Some new results related to the null controllability of the $1-d$ heat equation

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Abstract

We address three null controllability problems related to the $1-d$ heat equation. First we show that the $1-d$ heat equation with a rapidly oscillating density is uniformly null controllable as the period of the density tends to zero. We also prove that the same result holds for the finite-difference semi-discretization in space of the constant coefficient heat equation as the step size tends to zero. Finally, we prove that the null controllability of the constant coefficient heat equation can be obtained as limit of null controllability properties for singularly perturbed dissipative wave equations. The proofs combine results on sums of real exponentials, Carleman's inequalities for heat equations and sideways energy estimates for wave equations.

Résumé

Nous étudions trois problèmes liés à la contrôlabilité à zéro de l'équation de la chaleur $1-d$. Tout d'abord on montre que l'équation de la chaleur à dénsité périodique rapidement oscillante est uniformement contrôlable à zéro lorsque la période tend vers zéro. On démontre ensuite que les semi-discretisations en espace par différences finies de l'équation de la chaleur à coefficients constants sont uniformement contrôlables lorsque le pas de la discretization tend vers zéro. On montre aussi que la contrôlabilité à zéro de l'équation de la chaleur $1-d$ peut être obtenue comme une perturbation singulière des propriétés de contrôle à zéro d'équations d'ondes dissipatives.

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1 Introduction and main results

In this paper we present some recent results on the null controllability of $1-d$ heat equations.

In order to introduce the notion of null controllability, let us consider first the constant coefficient heat equation:

$$\left\{\begin{array}{l} u_t - u_{xx} = 0,\ 0 < x < 1,\ 0 < t < T \\ u(0,t) = 0,\ u(1,t) = v(t),\ 0 < t < T \\ u(x,0) = u^0(x),\ 0 < x < 1. \end{array}\right. \tag{1.1}$$

In (1.1) $u = u(x,t)$ is the state and $v = v(t)$ the control which is acting on the right extreme of the interval. System (1.1) is said to be null controllable in time $T > 0$ if for any $u^0 \in L^2(0,1)$ there exists a control $v \in L^2(0,T)$ such that the solution of (1.1) satisfies

$$u(x,T) = 0,\ 0 < x < 1. \tag{1.2}$$

It is by now well known that system (1.1) is null controllable for all $T > 0$. In fact the null controllability of the linear heat equation holds in a much more general multi-dimensional setting (see for instance [F], [FI] and [LR]). Moreover we have the following bound for the control of system (1.1):

$$\| v \|_{L^2(0,T)} \leq C \| u^0 \|_{L^2(0,1)},\ \forall u^0 \in L^2(0,1), \tag{1.3}$$

where C is a positive constant depending only on the time T.

The null controllability of system (1.1) is equivalent to an observability estimate for the adjoint system:

$$\left\{\begin{array}{ll} -\varphi_t - \varphi_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ \varphi(x,T) = \varphi^0(x), & 0 < x < 1 \\ \varphi(0,t) = \varphi(1,t) = 0 & 0 < t < T. \end{array}\right. \tag{1.4}$$

More precisely, system (1.1) is null controllable in time $T > 0$ if and only if there exists a positive constant $C > 0$ such that

$$\| \varphi(0) \|^2_{L^2(0,1)} \leq C \int_0^T |\varphi_x(1,t)|^2\, dt \tag{1.5}$$

for all solution of (1.4).

In this paper the stability of the null controllability property of system (1.1) is studied under three different perturbations:

a.- rapidly oscillating coefficients;

b.- singular perturbation;

c.- space semi-discretization.

1.1 Rapidly oscillating periodic density

Let us consider first the heat equation with a periodic rapidly oscillating density:

$$
\left\{
\begin{array}{ll}
\rho\left(\frac{x}{\varepsilon}\right) u_t - u_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\
u(0,t) = 0, \; u(1,t) = v(t), & 0 < t < T \\
u(x,0) = u^0(x), & 0 < x < 1,
\end{array}
\right.
\tag{1.6}
$$

where $0 < \varepsilon < 1$ and ρ is assumed to be a C^2, periodic function of period ξ such that

$$
0 < \rho_0 \le \rho(x) \le \rho_1 < \infty, \; \forall x \in I\!R \tag{1.7}
$$

for suitable positive constant $\rho_0, \rho_1 > 0$.

Passing to the limit as $\varepsilon \to 0$ in (1.6) formally we obtain the averaged system

$$
\left\{
\begin{array}{ll}
\bar{\rho} u_t - u_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\
u(0,t) = 0, \; u(1,t) = v(t), & 0 < t < T \\
u(x,0) = u^0(x), & 0 < x < 1,
\end{array}
\right.
\tag{1.8}
$$

where $\bar{\rho}$ is the average of ρ over a period: $\bar{\rho} = \frac{1}{\xi}\int_0^\xi \rho(x)dx$.

We have the following result:

Theorem 1.1 *Assume that ρ is C^2, periodic of period ξ and such that* (1.7) *holds. Then, for any $T > 0$, systems* (1.6) *are uniformly null controllable as $\varepsilon \to 0$. More precisely, for any $u^0 \in L^2(0,1)$ and $0 < \varepsilon < 1$ there exists a control $v_\varepsilon \in L^2(0,T)$ such that the solution of* (1.6) *satisfies* (1.2). *Moreover there exists a positive constant $C > 0$ such that*

$$
\| v \|_{L^2(0,T)} \le C \| u^0 \|_{L^2(0,1)}, \; \forall u^0 \in L^2(0,1), \; \forall 0 < \varepsilon < 1. \tag{1.9}
$$

Finally, the controls v_ε of (1.6) *may be built so that*

$$
v_\varepsilon \to v \text{ in } L^2(0,T) \text{ as } \varepsilon \to 0, \tag{1.10}
$$

$v \in L^2(0,T)$ being a null control for the limit system (1.8).

Some remarks are in order.

Remark 1.1

(a) Slightly changing the proof of Theorem 1.1 we shall present below, the C^2 assumption on ρ may be relaxed to assume that ρ is of class C^1. However, in this case the result holds only when $T > T_0$, for a suitable $T_0 > 0$ that depends on ρ.

(b) The uniform controllability result of Theorem 1.1 is equivalent to an uniform observability inequality for the corresponding adjoint systems

$$(1.11)\qquad \begin{cases} -\rho(x/\varepsilon)\varphi_t - \varphi_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ \varphi(0,T) = \varphi(1,t) = 0, & 0 < t < T \\ \varphi(x,T) = \varphi^0(x), & 0 < x < 1. \end{cases}$$

More precisely, as a consequence of Theorem 1.1 it follows that for any $T > 0$ there exists a positive constant $C(T) > 0$, independent of ε, such that

$$(1.12)\qquad \| \varphi(x,0) \|^2_{L^2(0,1)} \leq C(T) \int_0^T |\varphi_x(1,t)|^2 \, dt$$

for every solution of (1.11) and $0 < \varepsilon < 1$.

Note that, in this case, the uniform observability (1.12) is obtained as a consequence of the uniform null controllability result of Theorem 1.1.

(c) The analogue of Theorem 1.1 is false in the context of the wave equation with rapidly oscillating density. In that case, for fixed initial data the controls may blow up as $\varepsilon \to 0$. This is due to the existence of eigenfunctions of the underlying elliptic eigenvalue problem for which most of the energy is concentrated far away from the extreme $x = 1$. The corresponding eigenvalues are of the order of $\lambda \sim \varepsilon^{-2}$. In the context of the heat equation the existence of these eigenfunctions is not an obstacle to the uniform observability to hold because of the strong dissipativity of the system for these frequencies.

