ENERGY DECAY OF MAGNETOELASTIC WAVES IN A BOUNDED CONDUCTIVE MEDIUM

G. Perla Menzala[1] and E. Zuazua[2]

ABSTRACT

We study the decay of the energy of solutions of the system of magneto-elasticity in a bounded, three-dimensional conductive medium. We prove that all solutions do decay as $t \to \infty$ in the energy-space when the domain is simply connected. We also describe the large time behavior of solutions when the domain is not simply connected. Our results are similar to those of C. Dafermos on the system of thermoelasticity.

[1] National Laboratory of Scientific Computation, LNCC/CNPq and Institute of Mathematics, UFRJ, Brazil. E-mail: perla@lncc.br. Partially supported by a Grant of CNPq and PRONEX (MCT, Brasil)

[2] Universidad Complutense de Madrid, Departamento de Matemática Aplicada, 28040 Madrid, Spain. E-mail: zuazua@eucmax.sim.ucm.es. Supported by grants PB96-0663 of the DGES (Spain) and ERB FMRX CT960033 of the European Union.

1. INTRODUCTION

Let Ω be a bounded domain of $I\!R^3$ of class C^2. We consider the following system of magneto-elasticity:

$$\left.\begin{array}{l} \rho u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \text{div } u - \dfrac{\mu_0}{4\pi}\,[\text{curl } h]\times \widetilde{H} = 0 \qquad \text{in } \Omega\times(0,\infty) \\ \beta h_t + \text{curl curl } h - \beta \text{curl } [u_t\times \widetilde{H}] = 0 \qquad \text{in } \Omega\times(0,\infty) \\ \text{div } h = 0 \qquad \text{in } \Omega\times(0,\infty) \end{array}\right] \tag{1.1}$$

$$\left.\begin{array}{ll} u = 0 & \text{on } \partial\Omega\times(0,\infty) \\ h\cdot\eta = 0 & \text{on } \partial\Omega\times(0,\infty) \\ \text{curl } (h)\times\eta = 0 & \text{on } \partial\Omega\times(0,\infty) \end{array}\right] \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad h(x,0) = h_0(x) \qquad \text{in } \Omega. \tag{1.3}$$

In (1.1), the vector field u denotes the displacement: $u = (u_1, u_2, u_3)$, while $h = (h_1, h_2, h_3)$ denotes the magnetic field. A (known) constant magnetic field taken along the x_3-axis is denoted by $\widetilde{H}$. All other parameters stand as follows:

i) μ, λ are the Lamé constants with $\mu > 0$ and $(\lambda + 2\mu) > 0$;

ii) ρ is the mass density ($\rho > 0$);

iii) μ_0 is the magnetic permeability ($\mu_0 > 0$);

iv) β is a parameter which is proportional to the electric conductivity ($\beta > 0$).

By $\cdot$ we denote the scalar product in $I\!R^3$ while $\times$ denotes the vector product. Finally, $\eta = \eta(x)$ denotes the exterior unit normal at $x \in \partial\Omega$.

Related systems in magnetoelasticity were considered in [4] and [6] and there is a large literature on the subject. However, this paper is, to our knowledge, the first attempt on the analysis of the large time behavior in bounded domains.

System (1.1), (1.2) and (1.3) is well-posed in the energy space

$$X = (H_0^1(\Omega))^3 \times (L^2(\Omega))^3 \times H$$

where $H = \{f \in (L^2(\Omega))^3, \text{div } f = 0 \text{ in } \Omega \text{ and } f \cdot \eta = 0 \text{ on } \partial\Omega\}$. The norm in $(L^2(\Omega))^3$ and H is given by $||f||^2 = \int_\Omega |f|^2 dx$ while in $(H_0^1(\Omega))^3$ the norm is $||f||^2_{(H_0^1(\Omega))^3} = \int_\Omega (\mu|\nabla f|^2 + (\lambda+\mu)|\text{div } f|^2) dx$. Note that, under the assumptions $\lambda > 0$ and $\lambda + 2\mu > 0$, by Korn's inequality (see for instance [5]), this norm is equivalent to the one induced by $(H^1(\Omega))^3$. By $(\cdot,\cdot)_X$, $(\cdot,\cdot)_H$ we shall denote the inner product in X and H respectively.

Thus, for any $(u_0, u_1, h_0) \in X$, there exists a unique solution (u, h) of system (1.1), (1.2) and (1.3) such that $(u, u_t, h) \in C([0, +\infty);\ X)$. Furthermore, the energy

$$E(t) = \frac{1}{2} \int_\Omega \left[\rho|u_t|^2 + \mu|\nabla u|^2 + (\lambda+\mu)(\text{div } u)^2 + \frac{\mu_0}{4\pi}\, |h|^2\right] dx \tag{1.4}$$

satisfies

$$\frac{dE}{dt} = -\frac{\mu_0}{4\pi\beta} \int_\Omega |\text{curl } h|^2 dx. \tag{1.5}$$

Therefore all solutions of (1.1)–(1.3) are bounded in X and the energy $E(t)$ decreases along trajectories.

