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DISCRETE INGHAM INEQUALITIES AND APPLICATIONS*

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Abstract. In this paper we prove a discrete version of the classical Ingham inequality for nonharmonic Fourier series whose exponents satisfy a gap condition. Time integrals are replaced by discrete sums on a discrete mesh. We prove that, as the mesh becomes finer and finer, the limit of the discrete Ingham inequality is the classical continuous one. This analysis is partially motivated by control-theoretical issues. As an application we analyze the control/observation properties of numerical approximation schemes of the 1-d wave equation. The discrete Ingham inequality provides observability and controllability results which are uniform with respect to the mesh-size in suitable classes of numerical solutions in which the high frequency components have been filtered. We also discuss the optimality of these results in connection with the dispersion diagrams of the numerical schemes.

Key words. wave equation, numerical approximation schemes, nonharmonic analysis, discrete Fourier transform, Ingham inequalities, observability, controllability, dispersion, group velocity

AMS subject classifications. 42C99, 65T50, 65M06, 65N06

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1. Introduction.

Families of "nonharmonic" exponentials $\left\{e^{i\lambda_k t}\right\}$ appear in various fields of mathematics and signal processing. One of the central problems arising in all of these applications is the question of the Riesz basis property.

The following inequality for nonharmonic Fourier series due to Ingham is well known (see [9] and [26, p. 162]): *Assume that the strictly increasing sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ of real numbers satisfies the "gap" condition*

$$\lambda_{k+1} - \lambda_k \geq \gamma \quad \text{for all } k \in \mathbb{Z}, \tag{1.1}$$

for some $\gamma > 0$. Then, for all $T > 2\pi/\gamma$ there exist two positive constants C_1, C_2 depending only on γ and T such that

$$C_1(T,\gamma) \sum_{k=-\infty}^{\infty} |a_k|^2 \leq \int_0^T \left| \sum_{k=-\infty}^{\infty} a_k e^{it\lambda_k} \right|^2 dt \leq C_2(T,\gamma) \sum_{k=-\infty}^{\infty} |a_k|^2, \tag{1.2}$$

for every complex sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2$, where

$$C_1(T,\gamma) = \frac{2T}{\pi}\left(1 - \frac{4\pi^2}{T^2\gamma^2}\right) > 0, \tag{1.3}$$

$$C_2(T,\gamma) = \frac{8T}{\pi}\left(1 + \frac{4\pi^2}{T^2\gamma^2}\right) > 0, \tag{1.4}$$

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and ℓ^2 is the Hilbert space of square summable sequences,

$$\ell^2 = \left\{ \{a_k\} \ : \ \|a_k\|_{\ell^2}^2 = \sum_{k\in\mathbb{N}} |a_k|^2 < \infty \right\}. \tag{1.5}$$

This result shows that the sequence of exponentials $\left\{e^{i\lambda_k t}\right\}$ forms a Riesz basis of its span for $T > 2\pi/\gamma$ (see [26, Chapter 3, p. 112]).

As we have mentioned above, one of the main applications of Ingham's inequality and its variants is the control of wave-like equations and other closely related problems like observability or inverse problems. The problem of observability for wave equations consists of analyzing whether the energy of the waves propagating in a domain with suitable boundary conditions can be estimated in terms of the energy concentrated on a given subregion of the domain (or its boundary) where propagation occurs in a given time interval. On the other hand, the goal in controllability problems is to drive the solutions of a given dynamical system (continuous or discrete) to a given state at a given final time by means of a control acting on the system on that subregion (or its boundary). It is well known that the two problems are equivalent provided one chooses an appropriate functional setting, which depends on the equation (see, for instance, [17]).

In the context of partial differential equations, using the Fourier representation of the solutions, the problem of observability can be reduced to an application of Ingham's inequality in which the sequence $\{\lambda_k\}$ is constituted by the spectrum of the generator of the underlying semigroup. However, the gap condition (1.1) that is required to apply Ingham's inequality often limits the range of applicability of this technique to 1-d problems like strings and beams. This has led to a significant number of controllability results (see [15]) and also to far reaching generalizations of the Ingham theorem under weakened gap conditions (see [2], [4], [5], [8], [12], [13]). The most complete result in this direction has been obtained independently by Baiocchi, Komornik, and Loreti in [2], [3], [4], and Avdonin and Moran in [1].

In the numerical analysis of those observability inequalities and for studying the controllability properties of numerical schemes the need of a discrete version of this inequality arises naturally (see [7], [19], [20], [21]). This paper is devoted to proving a discrete version of that Ingham inequality.

The inequality we prove is uniform with respect to the mesh-size Δt in the time-discretization and, in the limit as $\Delta t \to 0$, yields the classical Ingham inequality above.

The discrete Ingham inequality we prove is the natural tool to prove observability/controllability properties for fully discrete schemes for the approximation of the 1-d wave equation and other closely related models (vibrating beams, Schrödinger equation, etc.) and to show that the controls of the limiting continuous model are the limit of the controls of the full discrete schemes. However, it is important to recall that, as it is by now well known [28], numerical approximation schemes often introduce spurious high frequency solutions that may be an obstacle for uniform (with respect to the mesh-size) observability/controllability results. Thus, one often needs to filter or cut-off those spurious numerical solutions. Our generalization of Ingham's inequality to the discrete context explains how this filtering has to be done in order to guarantee uniform results.

As an example of application of our discrete Ingham inequality we perform the analysis of the observability/controllability properties of the most standard centered fully discrete schemes for the wave equation.

The main reason for the lack of uniform observability/controllability of the numerical high frequency spurious solutions, is that they generate high frequency wave packets for which the group velocity is of the order of the mesh-size ([28]). Thus, as the mesh-size tends to zero, since the velocity becomes smaller and smaller, the time for observability/controllability increasing in a divergent way. This fact is related to the dispersion diagram associated to the numerical approximation scheme, since, roughly, the slope of the dispersion diagram is the group velocity of propagation of wave packets and also coincides with the spectral gap. Part of this article is devoted to explaining the connections of these notions and to show how combining the qualitative information that the dispersion diagram provides with the discrete Ingham inequality, one can get precise information on how the filtering should be implemented, if needed.

As proved in the original article by Ingham (see [9, p. 368]), an L^1-version of inequality (1.2) also holds. More precisely, for every increasing sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ of real numbers satisfying the "gap" condition (1.1) we also have

$$(1.6)\qquad C_1(T,\gamma)\,|a_k| \le \int_0^T \left|\sum_{k=-\infty}^{\infty} a_k e^{it\lambda_k}\right| dt \le C_2(T,\gamma)\,|a_k| \quad \text{for all } k\in\mathbb{Z}$$

for all $T > 2\pi/\gamma$.

In this paper we also prove a discrete version of this inequality.

Our proofs are strongly inspired in that by Ingham (see also [26]), which is based on the use of a suitable cut-off, nonnegative function, with compact support on the time interval $(0,T)$ and whose Fourier transform is "concentrated" around $\tau = 0$. We use the same function in the physical space, but its Fourier transform has to be replaced by the discrete one. One of the key points in the proof is a careful comparison between the continuous and discrete transforms of this weight function. This is done by using a key result by N. Trefethen [23].

This paper is organized as follows: in section 2 we state our discrete Ingham inequality (see Theorem 2.1), we analyze the necessity of its hypotheses and compare both the continuous and discrete inequalities. We also formulate a discrete version of the L^1 analogue (1.6) (see Theorem 2.2). In section 3 we discuss the application of this result to the study of the properties of the solutions of fully discrete approximations of the wave equation. In section 4 the controllability problem for the discrete system is addressed and the main results of existence, characterization, and convergence of the discrete controls are presented and proved. In section 5 we discuss these results in connection with the dispersion diagrams of the discrete equations under consideration. Finally, section 6 is devoted to proving the discrete Ingham inequality and its discrete L^1 version.

The discrete Ingham inequality we present in this paper has been announced in [21].

2. Main results. The main result of this paper is as follows.

THEOREM 2.1 (discrete Ingham inequality). *Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be an increasing sequence of real numbers satisfying for some $\gamma > 0$ the "gap" condition*

$$(2.1)\qquad \lambda_{k+1} - \lambda_k \ge \gamma > 0 \quad \text{for all } k \in \mathbb{Z}.$$

Let $T > 0$ and $0 < \Delta t \le 1$. Assume that $\{\lambda_k\}_{|k|\le N}$ satisfies the additional condition

$$(2.2)\qquad |\lambda_k - \lambda_l| \le \frac{2\pi - (\Delta t)^p}{\Delta t} \quad \text{for all } \ |k|\le N, \ \ |l|\le N, \ \text{ for some } \ 0\le p<1/2,$$

where $2N \leq M$ *and* $M = [T/\Delta t - 1]$. *Then, there exists a positive number* $\epsilon(\Delta t)$ *such that, for all* $T > T_0(\Delta t) := 2\pi/\gamma + \epsilon(\Delta t)$, *there exist two positive constants* $C_j(\Delta t, T, \gamma) > 0$, $j = 1, 2$, *such that*

(2.3)
$$C_1(\Delta t, T, \gamma) \sum_{k=-N}^{N} |a_k|^2 \leq \Delta t \sum_{n=0}^{M} \left| \sum_{k=-N}^{N} a_k e^{in\Delta t \lambda_k} \right|^2 \leq C_2(\Delta t, T, \gamma) \sum_{k=-N}^{N} |a_k|^2,$$

for every complex sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2$.

Moreover, if γ *and* p *in* (2.1) *and* (2.2) *are kept fixed, then* $\epsilon(\Delta t) = o(\Delta t)^{1-2p}$ *and the constants in* (2.3) *satisfy*

(2.4)
$$C_j(\Delta t, T, \gamma) = C_j(T, \gamma) + \delta_j(\Delta t), \quad j = 1, 2, \quad \text{with} \quad \delta_1(\Delta t) \leq 0 \quad \text{and} \quad \delta_2(\Delta t) \geq 0,$$

where $C_j(T, \gamma)$, $j = 1, 2$, *are the Ingham constants in* (1.3) *and* (1.4) *and* $\lim_{\Delta t \to 0} \delta_j(\Delta t) = 0$, $j = 1, 2$.

Concerning the L^1-version of Ingham inequality in (1.6), the following theorem holds.

THEOREM 2.2. *Under the hypotheses of Theorem* 2.1 *we also have the following discrete version of* (1.6):

(2.5)
$$C_1(\Delta t, T, \gamma)\, |a_k| \leq \Delta t \sum_{n=0}^{M} \left| \sum_{k=-N}^{N} a_k e^{in\Delta t \lambda_k} \right| \leq C_2(\Delta t, T, \gamma)\, |a_k| \quad \text{for all } |k| \leq N.$$

As in Theorem 2.1, *the time* T *and the constants in this inequality remain uniform as* $\Delta t \to 0$ *and converge to those of the continuous Ingham inequality* (1.6).

Remark 2.3. Condition $T > 2\pi/\gamma$ is optimal for the classical Ingham inequality (see [26, p. 163]). In this sense, the condition $T > 2\pi/\gamma + \epsilon(\Delta t)$ in Theorem 2.1 is asymptotically optimal since $\epsilon(\Delta t) \to 0$ as $\Delta t \to 0$.

It is important to emphasize that the time T and the constants C_j, $j = 1, 2$ in (2.3) are uniform in Δt. This is essential for the applications in numerical analysis in which $\Delta t \to 0$. The uniformity may be guaranteed because of the assumptions (2.1)–(2.2) on the sequence $\{\lambda_k\}_k$.

More precisely, when comparing the continuous and discrete inequalities, the following can be said:

- In both continuous and discrete cases, the sequence $\{\lambda_k\}_k$ is required to satisfy the so-called *gap condition* (2.1).
- The restriction (2.2) imposed on $\{\lambda_k\}_k$ in Theorem 2.1 is not needed in the classical continuous Ingham inequality (1.2).
- It is easy to see that, for every $N \in \mathbb{N}$ fixed, if we pass to the limit $\Delta t \to 0$ in (2.3), we get the classical Ingham inequality (1.2). Indeed, for (1.2) to be true for all sequences $(a_k)_{k\in\mathbb{Z}} \in \ell^2$ it is sufficient, by density, to prove it for sequences with only a finite number of nonzero components.

  In that case (1.2) is the limit of (2.3) because of the convergence of the minimal time T and the constants C_j, $j = 1, 2$, in (2.3) to those of (1.2).

  We also have a discrete Ingham inequality (2.3) for every sequence $(\lambda_k)_k$ verifying conditions (2.1) and (2.2), with $0 \leq p \leq 1$. But, if $p \geq 1/2$,

$\varepsilon(\Delta t) = o(\Delta t)^{1-2p} \to \infty$, so $T_0(\Delta t) \to \infty$, and this makes it of little use in practice because we are looking for a uniform (with respect to Δt) time T.

On the other hand, the restriction $2N \le M$, with $M = [T/\Delta t - 1]$, is sharp. Indeed, when $2N > M$ one can find nontrivial values of the coefficients $\{a_k\}_k$ such that

$$\sum_{k=-N}^{N} a_k e^{in\Delta t \lambda_k} = 0, \quad 0 \le n \le M \tag{2.6}$$

and

$$\sum_{k=-N}^{N} |a_k|^2 \neq 0. \tag{2.7}$$

Observe that (2.6) is a system of $M+1$ homogeneous linear equations with $2N+1$ unknown quantities a_k. If $2N > M$, this system necessarily has nontrivial solutions. This is in agreement with common sense. Indeed, in view of the fact that we only make $M+1$ measurements for $n = 0, \dots, M$ one cannot expect to recover more than $M+1$ coefficients of the solution.

When $2N \le M$, (2.6)–(2.7) do not hold. However if $\lambda_k - \lambda_l \in 2\pi\mathbb{Z}/\Delta t$ for certain values of k and l with $k \neq l$ the sequence $a_k = -a_l = 1$, $a_n = 0$, $n \neq k$, and l satisfies (2.6). Then, an inequality of type (2.3) is impossible. So, it is natural to impose on the sequence $\{\lambda_k\}_k$ the condition $\lambda_k - \lambda_l \notin 2\pi\mathbb{Z}/\Delta t$ for a discrete Ingham inequality (2.3) to hold.

In fact, to avoid *aliasing* one has to restrict the increasing sequence of real numbers $\{\lambda_k\}_k$ to be such that $\lambda_k - \lambda_l \in [2\pi m/\Delta t, 2\pi(m+1)/\Delta t]$, for some $m \in \mathbb{Z}$. Therefore, it is natural to impose the condition

$$|\lambda_k - \lambda_l| < \frac{2\pi}{\Delta t}.$$

In our theorem this latter condition is implied by the stronger one, (2.2), which is needed for the uniform estimates in (2.3) to hold. More precisely, the restriction $0 \le p < 1/2$ in (2.2) is needed to guarantee the asymptotically optimal time $T > 2\pi/\gamma + \epsilon(\Delta t)$, with $\epsilon(\Delta t) \to 0$ as $\Delta t \to 0$ since $\epsilon(\Delta t) = o(\Delta t)^{1-2p}$.

Remark 2.4. The condition $T > T_0(\Delta t)$ is necessary for the proof of the first inequality in (2.3) and in (1.6) (to have $C_1(\Delta t, T, \gamma) > 0$). The second inequality in (2.3) and (1.6), respectively, holds for all $T > 0$. In this respect the situation is the same as for the continuous inequalities (1.2).