■

The proof of Theorem 1.1 combines the following ingredients. First, using the results in [CZ] on the convergence of the spectrum as $\varepsilon \to 0$ and classical results on series of real exponentials (see [FR] and [K]) we show that the projection of the solutions over the subspace generated by the eigenfunctions with eigenvalues $\lambda \leq c\,\varepsilon^{-2}$, with $c > 0$ small enough, are uniformly controllable. On the other hand, using Carleman's inequalities as in [F] and [FI] one can show the existence of controls of the order of $\exp\left(C\,\varepsilon^{-4/3}\right)$ as $\varepsilon \to 0$. This two facts may be combined with the dissipativity in order to obtain the uniform null controllability of the system. For this, we apply the following control strategy in three steps: we divide the time interval $[0,T]$ in three consecutive subintervals: $[0,T] = I_1 \cup I_2 \cup I_3$. In the first one we control the proyection of the solution on the low frequencies. In I_2 we let the sytem evolve freely without control. In the last subinterval we control the whole solution. This leads to controls of the form $v_\varepsilon = v^1_\varepsilon 1_{I_1} + v^3_\varepsilon 1_{I_3}$, where 1_{I_j} denotes the characteristic function of the interval I_j. As we shall see, the controls obtained this way are uniformly bounded with respect to $\varepsilon \to 0$.

Finally there are two possibilities to study the convergence of the sequence of controls. First, we can analyze the behavior of the control $v_\varepsilon = v^1_\varepsilon 1_{I_1} + v^3_\varepsilon 1_{I_3}$ mentioned above. It is easy to see that v_ε^3 tends to zero exponentially while v_ε^1 converges to a control for the limit system (1.8) in the interval I_1. On the other hand we can also consider the controls of minimal L^2-norm given by the Hilbert Uniqueness Method (HUM) (see J.L. Lions [L]) in the whole interval $[0,T]$. It can be shown that, for a fixed initial datum $u^0 \in L^2(0,1)$, the control of minimal norm for (1.6) converges to the control of minimal norm for (1.8) as $\varepsilon \to 0$.

1.2 A singular perturbation problem

Let us consider the following damped wave equation

$$\text{(1.13)} \qquad \begin{cases} \varepsilon\, u_{tt} - u_{xx} + u_t = v 1_\omega, & 0<x<1, \quad 0<t<T \\ u(0,t) = u(1,t) = 0, & 0<t<T \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x), & 0<x<1 \end{cases}$$

with $0<\varepsilon<1$. In (1.13) $v = v(x,t)$ is the control and χ_ω denotes the characteristic function of an open non-empty subset ω of $(0,1)$ where the control is supported.

System (1.13) is null controllable, in fact it is exactly controllable, for any $\varepsilon > 0$ provided $T > 2\sqrt{\varepsilon}$. More precisely, for any $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ there exists a control $v_\varepsilon \in L^2(\omega \times (0,T))$ such that the solution of (1.13) satisfies

$$\text{(1.14)} \qquad u(x,T) = u_t(x,T) = 0,\ 0<x<1.$$

On the other hand, the formal limit of (1.13) as $\varepsilon \to 0$ is the heat equation

$$\text{(1.15)} \qquad \begin{cases} u_t - u_{xx} = v 1_\omega, & 0<x<1, \quad 0<t<T \\ u(0,t) = u(1,t) = 0, & 0<t<T \\ u(x,0) = u^0(x), & 0<x<1 \end{cases}$$

which is null controllable for all $T>0$.

The following result asserts that the null controllability of the heat equation may be obtained as limit of null controllability results for systems (1.13) as $\varepsilon \to 0$.

Theorem 1.2 *Let $T>0$. System* (1.13) *is uniformly null controllable in time T for any $0<\varepsilon<\frac{T^2}{4}$. More precisely, for any $(u^0,u^1) \in H_0^1(0,1) \times L^2(0,1)$ and $0<\varepsilon<\frac{T^2}{4}$ there exists a control $v_\varepsilon \in L^2(\omega \times (0,T))$ such that the solution u_ε of* (1.13) *verifies* (1.14) *and moreover there exists a constant $C>0$, independent of ε, such that*

$$\text{(1.16)} \qquad \| v \|_{L^2(\omega\times(0,T))} \leq C \left\| \left(u^0, u^1\right) \right\|_{H_0^1(0,1)\times L^2(0,1)}$$

for all $(u^0,u^1) \in H_0^1(0,1) \times L^2(0,1)$ and $0<\varepsilon<\frac{T^2}{4}$.

Finally, the controls v_ε of (1.13) *may be built so that*

$$\text{(1.17)} \qquad v_\varepsilon \to v \text{ in } L^2(\omega\times(0,T)) \text{ as } \varepsilon \to 0,$$

where v is a null control for the heat equation (1.15).

The proof of Theorem 1.2 is based on a control strategy in three steps similar to the one described above. First, using Fourier series, we prove the uniform controllability of the parabolic projections of the solution. On the other hand, using the sideways energy estimates in [Z] we can show that there exists a control of the order of $\exp\left(C\,\varepsilon^{-1/2}\right)$ driving the whole solution to the rest state. Then, taking advantage of the dissipativity of systems (1.13) and using a control strategy in three steps we deduce that Theorem 1.2 holds.

1.3 Space semi-discretizations

Given $N \in I\!N$ we set $h = 1/(N+1)$ and consider the following semi-discretizations of the heat equation

$$\left\{\begin{array}{ll} u_j'(t) - \left[\frac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{h^2}\right] = 0, & 0 < t < T,\, j = 1,\cdots,N \\ u_0(t) = 0,\, u_{N+1}(t) = v(t), & 0 < t < T \\ u_j(0) = u_j^0,\, j = 1,\cdots,N. & \end{array}\right. \tag{1.18}$$

System (1.8) is a finite-difference space semi-discretization of the heat equation with control on the extreme $x = 1$ (which corresponds to the value $N+1$ of the index j since u_j is an approximation of the value of the solution u of the continuous heat equation at the node $x_j = jh$).

The following result guarantees the uniform controllability of systems (1.18) as $h \to 0$:

Theorem 1.3 *Let $T > 0$. Systems* (1.18) *are uniformly null controllable as $h \to 0$. More precisely for any $\left\{u_j^0\right\}_{j=1}^N$ and $h > 0$ there exists a control $v_h \in L^2(0,T)$ such that the solution of system* (1.18) *satisfies*

$$u_j(T) = 0,\, j = 1,\cdots,N. \tag{1.19}$$

Moreover, there exists a constant $C > 0$, independent of h, such that

$$\| v \|^2_{L^2(0,T)} \leq Ch \sum_{j=1}^{N} | u_j^0 |^2, \tag{1.20}$$

for all $\left\{u_j^0\right\}_{j=1}^N$ and $h > 0$.

Finally, given $u^0 \in L^2(0,1)$ the controls v_h of system (1.18) *may be built such that*

$$v_h \to v \text{ in } L^2(0,T) \text{ as } h \to 0 \tag{1.21}$$

where v is a null control for the continuous heat equation provided the initial data in (1.18) *are chosen in an appropriate way.*

Some remarks are in order:

Remark 1.2

(a) The analogue of Theorem 1.3 is false for the semi-discrete wave equation (see [IZ]). This is due to the fact that the semi-discrete system presents spurious high frequency oscillations. In the case of the heat equation this spurious modes are damped out by the system. Therefore, the situation is similar to the problem discussed in section 1.1 concerning the heat equation with rapidly oscillatory coefficients.

