Large time behaviour for a simplified $N-$dimensional model of fluid-solid interaction

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Abstract

In this paper, we study the large time behaviour of solutions of a parabolic equation coupled with an ordinary differential equation (ODE). This system can be seen as a simplified N-dimensional model for the interactive motion of a rigid body (a ball) immersed in a viscous fluid in which the pressure of the fluid is neglected. Consequently, the motion of the fluid is governed by the heat equation and the standard conservation law of linear momentum determines the dynamics of the rigid body. In addition, the velocity of the fluid and that of the rigid body coincide on its boundary. The time variation of the ball position, and consequently of the domain occupied by the fluid, are not known a priori, so we deal with a free boundary problem. After proving the existence and uniqueness of a strong global in time solution, we get its decay rate in L^p $(1 \leq p \leq \infty)$, assuming the initial data to be integrable. Then, working in suitable weighted Sobolev spaces, and using the so-called similarity variables and scaling arguments, we compute the first term in the asymptotic development of solutions. We prove that the asymptotic profile of the fluid is the heat kernel with an appropriate total mass. The L^∞ estimates we get allow us to describe the

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asymptotic trajectory of the centre of mass of the rigid body as well. We compute also the second term in the asymptotic development in L^2 after establishing the decay rate of the Laplacian of the solution.

Keywords and Phrases: Fluid-solid interaction, heat-ODE coupled system, large time behaviour, similarity variables, heat kernel.

AMS Subject Classification: 35B40, 35K15, 35R35, 35K05, 34E05.

1 Introduction and main results

The aim of this paper is to describe the large time asymptotic behaviour for a coupled system of partial and ordinary differential equations. The system under consideration is a simplified $N-$dimensional model for the motion of a rigid body inside a fluid flow.

The governing equation for the fluid is merely the heat equation whereas the motion of the solid is governed by the balance equation for linear momentum. For the sake of simplicity, we assume the solid to be a moving ball of radius 1 occupying the domain $B(t)$ of $\mathbb{R}^N$ whose centre of mass lies in the point $\mathbf{h}(t)$. Thus, the system we shall deal with is the following one:

$$\begin{cases} \mathbf{u}_t - \Delta \mathbf{u} = \mathbf{0}, & \mathbf{x} \in \Omega(t), \quad t > 0, \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{h}'(t), & \mathbf{x} \in \partial B(t), \quad t > 0, \\ m\, \mathbf{h}''(t) = -\displaystyle\int_{\partial\Omega(t)} \mathbf{n} \cdot \nabla \mathbf{u} \, d\sigma_x, & \qquad t > 0, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega(0), & \mathbf{h}(0) = \mathbf{h}_0, \quad \mathbf{h}'(0) = \mathbf{h}_1, \end{cases} \tag{1.1}$$

where $\Omega(t) := \mathbb{R}^N / B(t)$ and $m > 0$ stands for the mass of the ball. The vector $\mathbf{n}(\mathbf{x}, t)$ is the unit normal to $\partial\Omega(t)$ at the point $\mathbf{x}$ directed to the interior of $B(t)$. In the above system the unknowns are $\mathbf{u}(\mathbf{x}, t)$ (that can be seen as the Eulerian velocity field of the fluid) and $\mathbf{h}(t)$. The coupling condition (1.1-ii) ensures that the velocity of the body is the same as the one of the fluid on its boundary. The equation (1.1-iii) results from the standard conservation law of linear momentum.

Let us stress the main differences between our model and a full model of fluid-structure interaction, namely:

$$\begin{cases} \mathbf{u}_t - \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{0}, & \mathbf{x} \in \Omega(t), \quad t > 0, \\ \operatorname{div} \mathbf{u} = 0, & \mathbf{x} \in \Omega(t), \quad t > 0, \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{h}'(t), & \mathbf{x} \in \partial B(t), \quad t > 0, \\ m\, \mathbf{h}''(t) = -\displaystyle\int_{\partial\Omega(t)} T \mathbf{n} \, d\sigma_x, & \qquad t > 0, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega(0), & \mathbf{h}(0) = \mathbf{h}_0, \quad \mathbf{h}'(0) = \mathbf{h}_1, \end{cases} \tag{1.2}$$