3. Application to the uniform observability of the full discretizations of the 1-d wave equation.

3.1. The wave equation. This section is motivated by the classical problem of control of waves. More precisely, it is related with the controllability of the 1-d wave equation: given $T > 0$ and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$, the problem is to find a control function $v \in L^2(0,T)$ such that the solution of the system

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1, \quad 0 < t < T, \\ u(0,t) = 0, \quad u(1,t) = v(t), & 0 < t < T, \\ u(x,0) = u^0(x), \qquad u_t(x,0) = u^1(x), & 0 < x < 1, \end{cases} \tag{3.1}$$

satisfies

$$u(T) = u_t(T) = 0, \qquad 0 < x < 1. \tag{3.2}$$

This property is well known to be true for $T \geq 2$. This problem has been studied and solved in a much more general setting and, in particular, for multidimensional wave equations [17]. Several approaches to the problem have been developed. In particular, the Hilbert uniqueness method (HUM) introduced by Lions in [17] offers a general way of reducing the problem to the so-called *observability problem* for the adjoint (up to an inversion in time) wave equation in the absence of control:

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0, & 0 < x < 1, \quad 0 < t < T, \\ \phi(0,t) = \phi(1,t) = 0, & 0 < t < T, \\ \phi(x,0) = \phi^0(x), \qquad \phi_t(x,0) = \phi^1(x), & 0 < x < 1. \end{cases} \tag{3.3}$$

It is well known that the energy

$$E(t) = \frac{1}{2} \int_0^1 \left(\mid \phi_x(x,t) \mid^2 + \mid \phi_t(x,t) \mid^2 \right) dx \tag{3.4}$$

of the solutions of (3.3) satisfies

$$\frac{dE(t)}{dt} = E'(t) = 0 \quad \text{for all } t \in [0,T]$$

and therefore is conserved in time.

The observability problem is as follows: *To find $T > 0$ such that there exists a constant $C(T) > 0$ for which*

$$E(0) \leq C(T) \int_0^T \mid \phi_x(1,t) \mid^2 dt \tag{3.5}$$

holds for every solution of (3.3).

HUM allows showing that, once the observability inequality (3.5) is satisfied for the adjoint system (3.3), system (3.1) is controllable in time T. Moreover, HUM provides a systematic method to build the control $v = \phi_x(1,t)$ of minimal $L^2(0,T)$-norm.

In the context of the 1-d wave equation (3.3), inequality (3.5) can be easily proved by several methods including *Fourier series, D'Alembert Formula*, *multiplier techniques*, and *Ingham's theorem* (1.2), provided $T \geq 2$.

In order to solve the problem (3.5) applying the classical Ingham inequality, one uses Fourier series techniques. Indeed, the solution of (3.3) admits the Fourier development

$$\phi(x,t) = \sum_{k \in \mathbb{Z} \setminus \{0\}} a_k e^{i\lambda_k t} \varphi_k(x), \tag{3.6}$$

with $\{\lambda_k\}_k$, $\lambda_k = k\pi = -\lambda_{-k}$, $k > 0$, being the sequence of eigenvalues of the system, $\varphi_k(x) = \sin(k\pi x)$, the corresponding eigenfunctions and $a_k \in \mathbb{C}$ the Fourier coefficients, which can be computed explicitly in terms of the initial data in (3.3).

By definition (3.4) of the conserved energy of the solution ϕ of (3.3) given by (3.6), we have

$$E_\phi = \frac{1}{2} \sum_{k \in \mathbb{Z} \setminus \{0\}} k^2 \pi^2 \left| a_k \right|^2. \tag{3.7}$$

On the other hand, in view of the explicit form of $\phi_x(1,t)$, inequality (3.5) may be written as:

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} k^2\pi^2 |a_k|^2 \leq C(T)\int_0^T \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} (-1)^k k\pi a_k e^{i\lambda_k t} \right|^2 dt. \tag{3.8}$$

According to Ingham's inequality (1.2), (3.8) holds for $T>2$, since the gap of the sequence $\{\lambda_k\}_k$ is constant, $\gamma=\pi$, and, consequently, the minimal observability time is $2\pi/\gamma = 2$. In this particular case the inequality holds also for the minimal time $T=2$. This is due to the orthogonality properties of the trigonometric polynomials. But, in general, i.e., for a general sequence $(\lambda_k)_{k\in\mathbb{Z}}$ satisfying the gap condition (1.1), it is well known that the Ingham inequality (1.2) may fail for the minimal time $T=2\pi/\gamma$ (see [26, p. 163]).

In order to obtain numerical approximations of the controls, it is natural to analyze the controllability and observability properties of numerical approximation schemes. We first recall some well-known facts about the space semi-discretization schemes to later address space-time discretizations.

3.2. Space semi-discretizations. First, we consider the semi-discrete version of the observability problem (3.5): Take $N\in\mathbb{N}$, set $h=1/(N+1)$ and consider the finite-difference space semi-discretization of (3.3):

$$\begin{cases} \phi_j'' = \dfrac{\phi_{j+1}-2\phi_j+\phi_{j-1}}{h^2}, & t>0, \quad j=1,\dots,N, \\ \phi_0=\phi_{N+1}=0, & t>0, \\ \phi(0)=\phi_{0,j}, \quad \phi_j'(0)=\phi_{1,j}, & j=1,\dots,N. \end{cases} \tag{3.9}$$

The energy of system (3.9) is given by

$$E_h(t) = \frac{h}{2}\sum_{j=1}^N |\ \phi_j(t)\ |^2 + \frac{h}{2}\sum_{j=0}^N \frac{|\ \phi_{j+1}(t)-\phi_j(t)\ |^2}{h^2} \tag{3.10}$$

and it is also conserved in time.

The semi-discrete version of (3.5) is

$$E_h(0) \leq C\int_0^T \left|\frac{\phi_N(t)}{h}\right|^2 dt. \tag{3.11}$$

More precisely, one seeks for a positive constant $C>0$ such that (3.11) holds.
The corresponding eigenvalue problem is of the form

$$\begin{cases} -[\varphi_{k+1}+\varphi_{k-1}-2\varphi_k]/h^2 = \lambda^2\varphi_k, \quad k=1,\dots,N, \\ \varphi_0=\varphi_{N+1}=0. \end{cases} \tag{3.12}$$

The eigenvalues and eigenvectors of (3.12) may be computed explicitly (see [10, p. 456]); one then has

$$\begin{cases} \lambda_k^2(h) = \dfrac{4}{h^2}\sin^2\left(\dfrac{\pi k h}{2}\right), & k=1,\dots,N, \\ \bar{\varphi}_k \equiv (\varphi_{k,1},\dots,\varphi_{k,N});\ \varphi_{k,j}=\sin(k\pi j h), & j,k=1,\dots,N. \end{cases} \tag{3.13}$$

The solutions of (3.9) in Fourier series are

$$\bar{\phi} = \sum_{k=-N, k\neq 0}^{N} a_k e^{i\lambda_k(h)t} \bar{\varphi}_k, \tag{3.14}$$

where $\bar{\phi} = (\phi_1, \ldots, \phi_N)$.

As pointed out in [11], (3.11) holds for all $T > 0$ and $h > 0$, but, the observability constant in (3.11) may not remain uniformly bounded as $h \to 0$, for any $T > 0$. More precisely,

$$\sup_{\bar{\phi} \in \mathcal{S}_h} \left[\frac{E_h(0)}{\int_0^T \mid \phi_N(t)/h \mid^2 \, dt} \right] \to \infty, \; as \; h \to 0, \tag{3.15}$$

where $\mathcal{S}_h$ is the set of all solutions of (3.9). This is due to the pathological behavior of the high frequency numerical solutions.

In the light of Ingham's inequality (1.2), the lack of uniform observability as h tends to zero may be explained because of the lack of gap between consecutive eigenvalues (see [11], [28]). In particular, the gap between the largest eigenvalues entering in the Fourier development of the solution of (3.9) may be bounded above as follows:

$$\lambda_N(h) - \lambda_{N-1}(h) \leq \frac{3\pi^2 h}{2} \to 0, \quad \text{as} \quad h \to 0. \tag{3.16}$$

As it was proved in [11], a suitable cut-off or filtering of the spurious numerical high frequencies may be a good cure for these pathologies. Given $0 < \alpha < 1$, we introduce the following classes of filtered solution of (3.9):

$$\mathcal{C}_\alpha(h) = \left\{ \bar{\phi} \text{ sol. of (3.9) } : \bar{\phi} = \sum_{|k| \leq \alpha N, \; k \neq 0} a_k e^{i\lambda_k t} \bar{\varphi}_k \right\}. \tag{3.17}$$

In the class $\mathcal{C}_\alpha(h)$ the high frequencies corresponding to the indexes $j > \alpha N$ have been cut-off. This guarantees a uniform gap condition

$$\lambda_{k+1}(h) - \lambda_k(h) \geq \pi \cos\left(\frac{\pi\alpha}{2}\right), \quad \text{for} \quad k \leq \alpha/h. \tag{3.18}$$

Consequently, applying Ingham's inequality, we may deduce the uniform observability in the class $\mathcal{C}_\alpha(h)$ for

$$T > T(\alpha) = 2/\cos(\pi\alpha/2). \tag{3.19}$$

Let us explain this in more detail.

By definition (3.10) of the conserved energy and taking into account the orthogonality properties of the eigenvectors (see [11], [20]), we have

$$E_h = \frac{1}{4} \sum_{k=-\alpha N, k\neq 0}^{\alpha N} |a_k|^2 \left(1 + \lambda_k^2(h)\right). \tag{3.20}$$

Then, inequality (3.11) in the class $\mathcal{C}_\alpha(h)$ may be rewritten as

$$\sum_{k=-\alpha N, k\neq 0}^{\alpha N} |a_k|^2 \left(1 + \lambda_k^2(h)\right) \leq C(T) \int_0^T \left| \sum_{k=-\alpha N, k\neq 0}^{\alpha N} \frac{\sin(Nk\pi h)}{h} a_k e^{i\lambda_k t} \right|^2 dt. \tag{3.21}$$

Applying now Ingham's theorem (1.2) for the real sequence $(\lambda_k(h))_{|k|\leq \alpha N}$, in view of (3.18), it follows that if $T > T(\alpha)$ with $T(\alpha)$ as in (3.19), there exists a constant $C > 0$ such that

$$\sum_{k=-\alpha N, k\neq 0}^{\alpha N} \left| a_k \frac{\sin(Nk\pi h)}{h} \right|^2 \leq C(T) \int_0^T \left| \sum_{k=-\alpha N, k\neq 0}^{\alpha N} \frac{\sin(Nk\pi h)}{h} a_k e^{i\lambda_k t} \right|^2 dt, \tag{3.22}$$

holds for every solution of (3.9) in the class $\mathcal{C}_\alpha(h)$. Finally, it is sufficient to observe that

$$\sum_{k=-\alpha N, k\neq 0}^{\alpha N} \left| a_k \frac{\sin(Nk\pi h)}{h} \right|^2 \sim E_h,$$

to obtain a uniform observability inequality (3.11) in each class $\mathcal{C}_\alpha(h)$ for all $0 < \alpha < 1$. Note, however, that the minimal time $T(\alpha)$ depends on the filtering parameter α and, in particular, $T(\alpha) \to 2$ as $\alpha \to 0$ and $T(\alpha) \to \infty$ as $\alpha \to 1$ (see [28] for a rigorous proof).

As a further step towards a complete theory of numerical approximations of controls it is natural to address the same issue for full space-time discretizations. This issue is addressed in the following section.

3.3. Fully discrete approximations. The main ingredient to derive the fully discrete analogue of (3.5) for a finite-difference full discretization of a homogeneous 1-d wave equation (3.3) is the Fourier representation of solutions combined, this time, with our discrete Ingham inequality in Theorem 2.1.

Given $M, N \in \mathbb{N}$ we set $\Delta x = 1/(N+1)$ and $\Delta t = T/(M+1)$ and introduce the nets

$$0 = x_0 < x_1 = \Delta x < \cdots < x_N = N\Delta x < x_{N+1} = 1,$$

$$0 = t_0 < t_1 = \Delta t < \cdots < t_M = M\Delta t < t_{M+1} = T$$

with $x_j = j\Delta x$ and $t_n = n\Delta t$, $j = 0, 1, \ldots, N+1$, $n = 0, 1, \ldots, M+1$.

We consider the following finite-difference discretization of (3.1):

(3.23)
$$\begin{cases} \dfrac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}, & j = 1, 2, \ldots, N;\ n = 1, 2, \ldots, M, \\ u_0^n = 0, \qquad u_{N+1}^n = v_{\Delta x}^n, & n = 1, 2, \ldots, M, \\ u_j^0 = u_{0j}, \quad u_j^1 = \Delta t u_{1j} + u_{0j}, & j = 1, 2, \ldots, N. \end{cases}$$

We shall denote by $\bar{u}^n = (u_1^n, \ldots, u_N^n)$ the solution at the time step n. As in the context of the continuous wave equation above, we consider the uncontrolled system

(3.24)
$$\begin{cases} \dfrac{\phi_j^{n+1} - 2\phi_j^n + \phi_j^{n-1}}{(\Delta t)^2} = \dfrac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{(\Delta x)^2}, & j = 1, 2, \ldots, N; \quad n = 1, 2, \ldots, M, \\ \phi_0^n = \phi_{N+1}^n = 0, & n = 1, 2, \ldots, M, \\ \phi_j^0 = \phi_{0j}, \quad \phi_j^1 = \phi_{0j} + \Delta t \phi_{1j}, & j = 1, 2, \ldots, N, \end{cases}$$

a central finite difference discretization of (3.3).

Under the stability condition $\mu = \Delta t/\Delta x \leq 1$ (μ is the Courant number), the scheme (3.24) is convergent of order 2.

However, as observed in [14], the resulting discrete sequence of controls $v^n_{\Delta x} = -\phi^n_N/\Delta x$ obtained with a discrete HUM method may have an unstable behavior as $(\Delta t, \Delta x) \to (0,0)$. More precisely, it is possible to exhibit initial conditions such that the discrete controls $v^n_{\Delta x}$ do not converge towards the control v for (3.1) (see [28]). Once more, filtering of high frequencies is an efficient cure for these instabilities and our discrete Ingham inequality is the tool to analyze how it behaves.

The energy of (3.24) is

$$E_n = \frac{\Delta x}{2}\sum_{j=0}^{N}\left[\left(\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}\right)^2 + \left(\frac{\phi_{j+1}^{n+1}-\phi_j^{n+1}}{\Delta x}\right)\left(\frac{\phi_{j+1}^{n}-\phi_j^{n}}{\Delta x}\right)\right] \geq 0, \tag{3.25}$$

which is a discretization of the continuous energy E in (3.4), and it is conserved in all the time steps $E_n = E_0$, $n = 1, \ldots, M$, for the solutions of (3.24) (see [20]).

Solutions of (3.24) admit the Fourier development (see [20])

$$\bar{\phi}^n = \sum_{k=-N, k\neq 0}^{N} a_k e^{i\lambda_k n\Delta t}\bar{\varphi}_{|k|}, \tag{3.26}$$

with $a_k \in \mathbb{C}$, $\bar{\varphi}_k = (\varphi_{k,1}, \ldots, \varphi_{k,N}) = (\sin(k\pi\Delta x), \ldots, \sin(Nk\pi\Delta x))$ and

$$\lambda_k = \operatorname{sgn}(k)\frac{2}{\Delta t}\arcsin\left(\frac{\Delta t}{\Delta x}\sin\frac{k\pi\Delta x}{2}\right). \tag{3.27}$$

Our goal is to analyze the discrete version of the observability inequality (3.5)

$$E_0 \leq C\left[\Delta t\sum_{n=0}^{M}\left|\frac{\phi_N^n}{\Delta x}\right|^2\right], \tag{3.28}$$

where E_0 is the conserved energy of the solutions of the discrete system (3.24). This inequality implies by HUM a controllability property of the discrete analogue (3.23) of the control system (3.1). Of course, we seek for a positive constant $C > 0$ in (3.28), independent on Δt and Δx. This will yield a family of controls that will be bounded as $\Delta t \to 0$, which constitutes a natural candidate to converge to the control of (3.1).

Inequality (3.28) is the discrete analogue of (3.5). In particular, note that, according to Taylor's formula $\phi_x(1,t) \sim (\phi(1,t) - \phi(1-\Delta x, t))/\Delta x$. Thus, at the discrete level and taking into account that, according to the boundary conditions, $\phi^n_{N+1} = 0$, we obtain $\phi_x(1,t) \sim -\phi^n_N/\Delta x$. Thus, the right-hand side of (3.28) represents a discrete version of the right-hand side term in the continuous observability inequality (3.5).