In this paper we are interested in studying whether

$$E(t) \to 0 \qquad \text{as} \qquad t \to +\infty \tag{1.6}$$

as long as (u_0, u_1, h_0) runs in X.

In the late sixties C. Dafermos [3] obtained important results in this direction concerning the system of linear thermoelasticity. Following the methods developed in [3] we use La Salle's invariance principle and reduce the problem of decay to the existence of non-trivial solutions of (1.1), (1.2) with $h \equiv 0$, provided Ω is simply connected. This amounts to say that we search for non-trivial solutions of

$$\begin{aligned} &\rho u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \text{div } u = 0 && \text{in} \quad \Omega \times (0,\infty) \\ &\text{curl } [u_t \times \widetilde{H}] = 0 && \text{in} \quad \Omega \times (0,\infty) \\ &u = 0 && \text{on} \quad \partial\Omega \times (0,\infty). \end{aligned} \tag{1.7}$$

As we shall see, whatever the domain Ω is, non trivial solutions of (1.7) do not exist. Therefore the energy of every solution (of (1.1)–(1.3)) does decay to zero as $t \to +\infty$.

Our analysis of system (1.7) follows the lines of a recent work by the second author [13] where a similar problem motivated by control theoretical issues was discussed. The results in [13] show that, generically with respect to the domain Ω, non-trivial solutions of (1.7) do not exist. More recently, in [10], using the methods of Clement and Sweers [2], it is proved that (1.7) has no non-trivial solution for any bounded Lipschitz domain Ω.

Let us compare our results with those of [3] on the system of thermoelasticity. In [3] the problem of decay was reduced to the existence of non-trivial solutions of the following system, instead of (1.7),

$$\begin{aligned} \rho u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla\text{div } u = 0 \qquad &\text{in} \quad \Omega \times (0,\infty) \\ \text{div } u = 0 \qquad &\text{in} \quad \Omega \times (0,\infty) \\ u = 0 \qquad &\text{on} \quad \partial\Omega \times (0,\infty). \end{aligned} \tag{1.8}$$

As it was observed in [3], when Ω is a ball of $\mathbb{R}^3$, non-trivial solutions of (1.8) do exist. While, generically with respect to Ω, the only solution of (1.8) is the trivial one. Our results are close to those of C. Dafermos [3]. Note however that in the context of magneto-elasticity there are no special domains in which solutions do not decay.

The Cauchy problem for a similar system in the whole space $\mathbb{R}^3$ was considered by Andreou and Dassios in [1]. In this work, it was proved that smooth solutions which vanish as $|x| \to +\infty$, do decay (in time) polynomially as $t \to +\infty$ with the same rate as for the system of thermoelasticity. Moreover, it was proved that the subspace of solutions for which the energy remains constant is empty. This is in contrast with the results we find in the context of bounded domains.

The rest of the paper is organized as follows: In Section 2 we discuss the well-posedness of system (1.1), (1.2), (1.3). In Section 3, applying La Salle's invariance principle we reduce the problem to the analysis of the existence of non-trivial solutions of (1.7) and then applying the non-existence results of [10] we conclude the decay. In Section 4 we discuss the case where Ω is not simply connected. Finally in Section 5 we mention some extensions of our results to the system of magneto-thermoelasticity. As we shall see, combining the dissipative effect of both thermal and magnetic components, the decay of the energy of every solution is guaranteed for every domain, as one could expect from the analysis of magneto-elastic waves.

We do not consider here the problem of whether the energy of solutions decays exponentially uniformly or not. At this respect, we mention that, in the context of thermoelasticity, it was recently proved by Lebeau and Zuazua [8] and H. Koch [7] that, even in the domains in which the energy decays to zero, very often the decay rate is not uniform. This is also probably the case in magneto-elasticity, but this is by now an open problem.

The notations we use in this paper are standard and can be found in A. Pazy [9] or R. Temam [11].

2. EXISTENCE AND UNIQUENESS

In order to prove existence and uniqueness of solutions of (1.1)–(1.3) we apply classical semigroup theory. To do that we write the system in the form

$$\begin{aligned} &\frac{dU}{dt} = \mathcal{A}U, \qquad t > 0 \\ &U(0) = (u_0, u_1, h_0/\alpha) \in X \end{aligned} \tag{2.1}$$

where

$$U(t) = (u, u_t, h/\alpha), \tag{2.2}$$

with $\alpha = \sqrt{4\pi\rho/\mu_0}$ and $\mathcal{A}$ is an unbounded operator in X of the form

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 \\ -A_1 & 0 & B \\ 0 & C & -A_2 \end{bmatrix} \tag{2.3}$$

The operators which appear in (2.3) are defined as follows:

Definition of A_1: A_1 is an unbounded operator in $(L^2(\Omega))^3$ with domain $\mathcal{D}(A_1) = [H^2(\Omega) \cap H_0^1(\Omega)]^3$ and such that

$$A_1 u = -\frac{\mu}{\rho}\,\Delta u - \frac{(\lambda+\mu)}{\rho}\,\nabla\,\mathrm{div}\ u.$$

Note that, under the conditions we have imposed on the Lamé coefficients, A_1 is also an isomorphism from $(H_0^1(\Omega))^3$ into its dual $(H^{-1}(\Omega))^3$.