where T is the stress tensor in the fluid whose components are defined by

$$T_{ij}(\mathbf{x},t) := -p(\mathbf{x},t)\,\delta_{ij} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and p stands for the pressure. When $N = 2$, on account of (1.2-ii) and (1.2-iii), the relation (1.2-iv) can be rewritten as:

$$m\,\mathbf{h}''(t) = -\int_{\partial\Omega(t)} \mathbf{n}\cdot\nabla\mathbf{u} - p\,\mathbf{n}\,d\sigma_x. \tag{1.3}$$

Indeed, from (1.2-ii), we deduce that $\mathbf{n}\cdot\nabla\mathbf{u}^T = (\mathbf{n}^\perp\cdot\nabla\mathbf{u})^\perp = 0$, because $\mathbf{u}$ is constant, equal to $\mathbf{h}'$, along $\partial B(t)$. Therefore, we obtain that $T\,\mathbf{n} = -p\,\mathbf{n} + \mathbf{n}\cdot\big(\nabla\mathbf{u} + \nabla\mathbf{u}^T\big)$ and so $T\,\mathbf{n} = -p\,\mathbf{n} + \mathbf{n}\cdot\nabla\mathbf{u}$. That yields relation (1.3). Note that the formulation (1.3) is quite close to (1.1-iii). In the sequel, the pressure term has been neglected. The same can be said about the equation for $\mathbf{u}$ in (1.1-i). Note that the convective quadratic non-linearity has also been neglected in (1.1-i). However, this simplification is less relevant since most of our developments can also be carried out in the presence of a non-linear convective term. Thus, the main difference between system (1.1) and the more realistic one is that in (1.1), we neglect the pressure term. Extending the results of this paper to the full system (1.2) is an interesting open problem.

The model (1.2), as well as more complete and complex ones involving several bodies with rotational motions, were extensively studied during the last years. Concerning the existence and uniqueness of weak solutions, see for example [4], [5] and [7], [8], [6] and [14], [13] and the references therein. Recently and independently of the present work, M. Tucsnak and T. Takahashi in [17] in the whole space and T. Takahashi in [16] for a bounded domain, proved the existence and uniqueness of a strong solution for the model (1.2), adding a rotational motion for the ball. Moreover, it was shown that the solution is global in time provided the ball does not collide with the boundary of the domain. Whether the ball may touch the boundary in finite time or, in the presence of various solid bodies, whether they may collide is certainly one of the most interesting open problems in this area.

Another source of motivation for the present paper was the article of J.L. Vázquez and E. Zuazua [18] on the large time behaviour for a simplified one dimensional model of fluid-structure interaction. In this paper, a sharp description of the asymptotic behaviour as time goes to infinity of a point particle, which floats in a fluid governed by the viscous Burgers equation is given. More precisely, it is proved that the velocity u of the fluid behaves, for t large, like the unique self-similar solution of the Burger's equation on $\mathbb{R}$ with source type initial data $M\delta_0$. The constant M is defined by $M := \int_{\mathbb{R}} u_0\,dx + m\,h_1$, the functions u_0 and h_1 being the initial velocities of the fluid and of the particle respectively. Our work is actually a natural extension of this one to the case of several space dimensions. However, in the present paper, the equation governing $\mathbf{u}$ is assumed to be linear although similar results could be proved for a model including a convective non-linearity in the parabolic equation.

Concerning the 1-d model analysed in [18], it is also worth mentioning that, recently, in [19] the lack of collision was proved in the case where several particles float on the fluid or in the presence of exterior boundaries.

It is also of interest to compare our results with the existing ones on the asymptotic behaviour of the Navier-Stokes equation (without rigid-bodies) in [2] and [3] (and references given there). In this paper, we prove that, roughly speaking, the first order approximation of the solution of our model is the heat kernel with an appropriate total mass. The same result holds for the solution of the Navier-Stokes equation in $\mathbb{R}^2$ and $\mathbb{R}^3$ (see [2] and [3]). One can expect the same result to be true for the Navier-Stokes equation coupled with the motion of a rigid body (as in (1.2)) but this result has not been proved so far.

