \le 1) \\
&\quad + \delta[R_0 + 2C_0^2k_1^2 + C_3'k_2^2R_0]\varphi'(aE(t))\sum_{i=1}^n \int_{\Gamma_2\cap[|u_i'|\le 1]} |u_i'|^{2q} d\Gamma \\
&\le -\delta\varphi'(aE(t))E(t) \\
&\quad + \Big[\eta\delta a C_1'\varphi''(aE(0))E(0) - \eta\Big]\sum_{i=1}^n \int_{\Gamma_2\cap[|u_i'|\le 1]} h(u_i')u_i' d\Gamma \\
&\quad + \delta[R_0 + 2C_0^2k_1^2 + C_3'k_2^2R_0]\varphi'(aE(t))\sum_{i=1}^n \int_{\Gamma_2\cap[|u_i'|\le 1]} |u_i'|^{2q} d\Gamma,
\end{aligned} \tag{4.60}$$

if δ is small enough. Let φ^* denote the dual of φ in the sense of Young (for definition, see [4, p.64]). Then, by Young's inequality [4, p.64] and Jensen's inequality [49], we deduce

$$\begin{aligned}
&\varphi'(aE(t))\int_{\Gamma_2\cap[|u_i'|\le 1]} |u_i'|^{2q} d\Gamma \\
&= \operatorname{mes}(\Gamma_2\cap[|u_i'|\le 1])\varphi'(aE)\frac{1}{\operatorname{mes}(\Gamma_2\cap[|u_i'|\le 1])}\int_{\Gamma_2\cap[|u_i'|\le 1]} |u_i'|^{2q} d\Gamma \\
&\le \operatorname{mes}(\Gamma_2\cap[|u_i'|\le 1])\Big[\varphi^*(\varphi'(aE)) + \varphi\Big(\frac{1}{\operatorname{mes}(\Gamma_2\cap[|u_i'|\le 1])}\int_{\Gamma_2\cap[|u_i'|\le 1]} |u_i'|^{2q} d\Gamma\Big)\Big] \\
&\le \operatorname{mes}(\Gamma_2)\varphi^*(\varphi'(aE)) + \int_{\Gamma_2\cap[|u_i'|\le 1]} \varphi(|u_i'|^{2q}) d\Gamma\Big) \\
&\le \operatorname{mes}(\Gamma_2)\varphi^*(\varphi'(aE)) + \int_{\Gamma_2\cap[|u_i'|\le 1]} u_i' h(u_i') dx.
\end{aligned} \tag{4.61}$$

Hence, if δ is small enough, then it follows from (4.60) and (4.61) that

$$V'(t) \leq -\delta\varphi'(aE(t))E(t) + n\delta[R_0 + 2C_0^2k_1^2 + C_3'k_2^2R_0]\text{mes}(\Gamma_2)\varphi^*(\varphi'(aE)). \tag{4.62}$$

By the definition of the dual $\varphi^*(t)$ of $\varphi(s)$, $\varphi^*(t)$ is the Legendre transform of $\varphi(s)$, which is given by (see [4, p.61-62])

$$\varphi^*(t) = t\varphi'^{-1}(t) - \varphi[\varphi'^{-1}(t)]. \tag{4.63}$$

It therefore follows from (4.62) that

$$\begin{aligned} V'(t) &\leq -\delta\varphi'(aE(t))E(t) \\ &\quad + n\delta[R_0 + 2C_0^2k_1^2 + C_3'k_2^2R_0]\text{mes}(\Gamma_2)[a\varphi'(aE(t))E(t) - \varphi(aE(t))] \\ &= -n\delta[R_0 + 2C_0^2k_1^2 + C_3'k_2^2R_0]\text{mes}(\Gamma_2)\varphi(aE(t)) \\ &\quad -\frac{\delta}{2}\varphi'(aE(t))E(t), \end{aligned} \tag{4.64}$$

with

$$a = \frac{1}{2n[R_0 + 2C_0^2k_1^2 + C_3'k_2^2R_0]\text{mes}(\Gamma_2)}. \tag{4.65}$$

On the other hand, since $\varphi(s)$ and $\varphi'(s)$ are positive and increasing on $[0,\infty)$, it follows from (4.10) that

$$[1 - \delta C_1'\varphi'(aE(0))]E(t) \leq V(t) \leq [1 + \delta C_1'\varphi'(aE(0))]E(t). \tag{4.66}$$

It therefore follows from (4.64) that

$$\begin{aligned} V'(t) \leq\ & -\frac{\delta V(t)}{2(1 + \delta C_1'\varphi'(aE(0)))}\varphi'\Big(\frac{aV(t)}{1 + \delta C_1'\varphi'(aE(0))}\Big) \\ & -n\delta[R_0 + 2C_0^2k_1^2 + C_3'k_2^2R_0]\text{mes}(\Gamma_2)\varphi\Big(\frac{aV(t)}{1 + \delta C_1'\varphi'(aE(0))}\Big). \end{aligned} \tag{4.67}$$

This is (2.115).