Inequality (3.28) may also be seen as a time-discretization of the semi-discrete observability inequality (3.11). Note that, in fact, the semi-discrete case corresponds to taking $\mu = 0$ in the fully discrete scheme.

According to Theorem 2.1, the spectral gap between two consecutive eigenvalues plays a very important role in the analysis of the uniform observability inequality (3.28).

It is important to distinguish two cases:

• In the particular case where $\Delta t = \Delta x := h$ ($\mu = 1$) we have

$$\lambda_k = \operatorname{sgn}(k)\frac{2}{h}\arcsin\left(\sin\frac{k\pi h}{2}\right) = \operatorname{sgn}(k)k\pi.$$

Thus,

$$\lambda_{k+1} - \lambda_k = \gamma = \pi.$$

But the condition (2.2) does not hold, because

$$\max_{k,l}|\lambda_k - \lambda_l| = \frac{2\pi - 2\pi\Delta t}{\Delta t}.$$

Note, however, that, in this particular case, due to the orthogonality properties of the family of complex discrete exponentials involved in the Fourier representation of solutions,

$$\sum_{n=0}^{M} e^{in\Delta t\pi(k-l)} = (M+1)\delta_{k,l},$$

where $\delta_{k,l}$ is Kronecker's delta, an inequality of type (2.3) holds immediately and the discrete Ingham inequality is not needed.

Indeed, denoting by $m_k = (-1)^k a_k \sin(k\pi\Delta x)/\Delta x$, the energy of the solutions (3.24) concentrated on the extreme $x = 1$ can be written as

$$\Delta t \sum_{n=0}^{M}\left|\frac{\phi_N}{\Delta x}\right|^2 = \Delta t \sum_{n=0}^{M}\left|\sum_{k=-N}^{N} m_k e^{in\Delta t\pi k}\right|^2 \tag{3.29}$$

and the total energy of the solutions is

$$E_0 = \frac{1}{2}\sum_{k=-N}^{N}|m_k|^2 \tag{3.30}$$

(see [20] for more details). Then, for $T = 2$ we have

$$\begin{aligned} h\sum_{n=0}^{M}\left|\frac{\phi_N}{h}\right|^2 &= h\sum_{n=0}^{M}\left|\sum_{k=-N}^{N} m_k e^{inh\pi k}\right|^2 \\ &= h\sum_{n=0}^{M}\sum_{k=-N}^{N}|m_k|^2 + h\sum_{n=0}^{M}\sum_{k=-N,k\neq l}^{N} m_k\bar{m}_l e^{inh\pi(k-l)} = 2\sum_{k=-N}^{N}|m_k|^2, \end{aligned}$$

and therefore

$$E_0 = \frac{1}{4}\left[h\sum_{n=0}^{M}\left|\frac{\phi_N^n}{h}\right|^2\right].$$

A similar identity holds for the continuous wave equation (3.3) in the minimal observability time $T = 2$. Namely

$$E = \frac{1}{4}\int_0^2 |\phi_x(1,t)|^2$$

for every solution ϕ of (3.3), where E is the energy of the solutions $\phi = \phi(x,t)$.

• In the case when $\mu < 1$ the gap between two consecutive eigenfrequencies decreases at high frequencies and it is of the order of Δx when $\Delta x \to 0$. Indeed, we have

$$\begin{aligned}
&|\lambda_{k+1} - \lambda_k| = \left| \frac{2}{\Delta t} \left[\arcsin\left(\frac{\Delta t}{\Delta x} \sin \frac{(k+1)\pi\Delta x}{2} \right) - \arcsin\left(\frac{\Delta t}{\Delta x} \sin \frac{k\pi\Delta x}{2} \right) \right] \right| \\
&\leq \left| \frac{\pi}{2} \frac{2}{\Delta t} \frac{\Delta t}{\Delta x} \left(\sin \frac{(k+1)\pi\Delta x}{2} - \sin \frac{k\pi\Delta x}{2} \right) \right| \\
&= \left| \frac{\pi}{2} \frac{2}{\Delta x} \left[\sin \frac{k\pi\Delta x}{2} \left(\cos \frac{\pi\Delta x}{2} - 1 \right) + \sin \frac{\pi\Delta x}{2} \cos \frac{k\pi\Delta x}{2} \right] \right| \\
&\leq \left| \frac{\pi}{2} \frac{2}{\Delta x} \left| 1 - \cos \frac{\pi\Delta x}{2} \right| + \frac{\pi^2}{2} \cos \frac{k\pi\Delta x}{2} \right| \\
&= \left| \frac{\pi}{2} \frac{2}{\Delta x} 2 \sin^2 \frac{\pi\Delta x}{4} + \frac{\pi^2}{2} \cos \frac{k\pi\Delta x}{2} \right| \leq \left| \frac{\pi^2}{2} \left[\frac{\pi\Delta x}{4} + \sin\left(\frac{((N+1)-k)\Delta x\pi}{2} \right) \right] \right|.
\end{aligned}$$

In particular, the gap for the highest frequencies satisfies

$$|\lambda_N - \lambda_{N-1}| \leq \frac{\pi^2}{2} \left(\frac{\pi\Delta x}{4} + \frac{\pi\Delta x}{2} \right) = \frac{3\pi^3\Delta x}{8} \to 0, \text{ when } \Delta x \to 0.$$

So the uniform gap condition (2.1) is not satisfied and we cannot directly apply Theorem 2.1 to prove inequality (3.28). Therefore, as soon as $\mu < 1$, we are in the same situation as for the semi-discrete equation (3.9) in which $\mu = 0$: the lack of spectral gap may produce the degeneracy of the observability constant.

To remedy this lack of uniform estimates, we need to introduce a subclass of solutions of system (3.24) where the high frequency components have been filtered. To do that, given $\alpha \in (0,1)$, the so-called *filtering parameter*, we consider the class $\mathcal{C}_\alpha(\Delta x)$,

$$\mathcal{C}_\alpha(\Delta x) = \left\{ \bar{\phi}^n \text{ sol. of (3.24) } : \bar{\phi}^n = \sum_{k=-\alpha N, k\neq 0}^{\alpha N} a_k e^{i\lambda_k n\Delta t} \bar{\varphi}_{|k|} \right\}, \tag{3.31}$$

of solutions involving the eigenvalues $\{\lambda_k\}_{k\in[-\alpha N,\alpha N]}$, $k \neq 0$:

$$\bar{\phi}^n = \sum_{k=-\alpha N, k\neq 0}^{\alpha N} a_k e^{i\lambda_k n\Delta t} \bar{\varphi}_{|k|}. \tag{3.32}$$

Let us first check the gap condition. We have

(3.33)
$$\begin{aligned}
\lambda_{k+1} - \lambda_k &= \frac{2}{\Delta t} \left[\arcsin\left(\frac{\Delta t}{\Delta x} \sin \frac{(k+1)\pi\Delta x}{2} \right) - \arcsin\left(\frac{\Delta t}{\Delta x} \sin \frac{k\pi\Delta x}{2} \right) \right] \\
&= \frac{\pi \cos \frac{\xi\Delta x}{2}}{\sqrt{1 - \left(\frac{\Delta t}{\Delta x} \sin \frac{\xi\Delta x}{2} \right)^2}} := \gamma_k,
\end{aligned}$$

for every $k \in [-\alpha N, \alpha N]$ and for some $\xi \in [k\pi, (k+1)\pi]$. Therefore, in particular,

$$\lambda_{k+1} - \lambda_k \geq \frac{\pi \cos \frac{N\alpha\pi\Delta x}{2}}{\sqrt{1 - \left(\frac{\Delta t}{\Delta x} \sin \frac{\xi \Delta x}{2}\right)^2}} \geq \pi \cos \frac{N\alpha\pi\Delta x}{2} \geq \pi(1-\alpha).$$

Consequently, for any filtering parameter $\alpha \in (0,1)$, the gap condition (2.1) holds with

$$\gamma_\alpha := \min_{|k| \leq \alpha N} (\gamma_k) \geq \gamma(\alpha) = \pi \cos\left(\frac{N\alpha\pi\Delta x}{2}\right) \geq \pi(1-\alpha). \tag{3.34}$$

On the other hand, by the mean value theorem,

(3.35)

$$\begin{aligned} |\lambda_k - \lambda_l| &= \left| \frac{2}{\Delta t} \left(\arcsin\left(\frac{\Delta t}{\Delta x} \sin\left(\frac{k\pi\Delta x}{2}\right)\right) - \arcsin\left(\frac{\Delta t}{\Delta x} \sin\left(\frac{l\pi\Delta x}{2}\right)\right)\right)\right| \\ &= \left| \frac{2}{\Delta t} \frac{\frac{\Delta t}{\Delta x} \frac{\pi\Delta x}{2} \cos\left(\frac{\xi\pi\Delta x}{2}\right)(k-l)}{\sqrt{1 - \left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2 \frac{\xi\pi\Delta x}{2}}} \right| \leq \left| \frac{2N\alpha\pi \cos\left(\frac{\xi\pi\Delta x}{2}\right)}{\sqrt{1 - \left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2 \frac{\xi\pi\Delta x}{2}}} \right| \\ &\leq \left| \frac{2N\alpha\pi \cos\left(\frac{\xi\pi\Delta x}{2}\right)}{\sqrt{\left(\frac{\Delta t}{\Delta x}\right)^2 - \left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2 \frac{\xi\pi\Delta x}{2}}} \right| = \frac{2N\alpha\pi\Delta x}{\Delta t} = \frac{2\alpha\pi - 2\alpha\pi\Delta x}{\Delta t} \\ &\leq \frac{2\pi\alpha(1-\Delta t)}{\Delta t}. \end{aligned}$$

In view of (3.35), by choosing conveniently the filtering parameter α such that

$$\alpha \leq \alpha^*(\Delta t) := \frac{2\pi - (\Delta t)^p}{2\pi(1-\Delta t)}, \tag{3.36}$$

with $0 \leq p < 1/2$, hypothesis (2.2) of Theorem 2.1 is verified.

In practice it is convenient to fix the filtering parameter $0 < \alpha < 1$, independent of Δt. In this way (3.36) is automatically satisfied for Δt small enough, which is the relevant case in numerical approximation problems. On the other hand the gap condition (3.34) is also automatically and uniformly satisfied for the truncated sequence $\{\lambda_k\}_{|k| \leq N\alpha}$.

Note that the gap γ_α (respectively, the minimal observability/ control time $2\pi/\gamma_\alpha$) tends to π (respectively, to 2) when $\alpha \searrow 0^+$ while it converges to zero (respectively, to infinity) when $\alpha \nearrow 1^-$.

Note also that the minimal observability/control time can be taken to be any $T > 2\pi/\gamma_\alpha$ since the minimal time $T(\alpha) = 2\pi/\gamma_\alpha + \epsilon(\Delta t)$ tends to $2\pi/\gamma_\alpha$ as Δt tends to zero.

More precisely, the following theorem holds.

THEOREM 3.1. *For all Courant numbers $0 < \mu < 1$ and all values of the filtering parameter $0 < \alpha < 1$, the observability inequality below holds that*

$$E_0 \leq \frac{1}{2\cos^2 \frac{\alpha\pi}{2} C_1(T, \gamma_\alpha)} \left[\Delta t \sum_{n=0}^{M} \left| \frac{\phi_N}{\Delta x} \right|^2 \right] \tag{3.37}$$

for every solution of (3.24) *in the class $\mathcal{C}_\alpha(\Delta x)$, uniformly as $(\Delta t, \Delta x) \to (0,0)$ for any $T > T(\alpha) = 2\pi/\gamma_\alpha$, with $C_1(T, \gamma_\alpha)$ given by* (1.3). *Moreover,*

1. *$T(\alpha) \nearrow \infty$ as $\alpha \nearrow 1^-$ and $T(\alpha) \searrow 2$ as $\alpha \searrow 0^+$.*
2. *$C_\alpha(T) := \dfrac{1}{2\cos^2\frac{\alpha\pi}{2} C_1(T,\gamma_\alpha)} \searrow C(T) = \dfrac{1}{2C_1(T,\gamma)}$ as $\alpha \searrow 0^+$ with $C_1(T,\gamma)$ given by* (1.3)*, where $C(T)$ is the constant of the continuous observability inequality* (3.5).

Remark 3.2. This theorem allows the recovery of the uniform observability of the original system (3.3) as the limit when $(\Delta t, \Delta x) \to (0,0)$ of the observability of the solutions of discrete one (3.24) in the classes (3.31) by means of Fourier filtering; the statements in this theorem coincide with the predictions one may deduce from the analysis of the dispersion diagram of the numerical scheme [28], as we shall see in the next section.

Proof (Sketch of the proof). The energy of the solutions (3.26) of the discrete system (3.24), concentrated on $x = 1$ is given by (3.29) and the total energy (3.25) of the solutions is

$$E_0 = \frac{2}{(\Delta x)^2} \sum_k a_k^2 \sin^2 \frac{k\pi\Delta x}{2} = \frac{2}{(\Delta x)^2} \sum_k a_k^2 \frac{\sin^2(k\pi\Delta x)}{4\cos^2\frac{k\pi\Delta x}{2}} = \frac{1}{2} \sum_k |\, m_k \,|^2 \, \frac{1}{\cos^2\frac{k\pi\Delta x}{2}},$$

where $m_k = \sin(Nk\pi\Delta x)/\Delta x$.

For all $k \in [-\alpha N, \alpha N]$ we have $\cos(\alpha\pi/2) \le \cos(\alpha N\pi\Delta x/2) \le \cos(k\pi\Delta x/2) \le 1$ and, in this case,

$$\frac{1}{2}\sum_k |\, m_k \,|^2 \le E_0 \le \frac{1}{2\cos^2\frac{N\alpha\pi\Delta x}{2}} \sum_k |\, m_k\,|^2 \le \frac{1}{2\cos^2\frac{\alpha\pi}{2}} \sum_k |\, m_k \,|^2. \tag{3.38}$$

Applying Theorem 2.1 and the Fourier representation (3.32) of the solutions we obtain that, for all $T > 2\pi/\gamma_\alpha + \epsilon(\Delta t)$, there exist positive constants $C_j(\Delta t, T, \gamma_\alpha)$, $j = 1, 2$, such that

$$C_1(\Delta t, T, \gamma_\alpha) \sum_{k=-\alpha N}^{\alpha N} |m_k|^2 \le \Delta t \sum_{n=0}^{M} \left| \sum_{k=-\alpha N}^{\alpha N} m_k e^{in\Delta t \lambda_k} \right|^2 \le C_2(\Delta t, T, \gamma_\alpha) \sum_{k=-N}^{N} |m_k|^2.$$

Therefore, for every α as in (3.36), by (3.38), the following inequalities hold:

$$2\cos^2\frac{\alpha\pi}{2} C_1(\Delta t, T, \gamma_\alpha) E_0 \le \Delta t \sum_{n=0}^{M} \left| \frac{\phi_N}{\Delta x} \right|^2 \le 2C_2(\Delta t, T, \gamma_\alpha) E_0, \tag{3.39}$$

with $C_j(\Delta t, T, \gamma_\alpha)$, $j = 1, 2$, defined by relations (2.4), for every truncated solution (3.32) of system (3.24) belonging to the class $\mathcal{C}_\alpha(\Delta x)$. ∎

The uniform observability inequality (3.39) implies uniform controllability results, as we shall prove in the next section, for the projection (over the subspace of unfiltered Fourier components) of solutions of the dual controlled system (3.23). In the limit as $\Delta t, \Delta x \to 0$ one recovers the sharp controllability results of the wave equation (3.1). For the details of the proof of convergence of controls we refer to [20] where the case $\Delta t = \Delta x$ was studied in detail. But, as mentioned above, for this particular one, because of the orthogonality of complex harmonic polynomials, the discrete Ingham inequality is not needed. We also refer to [16] where the convergence of controls for the semi-discretizations of the beam equation was analyzed in detail.

The usual centered finite-difference approximation of the wave equation we have considered here is only a simple example in which the discrete Ingham's theorem can be applied, together with some filtering mechanism, to get uniform observability inequalities. The discrete Ingham inequality can also be applied, for instance, to the implicit fully finite difference approximation of the wave equation, introduced in [18].