Let us go back now to system (1.1) we are dealing with. It is a linear, free boundary problem since the position of $B(t)$ is to be determined. Applying the change of variables:

$$\mathbf{v}(\mathbf{x},t) = \mathbf{u}(\mathbf{x}+\mathbf{h}(t),t), \qquad \mathbf{g}(t) = \mathbf{h}'(t),$$

we can rewrite system (1.1) using $\mathbf{v}$ and $\mathbf{g}$ as new unknown functions and the system turns out to be non-linear but in a fixed domain, independent of t. Indeed, we get

$$\begin{cases} \mathbf{v}_t - \Delta\mathbf{v} - \mathbf{g}\cdot\nabla\mathbf{v} = \mathbf{0}, & \mathbf{x}\in\Omega, \quad t>0, \\ \mathbf{v}(\mathbf{x},t) = \mathbf{g}(t), & \mathbf{x}\in\partial B, \quad t>0, \\ m\,\mathbf{g}'(t) = -\displaystyle\int_{\partial\Omega} \mathbf{n}\cdot\nabla\mathbf{v}\,d\sigma_x, & \quad t>0, \\ \mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, & \mathbf{g}(0) = \mathbf{h}_1. \end{cases} \tag{1.4}$$

Here B stands for the fixed ball of centre 0 and radius 1, $\Omega := \mathbb{R}^N/B$ and $\mathbf{n}(\mathbf{x})$ is the unit normal to $\partial\Omega$ at the point $\mathbf{x}$ directed to the interior of B.

In view of (1.4), it is clear that the N components of the fluid field ($\mathbf{u}$ or $\mathbf{v}$) are coupled through the unknowns ($\mathbf{h}$ or $\mathbf{g}$) describing the motion of the solid. Thus, although it might seem not to be the case, all the components of $\mathbf{u}$ are coupled in (1.1).

1.1 Notations

Throughout this article, we shall use bold print notations for $N-$dimensional vectors like $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v}$, $\boldsymbol{\zeta}$ whereas we keep the usual characters for real valued functions : u, v, ζ. The generic notation v will be used for any of the components v_i of the vector $\mathbf{v}$.

In the same way, $\mathbf{L}^2(\omega)$, $\mathbf{H}^1(\omega)$ will stand for $L^2(\omega)^N$ and $H^1(\omega)^N$ respectively, ω being a subset of $\mathbb{R}^N$.

A $N \times N$ matrix is denoted $[\mathbf{M}]$. Its entries are M_{ij}, $1 \leq i, j \leq N$ and $\mathbf{M}_i$ stands for the $i-$th row. However, to shorten notation, we sometimes drop the index i and denote generically by $\mathbf{M}$ any row of the matrix $[\mathbf{M}]$. For instance, according to these simplifications, the matrix identity $[\mathbf{M}] = \mathbf{U}\,\mathbf{V}^T$ leads to the vectors equality (equality of the rows of the matrices) $\mathbf{M} = U\,\mathbf{V}$ and can also be rewritten as N scalar equalities:

$$M_i = U\,V_i, \quad \forall i = 1, \ldots, N.$$

For vectors and matrices, the classical Eulerian norms are defined:

$$|\mathbf{V}| = \left(\sum_{i=1}^{N} V_i^2 \right)^{1/2} \quad \text{and} \quad |[\mathbf{M}]| = \left(\sum_{i,j=1}^{N} M_{ij}^2 \right)^{1/2}.$$

When $\mathbf{V}(\mathbf{x})$ and $[\mathbf{M}](\mathbf{x})$ are a vector valued function and a matrix valued function respectively, on an open set $\omega \subset \mathbb{R}^N$, we denote:

$$\|\mathbf{V}\|_p = \left(\sum_{i=1}^{N} \int_\omega V_i^p \, d\mathbf{x} \right)^{1/p} \quad \text{and} \quad \|[\mathbf{M}]\|_p = \left(\sum_{i,j=1}^{N} \int_\omega M_{ij}^p \, d\mathbf{x} \right)^{1/p},$$

for all $1 \leq p < \infty$.