Definition of A_2: A_2 is an unbounded operator in H with domain

$$\mathcal{D}(A_2) = \left\{ g \in (H^2(\Omega))^3 \cap H, \mathrm{curl}\ (g) \times \eta = 0 \text{ on } \partial\Omega \right\}$$

and such that

$$A_2 h = \frac{1}{\beta}\,\mathrm{curl}\ \mathrm{curl}\ h.$$

Definition of B: B is an unbounded operator from H into $(L^2(\Omega))^3$ with domain

$$\mathcal{D}(B) = \left\{ h \in H; \mathrm{curl}\ (h) \times \widetilde{H} \in (L^2(\Omega))^3 \right\}$$

and such that $Bh = \frac{\mu_0\alpha}{4\pi\rho}\mathrm{curl}\ (h) \times \widetilde{H}$.

Definition of C: The operator C is unbounded from $(L^2(\Omega))^3$ into H with domain

$$\mathcal{D}(C) = \left\{ v \in (L^2(\Omega))^3, \mathrm{curl}\ (v \times \widetilde{H}) \in H \right\}$$

and such that $Cv = \frac{\mu_0\alpha}{4\pi\rho}\mathrm{curl}\ [v \times \widetilde{H}]$. Note that $\frac{\mu_0\alpha}{4\pi\rho} = \frac{1}{\alpha}$.

We set

$$\mathcal{D}(\mathcal{A}) = \left[H^2(\Omega) \cap H_0^1(\Omega)\right]^3 \times \left(H_0^1(\Omega)\right)^3 \times \mathcal{D}(A_2).$$

The following holds.

Lemma 1.1. *$\mathcal{A}$ is dissipative and Range $(I - \mathcal{A}) = X$.*

Proof of Lemma 1.1. First of all we check the dissipativity of $\mathcal{A}$:

$$
\begin{aligned}
(\mathcal{A}U, U)_X &= (v, -A_1 u + B\omega, Cv - A_2\omega), (u, v, \omega))_X = \\
&= (v, u)_{(H_0^1(\Omega))^3} + (-A_1 u + B\omega, v)_{(L^2(\Omega))^3} + (Cv - A_2\omega, \omega)_H = \\
&= (v, u)_{(H_0^1(\Omega))^3} + (-A_1 u, v)_{(L^2(\Omega))^3} + (B\omega, v)_{(L^2(\Omega))^3} + (Cv, \omega)_H - (A_2\omega, \omega)_H = \\
&= -(A_2\omega, \omega)_H = -\frac{1}{\beta}\int_\Omega |\text{curl } \omega|^2 dx \le 0
\end{aligned}
$$

for all $U = (u, v, \omega) \in \mathcal{D}(\mathcal{A})$. In the proof we have used the fact that $(Cv, \omega)_H = -(B\omega, v)_{(L^2(\Omega))^3}$ when $U = (u, v, \omega) \in \mathcal{D}(\mathcal{A})$, which can easily be derived by integration by parts.

Now, let $f = (f_1, f_2, f_3) \in X$ and let us show the existence of (u, v, ω) belonging to $\mathcal{D}(\mathcal{A})$ such that

$$
(I - \mathcal{A}) \begin{pmatrix} u \\ v \\ \omega \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3. \end{pmatrix} \tag{2.4}
$$

In order to solve (2.4) it is sufficient to solve the system

$$
\begin{aligned}
u + A_1 u - B\omega &= g_1 \\
-Cu + \omega + A_2\omega &= g_2
\end{aligned} \tag{2.5}
$$

where

$$
\begin{aligned}
g_1 &= f_1 + f_2 \\
g_2 &= f_3 - Cf_1
\end{aligned} \tag{2.6}
$$

with

$$
v = u - f_1. \tag{2.7}
$$

Observe that $g_1 \in (L^2(\Omega))^3$. On the other hand, $g_2 \in H$. The latter is a consequence of the following lemma:

Lemma 1.2. *$H_0^1(\Omega))^3$ is continuously embedded into the domain of C.*

Proof of Lemma 1.2. For any $u \in (H_0^1(\Omega))^3$ it is clear that curl $[u \times \widetilde{H}]$ belongs to $(L^2(\Omega))^3$. On the other hand, it is also clear that div (curl $[u \times \widetilde{H}]) \equiv 0$. Therefore the trace curl $[u \times \widetilde{H}] \cdot \eta$ is well defined in $H^{-1/2}(\partial\Omega)$. Taking into account that compactly supported smooth functions are dense in $(H_0^1(\Omega))^3$, it is easy to see that this trace vanishes. This shows that $u \in \mathcal{D}(C)$ and completes the proof of Lemma 1.2. □