(b) In order to prove the convergence (1.21) the initial data of the semi-discrete system (1.18) have to be chosen in an appropriate way. Given $u^0 \in L^2(0,1)$ we develop it in Fourier series

$$u^0 = \sum_{k=1}^{\infty} a_k \sin(k\pi x) . \tag{1.22}$$

The initial data for system (1.17) can then be taken as follows:

$$u_j^0 = \sum_{k=1}^{N} a_k \sin(k\pi j h) ,\ j = 1, \cdots, N \tag{1.23}$$

which is the value at the node $x_j = jh$ of the Fourier series of u^0 truncated at $k = N$

■

The proof of Theorem 1.3 is simpler than those of Theorem 1.1 and 1.2. The spectrum of system (1.18) can be computed explicitly. Therefore, Theorem 1.3 can be proved directly using classical results on series of real exponentials.

■

The three problems we have considered here make sense in several space dimensions. All of then are widely open.

The rest of this paper is devoted to give a sketch of the proof of these results.

2 The heat equation with rapidly oscillatory coefficients

This section is devoted to give a sketch of the proof of Theorem 1.1. The first subsection is devoted to recall some results on the spectrum of the system. In the second one we obtain the uniform null-controllability of the low frequencies. In the third one we apply Carleman's inequalities to prove the null controllability of the whole solution with controls of exponentially growing size as $\varepsilon \to 0$. Finally, we show that a control strategy in three steps allows us to build controls that remain bounded as $\varepsilon \to 0$.

2.1 Preliminaries on the spectrum of the system

Let us consider the eigenvalue problem associated with system (1.6):

$$\begin{cases} -w_{xx} = \lambda\rho(x/\varepsilon)w, & 0 < x < 1 \\ w(0) = w(1) = 0. \end{cases} \tag{2.1}$$

For any $\varepsilon > 0$, system (2.1) admits a sequence of distinct eigenvalues

$$0 < \lambda_{1,\varepsilon} < \lambda_{2,\varepsilon} < \cdots < \lambda_{k,\varepsilon} < \cdots \to \infty.$$

On the other hand, the corresponding eigenfunctions $\{w_{j,\varepsilon}\}_{j\geq 1}$ may be chosen to constitute an orthonormal basis of $L^2\left((0,1), \rho(x/\varepsilon)\right)$.

The limit (as $\varepsilon \to 0$) eigenvalue problem is

$$\begin{cases} -w_{xx} = \lambda\bar{\rho}w, & 0 < x < 1 \\ w(0) = w(1) = 0 \end{cases} \tag{2.2}$$

where the eigenvalues and eigenfunctions may be computed explicitly

$$\lambda_k = \frac{k^2\pi^2}{\bar{\rho}},\ w_k(x) = \sin(k\pi x). \tag{2.3}$$

It is well known that, for each $j \geq 1$ fixed,

$$\lambda_{j,\varepsilon} \to \lambda_j \text{ as } \varepsilon \to 0, \tag{2.4}$$

but this fact is far from being sufficient to address the problem under consideration.

The following sharp result was proved in [CZ]:

Proposition 2.1 *There exist $c, \gamma > 0$ such that*

$$\min_{\lambda \leq c\,\varepsilon^{-2}} \left|\sqrt{\lambda_{j+1,\varepsilon}} - \sqrt{\lambda_{j,\varepsilon}}\right| \geq \gamma > 0. \tag{2.5}$$

Moreover, there exist positive constants C_1 and C_2, independents of ε, such that

$$C_1\left|\partial_x w_{j,\varepsilon}(1)\right|^2 \leq \int_0^1 \left|\partial_x w_{j,\varepsilon}\right|^2 dx \leq C_2\left|\partial_x w_{j,\varepsilon}(1)\right|^2 \tag{2.6}$$

for all eigenfunctions of (2.1) *such that $\lambda_{j,\varepsilon} \leq c\,\varepsilon^{-2}$ and all $0 < \varepsilon < 1$.*

Remark 2.1 The result of Proposition 2.1 is sharp. It is by now well known that there are eigenvalues λ of the order of $c\,\varepsilon^{-2}$, for a suitable $c > 0$, such that the energy concentrated on the extreme $x = 1$ of the boundary is exponentially small (as $\varepsilon \to 0$) with respect to its total energy (see [ABR] and [CZ]). The gap condition (2.5) also fails for eigenvalues of this order. However, given any $k \in \mathbb{N}$, if ρ is sufficiently smooth, (2.5) and (2.6) are again true for $\lambda \geq C\,\varepsilon^{-2-2/k}$. This indicates that the uniform gap condition (2.5) and the uniform observability property (2.6) for the eigenfunctions only fails when λ is of the order of ε^{-2}.

■

We develop solutions of (1.11) in Fourier series:

$$\varphi_\varepsilon(x,t) = \sum_{j\geq 1} a_{j,\varepsilon} e^{-\lambda_{j,\varepsilon}(T-t)} w_{j,\varepsilon}(x), \tag{2.7}$$

where $\{a_{j,\varepsilon}\}$ are the Fourier coefficients of the datum φ^0 in the basis $\{w_{j,\varepsilon}\}$ of $L^2((0,1), \rho(x/\varepsilon))$.

Let us consider the finite-dimensional subspace E_ε of $L^2((0,1), \rho(x,\varepsilon))$ generated by the eigenfunctions of (2.1) associated with the low frequencies $\lambda \leq c\varepsilon^{-2}$, where $c > 0$ is as in Proposition 2.1. In other words,

$$E_\varepsilon = \underset{\lambda_{j,\varepsilon} \leq c\varepsilon^{-2}}{\text{span}} \{w_{j,\varepsilon}\}. \tag{2.8}$$

We also denote by π_ε the orthogonal projection from $L^2((0,1), \rho(x,\varepsilon))$ over E_ε.

It is easy to see that if the initial datum φ^0 of (1.11) is such that $\pi_\varepsilon\varphi^0 = 0$ then the solution φ_ε of (1.11) satisfies $\pi_\varepsilon\varphi_\varepsilon(t) = 0$ for all $0 \leq t \leq T$. Moreover

$$\| \varphi_\varepsilon(t) \|^2_{L^2((0,1))} \leq Ce^{-c(t-T)/\varepsilon^2} \| \varphi^0 \|^2_{L^2(0,1)}, \tag{2.9}$$

for all $\varphi^0 \in L^2(0,1)$ such that $\pi_\varepsilon\varphi^0 = 0$, $0 \leq t \leq T$ and $0 < \varepsilon < 1$.