The non negative constants shall be denoted by C along the computations. The value of C can change from one line to the other. We sometimes use C_1 and C_2 when its values need to be followed along the computations. The notation C_p allows to emphasise the dependence with respect to p, the exponent of the Sobolev or L^p space we are working in. Finally, in some equalities, $C(t)$ will stand for a real valued function such that $|C(t)| \leq C$ for all $t > 0$.

$L^2(F, \omega)$ and $H^1(F, \omega)$ stand for weighted spaces where F is a positive function (the weight) on the subset ω of $\mathbb{R}^N$. They are endowed with the scalar products $\int_\omega u\,v\,F(\mathbf{x})\,d\mathbf{x}$ and $\int_\omega \nabla u \cdot \nabla v\,F(\mathbf{x})\,d\mathbf{x} + \int_\omega u\,v\,F(\mathbf{x})\,d\mathbf{x}$ respectively. To shorten notations, we will write $L^2(F)$ and $H^1(F)$ instead of $L^2(F, \mathbb{R}^N)$ and $H^1(F, \mathbb{R}^N)$ respectively when $\omega = \mathbb{R}^N$.

Finally, $\dot{H}^1(\omega)$ is the closure of $C_c^1(\omega)$ (the space of C^1 functions with compact support in ω) for the norm $\left(\int_\omega |\nabla u|^2 \, d\mathbf{x} \right)^{\frac{1}{2}}$.

1.2 The scalar version of system (1.1)

Any component (v^i, g^i), $i = 1, \ldots, N$ of the solution $(\mathbf{v}, \mathbf{g})$ of system (1.4), that we shall merely denote by v and g, is a vector valued function with two scalar components, which solves:

$$\begin{cases} v_t - \Delta v - \mathbf{g}\cdot\nabla v = 0, & \mathbf{x}\in\Omega, \quad t>0,\\ v(\mathbf{x},t) = g(t), & \mathbf{x}\in\partial B, \quad t>0,\\ m\, g'(t) = -\displaystyle\int_{\partial\Omega} \frac{\partial v}{\partial \mathbf{n}}\, d\sigma_x, & \quad t>0,\\ v(\mathbf{x},0) = v_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, \quad g(0)=h_1. \end{cases} \tag{1.5}$$

Note in particular that in (1.5), v satisfies a scalar heat equation. However, all the scalar equations satisfied by the components v^i, $i=1,\ldots,N$ are coupled through the convective term and in particular, through the vector field $\mathbf{g}$ describing the motion of the solid.

As far as the first term in the large time asymptotic development is concerned, we shall prove that the term $\mathbf{g}\cdot\nabla v$ can be neglected.

To simplify notations, we will sometimes work with these scalar functions (v,g).

1.3 Main results

Theorem 1.1 (Existence and uniqueness of solutions) *For any* $(\mathbf{v}_0,\mathbf{g}_0)\in \mathbf{L}^2(\Omega)\times\mathbb{R}^N$*, there exists a unique global strong solution* $(\mathbf{v},\mathbf{g})$ *of system* (1.4) *such that:*

$$\begin{aligned} &\mathbf{v}\in C\left([0,+\infty),\mathbf{L}^2(\Omega)\right)\cap L^2\left((0,\infty),\dot{\mathbf{H}}^1(\Omega)\right),\\ &\mathbf{g}\in C\left([0,+\infty)\right). \end{aligned}$$

The proof of this Theorem is quite classical and is given at the end of this paper, in the Appendix A.

If we integrate the first equation of system (1.4), use the Stokes formula and the transmission condition (1.4-iii) on the boundary of the ball, we deduce that

$$\mathbf{M}_1 := \int_\Omega \mathbf{v}\, d\mathbf{x} + m\,\mathbf{g}, \tag{1.6}$$

is independent of time. This *first momentum* plays a crucial role in the description of the large time behaviour of $\mathbf{v}$.This idea will be made more precise in the following Theorem.