Let us go back to system (2.5), taking into account that the right hand side (g_1, g_2) belongs to $(L^2(\Omega))^3 \times H$. We can use Lax-Milgram's lemma to obtain a weak solution of (2.5). Afterwords, elliptic regularity will conclude the assertion. We introduce the following bilinear form in $(H_0^1(\Omega))^3 \times V$ where $V = \{\omega \in H : \text{curl } \omega \in (L^2(\Omega))^3\}$:

$$a((u,\omega),(\hat{u},\hat{\omega})) = \int_\Omega [u \cdot \hat{u} + \mu \nabla u \cdot \nabla \hat{u} + (\lambda+\mu) \text{div } u \text{div } \hat{u} + \omega \cdot \hat{\omega}$$

$$+ \frac{1}{\beta} \text{curl } \omega \cdot \text{curl } \hat{\omega}] dx - (B\omega, \hat{u})_{(L^2(\Omega))^3} - (Cu, \hat{\omega})_H. \tag{2.8}$$

This bilinear form is continuous in $(H_0^1(\Omega))^3 \times V$. On the other hand, it is coercive since

$$a((u,\omega),(u,\omega)) = \int_\Omega [|u|^2 + \mu|\nabla u|^2 + (\lambda+\mu)|\text{div } u|^2 + |\omega|^2 + \frac{1}{\beta}|\text{curl } \omega|^2] dx. \tag{2.9}$$

Therefore, by Lax-Milgram's Lemma there exists a unique $(u,\omega) \in (H_0^1(\Omega))^3 \times V$ such that

$$a((u,\omega),(\hat{u},\hat{\omega})) = ((g_1,g_2),(\hat{u},\hat{\omega}))_{(H_0^1(\Omega))^3 \times V}, \quad \forall (\hat{u},\hat{\omega}) \in (H_0^1(\Omega))^3 \times V. \tag{2.10}$$

It is easy to see that the solution of (2.10) is a weak solution of (2.5). On the other hand, since $\omega \in V$, we know that $B\omega \in (L^2(\Omega))^3$. Thus, u is a weak solution of

$$u \in (H_0^1(\Omega))^3;\ u + A_1 u = g_3 = g_1 + B\omega \in (L^2(\Omega))^3.$$

By elliptic regularity we deduce that $u \in \mathcal{D}(A_1) = (H^2 \cap H_0^1(\Omega))^3$. The same argument allows to show that $\omega \in \mathcal{D}(A_2)$.

This concludes the proof of Lemma 1.1. □

As a direct consequence of Lemma 1.1, applying Lumer-Phillips' Theorem we have that $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup of contractions on X. More precisely:

Theorem 1.2. *Let us consider problem (1.1), (1.2) with initial conditions (1.3) where the constant magnetic field $\widetilde{H}$ and the parameters λ, μ, ρ, μ_0 and β stand as in the introduction.*

Let (u_0, u_1, h_0) be in the space $X = (H_0^1(\Omega))^3 \times (L^2(\Omega))^3 \times H$ where

$$H = \{f \in (L^2(\Omega))^3, \text{div } f = 0 \text{ in } \Omega \text{ and } f \cdot \eta = 0 \text{ on } \partial\Omega\}.$$

Then, problem (1.1), (1.2), (1.3) is globally well-posed and the (unique) weak solution (u, u_t, h) belongs to $C([0, \infty), X)$.

Moreover, when $(u_0, u_1, h_0) \in \mathcal{D}(\mathcal{A})$, the solution (u, u_t, h) belongs to $C([0, \infty), \mathcal{D}(\mathcal{A}))$ $\cap C^1([0, \infty), X)$, verifies (1.1)-(1.3) pointwise and

$$||(u(t), u_t(t), h(t)/\alpha)||_{\mathcal{D}(\mathcal{A})} \leq ||(u_0, u_1, h_0/\alpha)||_{\mathcal{D}(\mathcal{A})}, \; \forall t \geq 0. \tag{2.11}$$

Finally, for any solution with initial data in $\mathcal{D}(\mathcal{A})$ the energy E in (1.4) satisfies identity (1.5) for all $t \geq 0$.