2.2 Uniform controllability of the low frequencies

The following result on the sum of real exponentials can be derived from the results in [FR]. Similar results can also be found in [K]. Given $\xi > 0$ and a decreasing function $N : (0,\infty) \to I\!N$ such that $N(\delta) \to \infty$ as $\delta \to 0$, we introduce the class $\mathcal{L}(\xi, N)$ of increasing sequences of positive real numbers $\{\mu_j\}_{j\geq 1}$ such that

$$\mu_{j+1} - \mu_j \geq \xi > 0, \; \forall j \geq 1, \tag{2.10}$$

$$\sum_{k\geq N(\delta)} \mu_k^{-1} \leq \delta, \; \forall \delta > 0. \tag{2.11}$$

The following holds:

Proposition 2.2 *Given a class of sequences $\mathcal{L}(\xi, N)$ and $T > 0$ there exists a constant $C > 0$ (which depends on ξ, N and T) such that*

$$\int_0^T \left| \sum_{k=1}^\infty a_k e^{-\mu_k t} \right|^2 dt \geq \frac{C}{\left(\sum_{k\geq 1} \mu_k^{-1} \right)} \sum_{k\geq 1} \frac{|a_k|^2 e^{-2\mu_k T}}{\mu_k} \tag{2.12}$$

for all $\{\mu_j\} \in \mathcal{L}(\xi, N)$ and all bounded sequence of real numbers.

It is easy to see that, in view of Proposition 2.1, the eigenvalues $\{\lambda_{j,\varepsilon}\}$ of system (2.1) such that $\lambda \le c\,\varepsilon^{-2}$ are in the conditions of Proposition 2.2. To be more precise, we introduce the sequences

$$\mu_{j,\varepsilon} = \begin{cases} \lambda_{j,\varepsilon}, & \text{if} \qquad \lambda_{j,\varepsilon} \le c\,\varepsilon^{-2} \\ Cj^2, & \text{otherwise} \end{cases} \tag{2.13}$$

with $C > 0$ large enough so that for each $\varepsilon > 0$ all the eigenvalues $\mu_{j,\varepsilon}$ are distinct. Of course, this can be done since, taking into account that the density ρ satisfies (1.7), using Rayleigh cocient, it is easy to see that the eigenvalues $\lambda_{j,\varepsilon}$ satisfy

$$\alpha j^2 \le \lambda_{j,\varepsilon} \le \beta j^2 \tag{2.14}$$

for all $j \ge 1$, $0 < \varepsilon < 1$ and suitable positive constants α, β. Then, in view of (2.5) and (2.14) it is easy to see that there exists $\xi > 0$ and a decreasing function $N : (0, \infty) \to I\!N$ such that $N(\delta) \to \infty$ as $\delta \to 0$ and so $\{\mu_{j,\varepsilon}\} \in \mathcal{L}(\xi, N)$ for all $0 < \varepsilon < 1$.

We now consider an initial datum $\varphi^0 \in E_\varepsilon$. It can be written in the form

$$\varphi^0 = \sum_{\lambda_{j,\varepsilon} \le c\,\varepsilon^{-2}} a_{j,\varepsilon} w_{j,\varepsilon}(x) \tag{2.15}$$

and the corresponding solution φ_ε of (2.11) is given by

$$\varphi^\varepsilon = \sum_{\lambda_{j,\varepsilon} \le c\,\varepsilon^{-2}} a_{j,\varepsilon} e^{-\lambda_{j,\varepsilon}(T-t)} w_{j,\varepsilon}(x). \tag{2.16}$$

Therefore

$$\partial_x \varphi^\varepsilon(1,t) = \sum_{\lambda_{j,\varepsilon} \le c\,\varepsilon^{-2}} a_{j,\varepsilon} e^{-\lambda_{j,\varepsilon}(T-t)} \partial_x w_{j,\varepsilon}(1). \tag{2.17}$$

In view of Proposition 2.2 we deduce the following result:

Proposition 2.3 *Given any $T > 0$, there exists a positive constant $C(T) > 0$ such that*

$$\| \varphi_\varepsilon(\cdot, 0) \|^2_{L^2(0,1)} \le C(T) \int_0^T |\partial_x \varphi_\varepsilon(1,t)|^2 \, dt \tag{2.18}$$

for any solution of (1.11) *corresponding to initial data* $\varphi^0 \in E_\varepsilon$ *and any* $0 < \varepsilon < 1$.

Note that Proposition 2.3 guarantees the uniform observability of solutions of the adjoint system consisting only on the low frequencies. As a consequence of Proposition 2.3 the following uniform controllability result of the low frequencies holds:

Proposition 2.4 *For any $T > 0$, $0 < \varepsilon < 1$ and $u^0 \in L^2(0,1)$ there exists a control $v_\varepsilon \in L^2(0,T)$ such that the solution of* (1.6) *satisfies*

$$\pi_\varepsilon\,(u_\varepsilon(T)) = 0. \tag{2.19}$$

Moreover, there exists a constant $C(T) > 0$ independent of $0 < \varepsilon < 1$ such that

$$\| v_\varepsilon \|_{L^2(0,T)} \le C(T) \| u^0 \|_{L^2(0,1)} \tag{2.20}$$

for all $u^0 \in L^2(0,1)$.

2.3 Global non-uniform controllability

As we have explained in the introduction, to apply the control method in three steps we need a global controllability result even if it is not uniform on ε. We apply Carleman's inequalities as in [FI] and [F] to derive the corresponding observability inequality.

To do that we consider a more general equation of the form

$$\begin{cases} a(x)\theta_t + \theta_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ \theta(0,t) = \theta(1,t) = 0, & 0 < t < T \\ \theta(x,t) = \theta^0(x), & 0 < x < 1, \end{cases} \tag{2.21}$$

where a is a C^2 density such that

$$0 < a_0 \le a(x) \le a_1 < \infty, \, \forall x \in (0,1). \tag{2.22}$$

The following holds:

Proposition 2.5 *For any $T > 0$ there exist constants $C_1(T) > 0$ and $C_2 > 0$ such that*

$$\| \theta(\cdot,0) \|^2_{L^2(0,1)} \le C_1(T) \exp\left(C_2 \| a \|_{W^{1,\infty}} + C_1(T) \| a \|^{2/3}_{W^{2,\infty}} \right) \int_0^T |\theta_x(1,t)|^2 \, dt \tag{2.23}$$

for every solution of (2.21)*, and for all a as above.*

Sketch of the proof. By a classical change of variables equation (2.21) can be transformed into an equation of the form

$$\psi_t + \psi_{xx} + b(x)\psi = 0. \tag{2.24}$$

This can be done so that $b \in C([0,T])$ and

$$\| b \|_{L^\infty(0,1)} \le C \| a \|_{W^{2,\infty}(0,1)} .$$

We apply the Carleman's inequalities in [FI] and [F] to (2.24). Going back to the original variables we deduce (2.23).

■

Remark 2.2 Assume that $a \in C^1([0,1])$ instead of $a \in C^2([0,1])$. Then, equation (2.21) can be transformed into an equation of the form

$$\psi_t + \psi_{xx} + c(x)\psi_x = 0 \tag{2.25}$$

with $c \in C([0,1])$ such that $\| c \|_{L^\infty(0,1)} \le C \| a \|_{W^{1,\infty}(0,1)}$.

One can also apply Carleman's inequalities in (2.25). In that case we obtain an observability inequality of the form

$$\| \theta(\cdot,0) \|^2_{L^2(0,1)} \le C_1 \exp\left(C_2 \left(1 + \| a \|^2_{W^{1,\infty}} \right) \right) \int_0^T |\theta_x(1,t)|^2 \, dt. \tag{2.26}$$

■

We now apply Proposition 2.4 to system (1.11). Setting $a(x) = \rho(x/\varepsilon)$, it is easy to see that

$$\| a \|_{W^{1,\infty}} \sim 1/\varepsilon; \ \| a \|_{W^{2,\infty}} = 1/\varepsilon^2 \ \text{ as } \ \varepsilon \to 0.$$

Thus, the following holds:

Proposition 2.6 *For any $T > 0$ there exists a positive constant $C(T) > 0$ such that*

$$\| \varphi(\cdot, 0) \|^2_{L^2(0,1)} \leq C(T) \exp\left(C(T)\,\varepsilon^{-4/3}\right) \int_0^T |\partial_x \varphi(1,t)|^2 \, dt \tag{2.27}$$

for every solution of (1.11) *and all* $0 < \varepsilon < 1$.