4. Uniform controllability of the filtered solutions. In this section, we apply the uniform observability results obtained above to analyze the controllability properties of the fully discrete system (3.23).

Let us define the Hilbert spaces of square summable sequences $\hbar^1$ and $\hbar^{-1}$ as follows:

$$\hbar^1 = \left\{ \{a_k\} \in \ell^2 \ : \quad \|a_k\|^2_{\hbar^1} = \sum_{k\in\mathbb{N}} | \, k\pi a_k \, |^2 < \infty \right\}, \tag{4.1}$$

$$\hbar^{-1} = \left\{ \{a_k\} \in \ell^2 \ : \quad \|a_k\|^2_{\hbar^{-1}} = \sum_{k\in\mathbb{N}} \left| \frac{a_k}{k\pi} \right|^2 < \infty \right\}, \tag{4.2}$$

where the discrete space ℓ^2 is given by (1.5).

For every $\alpha \in (0,1)$, we introduce the space $\mathcal{S}_\alpha$ generated by the eigenvectors $(\bar{\varphi}_k)$ involved in $\mathcal{C}_\alpha(\Delta x)$ of the filtered solutions of the homogeneous system (3.24) with filtering parameter α:

$$\mathcal{S}_\alpha = span \left\{ \bar{\varphi}_k : |k| \leq \alpha N \right\}. \tag{4.3}$$

For every $s \in \mathbb{R}$, we denote by $\hbar^s_{\Delta x,\alpha}$ the space S_α endowed with the norm

$$\|v\|^2_{s,\Delta x} = \sum_{|k|\leq N\alpha} \lambda_k^s \left| a_k \right|^2, \quad \text{for} \quad v \in S_\alpha : v = \sum_{|k|\leq N\alpha} a_k \bar{\varphi}_k,$$

where λ_k are as in (3.27).

For every $\alpha \in (0,1)$ and $T > 0$, we consider the partial controllability problem for system (3.23) in the space $\ell^2 \times \hbar^{-1}$, which consists of finding a control $\bar{v}^n \in \mathbb{R}^M$ such that, for all initial data $(\bar{u}^0, \bar{u}^1) \in \ell^2 \times \hbar^{-1}$, the solution $\bar{u}^n$ of (3.23) satisfies

$$(\Pi_\alpha \bar{u}^M, \Pi_\alpha \bar{u}^{M+1}) = (0,0), \tag{4.4}$$

where Π_α is the orthogonal projection over S_α; i.e.,

$$\left(\Pi_\alpha \bar{u}^M, \Pi_\alpha \bar{u}^{M+1}\right) = \left(\sum_{|k|\leq N\alpha} c_k \bar{\varphi}_k, \sum_{|k|\leq N\alpha} d_k \bar{\varphi}_k \right),$$

where (c_k) and (d_k) are the Fourier coefficients of $(\bar{u}^M, \bar{u}^{M+1})$ in the basis of the eigenvectors $(\bar{\varphi}_k)_k$. Observe that we only require to control uniformly the projection Π_α of the solutions of the discrete system (3.23) over subspaces in which the high frequencies have been filtered.

As we shall see this result is a consequence of the partial observability results of the previous section in the class of filtered solutions $\mathcal{C}_\alpha(\Delta x)$.

Multiplying the first equation in (3.23) by an arbitrary solution $\bar{\phi}^n$ of (3.24) and adding in j and n, we get

$$\Delta t \sum_{n=1}^{M} v^n_{\Delta x} \frac{\phi^n_N}{\Delta x} + \frac{1}{\mu} \sum_{j=0}^{N} \left[u^1_j \phi^0_j - u^0_j \phi^1_j \right] = \frac{1}{\mu} \sum_{j=0}^{N} \left[u^{M+1}_j \phi^M_j - u^M_j \phi^{M+1}_j \right]. \tag{4.5}$$

The solution of system (3.23) may be characterized through a transposition argument based on the identity above. Indeed, given $M,\ N \in \mathbb{N}$, $\bar{v}_{\Delta t} \in \mathbb{R}^M$, and $(\bar{u}^0, \bar{u}^1) \in \mathbb{R}^N \times \mathbb{R}^N$, $\{\bar{u}^n\}$ solves (3.23) if for every $s \in [1, M]$ it holds that

$$L_s(\bar{\phi}^0, \bar{\phi}^1) = \frac{1}{\mu} \sum_{j=1}^{N} \left[u_j^s \phi_j^{s+1} - u_j^{s+1} \phi_j^s \right],$$

or equivalently

$$L_s(\bar{\phi}^0, \bar{\phi}^1) = \frac{1}{\mu} \left(\bar{u}^s, \bar{\phi}^{s+1} \right)_{\mathbb{R}^N} + \frac{1}{\mu} \left(\bar{u}^{s+1}, -\bar{\phi}^s \right)_{\mathbb{R}^N}, \tag{4.6}$$

for every solution $\{\bar{\phi}^n\}$ of the discrete problem (3.24), where the functional $L_s \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is such that

$$L_s(\bar{\phi}^0, \bar{\phi}^1) = \frac{1}{\mu} \sum_{j=1}^{N} \left[u_j^0 \phi_j^1 - u_j^1 \phi_j^0 \right] - \Delta t \sum_{n=1}^{s} v_{\Delta x}^n \left[\frac{\phi_N^n}{\Delta x} \right].$$

The projection $\Pi_\alpha \bar{u}^n$ may be characterized by the same variational formulation (4.6), with the only difference being that the test functions in (4.6) are solutions of (3.24) in the class $\mathcal{C}_\alpha(\Delta x)$ (3.31).

Remark 4.1. Identity (4.4) is equivalent, by (4.5), to

$$\Delta t \sum_{n=1}^{M} v_{\Delta x}^n \left[\frac{\phi_N^n}{\Delta x} \right] = \frac{1}{\mu} \sum_{j=0}^{N} \left[u_j^0 \phi_j^1 - u_j^1 \phi_j^0 \right], \tag{4.7}$$

where $(\bar{\phi}^0, \bar{\phi}^1)$ are the initial data corresponding to the solution $\bar{\phi}^n \in \mathcal{C}_\alpha(\Delta x)$ of the discrete system (3.24).

Now let $\Delta x = 1/q$, $\Delta t = \mu/q$, $N = q-1$, for some $q \in \mathbb{N}$ and $\mu < 1$. We have the following uniform (with respect to $(\Delta t, \Delta x) \to (0,0)$) partial controllability property.

THEOREM 4.2. *Let $0 < \mu < 1$ and let us fix an arbitrary value of the filtering parameter $0 < \alpha < 1$. For every $T > T(\alpha) = 2\pi/\gamma_\alpha$, the system (3.24) is partially controllable on $\ell^2 \times \hbar^{-1}$ with controls $\bar{v}_{\Delta t}^n \in \mathbb{R}^M$ when $M = [Tq/\mu - 1]$. Moreover, the controls of minimal norm are uniformly bounded with respect to Δt. More precisely*

$$\left[\Delta t \sum_{n=0}^{M} |v_{\Delta x}^n|^2 \right]^{1/2} \leq C \| (\bar{u}^1, -\bar{u}^0) \|_{\hbar^{-1} \times \ell^2}, \tag{4.8}$$

where $C = C(T, \gamma_\alpha) > 0$ is a constant independent of $\Delta t \in (0,1)$.

Proof. Let $(\bar{\phi}^n) \in \mathcal{C}_\alpha(\Delta x)$ be the solution of (3.24) with initial data $(\bar{\phi}^0, \bar{\phi}^1) \in S_\alpha \times S_\alpha$ and define the convex quadratic functional $J_{\Delta x} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, by

$$J_{\Delta x}(\bar{\phi}^0, \bar{\phi}^1) = \frac{\Delta t}{2} \sum_{n=0}^{M} \left| \frac{\phi_N^n}{\Delta x} \right|^2 - \frac{1}{\mu^2} \Delta x \sum_{j=0}^{N} \left(u_j^0 \frac{\phi_j^1 - \phi_j^0}{\Delta t} - \frac{u_j^1 - u_j^0}{\Delta t} \phi_j^0 \right). \tag{4.9}$$

For every $\bar{\phi}^n \in \mathcal{C}_\alpha(\Delta x)$ we have

$$\begin{aligned} \left| \sum_{j=0}^{N} \left(u_j^0 \phi_j^1 - u_j^1 \phi_j^0 \right) \right| &= \left| \left(\Pi_\alpha \bar{u}^1, \bar{\phi}^0 \right)_{\mathbb{R}^N} \right| + \left| \left(\Pi_\alpha \bar{u}^0, -\bar{\phi}^1 \right)_{\mathbb{R}^N} \right| \\ &\leq \| \Pi_\alpha \bar{u}^1 \|_{\hbar^{-1}} \| \bar{\phi}^0 \|_{\hbar^1} + \| \Pi_\alpha \bar{u}^0 \|_{\ell^2} \| \bar{\phi}^1 \|_{\ell^2} \\ &\leq \| \Pi_\alpha (\bar{u}^1, -\bar{u}^0) \|_{\hbar^{-1} \times \ell^2} \| (\bar{\phi}^0, \bar{\phi}^1) \|_{\hbar^1 \times \ell^2}. \end{aligned} \tag{4.10}$$

According to (4.10) and the direct observability inequality (the right-hand side term in (3.39)) we deduce that $J_{\Delta x}$ is continuous.

On the other hand, according to the observability inequality (3.37), $J_{\Delta x}$ is uniformly coercive in $\mathcal{C}_\alpha(\Delta x)$,

$$\left|J_{\Delta x}(\bar\phi^0,\bar\phi^1)\right| \ge \left\|(\bar\phi^0,\bar\phi^1)\right\|_{\hbar^1\times\ell^2}\left[C_1(T,\gamma_\alpha)\left\|(\bar\phi^0,\bar\phi^1)\right\|_{\hbar^1\times\ell^2} - \|\Pi_\alpha(\bar u^1,-\bar u^0)\|_{\hbar^{-1}\times\ell^2}\right]. \tag{4.11}$$

Thus, there exists a unique minimizer $(\hat\phi^0,\hat\phi^1)$ of $J_{\Delta x}$,

$$J_{\Delta x}(\hat\phi^0,\hat\phi^1) = \min_{(\bar\phi^0,\bar\phi^1)\in S_\alpha\times S_\alpha} J_{\Delta x}(\bar\phi^0,\bar\phi^1).$$

Let $\hat\phi^n \in \mathcal{C}_\alpha(\Delta x)$ be the solution of the adjoint problem (3.24) with this minimizer as initial datum.

The pair $(\hat\phi^0,\hat\phi^1)$ satisfies the Euler–Lagrange equation

$$\Delta t\sum_{n=0}^{M}\frac{\hat\phi_N^n}{\Delta x}\frac{\phi_N^n}{\Delta x} = \frac{1}{\mu}\sum_{j=0}^{N}\left[u_j^0\phi_j^1 - u_j^1\phi_j^0\right], \tag{4.12}$$

for every initial data $(\bar\phi^0,\bar\phi^1)\in S_\alpha\times S_\alpha$ associated to the solution $\bar\phi^n\in\mathcal{C}_\alpha(\Delta x)$ of (3.24). Therefore, according to (4.7), the control we were looking for is $v_{\Delta x}^n = \hat\phi_N^n/\Delta x$.

To conclude the proof we check the uniform boundedness of the controls $v_{\Delta x}^n$. We have

$$J_{\Delta x}((\hat\phi^0,\hat\phi^1)) \le J_{\Delta x}(0,0) = 0,$$

and, by (4.10), this implies

$$\frac{\Delta t}{2}\sum_{n=0}^{M}\left|\frac{\hat\phi_N^n}{\Delta x}\right|^2 \le \|\Pi_\alpha(\bar u^1,-\bar u^0)\|_{\hbar^{-1}\times\ell^2}\|(\bar\phi^0,\bar\phi^1)\|_{\hbar^1\times\ell^2}. \tag{4.13}$$

The discrete energy E_0 of a solution $\bar\phi^n$ of (3.24) with initial data $(\bar\phi^0,\bar\phi^1)$ satisfies

$$E_0 = \frac{1}{2}\|(\bar\phi^0,\bar\phi^1)\|^2_{\hbar^1\times\ell^2}.$$

Now, using the Fourier development (3.32) of the solution $\bar\phi^n$ and applying the observability inequality (3.37) we get

$$\begin{aligned}\|(\bar\phi^0,\bar\phi^1)\|^2_{\hbar^1\times\ell^2} = 2E_0 &\le \frac{1}{\cos^2\frac{N\alpha\pi\Delta x}{2}C_1(T,\gamma_\alpha,\Delta t)}\Delta t\sum_{n=0}^{M}\left|\frac{\hat\phi_N^n}{\Delta x}\right|^2\\ &\le \frac{1}{\cos^2\frac{\alpha\pi}{2}C_1(T,\gamma_\alpha)}\Delta t\sum_{n=0}^{M}\left|\frac{\hat\phi_N^n}{\Delta x}\right|^2.\end{aligned} \tag{4.14}$$

Therefore, in (4.13) we obtain

$$\left[\Delta t\sum_{n=0}^{M}\left|\frac{\hat\phi_N^n}{\Delta x}\right|^2\right]^{1/2} \le \frac{1}{\sqrt{\cos^2\frac{\alpha\pi}{2}C_1(T,\gamma_\alpha)}}\|\Pi_\alpha(\bar u^1,-\bar u^0)\|_{\hbar^{-1}\times\ell^2}, \tag{4.15}$$

and then, the discrete controls $v^n_{\Delta x} = \hat{\phi}^n_N/\Delta x$ satisfy

$$\left[\Delta t \sum_{n=0}^{M} |v^n_{\Delta x}|^2\right]^{1/2} \leq C(T, \gamma_\alpha)\|\Pi_\alpha(\bar{u}^1, -\bar{u}^0)\|_{\hbar^{-1}\times\ell^2}, \tag{4.16}$$

as stated above. ∎

Remark 4.3. Note that, with the notations (3.32), the controls $(v^n_{\Delta x})$ are of the form

$$v^n_{\Delta x} = -\frac{\mu}{\Delta t}\sum_{k=-N\alpha}^{N\alpha} \cos(k\pi)\sin(k\pi\Delta t/\mu)\hat{a}_k e^{i\lambda_k n\Delta t}, \tag{4.17}$$

where $(\hat{a}_k)_k$ are the Fourier coefficients of the solution $\hat{\phi}^n \in \mathcal{C}_\alpha(\Delta x)$ of the adjoint problem (3.24), with initial data $(\hat{\phi}^0, \hat{\phi}^1)$ being the minimizer of the functional $J_{\Delta x}$.

Now we show the convergence of the controls $v^n_{\Delta x}$ of the discrete system (3.23) to the HUM control of the continuous one (3.1), as $\Delta t, \Delta x \to 0$.

Given an initial state $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ of the continuous system (3.1), we develop it in Fourier series

$$(u^0, u^1) = \sum_{k=1}^{\infty}(c_k, d_k)\varphi_k(x), \tag{4.18}$$

with

$$\sum_{k\in\mathbb{N}}\left[|c_k|^2 + \left|\frac{d_k}{k\pi}\right|^2\right] < \infty. \tag{4.19}$$

We now construct the initial states for the discrete system (3.23) by setting

$$(\bar{u}^0, \bar{u}^1) = \sum_{k=1}^{N}(c_k, c_k\cos(\lambda_k n\Delta t) + \frac{d_k}{\lambda_k}\sin(\lambda_k n\Delta t))\bar{\varphi}_k, \tag{4.20}$$

with λ_k given by (3.27). They may be rewritten as

$$(\bar{u}^0, \bar{u}^1) = \sum_{k=1}^{\infty}(c^N_k, c^N_k\cos(\lambda_k n\Delta t) + \frac{d^N_k}{\lambda_k}\sin(\lambda_k n\Delta t))\bar{\varphi}_k, \tag{4.21}$$

where

$$c^N_k = c_k\chi_N(k), \qquad d^N_k = d_k\chi_N(k),$$

χ_N being the characteristic function of the set $\{1, \dots, N\}$.