3. DECAY FOR SIMPLY CONNECTED DOMAINS

We apply here the classical method based on La Salle's invariance principle (see for instance [3]). First of all, we observe, as proved in Section 2 that, when the initial data are smooth, say (u_0, u_1, h_0) belongs to $\mathcal{D}(\mathcal{A})$ then, the trajectory $(u, u_t, h)_{t\geq 0}$ remains bounded in $\mathcal{D}(\mathcal{A})$ (see (2.11)). By compactness of the Sobolev embedding we deduce that the trajectory $\{(u, u_t, h)\}_{t\geq 0}$ is relatively compact in the energy space $X = (H_0^1(\Omega))^3 \times (L^2(\Omega))^3 \times H$. Thus, the ω-limit set is not empty:

$$\omega(u_0, u_1, h_0) = \Big\{(\tilde{u}_0, \tilde{u}_1, \tilde{h}_0) \in X \text{ such that there exists } \{t_n\}_{n\geq 0}, t_n \to \infty \\ \text{with } (u(t_n), u_t(t_n), h(t_n)) \to (\tilde{u}_0, \tilde{h}_1, \tilde{h}_0) \text{ in } X \text{ as } t_n \to \infty\Big\}. \tag{3.1}$$

By La Salle's invariance principle and taking into account that the energy $E(t)$ constitutes a Lyapunov functional that decreases along trajectories with dissipation rate $-\frac{\mu_0}{4\pi\beta}\int_\Omega |\mathrm{curl}\ h|^2 dx$, we deduce that

$$\begin{aligned}\omega(u_0, u_1, h_0) \subseteq F = \Big\{ & (\tilde{u}_0, \tilde{u}_1, \tilde{h}_0) \in X \text{ such that the solution of} \\ & (1.1),\ (1.2),\ (1.3) \text{ satisfies curl } h = 0 \text{ in } \Omega \times (0, \infty) \Big\}.\end{aligned} \tag{3.2}$$

In order to conclude that solutions of (1.1)-(1.3) converge to zero in the energy space as $t \to +\infty$ it is sufficient to check whether F, the subspace of solutions of (1.1), (1.2), (1.3) such that curl $h = 0$ as in (3.2), reduces to the trivial initial datum $\tilde{u}_0 \equiv \tilde{u}_1 \equiv \tilde{h}_0 \equiv 0$ or not.

From now on in this section we shall assume that Ω is *simply connected.*

Let us analyse the structure of F. Since curl $h = 0$ in $\Omega \times (0, \infty)$ and Ω is simply connected then $h = \nabla p$ for some p. We also know that div $h = 0$ in $\Omega \times (0, \infty)$. Therefore $\Delta p = 0$ in $\Omega \times (0, \infty)$. Since $h \in H$ then we know that $h \cdot \eta = 0$ on $\partial\Omega \times (0, \infty)$ consequently $\frac{\partial p}{\partial \eta} = 0$ on $\partial\Omega \times (0, \infty)$ which implies that $p =$ constant and therefore $h = \nabla p \equiv 0$ in $\Omega \times (0, \infty)$. In view of the second equation of (1.1) we also deduce that

$$\mathrm{curl}\ [u_t \times \widetilde{H}] = 0 \text{ in } \Omega \times (0, \infty). \tag{3.3}$$

Therefore, F can be characterized as the subspace of initial data of finite energy (u_0, u_1, h_0) such that

a) $h = 0$ in $\Omega \times (0, \infty)$;

b) u is the solution of the Lamé system

$$\begin{bmatrix} \rho u_{tt} - \mu\Delta u - (\lambda + \mu)\nabla \mathrm{div}\ u = 0 & \text{in } \Omega \times (0, \infty) \\ u = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x) & \text{in } \Omega \end{bmatrix} \tag{3.4}$$

c) u satisfies (3.3).

We follow the arguments of [3] that use the expansion of solutions of (3.4) in Fourier series on the basis of the eigenfunctions of the elliptic Lamé operator and the following orthogonality property of complex exponentials:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i\alpha t}e^{i\beta t}dt=\delta_{\alpha,\beta}$$

with $\delta_{\alpha,\beta}=0$ if $\alpha,\ \beta\in I\!R$ with $\alpha\neq\beta$ and $\delta_{\alpha,\alpha}=1$. We deduce that (3.3)–(3.4) has a non-trivial solution if and only if, there exists a non-trivial eigenfunction $\omega=\omega(x)\in(H_0^1(\Omega))^3$ of the Lamé system

$$\left.\begin{array}{ll} -\mu\Delta\omega-(\lambda+\mu)\nabla\mathrm{div}\ \omega=\rho\gamma^2\omega & \text{in }\Omega\\ \omega=0 & \text{on }\partial\Omega \end{array}\right] \tag{3.5}$$

such that

$$\mathrm{curl}\ [\omega\times\widetilde{H}]=0\text{ in }\Omega. \tag{3.6}$$

When (3.5)–(3.6) admits a non-trivial solution, it is easy to see that the set F defined in (3.2) contains non-trivial elements too. In this case, obviously, there are solutions of (1.1), (1.2), (1.3) of the form $(u,h)=(e^{i\gamma t}\omega(x),0)$ whose energy remains constant as $t\to+\infty$.

When the only solution of (3.5)–(3.6) is the trivial one, we deduce that F is reduced to zero, that is, $F=\{(0,0,0)\}$. Thus, in view of (3.1), (3.2) allows us to conclude that the energy of smooth solutions of (1.1), (1.2), (1.3) tends to zero as $t\to+\infty$. Then, taking into account that $\mathcal{D}(\mathcal{A})$ is dense in the energy space X and the dissipativity of system (1.1), (1.2), (1.3) one may deduce easily that every solution of finite energy of (1.1), (1.2), (1.3) converges to zero in the energy space as $t\to+\infty$.