Note that Proposition 2.6 provides a observability inequality for all solutions of (1.11). However, the observability constant in (2.27) blows up exponentially as $\varepsilon \to 0$.

As a consequence of Proposition 2.6 the following non-uniform controllability result holds:

Proposition 2.7 *For any $T > 0$, $0 < \varepsilon < 1$ and $u^0 \in L^2(0,1)$ there exists a control $v_\varepsilon \in L^2(0,T)$ such that the solution u_ε of* (1.6) *satisfies* (1.2). *Moreover, there exists a constant $C(T) > 0$ such that*

$$\| v_\varepsilon \|_{L^2(0,T)} \leq C(T) \exp\left(C(T)\,\varepsilon^{-4/3}\right) \| u^0 \|_{L^2(0,1)} \tag{2.28}$$

for all $u^0 \in L^2(0,1)$ and all $0 < \varepsilon < 1$.

2.4 Conclusion: Global uniform controllability in three steps

Given $T > 0$, we divide the control interval $[0,T]$ in three subintervals $[0,T] = I_1 \cup I_2 \cup I_3$. For symplicity we are going to take $I_j = [(j-1)T/3, \ jT/3]$, $j = 1,2,3$.

We fix an initial datum $u^0 \in L^2(0,1)$. We now build a control $v_\varepsilon \in L^2(0,T)$ for all $0 < \varepsilon < 1$ such that the solution of (1.6) satisfies (1.2) and that remains bounded in $L^2(0,T)$ as $\varepsilon \to 0$. We proceed in three steps. In the first subinterval I_1 we apply Proposition 2.4. In this way we deduce the existence of a bounded (as $\varepsilon \to 0$) family of controls $v^1_\varepsilon \in L^2(0,T/3)$ such that the solution u_ε of (1.6) satisfy

$$\pi_\varepsilon\left(u_\varepsilon(T/3)\right) = 0. \tag{2.29}$$

Then, in the second time interval I_2 we let the equation (1.6) evolve freely without control. Of course, the condition (2.29) is kept, i.e.

$$\pi_\varepsilon\left(u_\varepsilon(t)\right) = 0, \ \forall T/3 \leq t \leq 2T/3.$$

On the other hand, in view of (2.20) and by uniform well-posedness of system (1.6) in $L^2(0,1)$ we deduce that

$$\| u_\varepsilon(T/3) \|_{L^2(0,1)} \leq C \| u^0 \|_{L^2(0,1)} \tag{2.30}$$

for a suitable $C > 0$ independent of $0 < \varepsilon < 1$. Combining (2.9) (which can be applied to u_ε in the interval I_2 since the control vanishes there), (2.29) and (2.30) we deduce that

$$\| u_\varepsilon(2T/3) \|_{L^2(0,1)} \leq Ce^{-CT/\varepsilon^2} \| u^0 \|_{L^2(0,1)} \tag{2.31}$$

for a suitable constant $C > 0$ independent of $0 < \varepsilon < 1$.

We then apply Proposition 2.7 in the time interval I_3. We deduce the existence of a control $v_\varepsilon^3 \in L^2(I_3)$ such that

$$\| v_\varepsilon^3 \|_{L^2(I_3)} \leq Ce^{C\,\varepsilon^{-4/3}} \| u(2T/3) \|_{L^2(0,T)} \tag{2.32}$$

and

$$u_\varepsilon(T) = 0. \tag{2.33}$$

According to (2.31)-(2.32) it follows that

$$\| v_\varepsilon^3 \|_{L^2(I_3)} \leq C \exp\left(C\,\varepsilon^{-4/3} - cT\,\varepsilon^{-2}\right) \| u^0 \|_{L^2(0,1)} \,. \tag{2.34}$$

Obviously

$$\| v_\varepsilon^3 \|_{L^2(I_3)} \to 0 \text{ exponentially as } \varepsilon \to 0. \tag{2.35}$$

The control $v_\varepsilon = v_\varepsilon^1 1_{I_1} + v_\varepsilon^3 1_{I_3}$ we have built is such that

$$\| v_\varepsilon \|_{L^2(0,T)} \leq \| u^0 \|_{L^2(0,1)}, \tag{2.36}$$

with $C > 0$ independent of $0 < \varepsilon < 1$.

2.5 Passage to the limit

There are two ways of passing to the limit in the controls v_ε.

First, we can pass to the limit in the controls we have built in the previous section. As indicated in (2.35) the control v_ε^3 tends to zero exponentially. On the other hand one can show that the control v_ε^1 converges in $L^2(0, T/3)$ to a null control $v \in L^2(0, T/3)$ of the limit heat equation (1.8). This control is such that the solution of (1.8) satisfies

$$u(T/3) = 0 \tag{2.37}$$

and therefore, by taking $v \equiv 0$ for $t \geq T/3$, it also satisfies $u(t) = 0$, for all $t \geq T/3$.

In order to pass to the limit in the controls we can also consider the controls v_ε of minimal $L^2(0,T)-$norm for (1.6). This controls may be characterized by HUM (see [L]) as follows. Given $u^0 \in L^2(0,1)$ we define the functional $J_\varepsilon : H_\varepsilon \to I\!R$ where

$$J_\varepsilon\left(\varphi^0\right) = \frac{1}{2}\int_0^T |\partial_x \varphi_\varepsilon(1,t)|^2\,dt - \int_0^1 \rho(x/\varepsilon) u^0(x) \varphi_\varepsilon(x,0) dx, \tag{2.38}$$

φ_ε being the solution of (1.11) with initial datum φ^0 and H_ε the Hilbert space

$$H_\varepsilon = \left\{ \varphi^0 \in L^2(0,1) : \text{ the solution of (1.11) satisfies } \int_0^T |\partial_x \varphi_\varepsilon(1,t)|^2\,dt < \infty \right\} \tag{2.39}$$

endowed with the natural norm

$$\| \varphi^0 \|_{H_\varepsilon} = \left(\int_0^T |\partial_x \varphi_\varepsilon(1,t)|^2 \, dt \right)^{1/2}.$$

This functional is continuous and convex. Moreover it is uniformly (with respect to $\varepsilon \to 0$) coercive. Indeed, it is easy to see by duality that the uniform bound (2.36) on the control implies the uniform observability inequality (1.12) for the adjoint system (1.11). Then, each J_ε has a unique minimizer φ_ε^0 in H_ε. The uniform coercivity implies that

$$\| \varphi_\varepsilon^0 \|_{H_\varepsilon} \leq C, \ \forall 0 < \varepsilon < 1, \tag{2.40}$$

which, in view of the definition of the norm $\| \cdot \|_{H_\varepsilon}$, implies that the corresponding solutions of (1.11) satisfy

$$\int_0^T |\partial_x \varphi_\varepsilon(1,t)|^2 \, dt \leq C, \ \forall 0 < \varepsilon < 1. \tag{2.41}$$