It remains to prove (2.118). We argue by contradiction. Suppose that $E(t)$ does not tend to zero as $t \to \infty$. Since $E(t)$ is decreasing on $[0,\infty)$, we have

$$E(t) \geq \sigma > 0, \quad \forall t \geq 0, \tag{4.68}$$

and by (4.66), we have

$$V(t) \geq \beta > 0, \quad \forall t \geq 0. \tag{4.69}$$

Thus we have

$$\varphi'\Big(\frac{aV(t)}{b}\Big) \geq \gamma > 0, \quad \forall t \geq 0. \tag{4.70}$$

It therefore follows from (2.115) that

$$V'(t) \leq -\frac{\delta\gamma}{2b}V(t), \ \ \forall t \geq 0, \tag{4.71}$$

which is in contradiction with (4.69). This completes the proof. □

Proof of Corollary 2.9. By Theorem 2.7, it suffices to prove that, for t large enough, (2.115) becomes (2.121). Since h is odd in $[-1,1]$ and $s^{1/(2q)}h(s^{1/(2q)})$ is convex in $[0,1]$, we can take for $0 \le s \le 1$

$$\varphi(s) = s^{1/(2q)}h(s^{1/(2q)}), \tag{4.72}$$

and then

$$\varphi'(s) = \frac{1}{2q}[s^{(1-2q)/(2q)}h(s^{1/(2q)}) + s^{(1-q)/q}h'(s^{1/(2q)})]. \tag{4.73}$$

On the other hand, by (2.118), there exists $T > 0$ such that

$$V(t) \le 1, \quad \forall t \ge T. \tag{4.74}$$

Thus, for $t \ge T$, substituting (4.72) and (4.73) into (2.115), we obtain (2.121). □

Proof of Corollary 2.10. If h satisfies (2.95), then we can take

$$\varphi(s) = k_3 s^{(p+1)/(2q)}, \quad \text{for } s \ge 0. \tag{4.75}$$

Thus, (2.115) becomes

$$V'(t) = -K(V(t))^{(p+1)/(2q)}, \tag{4.76}$$

where K is a positive constant independent of V. Hence, the decay properties (2.52) and (2.53) follow from (4.76). □

5 Further Comments

In the proof of Theorems 2.2-2.7, inequality (2.25) of Poincaré type plays a key role. This inequality is guaranteed by assumption (2.16). If this assumption does not hold, i.e., $\Gamma_1 = \emptyset$ and $a(x) \equiv 0$, then this inequality is no longer true for all $u \in H^1(\Omega)$. When $\Gamma_1 = \emptyset$ and $a(x) \equiv 0$, in order to guarantee the coercivity of the energy , it is natural to look for a closed subspace or subset $\mathcal{W}$ of $(H^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega)$ invariant under the flow generated by the semigroup such that the energy norm (2.20) on $\mathcal{W}$ is equivalent to the usual one induced by $(H^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega)$.

Naturally, one is tempted to consider the following space of functions with zero average

$$\begin{aligned}\mathcal{H}_0 = \Big\{(u,v,\theta) \in (H^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega) \ :\\ \textstyle\int_\Omega u(x)dx = \int_\Omega v(x)dx = 0\Big\},\end{aligned} \tag{5.1}$$

on which the energy norm is really equivalent to the usual one. Unfortunately, $\mathcal{H}_0$ is not invariant. To see this, we define the function

$$f(t) = \int_\Omega u(t)dx. \tag{5.2}$$

We look at the special case where $g(u) = u$. Take sufficiently regular initial condition $(u^0, u^1, \theta^0) \in \mathcal{H}_0$ such that

$$\int_\Gamma m \cdot \nu u^1 d\Gamma \neq 0, \tag{5.3}$$

and let u, θ be the solution of (2.7) corresponding to this initial data. By the continuity of $u'(t)$ with respect to t, we have that

$$\int_\Gamma m \cdot \nu u'(t) d\Gamma \neq 0 \tag{5.4}$$

for $t > 0$ small enough. It therefore follows that

$$\begin{aligned} f''(t) &= \int_\Omega u''(t)dx \\ &= \int_\Omega [\mu\Delta u + (\lambda+\mu)\nabla \text{div } u - \alpha\nabla\theta]dx \\ &= \int_\Gamma [\mu\frac{\partial u}{\partial \nu} + (\lambda+\mu)\text{div}(u)\nu - \alpha\theta\nu]d\Gamma \\ &= -\int_\Gamma m \cdot \nu u'(t) d\Gamma \\ &\not\equiv 0. \end{aligned} \tag{5.5}$$

Hence, $\int_\Omega u(t)dx$ and $\int_\Omega u'(t)dx$ are not always equal to zero along the solution trajectories of (2.7).

On the other hand, we do have the following conserved quantity

$$\int_\Omega u'(t)dx + \int_0^t \int_\Gamma m \cdot \nu g(u'(t)) d\Gamma dt \equiv C \text{ (a constant)}, \tag{5.6}$$

since, by (2.7), we have

$$\int_\Omega u''(t)dx + \int_\Gamma m \cdot \nu g(u'(t)) d\Gamma \equiv 0. \tag{5.7}$$

If $g(u)$ is linear, i.e., $g(u) = ku$, then we can easily find an invariant subspace $\mathcal{W}$ as follows

$$\mathcal{W} = V \times L^2(\Omega), \tag{5.8}$$

where

$$V = \{(u,v) \in (H^1(\Omega))^n \times (L^2(\Omega))^n \ : \ \int_\Gamma km \cdot \nu u d\Gamma + \int_\Omega v dx = 0\}. \tag{5.9}$$

Moreover, the energy norm on $\mathcal{W}$ is equivalent to the usual one. However, for the general nonlinear boundary feedbacks, it is difficult to find such an invariant closed subset. Thus, the case that $\Gamma_1 = \emptyset$ and $a(x) \equiv 0$ with g nonlinear is open.

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