We have proved the following result:

Theorem 3.1. *The following two statements are equivalent:*

(a) *Every finite energy solution of (1.1), (1.2), (1.3) is such that* $E(t)\to 0$ *as* $t\to+\infty$;

(b) *The domain* Ω *and the Lamé coefficients* λ, μ *are such that the only eigenfunction of the Lamé system (3.5) such that (3.6) holds, is the trivial one,* $\omega \equiv 0$.

From (3.5)–(3.6) further and more explicit conditions on ω can easily be obtained. Indeed, we write $\widetilde{H} = (0, 0, c)$ where c is a constant. Then, (3.6) can be written as

$$\frac{\partial \omega_1}{\partial x_3} = \frac{\partial \omega_2}{\partial x_3} = \frac{\partial \omega_1}{\partial x_1} + \frac{\partial \omega_2}{\partial x_2} = 0 \qquad \text{in } \Omega.$$

From the vanishing of $\frac{\partial \omega_1}{\partial x_3}$ and $\frac{\partial \omega_2}{\partial x_3}$ and the fact that $\omega_1 = \omega_2 = 0$ on $\partial\Omega$, we deduce that $\omega_1 \equiv \omega_2 \equiv 0$ in Ω. Thus, the problem is reduced to the analysis of whether $\omega_3 \equiv 0$ or not. From the first two equations in (3.5) and the fact that $\omega_1 \equiv \omega_2 \equiv 0$ in Ω we deduce that

$$\frac{\partial^2 \omega_3}{\partial x_1 \partial x_3} = \frac{\partial^2 \omega_3}{\partial x_2 \partial x_3} = 0 \qquad \text{in } \Omega \tag{3.7}$$

while the last equation of (3.5) is equivalent to

$$\begin{aligned} & -\mu\Delta\omega_3 - (\lambda + \mu)\,\frac{\partial^2 \omega_3}{\partial x_3^2} = \rho\gamma^2\omega_3 && \text{in } \Omega \\ & \omega_3 = 0 && \text{on } \partial\Omega \end{aligned} \tag{3.8}$$

Moreover, (3.7) is equivalent to the fact that ω_3 can be written as

$$\omega_3 = \omega_3(x_1, x_2, x_3) = a(x_3) + b(x_1, x_2) \tag{3.9}$$

for suitable H^1-functions $a(\cdot)$ and $b(\cdot, \cdot)$ of one and two variables respectively. In this way we see that condition (b) of Theorem 1.3 is equivalent to the existence of eigenfunctions ω_3 of the scalar elliptic equation (3.8) of the form (3.9).

We have proved the following result:

Theorem 3.2. *The statements (a) and (b) of Theorem 1.3 are equivalent to the following one:*

(c) There is no non-trivial eigenfunction $\omega_3 \in H_0^1(\Omega)$ *of (3.8) of the form (3.9).*

Condition (c) of Theorem 2.3 was analysed in detail in [13]. In that article this question was motivated by a completely different problem: The approximate controllability of the

Lamé system with plannar volume forces. But surprisingly, in both cases, the problems reduce to the analysis of condition (c). It was proved that, under some conditions on the domain Ω, condition (c) was satisfied. It was also shown this conditions to hold generically with respect to the domain Ω. More recently this problem was analyzed in [10] with the techniques in [2]. It was proved that condition (c) holds for any Lipschtiz domain Ω. Therefore, the following holds:

Theorem 3.3. *Let Ω be any bounded, simply connected smooth domain of $\mathbb{R}^3$. Then, every finite energy solution of (1.1), (1.2), (1.3) is such that $E(t) \to 0$ as $t \to +\infty$.*

Remark. Note that, as condition (c) holds for any bounded Lipschtiz domain. Thus Theorem 3.3 can be easily extended to that class of domains. The regularity of the domain Ω was only used when identifying the domains of the operators involved in the semigroup formulation of the system under consideration. □

4. MULTIPLY CONNECTED DOMAINS

Let us discuss the case where Ω is not simply connected. Existence, uniqueness and compactness of trajectories can be proved in the same way as before. Then, ω-limit sets are well defined and they are constituted by solutions such that the magnetic field h satisfies

$$\begin{aligned} &\text{curl } h = 0 \qquad \text{in } \Omega \\ &\text{div } h = 0 \qquad \text{in } \Omega \\ &h \cdot \eta = 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

Suppose that

$$\Omega = \Omega_0 \backslash [\overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_N], \tag{4.2}$$

Ω_0 being an open bounded and connected set of $\mathbb{R}^3$ whose boundary $\partial\Omega_0$ is of class C^2 and Ω_j $(1 \le j \le N)$ are open subsets of Ω_0 with boundaries $\partial\Omega_j$ of class C^2. According to Lemma 1.3 in Temam's book [11], if h solves problem (4.1) then, it belongs to a finite dimensional space of dimension N generated, say, by $\hat{h}_1, \hat{h}_2, \cdots, \hat{h}_N$. Let us briefly recall how the functions $\hat{h}_j$ can be found.