The control v_ε of minimal $L^2(0,T)-$norm for (1.6) is precisely

$$v_\varepsilon(t) = \partial_x \varphi_\varepsilon(1,t). \tag{2.42}$$

One can then show that

$$\partial_x \varphi_\varepsilon(1,t) \to \partial_x \varphi(1,t) \text{ in } L^2(0,T), \tag{2.43}$$

where φ is the solution of the limit adjoint system

$$\begin{cases} \bar{\rho}\varphi_t + \varphi_{xx} = 0, & 0 < x < 1, \quad 0 < t < T, \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T, \\ \varphi(x,T) = \varphi^0(x) \end{cases} \tag{2.44}$$

with the initial datum minimizing the limit functional

$$J(\varphi^0) = \frac{1}{2} \int_0^T |\partial_x \varphi(1,t)|^2 \, dt - \bar{\rho} \int_0^1 u^0(x) \varphi(x,0) dx \tag{2.45}$$

in the Hilbert space

$$H = \left\{ \varphi^0 : \text{ the solution } \varphi \text{ of (2.44) satisfies } \int_0^T |\varphi_x(1,t)|^2 \, dt < \infty \right\} \tag{2.46}$$

endowed with the natural norm.

This limit control v turns out to be the one of minimal $L^2(0,T)-$norm for the limit system (1.8).

3 The heat equation as singular limit of dissipative wave equations

This section is devoted to prove Theorem 1.2. We follow the scheme of section 2. The main differences with the proof of Theorem 1.1 are the following:

(a) The uniform controllability of the low frequencies is proved by the iterative method introduced by G. Lebeau and L. Robbiano in [LR]. An alternative proof can be given using nonharmonic Fourier series results, see [FR] and [K].

(b) The non-uniform global controllability is proved by means of sideways energy estimates instead of using Carleman's inequalities.

3.1 Preliminaries on the spectrum of the system

Let us consider the eigenvalue problem associated with system (1.13):

$$\begin{cases} -w_{xx} = -\varepsilon\,\lambda^2 w - \lambda w, & 0 < x < 1 \\ w(0) = w(1) = 0. \end{cases} \tag{3.1}$$

The eigenvalues can be computed explicitly

$$\lambda^{\varepsilon}_{k,\pm} = -\frac{1 \pm \sqrt{1 - 4\pi^2 k^2\,\varepsilon}}{2\,\varepsilon}. \tag{3.2}$$

The eigenfunctions are also explicit and independent of ε:

$$w^{\varepsilon}_{k,\pm} = \sin(k\pi x). \tag{3.3}$$

The solution of (1.13) without control (i.e. $v \equiv 0$) may be developed in Fourier series as follows:

$$u_{\varepsilon}(x,t) = \sum_{k=1}^{\infty} C^{\varepsilon}_{k}(t)\sin(k\pi x) \tag{3.4}$$

where C^{ε}_{k} is the solution of

$$\varepsilon\,\frac{d^2}{dt^2}C^{\varepsilon}_{k} + k^2\pi^2 C^{\varepsilon}_{k} + \frac{d}{dt}C^{\varepsilon}_{k} = 0;\ C^{\varepsilon}_{k}(0) = a_k,\ \frac{d}{dt}C^{\varepsilon}_{k}(0) = b_k, \tag{3.5}$$

and the coefficients a_k and b_k are such that

$$u^0(x) = \sum_{k\geq 1} a_k \sin(k\pi x);\ u^1(x) = \sum_{k\geq 1} b_k \sin(k\pi x). \tag{3.6}$$

Observe that the eigenvalues $\lambda^{\varepsilon}_{k,\pm}$ are real when $k \leq 1/(2\pi\sqrt{\varepsilon})$ and complex when $k > 1/(2\pi\sqrt{\varepsilon})$. We denote by $\pi_p u^{\varepsilon}$ the parabolic projection of the solution u^{ε} constituted by the Fourier components associated with real eigenvalues:

$$\pi_p u^{\varepsilon} = \sum_{k\leq 1/(2\pi\sqrt{\varepsilon})} C^{\varepsilon}_{k}(t)\sin(k\pi x). \tag{3.7}$$

In a similar way, we denote by $\pi_h u^\varepsilon$, the hyperbolic projection of u^ε, which corresponds to the complex eigenvalues:

$$\pi_h u^\varepsilon = \sum_{k>1/(2\pi\sqrt{\varepsilon})} C_k^\varepsilon(t)\sin(k\pi x). \tag{3.8}$$

The following result provides a decay rate for purely hyperbolic solutions that increases exponentially as $\varepsilon \to 0$:

Proposition 3.1 *Let u_ε be a solution of* (1.13) *with initial data* $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ *such that*

$$\pi_p u^0 = \pi_p u^1 = 0, \tag{3.9}$$

and without control, i.e.,

$$v \equiv 0. \tag{3.10}$$

Then, the solution u^ε of (1.13) *satisfies*

$$\pi_p u^\varepsilon(t) = 0, \ \forall t > 0 \tag{3.11}$$

and

$$\| u^\varepsilon(t) \|^2_{H_0^1(0,1)} + \varepsilon \| u_t^\varepsilon(t) \|^2_{L^2(0,1)} \leq 8e^{-\frac{t}{8\varepsilon}} \left[\| u^0 \|^2_{H_0^1(0,1)} + \varepsilon \| u^1 \|^2_{L^2(0,1)} \right] \tag{3.12}$$

for all $t > 0$ and all $0 < \varepsilon < 1$.

This result may be proved using the Fourier development of solutions and taking into account that the real part of all eigenvalues in the Fourier development of u^ε when (3.11) holds are such that $Re\left(\lambda^\varepsilon_{k,\pm}\right) = -1/(2\varepsilon)$.

3.2 Uniform controllability of the parabolic projections

Using the iterative argument introduced in [LR] to derive the null controllability of the heat equation one can prove the following result on the uniform (with respect to ε) null controllability of the parabolic projection of solutions:

Proposition 3.2 *Let $T > 0$. Then there exist $\varepsilon_0 > 0$, such that for any* $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ *there exists a control $v^\varepsilon \in L^2(\omega \times (0,T))$ such that the solution u^ε of* (1.13) *satisfies*

$$\pi_p u^\varepsilon(T) = \pi_p u_t^\varepsilon(T) = 0. \tag{3.13}$$

Moreover there exists a constant $C > 0$, independent of ε, such that

$$\| v^\varepsilon \|_{L^2(\omega\times(0,T))} \leq C \left[\| u^0 \|^2_{H_0^1(0,1)} + \varepsilon \| u^1 \|^2_{L^2(0,1)} \right]^{1/2} \tag{3.14}$$

for all $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ *and all* $0 < \varepsilon < \varepsilon_0$.

3.3 Non-uniform null controllability

In the following Proposition we state a result guaranteeing the null controllability of solutions with a control that may grow exponentially as $\varepsilon \to 0$.