In view of Theorem 4.2, there exists a HUM control $(v^n_{\Delta x})$ for the discrete system (3.23), satisfying (4.4), with initial data (4.20).

Let us now prove that the sequence $(v^n_{\Delta x})$ converges (in a sense to be more precise below) to $v \in L^2(0,T)$, which is the HUM control for system (3.1) with initial data (4.18).

To better analyze the convergence of controls, we define the continuous extension of the discrete controls by setting

$$v_{\Delta x}(t) = -\frac{\mu}{\Delta t}\sum_{k\in\mathbb{Z}}\cos(k\pi)\sin(k\pi\Delta t/\mu)\hat{a}_k e^{i\lambda_k t},$$

where $\hat{a}_k$ are taken to be zero for $|k| > \alpha N$. This function, when restricted to the mesh, coincides with $(v^n_{\Delta x})$ (recall that $v^n_{\Delta x}$ is given by (4.17)).

The following convergence result holds.

THEOREM 4.4. *Let $\mu = \Delta t/\Delta x \leq 1$. Consider M, N as in Theorem 4.2. Fix $(u_0, u_1) \in L^2(0,1) \times H^{-1}(0,1)$ and consider the continuous and discrete controls v and $v_{\Delta x}$ as above, with the filtering parameter $\alpha \in (0,1)$ and $T > T_\alpha$. Then,*

$$v_{\Delta x}(\cdot) \to v(\cdot) \ \textit{strongly} \text{ in } \ L^2(0,T) \ \text{ as } \ \Delta t \to 0. \tag{4.22}$$

Proof (*Sketch of the proof*). In view of (4.8) it is easy to see that

$$\Delta t \sum_{n=0}^{M} |v^n_{\Delta x}|^2 \leq C, \tag{4.23}$$

and therefore

$$\int_0^T |v_{\Delta x}|^2 \, dt \leq C. \tag{4.24}$$

Then, up to the extraction of a subsequence that we still denote by $\{v_{\Delta x}\}_{\Delta t}$, we have

$$v_{\Delta x}(t) \rightharpoonup v(t) \ \text{ in } \ L^2(0,T) \ \text{ as } \ \Delta t \to 0. \tag{4.25}$$

By a Γ-convergence argument it can also be seen that the limit v is given by

$$v(t) = -\partial_x \hat{\phi}(1,t), \tag{4.26}$$

where $\hat{\phi}$ is the solution of the adjoint problem (3.3) with initial data $(\hat{\phi}^0, \hat{\phi}^1) \in H^1_0(0,1) \times L^2(0,1)$, the unique minimizer of the functional

$$J(\phi^0,\phi^1) = \frac{1}{2}\int_0^T \mid \partial_x\phi(1,t) \mid^2 dt - \int_0^1 u^0\phi^1 - < u^1, \phi^0 >_{-1,1} \tag{4.27}$$

in the energy space $H^1_0(0,1) \times L^2(0,1)$. By taking limits in (4.7) and thanks to the construction of the initial data to be controlled for the discrete system we obtain

$$0 = \int_0^1 \left[u^1(x)\phi^0(x) - u^0(x)\phi^1(x) dx\right] + \int_0^T v(t)\partial_x\phi(1,t)dt \tag{4.28}$$

and this latter condition is equivalent to the fact that v, the limit in (4.25), is a control for system (3.1), driving the initial data (u^0, u^1) to rest; i.e., $v \in L^2(0,T)$ is the control of minimal L^2-norm.

The limit v being identified in a unique way, we deduce that the whole sequence $v_{\Delta x}$ converges.

Moreover, by the hypotheses of Theorem 4.4, the linear term of the discrete functional $J_{\Delta x}$ in (4.9) converges to the linear term of the functional defined in (4.27). Therefore, proving (4.22) is equivalent to proving that

$$J_{\Delta x}(\hat{\phi}^0_{\Delta x}, \hat{\phi}^1_{\Delta x}) \to J(\hat{\phi}^0, \hat{\phi}^1), \ \text{ as } \ \Delta x \to 0,$$

where $(\hat{\phi}^0_{\Delta x}, \hat{\phi}^1_{\Delta x}) \in S_\alpha \times S_\alpha$ minimizes (4.9) and $(\hat{\phi}^0, \hat{\phi}^1) \in H^1_0(0,1) \times L^2(0,1)$ minimizes (4.27). Indeed, taking into account the convergence of the linear terms in this functional, and the structure of the functionals (4.9) and (4.27), we deduce the convergence of the norms of the controls that, together with the weak convergence, ensure strong convergence.

Thus, the controls $v_{\Delta x}$ and the controlled discrete solutions $u_{\Delta x}$ converge to the control and the controlled solution of the wave equation (3.1). It is important to note that the projections of the solutions of the controlled system end up covering the whole range of frequencies so that, in the limit, we recover the exact controllability property (3.2) of the continuous wave equation.

The details of the several steps of the proof are given in [19] and we omit them for brevity. □

5. Discrete Ingham inequalities and dispersion diagrams. In this section we discuss the observability results obtained in section 3 applying discrete Ingham inequalities in connection with the dispersion diagrams of the equations and numerical schemes under consideration. We also discuss the optimality of these results. First of all, we introduce and recall some classical concepts and notations.

Any time-dependent scalar, linear partial differential equation with constant coefficients admits plane wave solutions

$$\phi(x,t) = e^{i(\omega t - \xi x)}, \quad \xi \in \mathbb{R},\ \omega \in \mathbb{C}, \tag{5.1}$$

where ξ is the *wave number* and ω is the *frequency*. The relationship

$$\omega = \omega(\xi) \tag{5.2}$$

is known as the *dispersion relation* for the equation.

Any individual *"monochromatic wave"* (involving only one Fourier component) of (5.1) moves at the *phase velocity*

$$c(\xi,\omega) = \frac{\omega(\xi)}{\xi}. \tag{5.3}$$

When one superimposes two waves with nearby propagation velocities, there appear wave packets which can propagate with different velocities. The energy of wave packets propagates at the so-called *group velocity*

$$C(\xi,\omega) = \frac{d\omega(\xi)}{d\xi}. \tag{5.4}$$

In general, the dispersion relation for a partial differential equation is a polynomial relation between ξ and ω, while a discrete model amounts to a trigonometric approximation.

• *Continuous problem.* For the continuous wave equation (3.3) we have $\omega(\xi) = \xi$ and therefore $c(\xi) = C(\xi) = 1$.

• *Semi-discrete problem.* For the semi-discrete scheme (3.9) the dispersion relation is

$$\omega(\xi) = \frac{2}{\Delta x} \sin\frac{\xi \Delta x}{2}, \quad \xi \in \left[-\frac{\pi}{\Delta x}, \frac{\pi}{\Delta x}\right]. \tag{5.5}$$

Note that, at the semi-discrete level, each dispersion relation is $2\pi/\Delta x$–periodic in ξ, and it is natural to take $\xi \in [-\pi/\Delta x, \pi/\Delta x]$ as a fundamental domain.

The phase velocity is in this case

$$c(\xi,\omega)=\frac{2}{\xi\Delta x}\sin\frac{\xi\Delta x}{2}. \tag{5.6}$$

The corresponding group velocity is

$$C(\xi,\omega)=\frac{d\omega(\xi)}{d\xi}=\cos\frac{\xi\Delta x}{2}. \tag{5.7}$$

• *Discrete problem.* The same analysis can be developed for fully discrete schemes. Considering numerical plane waves $\phi_j^n=e^{i(\omega n\Delta t-\xi j\Delta x)}$, for system (3.24), one obtains the dispersion relation

$$\omega(\xi)=\frac{2}{\Delta t}\arcsin\left(\frac{\Delta t}{\Delta x}\sin\frac{\xi\Delta x}{2}\right). \tag{5.8}$$

It is $2\pi/\Delta x$-periodic in ξ and $2\pi/\Delta t$-periodic in ω.

• When $\Delta t=\Delta x$ we obtain

$$\omega(\xi)=\xi. \tag{5.9}$$

This case is particularly interesting since (5.9) coincides with the dispersion relation for the continuous weave equation. In this case, $c(\xi,\omega)=C(\xi,\omega)=1$ and the discrete waves propagate at a constant velocity identically equal to one, like in the continuous case. But, as we shall see, this is a completely exceptional situation.

• When $\mu<1$, the phase velocity is given by

$$c(\xi,\omega)=\frac{2}{\xi\Delta t}\arcsin\left(\frac{\Delta t}{\Delta x}\sin\frac{\xi\Delta x}{2}\right) \tag{5.10}$$

and the group velocity is

$$C(\xi,\omega)=\frac{d\omega(\xi)}{d\xi}=\frac{\cos\frac{\xi\Delta x}{2}}{\sqrt{1-\left(\frac{\Delta t}{\Delta x}\sin\frac{\xi\Delta x}{2}\right)^2}}. \tag{5.11}$$

For $\Delta t=0$ the phase and group velocities in (5.10) and (5.11), which depend on ξ, coincide with those of the semi-discrete case (5.6) and (5.7), respectively, as expected.

Note that, as $\Delta x\to 0$, for all ξ we have

$$C(\xi,\omega)\le\frac{\cos\frac{\xi\Delta x}{2}}{\sqrt{1-\left(\frac{\Delta t}{\Delta x}\right)^2}}\to 0$$

when $\xi=\pi/\Delta x$.

In Figures 1–4 we describe the evolution of the group velocity diagrams starting with the semi-discrete case ($\mu=0$) up to $\mu=1$, for fixed $\Delta x=0.001$.

In general, any discrete dynamics generates spurious high-frequency oscillations that do no exist at the continuous level [23, 25]. Moreover, the interaction of waves with the grid produces a dispersion phenomenon and the velocity of propagation of these high frequency numerical waves may converge to zero when the mesh-size tends to zero. These spurious oscillations weakly converge to zero. Consequently, their existence is compatible with the convergence of the numerical scheme for solving the initial-value problem. However, when we are dealing with the exact controllability or observability problems, a uniform time for the control of all numerical waves is needed. Since the velocity of propagation of some high frequency numerical waves may tend to zero as the mesh becomes finer and finer, uniform observability and therefore controllability properties of the discrete model may fail for all $T > 0$.

According to Theorem 2.1, the uniform gap between two consecutive eigenvalues is a sufficient (and actually also necessary) property for uniform (with respect to Δx and Δt) observability. On the other hand, the group velocity is the derivative of the eigenfrequencies λ_k and the spectral gap is, as we have seen, $\lambda_{k+1} - \lambda_k$. Both magnitudes are similar, and they become closer as $\Delta x \to 0$.

Thus, to efficiently observe at the point $x = 1$ a wave packet concentrated to the left of $x = 1$ that moves to the left (in the space variable) as t increases, and bounces back at $x = 0$ to eventually reach the observation point $x = 1$, the time needed is

$$T \geq 2/\min_{\xi}\{C(\xi,\omega)\}. \tag{5.12}$$

In the continuous case, (5.12) reduces to the well-known condition for observability $T \geq 2$ and it is uniform for all the frequencies. The minimal time $T = 2$ is the one one obtains in view of Ingham's theorem (1.2) because the gap is $\gamma = \pi$ in this case.

For the semi-discrete case, the observation time is

$$T \geq 2/\min_{\xi}(\cos(\xi\Delta x/2)). \tag{5.13}$$

But $\min_\xi(\cos(\xi\Delta x/2))$ is of the order of Δx, the same order as we have obtained in (3.18) for the spectral gap for the highest frequencies. Consequently, the observation time (5.13) diverges, $T \to \infty$, as $\Delta x \to 0$.

These facts confirm the necessity of filtering the high frequencies. Relation (5.13) shows that the time grows with the high frequencies, in the points where $\cos\xi\Delta x/2 \sim 0$ ($\xi \sim \pi/\Delta x$) and the same result is obtained applying the Ingham inequality.

For the fully discrete problem (3.24) the time needed for observation is

$$T \geq \max_{\xi} \frac{2\sqrt{1-\left(\frac{\Delta t}{\Delta x}\sin\frac{\xi\Delta x}{2}\right)^2}}{\cos\frac{\xi\Delta x}{2}}. \tag{5.14}$$

Passing to the limit in (5.14) as $\Delta t \to 0$ for fixed Δx, one obtains the same time as in the semi-discrete case (5.13). The observation time grows with the high frequencies, except for the very particular case $\Delta t = \Delta x$, where the time obtained in the previous section, using the orthogonality of the time exponentials, is $T = 2$, which coincides with the observation time given by the group velocity (5.14).

Summarizing, when $0 < \mu < 1$, the sequence of eigenvalues has no uniform gap and the observability time (5.14) tends to infinity. Therefore, as in the semi-discrete case, a suitable filtering of the spurious numerical high frequencies is necessary. Theorem 2.1 provides a sharp result in this direction and its main result coincides with the predictions one may do in view of the structure of the dispersion diagram.

6. Proof of the discrete Ingham inequality. The proof of Theorem 2.1 uses in an essential way some classical properties of the discrete Fourier transform. We recall these properties in subsection 6.1 following [23].

FIG. 1. *Group velocity for the semi-discrete (–) and discrete (- -) cases with* $\mu = \Delta t/\Delta x = 0.1$ *(left),* $\mu = \Delta t/\Delta x = 0.3$ *(middle),* $\mu = \Delta t/\Delta x = 0.5$ *(right),* $\Delta x = 0.001$.

FIG. 2. *Group velocity for the semi-discrete (–) and discrete (- -) cases with* $\mu = \Delta t/\Delta x = 0.9$ *(left),* $\mu = \Delta t/\Delta x = 0.999$ *(middle),* $\mu = \Delta t/\Delta x = 1$ *(right),* $\Delta x = 0.001$.

6.1. The discrete Fourier transform. Let $h > 0$ be a real number and let $\dots, x_{-1}, x_0, x_1, \dots$ be defined by $x_j = jh$. Thus $\{x_j\} = h\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. The ℓ_h^2-norm of a discrete function $\{v_j\}$ is defined as

$$\|v\|_h = \left[h \sum_{j=-\infty}^{\infty} |v_j|^2 \right]^{1/2}.$$

We denote by l_h^2 the Hilbert space $l_h^2 = \{v : \|v\|_h < \infty\}$, the space of discrete functions of finite $\|\cdot\|_h$ norm.

For any $v \in \ell_h^2$, the discrete Fourier transform of v is the function $\hat{v}$ defined by

$$\hat{v}(\xi) = h \sum_{j=-\infty}^{\infty} e^{-i\xi x_j} v_j, \qquad \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h} \right]. \tag{6.1}$$

This can be viewed as a discrete approximation of the continuous Fourier transform

$$\hat{u}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} u(x) dx, \qquad \xi \in \mathbb{R},$$

if $u = u(x)$ is a sufficiently smooth function such that $u(x_j) = v_j$.

A priori, the sum in (6.1) defines a function $\hat{v}(\xi)$ for all $\xi \in \mathbb{R}$. The function $\hat{v}(\xi)$ is $2\pi/h$ periodic on $\mathbb{R}$ and therefore we analyze it only for $\xi \in [-\pi/h, \pi/h]$ to avoid aliasing.

Let us recall a standard definition. A function u defined on $\mathbb{R}$ is said to have *bounded variation* if there is a constant M such that for any finite m and any points

FIG. 3. *Dispersion relation for the continuous (-.-), semi-discrete (- -) and discrete (–) cases with $\mu = \Delta t/\Delta x = 0.1$ (left) $\mu = \Delta t/\Delta x = 0.3$ (middle), $\mu = \Delta t/\Delta x = 0.5$ (right), $\Delta x = 0.001$.*

FIG. 4. *Dispersion relation for the continuous (-.-), semi-discrete (- -) and discrete (–) cases with $\mu = \Delta t/\Delta x = 0.9$ (left), $\mu = \Delta t/\Delta x = 0.999$ (middle), $\mu = \Delta t/\Delta x = 1$ (right), $\Delta x = 0.001$.*

$x_0 < x_1 < \cdots < x_m$,

$$\sum_{j=1}^{m} | u(x_j) - u(x_{j-1}) | \leq M.$$

Now we give a fundamental result (see [23, p. 96]) which describes the effect of discretization in the Fourier transform.