We have to make Ω to become simply connected with a finite number of smooth cuts. More precisely, we can find $\Sigma_1, \cdots, \Sigma_N$, N manifolds of dimension 2 of class C^2 such that $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$ and such that the open set $\dot{\Omega} = \Omega - \Sigma$, where $\Sigma = \cup_{j=1}^N \Sigma_j$, is simply connected and Lipschitzian (i. e., the $\Sigma_i's$ are not tangent to $\partial\Omega$). If h satisfies (6.1) then $h = \nabla p$ in $\dot{\Omega}$ and div $h = \Delta p = 0$ therefore p is C^∞ in $\dot{\Omega}$. In $\overline{\dot{\Omega}}$, p is C^∞ except in a neighbourhood of $\partial\Omega \cap \Sigma$. Since $h \cdot \eta = 0$ on $\partial\Omega$ then $\frac{\partial p}{\partial \eta} = 0$ on $\partial\Omega$. Let us denote by Σ_i^+ and Σ_i^- the two sides of Σ_i and η_i the unit normal on Σ_i; if a function g takes different values on Σ_i^+ and Σ_i^-, then we set

$$[g]_i = g \mid_{\Sigma_i^+} - g \mid_{\Sigma_i^-} .$$

We then set $\hat{h}_j = \nabla \hat{g}_j$, $1 \leq j \leq N$ where the functions $\hat{g}_j$ are constructed in the following way (see [9], Lemma 1.2, p. 461). Each $\hat{g}_j$, $1 \leq j \leq N$ is the unique solution (up to an additive constant) of the problem

$$\begin{cases} \Delta \hat{g}_j = 0 \text{ in } \dot{\Omega} \\ \dfrac{\partial \hat{g}_j}{\partial \eta} = 0 \text{ on } \partial\Omega \\ \left[\dfrac{\partial \hat{g}_j}{\partial \eta_i}\right]_i = 0 \ \text{ on } \Sigma_i, \qquad i = 1, \cdots, N \\ [\hat{g}_j]_i = \delta_{ij} \ \text{ on } \Sigma_i, \qquad i = 1, \cdots, N, \end{cases}$$

where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ otherwise.

Therefore, our problem (1.1)-(1.3) in this case has non-trivial stationary solutions of the form $u = 0$, $h = \hat{h}_j$, $1 \leq j \leq N$. In order to formulate the result about the asymptotic behavior we define the following closed subspace of X:

$$X_1 = \text{span } \left[(0, 0, \hat{h}_1), (0, 0, \hat{h}_2), \cdots, (0, 0, \hat{h}_N)\right]$$

and by X_2 its orthogonal complement in the energy space X. Thus $X = X_1 \oplus X_2$.

Let π be the orthogonal projection $\pi : X \to X_1$. Then, we decompose the solution (u, h) of (1.1)–(1.3) as

$$(u, u_t, h) = (v, v_t, g) + \pi(u, u_t, h)$$

with $(v, v_t, g) = (u, u_t, h) - \pi(u, u_t, h)$.

It is important to observe that $(v, v_t, g) \in X_2$ for all $t > 0$ and that it is a solution of (1.1)-(1.3). On the other hand, $\pi(u, u_t, h) = \pi(u_0, u_1, h_0)$ for all $t > 0$. Indeed, multiplying the equation satisfied by h by $\hat{h}_j$ (the stationary solution of (4.1)) we deduce that

$$\frac{d}{dt} \int_\Omega h\hat{h}_j dx = 0, \qquad \forall t > 0,\ \forall j = 1, 2, \cdots, N.$$

Applying La Salle's invariance principle in X_2 with the energy $E(t)$ as Lyapunov function, and taking into account that the only solution of (4.1) in X_2 is the trivial one, we deduce that (v, v_t, g) converges to zero in X as $t \to +\infty$. Thus, the following holds.

Theorem 4.1. *Consider problem (1.1), (1.2), (1.3) in a multiply connected bounded domain Ω with smooth boundary. Let $(u_0, u_1, h_0) \in X = (H_0^1)^3 \times (L^2(\Omega))^3 \times H$. Then, the solution (u, h) has the following property*

$$u \to 0 \ in \ (H_0^1(\Omega))^3 \tag{4.3}$$

$$u_t \to 0 \ in \ (L^2(\Omega))^3 \tag{4.4}$$

$$h \to \hat{h} \ in \ (L^2(\Omega))^3 \tag{4.5}$$

as $t \to +\infty$ where $\hat{h}$ belongs to the subspace generated by $\hat{h}_1, \hat{h}_2, \cdots, \hat{h}_N$ obtained above. More precisely, $(0, 0, \hat{h}) = \pi(u_0, u_1, h_0) \in X_1$.