Proposition 3.3 *Let $T > 0$. Then there exist positive constants $A, B > 0$ such that for every $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ and $0 < \varepsilon < \frac{T^2}{4}$ there exists a control $v^\varepsilon \in L^2(\omega \times (0,T))$ such that the solution of* (1.13) *satisfies*

$$u^\varepsilon(T) = \partial_t u^\varepsilon(T) = 0 \tag{3.15}$$

with the estimate

$$\| v^\varepsilon \|_{L^2(\omega\times(0,T))} \leq A e^{B/\sqrt{\varepsilon}} \left[\| u^0 \|^2_{H_0^1(0,1)} + \varepsilon \| u^1 \|^2_{L^2(0,1)}\right]^{1/2}. \tag{3.16}$$

This result may be proved using HUM as a direct consequence of the following observability estimate for the adjoint system:

$$\begin{cases} \varepsilon \varphi_{tt} - \varphi_{xx} - \varphi_t = 0, & 0 < x < 1, \quad 0 < t < T \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T, \\ \varphi(x,T) = \varphi^0(x),\ \varphi_t(x,T) = \varphi^1(x), & 0 < x < 1. \end{cases} \tag{3.17}$$

Proposition 3.4 *Let $T > 0$. Then there exist positive constants $A, B > 0$ such that*

$$\| \varphi^0 \|^2_{L^2(0,1)} + \varepsilon \| \varphi^1 \|^2_{H^{-1}(0,1)} \leq A e^{B/\sqrt{\varepsilon}} \int_0^T \int_\omega \varphi^2 dx dt \tag{3.18}$$

for every solution of (3.17) *and all $0 < \varepsilon < \frac{T^2}{4}$.*

The observability estimate of Proposition 3.4 follows in a straightforward way from the sideways energy estimates in [Z].

3.4 Conclusion: Global uniform controllability in three steps

Now we use a control strategy similar to the one used in Section 2.4. Given $T > 0$ and $\varepsilon_0 > 0$ small enough we divide the control interval $[0,T]$ in three pieces:$[0,T] = [0, \frac{T}{3}] + [\frac{T}{3}, \frac{2T}{3}] + [\frac{2T}{3}, T]$.

Given $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ we build the control in the form:

$$v^\varepsilon = v_1^\varepsilon 1_{[0,\frac{T}{3}]} + v_3^\varepsilon 1_{[\frac{2T}{3},T]}, \tag{3.19}$$

where 1_I is the characteristic function of the interval I. Thus the control is identically zero in the interval $[\frac{T}{3}, \frac{2T}{3}]$.

According to Proposition 3.2, for $\varepsilon_0 > 0$ small enough, there exists a control $v_1^\varepsilon \in L^2(\omega \times [0, \frac{T}{3}])$ such that

$$\| v_1^\varepsilon \|_{L^2(\omega\times[0,\frac{T}{3}])} \leq C \left[\| u^0 \|^2_{H_0^1(0,1)} + \varepsilon \| u^1 \|^2_{L^2(0,1)}\right]^{1/2}, \ \forall 0 < \varepsilon < \varepsilon_0 \tag{3.20}$$

and

$$\pi_p u^\varepsilon \left(\frac{T}{3}\right) = \pi_p u_t^\varepsilon \left(\frac{T}{3}\right) = 0, \; \forall 0 < \varepsilon < \varepsilon_0 \,. \tag{3.21}$$

In view of (3.20) and by classical energy estimates it is easy to see that

$$\| u^\varepsilon \left(\frac{T}{3}\right) \|^2_{H_0^1(0,1)} + \varepsilon \| u_t^\varepsilon \left(\frac{T}{3}\right) \|^2_{L^2(0,1)} \leq C \left[\| u^0 \|^2_{H_0^1(0,1)} + \varepsilon \| u^1 \|^2_{L^2(0,1)} \right] \tag{3.22}$$

with $C > 0$ independent of $0 < \varepsilon < \varepsilon_0$.

During the time interval $[\frac{T}{3}, \frac{2T}{3}]$ we let the solution of (1.13) evolve freely without control, i.e., $v = 0$ in this interval. As a consecuence of Proposition 3.1 we deduce that

$$\| u^\varepsilon \left(\frac{2T}{3}\right) \|^2_{H_0^1(0,1)} + \varepsilon \| u_t^\varepsilon \left(\frac{2T}{3}\right) \|^2_{L^2(0,1)} \leq C e^{-\frac{T}{24\varepsilon}} \left[\| u^0 \|^2_{H_0^1(0,1)} + \varepsilon \| u^1 \|^2_{L^2(0,1)} \right]. \tag{3.23}$$

We then apply the controllability result of Proposition 3.3. It follows that there exists a control $v_3^\varepsilon \in L^2(\omega \times [\frac{2T}{3}, T])$ such that

$$u^\varepsilon(T) = u_t^\varepsilon(T) = 0 \tag{3.24}$$

with

$$\begin{aligned} \| v_3^\varepsilon \|_{L^2(\omega \times [\frac{2T}{3}, T])} &\leq A e^{B/\sqrt{\varepsilon}} \left[\| u^\varepsilon \left(\frac{2T}{3}\right) \|^2_{H_0^1(0,1)} + \varepsilon \| u_t^\varepsilon \left(\frac{2T}{3}\right) \|^2_{L^2(0,1)} \right] \\ &\leq A \exp\left[\frac{B}{\sqrt{\varepsilon}} - \frac{T}{24\,\varepsilon}\right] \left[\| u^0 \|^2_{H_0^1(0,1)} + \varepsilon \| u^1 \|^2_{L^2(0,1)} \right]. \end{aligned} \tag{3.25}$$

In view of (3.25) it follows immediately that

$$\| v_3^\varepsilon \|_{L^2(\omega \times [\frac{2T}{3}, T])} \to 0 \text{ exponentially as } \varepsilon \to 0. \tag{3.26}$$

This fact combined with (3.20) shows that v^ε is bounded in $L^2(\omega \times (0,T))$.

3.5 Uniform observability

As an immediate corollary of the uniform controllability result of Theorem 1.2 the following uniform observability result holds for the adjoint systems (3.17):

Proposition 3.5 *Let $T > 0$. Then, there exists a positive constant $C > 0$ such that*

$$\| \varphi(0) - \varepsilon \varphi_t(0) \|^2_{H^{-1}(0,1)} + \varepsilon \| \varphi(0) \|^2_{L^2(0,1)} \leq C \int_0^T \int_\omega \varphi^2 dx dt \tag{3.27}$$

holds for all solution of (3.17) *and all* $0 < \varepsilon < \frac{T^2}{4}$.

3.6 Passage to the limit

As in the context of Theorem 1.1 we can pass to the limit in the controls in two ways. First, in what concerns to the controls constructed in three steps in paragraph 3.4 above, we observe that v_3^ε tends to zero as $\varepsilon \to 0$. On the other hand, it can be seen that the control $v_1^\varepsilon 1_{[0,\frac{T}{3}]}$ converges as $\varepsilon \to 0$ in $L^2(\omega \times (0, \frac{T}{3}))$ to a control $v \in L^2(0, \frac{T}{3})$ for the limit heat equation (1.8). Thus, the whole control v^ε converges as $\varepsilon \to 0$ in $L^2(\omega \times (0, T))$ to a control of (1.8).

We can also analyze the behavior of the null controls v^ε of minimal $L^2(\omega \times (0,T))-$norm. These controls may be characterized as follows. Given $T > 0$, $0 < \varepsilon < \frac{T^2}{4}$ and initial data $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ we consider the functional

$$J_\varepsilon : L^2(0,1) \times H^{-1}(0,1) \to \mathbb{R} \tag{3.28}$$

defined as

$$J_\varepsilon\left(\varphi^0, \varphi^1\right) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_0^1 \left(u^0 + \varepsilon\, u^1\right)\varphi(0)dx - \varepsilon\langle u^0, \varphi_t(0)\rangle, \tag{3.29}$$

φ being the solution of (3.17) with data (φ^0, φ^1).