Let us introduce the weight function $K(\mathbf{x}) := \exp\left(\frac{|x|^2}{4}\right)$ and the constant σ_N, the area of the unit sphere, necessary to state the main results of this paper. Note that $\frac{\sigma_N}{N}$ is therefore the volume of the unit sphere:

Theorem 1.2 (First term in the asymptotic development) *Let* $\mathbf{v}_0 \in \mathbf{L}^2(K,\Omega)$ *and* $\mathbf{g}_0 \in \mathbb{R}^N$ *and define, for all* $N \geq 2$ *and* $1 \leq p \leq \infty$*:*

$$\theta(N,p) := \frac{N}{2}\,\frac{(p-1)(p-N)}{p\,(2\,p+N(p-1))}. \tag{1.7}$$

Then there exist constants $C_p > 0$ *depending on the dimension* N*, on the mass* m *of the solid and on* p *such that the following inequalities hold:*

- *When* $m \neq \frac{\sigma_N}{N}$*:*
  - *When* $N = 2$*,*
    - *for all* p *such that* $1 \leq p \leq 2$*:*

$$\|\mathbf{v}(t) - \mathbf{M}_1\, G(t)\|_{\mathbf{L}^p(\Omega)} \leq C_p\,|\log(1+t)|\, t^{-\left(1-\frac{1}{p}\right)-\frac{1}{2}}, \tag{1.8}$$

    - *for all* p *such that* $2 < p \leq \infty$*:*

$$\|\mathbf{v}(t) - \mathbf{M}_1\, G(t)\|_{\mathbf{L}^p(\Omega)} \leq C_p\,|\log(1+t)|^{\frac{p}{2\,p-1}}\, t^{-\left(1-\frac{1}{p}\right)-\frac{1}{2}+\theta(2,p)}, \tag{1.9a}$$

$$|\mathbf{g}(t) - \mathbf{M}_1\,(4\pi t)^{-1}| \leq C_\infty\,|\log(1+t)|^{\frac{1}{2}}\, t^{-\frac{5}{4}}, \quad \forall\, t \geq 1. \tag{1.9b}$$

  - *When* $N \geq 3$*,*
    - *for all* p *such that* $1 \leq p \leq N$*:*

$$\|\mathbf{v}(t) - \mathbf{M}_1\, G(t)\|_{\mathbf{L}^p(\Omega)} \leq \; C_p\, t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}, \tag{1.10}$$

    - *for all* p *such that* $N < p \leq \infty$*:*

$$\|\mathbf{v}(t) - \mathbf{M}_1\, G(t)\|_{\mathbf{L}^p(\Omega)} \leq C_p\, t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}+\theta(N,p)}, \tag{1.11a}$$

$$|\mathbf{g}(t) - \mathbf{M}_1\,(4\pi t)^{-\frac{N}{2}}| \leq C_\infty\, t^{-\frac{N}{2}-\frac{1}{N+2}}, \quad \forall\, t \geq 1. \tag{1.11b}$$

- *When* $m = \frac{\sigma_N}{N}$ *:*
  - *When* $N = 2$ *and for all* $1 \leq p \leq \infty$*:*

$$\|\mathbf{v}(t) - \mathbf{M}_1\, G(t)\|_{\mathbf{L}^p(\Omega)} \leq C_p\,|\log(1+t)|\, t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}, \tag{1.12}$$

$$|\mathbf{g}(t) - \mathbf{M}_1\,(4\pi t)^{-\frac{N}{2}}| \leq C_\infty\,|\log(1+t)|\, t^{-\frac{N}{2}-\frac{1}{2}}, \quad \forall\, t \geq 1. \tag{1.13}$$

  - *When* $N \geq 3$ *and for all* $1 \leq p \leq \infty$*:*

$$\|\mathbf{v}(t) - \mathbf{M}_1\, G(t)\|_{\mathbf{L}^p(\Omega)} \leq C_p\, t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}, \tag{1.14}$$

$$|\mathbf{g}(t) - \mathbf{M}_1\,(4\pi t)^{-\frac{N}{2}}| \leq C_\infty\, t^{-\frac{N}{2}-\frac{1}{2}}, \quad \forall\, t \geq 1. \tag{1.15}$$