THEOREM 6.1. *Suppose that $u \in L^2(\mathbb{R})$ is a sufficiently smooth function defined on $\mathbb{R}$ and let $v \in \ell_h^2$ be the discretization obtained by sampling u at the grid points x_j, i.e., $u(x_j) = v_j$.*

Then, if u has $p-1$ continuous derivatives in $L^2(\mathbb{R})$ for some $p \geq 1$ and a p*th derivative in L^2 that has bounded variation, it follows that*

$$|\hat{v}(\xi) - \hat{u}(\xi)| = o(h^{p+1}), \quad \text{when } h \to 0, \tag{6.2}$$

uniformly on $\xi \in [-\pi/h, \pi/h]$.

Proof. Since u is continuous, apply *Poisson formula*

$$\hat{v}(\xi) = \sum_{j=-\infty}^{\infty} \hat{u}(\xi + 2\pi j/h), \qquad \xi \in [-\pi/h, \pi/h].$$

Thus, for every $u \in L^2(\mathbb{R})$ and $v \in \ell^2$, we obtain

$$\hat{v}(\xi) - \hat{u}(\xi) = \sum_{j=-\infty}^{\infty} \hat{u}(\xi + 2\pi j/h) - \hat{u}(\xi) = \sum_{j=1}^{\infty} \left[\hat{u}(\xi + 2\pi j/h) + \hat{u}(\xi - 2\pi j/h)\right],$$

with $\hat{u}$ and $\hat{v}$ the Fourier transforms of u and v, respectively.

If u verifies the hypothesis of Theorem 6.1, then

$$|\hat{u}(\xi)| \leq C_1 |\xi|^{-p-1}, \quad \text{when}\, \xi \to \infty$$

for some constant C_1. Therefore

$$|\hat{v}(\xi) - \hat{u}(\xi)| \leq C_1 \sum_{j=1}^{\infty} (j\pi/h)^{-p-1} = C_2 h^{p+1} \sum_{j=1}^{\infty} j^{-p-1}.$$

For every $p \geq 1$ this sum converges, which implies (6.2), as required. □

6.2. The discrete Fourier transform of Ingham's cut-off function. We study some general properties of a discrete Fourier transform that we shall use in the proof of the discrete Ingham inequality.

Given $M \in \mathbb{N}$ and $T > 0$ we consider the function $g : \mathbb{R} \to \mathbb{R}$

$$g(t) = \sin\left(\frac{t\pi}{T}\right) \chi_{(0,T)}, \tag{6.3}$$

where $\chi_{(0,T)}$ is the characteristic function of the interval $(0, T)$. Function (6.3) is precisely the same that Ingham [9] used in the proof of the continuous inequality (1.2). Its Fourier transform $G : \mathbb{R} \to \mathbb{R}$ is

$$G(\tau) = \int_{-\infty}^{\infty} g(t) e^{it\tau} dt = -2 \cos\frac{T\tau}{2} e^{\frac{iT\tau}{2}} \frac{\pi T}{(T^2\tau^2 - \pi^2)}. \tag{6.4}$$

We define the restriction of g to the grid

$$\ldots < t_{-1} < t_0 = 0 < t_1 < \cdots < t_{M+1} = T < \ldots,$$

with $t_n = n\Delta t$, i.e.,

$$h(n\Delta t) = g(t_n) = \sin(n\Delta t\pi/T)\chi_{M+1}(n),$$

χ_{M+1} being the characteristic function of the set $\{0, \ldots, M+1\}$.

For any $\tau \in \mathbb{R}$ we define the discrete Fourier transform of the discrete function h

$$H(\tau) := \Delta t \sum_{n=-\infty}^{\infty} h(n\Delta t) e^{in\Delta t\tau} \tag{6.5}$$

for all $\tau \in [-\pi/\Delta t, \pi/\Delta t]$.

Lemma 6.2. *For all Δt, $T > 0$ and $k \in \mathbb{Z}$ we have*

$$H(\tau) = -\frac{\Delta t \cos\frac{T\tau}{2} e^{\frac{iT\tau}{2}} \sin\frac{\Delta t\pi}{T}}{2\sin(\frac{\Delta t}{2T}(T\tau + \pi))\sin(\frac{\Delta t}{2T}(T\tau - \pi))} \tag{6.6}$$

for any $\tau \neq (2k\pi)/\Delta t \pm \pi/T$ with $k \in \mathbb{Z}$ and

$$H\left(\frac{2k\pi}{\Delta t} \pm \frac{\pi}{T}\right) = \mp\frac{T}{2i},\ k \in \mathbb{Z}. \tag{6.7}$$

The function H defined in (6.5) *is continuous and*

$$\lim_{\tau \to \frac{2k\pi}{\Delta t} \pm \frac{\pi}{T}} H(\tau) = \mp \frac{T}{2i}. \tag{6.8}$$

Proof (*Proof of Lemma* 6.2). We divide the proof into two steps: first, we prove that the explicit expression (6.5) of H is (6.6) and then we study the continuity of H.

• *Step* 1. *From the definition of the function H for all $\tau \neq 2k\pi/\Delta t \pm \pi/T$, $k \in \mathbb{Z}$, we have*

$$H(\tau) \quad = \Delta t \sum_{n=0}^{M} \sin \frac{n\pi\Delta t}{T} e^{in\Delta t \tau} = \Delta t \sum_{n=0}^{M} \frac{e^{\frac{in\pi\Delta t}{T}} - e^{-\frac{in\pi\Delta t}{T}}}{2i} e^{in\Delta t\tau}.$$

Hence

$$H(\tau) = \frac{\Delta t}{2i} \sum_{n=0}^{M} e^{\frac{in\Delta t}{T}(T\tau+\pi)} - \frac{\Delta t}{2i} \sum_{n=0}^{M} e^{\frac{in\Delta t}{T}(T\tau-\pi)}. \tag{6.9}$$

In order to obtain identity (6.5), it is useful to prove it only for any $|\tau| < 2k\pi/\Delta t - \pi/T$ with $k = 1$. Then, taking the periodicity properties of the complex exponentials into account, it is easy to obtain the same result for all $k \in \mathbb{Z}$, $\tau \neq 2k\pi/\Delta t \pm \pi/T$.

The first term on the right-hand side of (6.9) is

(6.10)

$$\begin{aligned}
\sum_{n=0}^{M} e^{\frac{in\Delta t}{T}(T\tau+\pi)} &= \frac{e^{\frac{i(M+1)\Delta t}{T}(T\tau+\pi)} - 1}{e^{\frac{i\Delta t}{T}(T\tau+\pi)} - 1} = \frac{e^{\frac{i(M+1)\Delta t}{T}(T\tau+\pi)} - 1}{\cos(\frac{\Delta t}{T}(T\tau+\pi)) + i\sin(\frac{\Delta t}{T}(T\tau+\pi)) - 1} \\
&= \frac{e^{i(T\tau+\pi)} - 1}{1 - 2\sin^2(\frac{\Delta t}{2T}(T\tau+\pi)) + 2i\sin(\frac{\Delta t}{2T}(T\tau+\pi))\cos(\frac{\Delta t}{2T}(T\tau+\pi)) - 1} \\
&= \frac{-e^{iT\tau} - 1}{2i\sin(\frac{\Delta t}{2T}(T\tau+\pi))(\cos(\frac{\Delta t}{2T}(T\tau+\pi)) + i\sin(\frac{\Delta t}{2T}(T\tau+1\pi)))} \\
&= \frac{-(e^{iT\tau}+1)e^{-\frac{i\Delta t}{2T}(T\tau+\pi)}}{2i\sin(\frac{\Delta t}{2T}(T\tau+\pi))}.
\end{aligned}$$

For the second one we have

(6.11)

$$\begin{aligned}
\sum_{n=0}^{M} e^{\frac{in\Delta t}{T}(T\tau-\pi)} &= \frac{e^{\frac{i(M+1)\Delta t}{T}(T\tau-\pi)} - 1}{e^{\frac{i\Delta t}{T}(T\tau-\pi)} - 1} = \frac{e^{\frac{i(M+1)\Delta t}{T}(T\tau-\pi)} - 1}{\cos\frac{\Delta t}{T}(T\tau-\pi) + i\sin\frac{\Delta t}{T}(T\tau-\pi) - 1} \\
&= \frac{e^{i(T\tau-\pi)} - 1}{-2\sin^2\frac{\Delta t}{2T}(T\tau-\pi) + 2i\sin\frac{\Delta t}{2T}(T\tau-\pi)\cos\frac{\pi\Delta t}{2T}(T\tau-\pi)} \\
&= \frac{-e^{iT\tau} - 1}{2i\sin\frac{\Delta t}{2T}(T\tau-\pi)e^{\frac{i\Delta t}{2T}(T\tau-\pi)}} = \frac{-(e^{iT\tau}+1)e^{-\frac{i\Delta t}{2T}(T\tau-\pi)}}{2i\sin\frac{\Delta t}{2T}(T\tau-\pi)}.
\end{aligned}$$

Substituting (6.10) and (6.11) into (6.9) we obtain

(6.12)

$$\begin{aligned}
&H(\tau)\\
&= \frac{-\Delta t}{4}\left[\frac{(e^{iT\tau}+1)e^{-\frac{i\Delta t}{2T}(T\tau-\pi)}}{\sin\frac{\Delta t}{2T}(T\tau-\pi)} - \frac{(e^{iT\tau}+1)e^{-\frac{i\Delta t}{2T}(T\tau+\pi)}}{\sin(\frac{\Delta t}{2T}(T\tau+\pi))}\right]\\
&= \frac{-\Delta t(e^{iT\tau}+1)}{4}\left(\frac{\cos\frac{\Delta t}{2T}(T\tau-\pi) - i\sin\frac{\Delta t}{2T}(T\tau-\pi)}{\sin\frac{\Delta t}{2T}(T\tau-\pi)}\right.\\
&\left.-\frac{\cos\frac{\Delta t}{2T}(T\tau+\pi) - i\sin(\frac{\Delta t}{2T}(T\tau+\pi))}{\sin(\frac{\Delta t}{2T}(T\tau+\pi))}\right)\\
&= \frac{-\Delta t(e^{iT\tau}+1)}{4}\left(\frac{\cos\frac{\Delta t}{2T}(T\tau-\pi)}{\sin\frac{\Delta t}{2T}(T\tau-\pi)} - \frac{\cos\frac{\Delta t}{2T}(T\tau+\pi)}{\sin(\frac{\Delta t}{2T}(T\tau+\pi))}\right)\\
&= \frac{-\Delta t(e^{iT\tau}+1)}{4}\left(\frac{\cos\frac{\Delta t}{2T}(T\tau-\pi)\sin\frac{\Delta t}{2T}(T\tau+\pi) - \cos\frac{\Delta t}{2T}(T\tau+\pi)\sin\frac{\Delta t}{2T}(T\tau-\pi)}{\sin\frac{\Delta t}{2T}(T\tau-\pi)\sin(\frac{\Delta t}{2T}(T\tau+\pi)}\right)\\
&= \frac{-\Delta t(e^{iT\tau}+1)}{4\sin\frac{\Delta t}{2T}(T\tau-\pi)\sin\frac{\pi\Delta t}{2T}(T\tau+\pi)}\sin\frac{\pi\Delta t}{T}.
\end{aligned}$$

Therefore, applying Euler's formula in (6.12) we obtain the following expression for H*:*

$$\begin{aligned}
H(\tau) &= \frac{-\Delta t\,(\cos(T\tau)+i\sin(T\tau)+1)\sin\frac{\pi\Delta t}{T}}{4\sin\frac{\Delta t}{2T}(T\tau-\pi)\sin(\frac{\Delta t}{2T}(T\tau+\pi))}\\
&= \frac{-\Delta t\left(2\cos^2\frac{T\tau}{2}-1+2i\sin\frac{T\tau}{2}\cos\frac{T\tau}{2}+1\right)\sin\frac{\pi\Delta t}{T}}{4\sin\frac{\Delta t}{2T}(T\tau-\pi)\sin(\frac{\Delta t}{2T}(T\tau+\pi))}\\
&= \frac{-2\Delta t\cos\frac{T\tau}{2}\left(\cos\frac{T\tau}{2}+i\sin\frac{T\tau}{2}\right)\sin\frac{\pi\Delta t}{T}}{4\sin\frac{\Delta t}{2T}(T\tau-\pi)\sin(\frac{\Delta t}{2T}(T\tau+\pi)}\\
&= \frac{-\Delta t\cos\frac{T\tau}{2}e^{\frac{iT\tau}{2}}\sin\frac{\pi\Delta t}{T}}{2\sin\frac{\Delta t}{2T}(T\tau-\pi)\sin(\frac{\pi\Delta t}{2T}(T\tau+\pi))}.
\end{aligned} \tag{6.13}$$

Moreover, if $\tau = 2k\pi/\Delta t + \pi/T$*, with* $k \in \mathbb{Z}$*, using the definition (6.5) of* H*, we deduce that*

$$\begin{aligned}
H(\tau) &= \Delta t\sum_{n=0}^{M}\sin\frac{n\pi\Delta t}{T}e^{in\Delta t(\frac{2k\pi}{\Delta t}+\frac{\pi}{T})}\\
&= \Delta t\sum_{n=0}^{M}\frac{e^{\frac{in\Delta t\pi}{T}} - e^{-\frac{in\Delta t\pi}{T}}}{2i}e^{2k\pi in}e^{\frac{in\Delta t\pi}{T}} = \frac{\Delta t}{2i}\sum_{n=0}^{M}e^{\frac{2in\Delta t\pi}{T}} - \frac{\Delta t}{2i}\sum_{n=0}^{M}1\\
&= \frac{\Delta t}{2i}\frac{e^{\frac{2i(M+1)\Delta t\pi}{T}}-1}{e^{\frac{2i\Delta t\pi}{T}}-1} - \frac{\Delta t}{2i}(M+1) = \frac{\Delta t}{2i}\frac{e^{2\pi i}-1}{e^{\frac{2i\Delta t\pi}{T}}-1} - \frac{T}{2i} = -\frac{T}{2i}.
\end{aligned}$$

For every $\tau = 2k\pi/\Delta t - \pi/T$, *with* $k \in \mathbb{Z}$,

$$\begin{aligned} H(\tau) &= \Delta t \sum_{n=0}^{M} \sin \frac{n\pi\Delta t}{T} e^{in\Delta t\left(\frac{2k\pi}{\Delta t} - \frac{\pi}{T}\right)} \\ &= \Delta t \sum_{n=0}^{M} \frac{e^{\frac{in\Delta t\pi}{T}} - e^{-\frac{in\Delta t\pi}{T}}}{2i} e^{2k\pi in} e^{\frac{-in\Delta t\pi}{T}} \\ &= \frac{\Delta t}{2i} \sum_{n=0}^{M} 1 - \frac{\Delta t}{2i} e^{\frac{-2i(M+1)\Delta t\pi}{T}} \sum_{n=0}^{M} e^{\frac{2in\Delta t\pi}{T}} \\ &= \frac{\Delta t}{2i}(M+1) - \frac{\Delta t}{2i} \frac{e^{\frac{2i(M+1)\Delta t\pi}{T}} - 1}{e^{\frac{2i\Delta t\pi}{T}} - 1} = \frac{T}{2i} - \frac{\Delta t}{2i} \frac{e^{2\pi i} - 1}{e^{\frac{2i\Delta t\pi}{T}} - 1} = \frac{T}{2i}. \end{aligned}$$

• *Step* 2. *It is easy to see that* H *is continuous on* $\mathbb{R}\backslash\{\tau : \tau = 2k\pi/\Delta t \pm \pi/T\}$, $k \in \mathbb{Z}$. *We now study the continuity of* H *at the singularities* $\tau = 2k\pi/\Delta t \pm \pi/T$. *For every* $\tau \to 2k\pi/\Delta t \pm \pi/T$, *we have* $\tau = 2k\pi/\Delta t \pm \pi/T + \varepsilon\pi$, *with* $\varepsilon \to 0$.