It is easy to see that J_ε is continuous and strictly convex. By (3.27) it can be seen that J_ε is also coercive. Thus J_ε has a unique minimizer $(\varphi_\varepsilon^0, \varphi_\varepsilon^1)$. Let φ^ε be the solution of (3.17) with this minimizer as initial datum. Then $v^\varepsilon = \varphi^\varepsilon 1_{\omega\times(0,T)}$ is the null control of (1.13) with minimal $L^2(\omega \times (0,T))-$norm. In view of the uniform observability (3.27) it is easy to see that $\{v_\varepsilon\}_{0<\varepsilon<\frac{T^2}{4}}$ is uniformly bounded in $L^2(\omega \times (0,T))$.

One can show that

$$v_\varepsilon \to v \text{ in } L^2(\omega \times (0,T)) \text{ as } \varepsilon \to 0 \tag{3.30}$$

where

$$v = \varphi \text{ in } \omega \times (0,T), \tag{3.31}$$

φ being the solution of the limit adjoint system

$$\begin{cases} \varphi_t + \varphi_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T \\ \varphi(x,T) = \varphi^0(x), & 0 < x < 1 \end{cases} \tag{3.32}$$

with the initial datum being the minimizer of the functional

$$J\left(\varphi^0\right) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_0^1 u^0(x)\varphi(x,0)dx, \tag{3.33}$$

in the Hilbert space

$$H = \left\{\varphi^0 : \varphi \text{ solution of (3.32) satisfies } \int_0^T \int_\omega \varphi^2 dxdt < \infty\right\}. \tag{3.34}$$

The control v turns out to be the null control of the limit heat equation (1.15) of minimal $L^2(\omega \times (0,T))-$norm.

4 Space semi-discretizations of the heat equation

This section is devoted to prove Theorem 1.3. The proof is much simpler than those of Theorems 1.1 and 1.2. The uniform controllability result is this time a direct consequence of Proposition 2.2 and the structure of the spectrum of the semi-discrete problems.

Let us analyze the spectral problem associated with the semi-discrete equation in the absence of control (i.e., with $v \equiv 0$):

$$\begin{cases} -\left[\frac{w_{j+1}+w_{j-1}-2w_j}{h^2}\right] = \lambda w_j, \; j = 1, \cdots, N \\ w_0 = w_{N+1} = 0. \end{cases} \tag{4.1}$$

For any $h > 0$ the eigenvalues and eigenvectors of (4.1) may be computed explicitly. We have

$$\lambda_k(h) = \frac{4}{h^2} \sin^2\left(\frac{\pi h k}{2}\right), \; k = 1, \cdots, N \tag{4.2}$$

and

$$w_k(h) = (w_{k,1}(h), \cdots, w_{k,N}(h))\,;\; w_{k,j}(h) = \sin(j\pi hk), \; k, j = \lambda, \cdots, N. \tag{4.3}$$

Let us consider the semi-discrete adjoint systems

$$\begin{cases} \varphi_j' + \frac{[\varphi_{j+1}-2\varphi_j+\varphi_{j-1}]}{h^2} = 0, & 0 < t < T, j = 1, \cdots, N \\ \varphi_0 = \varphi_{N+1} = 0, & 0 < t < T \\ \varphi_j(T) = \varphi_j^0, \; j = 1, \cdots, N. \end{cases} \tag{4.4}$$

Solution of (4.4) can be developed in Fourier series as follows:

$$\varphi_j(h,t) = \sum_{k=1}^{N} a_k e^{-\lambda_k(h)(T-t)} w_{k,j}(h). \tag{4.5}$$

The uniform null-controllability for systems (1.18) stated in Theorem 1.3 is equivalent to the following uniform observability inequality for the adjoint systems (4.4): For every $T > 0$ there exists $C > 0$ such that

$$h \sum_{j=1}^{N} |\varphi_j(h,0)|^2 \le C \int_0^T \left| \frac{\varphi_N(h,t)}{h} \right|^2 dt \tag{4.6}$$

for every solution of (4.4) and all $h > 0$.

In view of the explicit form (4.5) of solutions of (4.4), the fact that

$$h \sum_{j=1}^{N} |w_{k,j}(h)|^2 = \frac{1}{2}$$

for all $h > 0$ and $k = 1, \cdots, N$ and Proposition 2.2 it is sufficient to see that the sequences $\{\lambda_k(h)\}$, extended by $\lambda_k(h) = k^2\pi^2$ for $k \ge N+1$, belong to the same class $\mathcal{L}(\xi, N)$ for suitable

$\xi > 0$ and function $N : (0, \infty) \to I\!N$. To see this it is sufficient to check that (2.10) and (2.11) hold uniformly on $h > 0$. This is an explicit computation.

In order to pass to the limit on the controls as $h \to 0$, we first observe that the control $v \in L^2(0,T)$ of minimal $L^2(0,T)-$norm for (1.18) may be characterized by minimizing the functional

$$J_h\left(\varphi^0\right) = \frac{1}{2}\int_0^T \left|\frac{\varphi_N}{h}\right|^2 dt - h\sum_{j=1}^N \varphi_j(0)u_j^0$$

where $\varphi = (\varphi_1, \cdots, \varphi_N)$ is the solution of the adjoint system (4.4) with initial datum $\varphi^0 = (\varphi_1^0, \cdots, \varphi_N^0)$ and $u^0 = (u_1^0, \cdots, u_N^0)$ is the initial datum of (1.18) to be controlled.

It is easy to see that $J_h : I\!R^N \to I\!R$ is continuous, coercive and strictly convex. Thus J_h has a unique minimizer φ_h^0. The control v_h for (1.18) of minimal $L^2(0,T)-$norm is given by

$$v_h(t) = -\frac{\varphi_{h,N}(t)}{h},$$

where φ_h is the solution of (4.4) with this minimizer as initial datum. In view of the uniform observability inequality (4.6) it follows that $\{v_h\}_{h>0}$ is bounded in $L^2(0,T)$. It can then be seen that

$$v_h \to v \text{ as } h \to 0 \text{ in } L^2(0,T)$$

where $v \in L^2(0,T)$ is given by

$$v = -\partial_x \varphi(1,t)$$

where $\varphi = \varphi(x,t)$ is the solution of (3.32) with an initial datum characterized as the minimizer of the functional

$$J\left(\varphi^0\right) = \frac{1}{2}\int_0^T |\varphi_x(1,t)|^2 \, dt - \int_0^1 \varphi(x,0)u^0(x)dx$$

in the Hilbert space

$$H = \left\{\varphi^0 : \text{ the solution } \varphi \text{ of (3.32) satisfies } \varphi_x(1,t) \in L^2(0,T)\right\}.$$

This limit control v turns out to be the control of minimal $L^2(0,T)-$norm.

Remark 4.1 The same problem for the wave equation was analyzed in [IZ]. There it was seen that the uniform controllability fails due to the fact that the spectral gap for the roots of the eigenvalues $\left\{\sqrt{\lambda_k(h)}\right\}$ is not uniformly bounded from below as $h \to 0$. Note that, in the context of the heat equation, the relevant gap is the one associated to the eigenvalues $\{\lambda_k(h)\}$ which remains uniformly bounded from below as $h \to 0$.

■

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