In these estimates, G stands for the heat kernel on $\mathbb{R}^N$ defined by

$$G(t,\mathbf{x}) := (4\pi\, t)^{-\frac{N}{2}} \exp\left(-\frac{|\mathbf{x}|^2}{4t}\right),$$

and the first asymptotic momentum $\mathbf{M}_1$ is given by $\mathbf{M}_1 := \displaystyle\int_\Omega \mathbf{v}_0\, d\mathbf{x} + m\, \mathbf{g}_0$.

Remark 1.1 *The embedding $\mathbf{L}^2(K,\Omega) \subset \mathbf{L}^1(\Omega)$ ensures the existence of $\mathbf{M}_1$.*

Remark 1.2 *Note that the decay rates we obtain for $\mathbf{g}$ are the same as those for $\mathbf{v}$ in the $\mathbf{L}^\infty-$norm. This is perfectly natural in view of the coupling condition in (1.4-ii).*

Remark 1.3 *Considering the functions $\bar{\mathbf{v}} := \mathbf{v} - \mathbf{M}_1\, G$ and $\bar{\mathbf{g}} := \mathbf{g} - \mathbf{M}_1\, G|_{\partial\Omega}$, we shall remark in the sequel that $m\,\bar{\mathbf{g}}' \neq -\int_\Omega \mathbf{n}\cdot\nabla\bar{\mathbf{v}}\, d\mathbf{x}$. Moreover, the quantity $|m\,\bar{\mathbf{g}}' + \int_\Omega \mathbf{n}\cdot\nabla\bar{\mathbf{v}}\, d\mathbf{x}|$ depends on the mass m of the solid and has a decay rate of order $t^{-N/2-2}$ when $m = \frac{\sigma_N}{N}$ and only of order $t^{-N/2-1}$ when $m \neq \frac{\sigma_N}{N}$. This difference will be relevant in the computations and will lead to different decay rates for $\bar{\mathbf{v}}$ and $\bar{\mathbf{g}}$, as stated in Theorem 1.2 .*

According to Theorem 1.2, in a first approximation, $\mathbf{v}$ behaves, as $t \to \infty$, as the fundamental solution G of the heat equation. Note that this Gaussian profile is multiplied by $\mathbf{M}_1 := \int_\Omega \mathbf{v}_0\, dx + m\,\mathbf{g}_0$ which indicates that the fluid component of the system absorbs the initial momentum introduced by the solid mass.

The estimates (1.8) and (1.10) of Theorem 1.2 are sharp for $p = 2$ and all $N \geq 2$. This clearly appears when exhibiting the second term in the asymptotic development of $\mathbf{v}$ in $\mathbf{L}^2(\Omega)$ in the following Theorem. We do not know yet if the estimates (1.9) and (1.11) for the $\mathbf{L}^p-$norms with $p > N$ are sharp or not. At this respect it is important to observe that, despite the fact that the estimates we obtain for $p \leq N$ are similar to those that one obtains for the linear heat equation where one gains an extra $t^{-\frac{1}{2}}$ of decay when subtracting the fundamental solution, they deteriorate as p increases beyond the exponent $p = N$ due to the extra factor $t^{\theta(N,p)}$.

Remark 1.4 *In Theorem 1.2 the dynamics of $\mathbf{g}$ is simple since, for t large, the action of the fluid on the ball can be neglected. This can be easily predicted by a scaling argument. According to the scaling properties of the heat equation, given $(\mathbf{v},\mathbf{g})$ solution of (1.4), it is natural to introduce:*

$$\mathbf{v}_\lambda(\mathbf{x},t) := \lambda^N\, \mathbf{v}(\lambda\,\mathbf{x}, \lambda^2\, t), \quad \mathbf{g}_\lambda(t) := \lambda^N\, \mathbf{g}(\lambda^2\, t),$$