1. *The case* $\tau = 2k\pi/\Delta t + \pi/T + \varepsilon\pi$ *with* $\varepsilon \to 0$.
Using the definition of H *we have*

$$\begin{aligned} H\left(\frac{2k\pi}{\Delta t} + \frac{\pi}{T} + \varepsilon\pi\right) &= \Delta t \sum_{n=0}^{M} \sin \frac{n\pi\Delta t}{T} e^{in\Delta t\pi\left(\frac{2k}{\Delta t} + \frac{1}{T} + \varepsilon\right)} \\ &= \Delta t \sum_{n=0}^{M} \frac{e^{\frac{in\Delta t\pi}{T}} - e^{-\frac{in\Delta t\pi}{T}}}{2i} e^{in\pi 2k} e^{in\Delta t\pi\left(\frac{1}{T} + \varepsilon\right)}. \end{aligned}$$

Hence,

$$H\left(\frac{2k\pi}{\Delta t} + \frac{\pi}{T} + \varepsilon\pi\right) = \frac{\Delta t}{2i} \sum_{n=0}^{M} e^{\frac{in\Delta t\pi}{T}(2+T\varepsilon)} - \frac{\Delta t}{2i} \sum_{n=0}^{M} e^{in\Delta t\pi\varepsilon}. \tag{6.14}$$

For every $|x| < 2T/\Delta t$, *according to the classical formula for the sum of a geometric series, the following identity holds:*

$$\sum_{n=0}^{M} e^{\frac{in\Delta t\pi}{T}x} = \frac{e^{\frac{i(M+1)\Delta t\pi}{T}x} - 1}{e^{\frac{i\Delta t\pi}{T}x} - 1} = \frac{e^{i\pi x} - 1}{e^{\frac{i\Delta t\pi}{T}x} - 1}. \tag{6.15}$$

For the first sum entering on the right-hand term of (6.14) we have

$$\sum_{n=0}^{M} e^{\frac{in\Delta t\pi}{T}(2+T\varepsilon)} = \frac{e^{2\pi i}e^{iT\pi\varepsilon} - 1}{e^{\frac{i\Delta t\pi}{T}(2+T\varepsilon)} - 1} = \frac{e^{iT\pi\varepsilon} - 1}{e^{\frac{i\Delta t\pi}{T}(2+T\varepsilon)} - 1}. \tag{6.16}$$

For the second sum on the right-hand term (6.14), using (6.15) and the fact that $\Delta t\pi/T(2+T\varepsilon) < 2\pi$, *we have*

$$\sum_{n=0}^{M} e^{in\Delta t\pi\varepsilon} = \frac{e^{iT\pi\varepsilon} - 1}{e^{i\Delta t\pi\varepsilon} - 1}. \tag{6.17}$$

Finally, replacing (6.16) and (6.17) in (6.14) and taking the limit $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon\to 0} H\left(\frac{2k\pi}{\Delta t}+\frac{\pi}{T}+\varepsilon\pi\right)= \frac{\Delta t}{2i}\lim_{\varepsilon\to 0}\frac{e^{iT\pi\varepsilon}-1}{e^{\frac{i\Delta t\pi}{T}(2+T\varepsilon)}-1}-\frac{\Delta t}{2i}\lim_{\varepsilon\to 0}\frac{e^{iT\pi\varepsilon}-1}{e^{i\Delta t\pi\varepsilon}-1}$$

$$= 0-\frac{\Delta t}{2i}\lim_{\varepsilon\to 0}\frac{e^{iT\pi\varepsilon}-1}{e^{i\Delta t\pi\varepsilon}-1}=-\frac{\Delta t}{2i}\lim_{\varepsilon\to 0}\frac{i\pi T e^{iT\pi\varepsilon}}{i\Delta t\pi e^{i\Delta t\pi\varepsilon}}=-\frac{T}{2i}.$$

2. The case $\tau = 2k\pi/\Delta t - \pi/T + \varepsilon\pi$ with $\varepsilon \to 0$.
We have

$$H\left(\frac{2k\pi}{\Delta t}-\frac{\pi}{T}+\varepsilon\pi\right)=\Delta t\sum_{n=0}^{M}\sin\frac{n\pi\Delta t}{T}e^{in\Delta t\pi\left(\frac{2k}{\Delta t}-\frac{1}{T}+\varepsilon\right)}$$
$$=\Delta t\sum_{n=0}^{M}\frac{e^{\frac{in\Delta t\pi}{T}}-e^{-\frac{in\Delta t\pi}{T}}}{2i}e^{in\pi 2k}e^{in\Delta t\pi\varepsilon}e^{-\frac{in\Delta t\pi}{T}}.$$

Hence

$$H\left(\frac{2k\pi}{\Delta t}-\frac{\pi}{T}+\varepsilon\pi\right)=\frac{\Delta t}{2i}\sum_{n=0}^{M}e^{in\Delta t\pi\varepsilon}-\frac{\Delta t}{2i}\sum_{n=0}^{M}e^{\frac{in\Delta t\pi}{T}(T\varepsilon-2)}. \tag{6.18}$$

By (6.17), for the first sum entering on the right-hand term of (6.18) we have

$$\sum_{n=0}^{M}e^{in\Delta t\pi\varepsilon}=\frac{e^{iT\pi\varepsilon}-1}{e^{i\Delta t\pi\varepsilon}-1}. \tag{6.19}$$

Moreover, in (6.17), applying the identity (6.15) for the second sum on the right-hand term of (6.18), we obtain

$$\sum_{n=0}^{M}e^{-\frac{in\Delta t\pi}{T}(2-T\varepsilon)}=e^{-\frac{i(M+1)\Delta t\pi}{T}(2-T\varepsilon)}\sum_{n=0}^{M}e^{\frac{in\Delta t\pi}{T}(2-T\varepsilon)}$$
$$==e^{-2\pi i}e^{iT\pi\varepsilon}\frac{e^{\frac{i(M+1)\Delta t\pi}{T}(2-T\varepsilon)}-1}{e^{\frac{i\Delta t\pi}{T}(2-T\varepsilon)}-1}=e^{iT\pi\varepsilon}\frac{e^{2\pi i}e^{-iT\pi\varepsilon}-1}{e^{\frac{i\Delta t\pi}{T}(2-T\varepsilon)}-1}$$
$$=\frac{1-e^{iT\pi\varepsilon}}{e^{\frac{i\Delta t\pi}{T}(2-T\varepsilon)}-1}\to 0,\quad \varepsilon\to 0. \tag{6.20}$$

Hence,

$$\lim_{\varepsilon\to 0}H\left(\frac{2k\pi}{\Delta t}-\frac{\pi}{T}+\varepsilon\pi\right)=\frac{T}{\Delta t}\frac{\Delta t}{2i}=\frac{T}{2i}.$$

This concludes the proof of Lemma 6.2. ∎

Remark 6.3. For $\tau = 0$ in (6.6) we have

$$\begin{aligned} H(0) &= -\frac{\Delta t\sin\frac{\Delta t\pi}{T}}{2\sin\frac{\pi\Delta t}{2T}\sin\left(-\frac{\pi\Delta t}{2T}\right)}=-\frac{\Delta t\sin\frac{\Delta t\pi}{T}}{-2\sin^2\frac{\pi\Delta t}{2T}} \\ &=\frac{\Delta t 2\sin\frac{\Delta t\pi}{2T}\cos\frac{\Delta t\pi}{2T}}{2\sin^2\frac{\pi\Delta t}{2T}}=\frac{\Delta t\cos\frac{\Delta t\pi}{2T}}{\sin\frac{\pi\Delta t}{2T}}=\Delta t\cot\frac{\Delta t\pi}{2T}.\end{aligned} \tag{6.21}$$

Taking the limit $\Delta t \to 0$ in (6.6), for every τ fixed, we obtain

$$\lim_{\Delta t \to 0} H(\tau) = -2\cos\frac{T\tau}{2} e^{\frac{iT\tau}{2}} \frac{\pi T}{T^2\tau^2 - \pi^2} = G(\tau) \tag{6.22}$$

and this is the classical Fourier transform of g given by (6.4).

6.3. Proof of Theorem 2.1. *This section is devoted to the proof of the main result of this paper. The proof of the discrete inequality (2.3) follows the strategy used in [26, (pp. 162–163)] to prove the classical Ingham inequality (1.2).*

Proof (Proof of the first (so-called inverse) inequality in (2.3)). We prove the first inequality in (2.3), namely,

$$C_1(\Delta t, T, \gamma) \sum_{k=-N}^{N} |a_k|^2 \leq \Delta t \sum_{n=0}^{M} \left| \sum_{k=-N}^{N} a_k e^{in\Delta t \lambda_k} \right|^2.$$

Taking into account that $\sin(n\Delta t\pi/T) \leq 1$, *we have*

$$\begin{aligned}\Delta t \sum_{n=0}^{M} \left| \sum_k a_k e^{in\Delta t\lambda_k} \right|^2 &\geq \Delta t \sum_{n=0}^{M} \sin\frac{n\Delta t\pi}{T} \left| \sum_k a_k e^{in\Delta t\lambda_k} \right|^2 \\ &= \Delta t \sum_{n=0}^{M} \sin\frac{n\Delta t\pi}{T} \sum_k \sum_l a_k \bar{a}_l e^{in\Delta t(\lambda_k - \lambda_l)}.\end{aligned}$$

The function H defined by (6.6) is continuous, hence

$$\begin{aligned}&\Delta t \sum_{n=0}^{M} \sin\frac{n\Delta t\pi}{T} \sum_k \sum_l a_k \bar{a}_l e^{in\Delta t(\lambda_k - \lambda_l)} \\ &= \sum_k \sum_l a_k \bar{a}_l H(\lambda_k - \lambda_l) = H(0) \sum_k |a_k|^2 + \sum_k \sum_{l, l\neq k} a_k \bar{a}_l H(\lambda_k - \lambda_l) \\ &\geq H(0) \sum_k |a_k|^2 - \frac{1}{2} \sum_k \sum_{l, l\neq k} \left(|a_k|^2 + |a_l|^2\right) |H(\lambda_k - \lambda_l)| \\ &= H(0) \sum_k |a_k|^2 - \sum_k |a_k|^2 \sum_{l, l\neq k} |H(\lambda_k - \lambda_l)|.\end{aligned} \tag{6.23}$$

In the last term in (6.23) we have

$$\sum_{l, k\neq l} |H(\lambda_k - \lambda_l)| = \sum_{\substack{l, k\neq l \\ |\lambda_k - \lambda_l| \leq \frac{\pi}{\Delta t}}} |H(\lambda_k - \lambda_l)| + \sum_{\substack{l, k\neq l \\ |\lambda_k - \lambda_l| > \frac{\pi}{\Delta t}}} |H(\lambda_k - \lambda_l)|. \tag{6.24}$$

Moreover, the function H is periodic with period $2\pi/\Delta t$. Consequently, for every $k, l \in \mathbb{Z}$ with $\pi/\Delta t < |\lambda_k - \lambda_l| < 2\pi/\Delta t$, there exist $m_{k,l} \in [-\pi/\Delta t, \pi/\Delta t]$ such that $|m_{k,l}| = 2\pi/\Delta t - |\lambda_k - \lambda_l|$ with the property $H(\lambda_k - \lambda_l) = H(m_{k,l})$. Therefore, using this periodicity property and applying (6.2) from Theorem 6.1 and (6.24) in (6.23), *we obtain*

(6.25)

$$\Delta t\sum_{n=0}^{M}\left|\sum_{k}a_k e^{in\Delta t\lambda_k}\right|^2$$

$$\geq H(0)\sum_k|a_k|^2-\sum_k|a_k|^2\left(\sum_{\substack{l,k\neq l\\ |\lambda_k-\lambda_l|\leq\pi/\Delta t}}|H(\lambda_k-\lambda_l)|+\sum_{\substack{l,k\neq l\\ |m_{k,l}|\leq\pi/\Delta t}}|H(m_{k,l})|\right)$$

$$\geq H(0)\sum_k|a_k|^2-\sum_k|a_k|^2\left(\sum_{\substack{l,k\neq l\\ |\lambda_k-\lambda_l|\leq\pi/\Delta t}}|G(\lambda_k-\lambda_l)|+\sum_{\substack{l,k\neq l\\ |m_{k,l}|\leq\pi/\Delta t}}|G(m_{k,l})|\right)$$

$$+CN(\Delta t)^2\big).$$

On the other hand, as pointed out in [26, p. 162], for every sequence $\{\lambda_k\}$ *satisfying the gap condition (2.1), the function* G *satisfies*

(6.26)

$$\sum_{l\neq k,l=-N}^{N}|G(\lambda_k-\lambda_l)|\leq 2\pi T\sum_{l=-\infty,l\neq k}^{\infty}\frac{1}{T^2\left(\lambda_k-\lambda_l\right)^2-\pi^2}$$

$$\leq 2\pi T\sum_{l=-\infty,l\neq k}^{\infty}\frac{1}{T^2\gamma^2\left(k-l\right)^2-\pi^2}=4\pi T\sum_{r\geq1}\frac{1}{\frac{T^2\gamma^2}{4\pi^2}4\pi^2r^2-\pi^2}\leq\frac{16\pi}{T\gamma^2}\sum_{r\geq1}\frac{1}{4r^2-1}$$

$$=\frac{8\pi}{T\gamma^2}\sum_{r\geq1}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)=\frac{8\pi}{T\gamma^2}.$$

Further, for the terms of the sequence $\{\lambda_k\}$ *satisfying* $\pi/\Delta t <\mid \lambda_k-\lambda_l\mid< (2\pi-(\Delta t)^p)/\Delta t$, *(and then,* $(\Delta t)^{p-1}\leq|m_{k,l}|\leq\pi/\Delta t$*),* $k\neq l$, *we have*

$$|G(m_{k,l})|\leq 2\pi T\frac{1}{T^2\left(m_{k,l}\right)^2-\pi^2}=2\pi T\frac{1}{T^2\left(\frac{2\pi}{\Delta t}-(\lambda_l-\lambda_l)\right)^2-\pi^2}$$

$$\leq 2\pi T\frac{1}{T^2\left(\frac{2\pi}{\Delta t}-\frac{2\pi-(\Delta t)^p}{\Delta t}\right)^2-\pi^2}=2\pi T\frac{\Delta t^2}{T^2(\Delta t)^{2p}-\pi^2\Delta t^2}$$

and it follows that

$$\sum_{l\neq k,l=-N}^{N}|G(m_{k,l})|\leq(N\Delta t)2\pi T\frac{(\Delta t)^{1-2p}}{T^2-\pi^2(\Delta t)^{2-2p}}. \tag{6.27}$$

Using the relations (6.26) and (6.27) in (6.25) we obtain

$$\begin{aligned}&\Delta t\sum_{n=0}^{M}\sin\frac{n\Delta t\pi}{T}\sum_k\sum_l a_k\bar{a}_l e^{in\Delta t(\lambda_k-\lambda_l)}\\&\geq H(0)\sum_k|a_k|^2-\sum_k|a_k|^2\left[\frac{8\pi}{T\gamma^2}+NC\Delta t^2+CN\Delta t(\Delta t)^{1-2p}\right]\end{aligned} \tag{6.28}$$

when $\Delta t \to 0$. *For the function* $H(0)$ *given by (6.21) we have* $\lim_{\Delta t \to 0} H(0) = 2T/\pi$, *which is equivalent to* $2T/\pi - \theta \leq H(0) \leq 2T/\pi + \theta$, *with* $\theta \to 0$ *when* $\Delta t \to 0$. *In order to ensure the positivity of all the coefficients* $|a_k|^2$ *in (6.28) it is necessary and sufficient to have*

$$C_1(\Delta t, T, \gamma) := H(0) - \frac{8\pi}{T\gamma^2} - (NC\Delta t^2 + CN\Delta t(\Delta t)^{1-2p}) > 0, \tag{6.29}$$

which is equivalent to

$$T^2 - \frac{T\pi}{2}(\theta + \varepsilon_1) - \frac{4\pi^2}{\gamma^2} > 0,$$

where $\varepsilon_1 = NC\Delta t^2 + CN\Delta t(\Delta t)^{1-2p}$, $C > 0$. *This condition holds for every*

$$T(\Delta t) > T_0(\Delta t) = \frac{\frac{\pi}{2}(\varepsilon_1 + \theta) + \sqrt{\frac{\pi^2}{4}(\theta + \varepsilon_1)^2 + \frac{16\pi^2}{\gamma^2}}}{2} := \frac{2\pi}{\gamma} + \epsilon(\Delta t)$$

with $\epsilon(\Delta t) = C(\Delta t + N\Delta t(\Delta t)^{1-2p})$. *Hence, the inequality (2.3) holds with the constant* $C_1(\Delta t, T, \gamma)$ *defined by the relation (6.29) where* $\delta_1(\Delta t) = -(\varepsilon_1 + \theta)$.