Furthermore, $E(t) \to 0$ as $t \to \infty$.

Remark. The main difference with respect to the case in which Ω is simply connected, relies in the asymptotic behavior of h. When Ω is simply connected h tends to zero. However, when Ω is multiply connected the limit of h may be non trivial. □

5. THE SYSTEM OF MAGNETO-THERMOELASTICITY

Along this paper we have neglected thermal effects. The more complete and realistic system of magneto-thermo-elasticity is as follows:

$$\begin{aligned}
&\rho u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \text{div } u - \frac{\mu_0}{4\pi}[\text{curl } h] \times \widetilde{H} + \gamma_1 \nabla\theta = 0 \\
&\beta h_t + \text{curl curl } h - \beta \text{curl } [u_t \times \widetilde{H}] = 0 \\
&\theta_t - \Delta\theta + \gamma_2 \text{div } u_t = 0 \\
&\text{div } h = 0
\end{aligned} \tag{5.1}$$

in $\Omega \times (0,\infty)$,

$$\left.\begin{aligned}
&u = 0 && \text{on } \partial\Omega \times (0,\infty) \\
&\theta = 0 && \text{on } \partial\Omega \times (0,\infty) \\
&h \cdot \eta = 0 && \text{on } \partial\Omega \times (0,\infty) \\
&\text{curl } h \times \eta = 0 && \text{on } \partial\Omega \times (0,\infty)
\end{aligned}\right] \tag{5.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad h(x,0) = h_0(x), \quad \theta(x,0) = \theta_0(x). \tag{5.3}$$

Here the coupling constants γ_1, γ_2 are assumed to be positive.

System (5.1) is a linearization around the constant field $\widetilde{H}$ of the original nonlinear system and it was considered in the work of A.J. Willson [12], where the main concern was the propagation of plane waves.

The total energy in this case will be the sum of the elastic, magnetic and thermal energies:

$$E(t) = \frac{1}{2} \int_\Omega \left[\rho|u_t|^2 + \mu|\nabla u|^2 + (\lambda+\mu)(\text{div } u)^2 + \frac{\mu_0}{4\pi}|h|^2 + \frac{\gamma_1}{\gamma_2}\theta^2 \right] dx. \tag{4.4}$$

A straightforward calculation shows that

$$\frac{dE}{dt} = -\frac{\mu_0}{4\pi\beta} \int_\Omega |\text{curl } h|^2 dx - \frac{\gamma_1}{\gamma_2} \int_\Omega |\nabla\theta|^2 dx.$$

We consider the problem of whether $E(t) \to 0$ as $t \to \infty$ for any finite energy solution of (5.1)–(5.2)–(5.3). Following the arguments of Section 3 the decay problem is reduced to the existence of fields u of the form $u = (0, 0, u_3)$ with $u_3 = a(x_3) + b(x_1, x_2)$ where u_3 is a solution of

$$-\mu \Delta u_3 - (\lambda + \mu) \frac{\partial^2 u_3}{\partial x_3^2} = \gamma^2 u_3 \qquad \text{in } \Omega \tag{5.5}$$

$$\text{div } (0, 0, u_3) = 0 \qquad \text{in } \Omega \tag{5.6}$$

$$u_3 = 0 \qquad \text{on } \partial\Omega. \tag{5.7}$$

Observe that (5.6) arises now since, in view of the dissipation rate of the energy, in the ω-limit set $\theta \equiv 0$. This implies (5.6) by the third equation of (5.1). However div $(0, 0, u_3) = 0$ means $\frac{\partial u_3}{\partial x_3} = 0$ i.e., u_3 is independent of x_3 that is $u_3 = b(x_1, x_2)$. This is impossible unless $u_3 \equiv 0$, that is $u \equiv 0$, since u vanishes on the boundary. Thus, we have the following:

Theorem 5.1. *Let Ω be a bounded domain of $I\!R^3$ of class C^2. Then, every finite energy solution of (5.1), (5.2), (5.3) is such that*

$$E(t) \to 0 \qquad as \qquad t \to +\infty.$$

Moreover,

(a) If Ω is simply connected (u, u_t, h, θ) tends to zero in $(H_0^1(\Omega))^3 \times (L^2(\Omega))^3 \times (L^2(\Omega))^3 \times L^2(\Omega)$ as $t \to \infty$;

(b) If Ω is multiply connected (u, u_t, θ) tends to zero in $(H_0^1(\Omega))^3 \times (L^2(\Omega))^3 \times L^2(\Omega)$ while $h \to \hat{h}$ as $t \to \infty$, where $\hat{h}$ is as described in section 4.

Remark. In contrast to the known results in thermoelasticity [3] where the existence of domains where there is no decay is proved, the above theorem says that when all effects (elastic, magnetic and thermal) are combined there are no singular domains in which solutions may fail to decay. This result is an agreement with those of the previous sections that guarantee that the magnetic effect suffices for the decay of waves to hold. □

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