for all $\lambda > 0$. Then, $(\mathbf{v}_\lambda, \mathbf{g}_\lambda)$ is a solution of the following system:

$$\begin{cases} \mathbf{v}_{\lambda,t} - \Delta\mathbf{v}_\lambda - \lambda^{-N+1}\mathbf{g}_\lambda\cdot\nabla\mathbf{v}_\lambda = \mathbf{0}, & \mathbf{x}\in\Omega_\lambda, \qquad t>0,\\ \mathbf{v}_\lambda(\mathbf{x},t) = \mathbf{g}_\lambda(t), & \mathbf{x}\in\partial B_\lambda, \qquad t>0,\\ (m/\lambda)\,\mathbf{g}_\lambda'(t) = -\displaystyle\int_{\partial\Omega_\lambda} \mathbf{n}\cdot\nabla\mathbf{v}_\lambda\, d\sigma_x, & \qquad\qquad t>0,\\ \mathbf{v}_\lambda(\mathbf{x},0) = \lambda^N\mathbf{v}_0(\lambda\,\mathbf{x}), \quad \mathbf{x}\in\Omega_\lambda, & \mathbf{g}_\lambda(0) = \lambda^N\mathbf{h}_1, \end{cases} \tag{1.16}$$

where B_λ is the ball centred at the origin and of radius $1/\lambda$ and $\Omega_\lambda = \mathbb{R}^N \setminus B_\lambda$. Formally, as $\lambda \to \infty$ the convective term in the first equation vanishes, and the equation for the acceleration of the ball tends to the trivial identity. Taking this into account, the rescaled solution of the heat equation can be shown to converge to the Gaussian kernel with an appropriate mass. Thus,denoting by $\widetilde{\mathbf{v}}$ and $\widetilde{\mathbf{g}}$ the limits of $\mathbf{v}$ and $\mathbf{g}$ as $\lambda \to \infty$, one expects as well that $\widetilde{\mathbf{v}}(\mathbf{x},t) = \mathbf{M}_1\, G(\mathbf{x},t)$ and $\widetilde{\mathbf{g}}(t) = \mathbf{M}_1\, G(\mathbf{0},t)$, where $\mathbf{M}_1$ can be identified by conservation of momentum.

In view of Theorem 1.2, the initial function $\mathbf{u}$ of system 1.1 behaves as follows:

$$\mathbf{u}(\mathbf{x},t) \to \mathbf{M}_1\, G(\mathbf{x}-\mathbf{h}(t),t), \quad \text{as } t \to \infty,$$

in all the L^p spaces. Moreover, Theorem 1.2 yields precise estimates of the velocity of the ball, $\mathbf{g} := \mathbf{h}'$. Integrating these relations, we get:

- When $m \neq \dfrac{\sigma_N}{N}$:
  - When $N = 2$:
  $$t^{-\alpha}\left|\mathbf{h}(t) - \mathbf{M}_1\,(4\pi)^{-1}\,\log(t)\right| \to 0, \quad \text{as } t \to \infty,$$
  for all $\alpha < \dfrac{1}{4}$.
  - When $N \geq 3$:
  $$t^{-\alpha}\left|\mathbf{h}(t) - \frac{2}{2-N}\mathbf{M}_1\,(4\pi)^{-N/2}\,t^{-N/2+1}\right| \to 0, \quad \text{as } t \to \infty,$$
  for all $\alpha < \dfrac{N}{2} - \dfrac{N+1}{N+2}$.
- When $m = \dfrac{\sigma_N}{N}$:
  - When $N = 2$:
  $$t^{-\alpha}\left|\mathbf{h}(t) - \mathbf{M}_1\,(4\pi)^{-1}\,\log(t)\right| \to 0, \quad \text{as } t \to \infty,$$
  for all $\alpha < \dfrac{1}{2}$.
  - When $N \geq 3$:
  $$t^{-\alpha}\left|\mathbf{h}(t) - \frac{2}{2-N}\mathbf{M}_1\,(4\pi)^{-N/2}\,t^{-N/2+1}\right| \to 0, \quad \text{as } t \to \infty,$$
  for all $\alpha < \dfrac{N}{2} - \dfrac{1}{2}$.