Proof of the second (so-called direct) inequality. We now prove the inequality

$$\Delta t \sum_{n=0}^{M} \left| \sum_{k=-N}^{N} a_k e^{in\Delta t \lambda_k} \right|^2 \leq C_2(\Delta t, T, \gamma) \sum_{k=-N}^{N} |a_k|^2 .$$

We have

$$\Delta t \sum_{n=0}^{M} \left| \sum_k a_k e^{in\Delta t\lambda_k} \right|^2 = \Delta t \sum_{n=0}^{\left[\frac{M}{2}\right]} \left| \sum_k a_k e^{in\Delta t\lambda_k} \right|^2 + \Delta t \sum_{n=\left[\frac{M}{2}\right]+1}^{M} \left| \sum_k a_k e^{in\Delta t\lambda_k} \right|^2 . \tag{6.30}$$

Consider the first term on the right-hand side of (6.30),

$$\begin{aligned} \Delta t \sum_{n=0}^{\left[\frac{M}{2}\right]} \left| \sum_k a_k e^{in\Delta t\lambda_k} \right|^2 &= \Delta t \sum_{n=\left[\frac{M+1}{4}\right]+1}^{\left[\frac{M}{2}\right]+\left[\frac{M+1}{4}\right]+1} \left| \sum_k a_k e^{i\left(n-\left[\frac{M+1}{4}\right]-1\right)\Delta t\lambda_k} \right|^2 \\ &= \Delta t \sum_{n=\left[\frac{M+1}{4}\right]+1}^{\left[\frac{M}{2}\right]+\left[\frac{M+1}{4}\right]} \left| \sum_k a_k e^{i\left(n-\left[\frac{M+1}{4}\right]-1\right)\Delta t\lambda_k} \right|^2 + \Delta t \left| \sum_k a_k e^{i\left[\frac{M}{2}\right]\Delta t\lambda_k} \right|^2 . \end{aligned} \tag{6.31}$$

Using the properties of the entire part of a real number we have

$$\begin{aligned} &\left[\frac{M+1}{4}\right] \leq \frac{M+1}{4} \leq \left[\frac{M+1}{4}\right] + 1, \\ &\left[\frac{M}{2}\right] + \left[\frac{M+1}{4}\right] \leq \left[\frac{3M+1}{4}\right], \\ &\left[\frac{3M+1}{4}\right] \leq \frac{3M+1}{4} \leq \left[\frac{3M+1}{4}\right] + 1. \end{aligned}$$

For every $n \in \mathbb{N}$ *with* $\left[\frac{M+1}{4}\right] + 1 \leq n \leq \left[\frac{3M+1}{4}\right]$, *we have*

$$\frac{M+1}{4} \leq n \leq \frac{3M+1}{4} \tag{6.32}$$

and

$$\frac{\pi}{4} \leq \frac{n\pi\Delta t}{T} \leq \frac{3\pi}{4},$$

due to the fact that $(M+1)\Delta t = T$.

Therefore, for every $n \in \mathbb{N}$ *as in (6.32) we have* $\sin(n\pi\Delta t/T) \geq \sqrt{2}/2$ *and*

$$\begin{aligned}
&\Delta t \sum_{n=\left[\frac{M+1}{4}\right]+1}^{\left[\frac{M}{2}\right]+\left[\frac{M+1}{4}\right]} \left| \sum_k a_k e^{i\left(n-\left[\frac{M+1}{4}\right]-1\right)\Delta t \lambda_k} \right|^2 + \Delta t \left| \sum_k a_k e^{i\left[\frac{M}{2}\right]\Delta t \lambda_k} \right|^2 \\
&\leq 2\Delta t \sum_{n=\left[\frac{M+1}{4}\right]+1}^{\left[\frac{M}{2}\right]+\left[\frac{M+1}{4}\right]} \sin\frac{n\pi\Delta t}{T} \left| \sum_k a_k e^{i\left(n-\left[\frac{M+1}{4}\right]-1\right)\Delta t \lambda_k} \right|^2 + \Delta t \left| \sum_k a_k e^{i\left[\frac{M}{2}\right]\Delta t \lambda_k} \right|^2 \\
&\leq 2\Delta t \sum_{n=0}^{M} \sin\frac{n\pi\Delta t}{T} \left| \sum_k a_k e^{in\Delta t\lambda_k} e^{-i\left(\left[\frac{M+1}{4}\right]+1\right)\Delta t \lambda_k} \right|^2 + \Delta t \left| \sum_k a_k e^{i\left[\frac{M}{2}\right]\Delta t \lambda_k} \right|^2 \\
&= 2\Delta t \sum_{n=0}^{M} \sin\frac{n\pi\Delta t}{T} \sum_k \sum_l a_k \bar{a}_l e^{in\Delta t(\lambda_k-\lambda_l)} e^{-i\left(\left[\frac{M+1}{4}\right]+1\right)\Delta t(\lambda_k-\lambda_l)} \\
&+\Delta t \sum_k \sum_l a_k \bar{a}_l e^{2i\left[\frac{M}{2}\right]\Delta t(\lambda_k-\lambda_l)} \\
&= 2\sum_k \sum_l a_k \bar{a}_l H(\lambda_k-\lambda_l) e^{-i\left(\left[\frac{M+1}{4}\right]+1\right)\Delta t(\lambda_k-\lambda_l)} + \Delta t \sum_k \sum_l a_k \bar{a}_l e^{i\left[\frac{M}{2}\right]\Delta t(\lambda_k-\lambda_l)} \\
&= 2H(0)\sum_k |a_k|^2 + 2\sum_k \sum_{l, l\neq k} a_k \bar{a}_l H(\lambda_k-\lambda_l) e^{-i\left(\left[\frac{M+1}{4}\right]+1\right)\Delta t(\lambda_k-\lambda_k)} \\
&+\Delta t \sum_k |a_k|^2 + \Delta t \sum_k \sum_{l,l\neq k} a_k \bar{a}_l e^{i\left[\frac{M}{2}\right]\Delta t(\lambda_k-\lambda_k)} \\
&\leq 2H(0)\sum_k |a_k|^2 + \sum_k \sum_{l,l\neq k} \left(|a_k|^2 + |a_l|^2\right) |H(\lambda_k-\lambda_l)| + \Delta t \sum_k |a_k|^2
\end{aligned}$$

$$\begin{aligned}
&+\frac{N\Delta t}{2} \sum_k \sum_{l,l\neq k} \left(|a_k|^2 + |a_l|^2\right) \\
&\leq 2H(0)\sum_k |a_k|^2 + 2\sum_k |a_k|^2 \sum_{l,l\neq k} |H(\lambda_k-\lambda_l)| + \Delta t \sum_k |a_k|^2 + 2N\Delta t \sum_k |a_k|^2 .
\end{aligned} \tag{6.33}$$

Using the same argument (6.2) as in the proof of the inverse inequality, for every $C > 0$*, we have*

$$\sum_{l,k\neq l} |H(\lambda_k - \lambda_l)| \leq \sum_{l\neq k, l=-N}^{N} |G(\lambda_k - \lambda_l)| + CN(\Delta t)^2, \tag{6.34}$$

when Δt *is small enough, with* G *the Fourier transform (6.4) satisfying (6.26) and (6.27).*

Therefore, for every k*,*

$$\begin{aligned}
&2H(0)\sum_k |a_k|^2 + \sum_k \sum_{l,l\neq k} \left(|a_k|^2 + |a_l|^2\right) |H(\lambda_k - \lambda_l)| + \Delta t \sum_k |a_k|^2 \\
&+\frac{N\Delta t}{2} \sum_k \sum_{l,l\neq k} \left(|a_k|^2 + |a_l|^2\right) \leq 2\Delta t \cot \frac{\Delta t \pi}{2T} \sum_k |a_k|^2 \\
&+2\sum_k |a_k|^2 \left(\frac{8\pi}{T\gamma^2} + NC\Delta t^2 + CN\Delta t(\Delta t)^{1-2p} + 2\Delta t\right) + 2N\Delta t \sum_k |a_k|^2 .
\end{aligned}$$

Hence

$$\Delta t \sum_{n=0}^{\left[\frac{M}{2}\right]} \left| \sum_k a_k e^{in\Delta t \lambda_k} \right|^2 \leq \sum_k |a_k|^2 \left(2\Delta t \cot \frac{\Delta t \pi}{2T} + \frac{16\pi}{T\gamma^2} + \varepsilon(\Delta t)\right), \tag{6.35}$$

with $\varepsilon(\Delta t) = 2NC\Delta t^2 + 2CN\Delta t(\Delta t)^{1-2p} + 2\Delta t + 2N\Delta t$.

For the second right-hand term of (6.30) we have

$$\begin{aligned}
&\Delta t \sum_{n=\left[\frac{M}{2}\right]+1}^{M} \left| \sum_k a_k e^{in\Delta t \lambda_k} \right|^2 = \Delta t \sum_{n=\left[\frac{M}{2}\right]-\left[\frac{M+1}{4}\right]}^{M-\left[\frac{M+1}{4}\right]-1} \left| \sum_k a_k e^{i(n+\left[\frac{M+1}{4}\right]+1)\Delta t \lambda_k} \right|^2 \\
&= \Delta t \sum_{n=\left[\frac{M}{2}\right]+1-\left[\frac{M+1}{4}\right]}^{M-\left[\frac{M+1}{4}\right]-1} \left| \sum_k a_k e^{i(n+\left[\frac{M+1}{4}\right]+1)\Delta t \lambda_k} \right|^2 + \Delta t \sum_k \left| a_k e^{i(\left[\frac{M}{2}\right]+1)\Delta t \lambda_k} \right|^2 .
\end{aligned}$$

Taking into account that

$$M - \left[\frac{M+1}{4}\right] - 1 \leq \frac{3M+1}{4} \ \textit{and}\ \left[\frac{M}{2}\right] - \left[\frac{M+1}{4}\right] + 1 \geq \frac{M+1}{4},$$

for every $n \in \mathbb{N}$*, with* $(M+1)/4 \leq n \leq (3M+1)/4$ *we have* $\sin(n\pi\Delta t/T) \geq \sqrt{2}/2$.

Thus,

$$\begin{aligned}
&\Delta t \sum_{n=\left[\frac{M}{2}\right]+1}^{M} \left| \sum_k a_k e^{in\Delta t\lambda_k} \right|^2 \\
&\leq 2\Delta t \sum_{n=\left[\frac{M}{2}\right]+1-\left[\frac{M+1}{4}\right]}^{M-\left[\frac{M+1}{4}\right]-1} \sin \frac{n\pi\Delta t}{T} \left| \sum_k a_k e^{i(n+\left[\frac{M+1}{4}\right]+1)\Delta t\lambda_k} \right|^2 \\
&+\Delta t \sum_k \left| a_k e^{i(\left[\frac{M}{2}\right]+1)\Delta t\lambda_k} \right|^2 \\
&\leq 2\Delta t \sum_{n=0}^{M} \sin \frac{n\pi\Delta t}{T} \left| \sum_k a_k e^{in\Delta t\lambda_k} e^{i(\left[\frac{M+1}{4}\right]+1)\Delta t\lambda_k} \right|^2 + \Delta t \sum_k \left| a_k e^{i(\left[\frac{M}{2}\right]+1)\Delta t\lambda_k} \right|^2 \\
&= 2\Delta t \sum_{n=0}^{M} \sin \frac{n\pi\Delta t}{T} \sum_k \sum_l a_k \bar{a}_l e^{in\Delta t(\lambda_k-\lambda_l)} e^{i\left(\left[\frac{M+1}{4}\right]+1\right)\Delta t(\lambda_k-\lambda_k)} \\
&+\Delta t \sum_k \left| a_k e^{i(\left[\frac{M}{2}\right]+1)\Delta t\lambda_k} \right|^2
\end{aligned}$$

and we obtain the estimate

(6.36)

$$\begin{aligned}
&\Delta t \sum_{n=\left[\frac{M}{2}\right]+1}^{M} \left| \sum_k a_k e^{in\Delta t\lambda_k} \right|^2 \\
&\leq 2H(0) \sum_k |a_k|^2 + 2 \sum_k \sum_{l,l\neq k} |a_k|^2 |H(\lambda_k-\lambda_l)| + 2N\Delta t \sum_k |a_k|^2 \\
&\leq \left(2\Delta t \cot \frac{\Delta t\pi}{2T} + \frac{16\pi}{T\gamma^2} + NC\Delta t^2 + CN\Delta t(\Delta t)^{1-2p} + 2N\Delta t \right) \sum_k |a_k|^2 .
\end{aligned}$$

For the function $H(0)$ *defined by the relation (6.21) we have* $\lim_{\Delta t\to 0} H(0) = 2T/\pi$*.*

From (6.30) and (6.35) we get

$$\Delta t \sum_{n=0}^{M} \left| \sum_k a_k e^{in\Delta t\lambda_k} \right|^2 \leq \left(\frac{8T}{\pi} + \frac{32\pi}{T\gamma^2} + \delta_2(\Delta t) \right) \sum_k |a_k|^2 , \tag{6.37}$$

with

$$\delta_2(\Delta t) = 4\varepsilon(\Delta t) + \theta,$$

($2T/\pi - \theta \leq H(0) \leq 2T/\pi + \theta$*, with* $\theta \to 0$ *when* $\Delta t \to 0$*).*

This concludes the proof of Theorem 2.1. □

Proof (Proof of Theorem 2.2). Following the same steps of the above proof we obtain the discrete version (2.5) of the L^1 Ingham's inequality (1.6) given by Theorem 2.2.

More precisely, we have

$$\Delta t \sum_{n=0}^{M} \sin \frac{n\Delta t\pi}{T} \left(\sum_k a_k e^{in\Delta t\lambda_k} \right) e^{-in\Delta t\lambda_k} = \sum_l a_l H(\lambda_k - \lambda_l), \tag{6.38}$$

where function H is defined by (6.6).

Taking $l = \nu$ in (6.38), where $|a_\nu|$ is the greatest $|a_n|$, we deduce

$$\left| \Delta t \sum_{n=0}^{M} \sin \frac{n\Delta t\pi}{T} \left(\sum_k a_k e^{in\Delta t\lambda_k} \right) e^{-in\Delta t\lambda_\nu} \right| \geq |a_\nu H(0)| - |a_\nu| \sum_{k,k\neq\nu} |H(\lambda_k - \lambda_\nu)| . \tag{6.39}$$

Function H is exactly the discrete Fourier transform (6.5) used in the proof of discrete Ingham's inequality (2.3). By (6.24) and the estimates for $|H(\lambda_k - \lambda_\nu)|$ used in the proof of Theorem 2.1, i.e.,

$$\sum_{k,k\neq\nu} |H(\lambda_k - \lambda_\nu)| \leq \left[\frac{8\pi}{T\gamma^2} + NC\Delta t^2 + CN\Delta t(\Delta t)^{1-2p} \right],$$

as in (6.28), we deduce that

$$\Delta t \sum_{n=0}^{M} \left| \sum_{k=-N}^{N} a_k e^{in\Delta t\lambda_k} \right| \geq C_1(\Delta t, T, \gamma) \max |a_n| . \tag{6.40}$$

The direct inequality in (2.5) may be obtained using the same arguments and estimates and we omit the details. ∎

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