Theorem 1.3 (Second term in the asymptotic development) *Let* $(\mathbf{v}_0, \mathbf{g}_0) \in \mathbf{H}^2(\Omega, K) \times \mathbb{R}^N$ *s.t.* $\mathbf{v}_0|_{\partial\Omega} = \mathbf{g}_0$*. Then, as far as the* $\mathbf{L}^2-$*norm is concerned, we can improve Theorem 1.1:*

- *When* $N = 2$,

$$\lim_{t\to\infty} \frac{t^{\frac{N}{4}+\frac{1}{2}}}{|\log t|} \, \|\mathbf{v}(t) - \mathbf{M}_1 \, G(t) - |\log(1+t)| \, [\mathbf{M}_2^1] \, \nabla G - \, [\mathbf{M}_2^2] \, \nabla G\|_{\mathbf{L}^2(\Omega)} = 0. \tag{1.17a}$$

- *When* $N \geq 3$,

$$\lim_{t\to\infty} t^{\frac{N}{4}+\frac{1}{2}} \, \|\mathbf{v}(t) - \mathbf{M}_1 \, G(t) - [\mathbf{M}_2]\nabla G\|_{\mathbf{L}^2(\Omega)} = 0, \tag{1.17b}$$

where the second asymptotic momenta $[\mathbf{M}_2^1]$, $[\mathbf{M}_2^2]$ *and* $[\mathbf{M}_2]$ *are* $N \times N$ *matrices defined by*

- *When* $N = 2$,

$$[\mathbf{M}_2^1] := (4\pi)^{-1} \, \mathbf{M}_1 \, \mathbf{M}_1^T, \tag{1.18a}$$

and

$$\begin{aligned}[\mathbf{M}_2^2] := -\int_\Omega \mathbf{v}_0 \, \mathbf{x}^T \, d\mathbf{x} - m \, (4\pi)^{-2} \, \mathbf{M}_1 \, \mathbf{M}_1^T - \int_0^\infty \left(\int_{\partial\Omega} (\mathbf{n} \cdot \nabla \mathbf{v}) \, \mathbf{x}^T \, d\sigma_x \right) dt \\ + \mathbf{M}_1 \int_0^\infty (1+t)^{-\frac{3}{4}} \boldsymbol{\beta}^T \, dt - m \, (4\pi)^{-1} \int_0^\infty (1+t)^{-\frac{7}{4}} (\mathbf{M}_1 \, \boldsymbol{\beta}^T + \boldsymbol{\beta} \, \mathbf{M}_1^T) \, dt \\ - m \int_0^\infty (1+t)^{-2} \boldsymbol{\beta} \, \boldsymbol{\beta}^T \, dt.\end{aligned} \tag{1.18b}$$

- *When* $N \geq 3$,

$$\begin{aligned}[\mathbf{M}_2] := -\int_\Omega \mathbf{v}_0 \, \mathbf{x}^T \, d\mathbf{x} - \mathbf{M}_1 \, \mathbf{M}_1^T \, (4\pi)^{-N} \left[\frac{m}{N-1} - (4\pi)^{\frac{N}{2}} \frac{2}{N-2} \right] \\ - \int_0^\infty \left(\int_{\partial\Omega} (\mathbf{n} \cdot \nabla \mathbf{v}) \, \mathbf{x}^T \, d\sigma_x \right) dt + \mathbf{M}_1 \int_0^\infty (1+t)^{-\frac{N-1}{2} - \frac{1}{2+N}} \boldsymbol{\beta}^T \, dt \\ - m \, (4\pi)^{-\frac{N}{2}} \int_0^\infty (1+t)^{-\frac{2N-1}{2} - \frac{1}{2+N}} (\mathbf{M}_1 \, \boldsymbol{\beta}^T + \boldsymbol{\beta} \, \mathbf{M}_1^T) \, dt \\ - m \int_0^\infty (1+t)^{-\frac{2N-1}{2} - \frac{2}{2+N}} \boldsymbol{\beta} \, \boldsymbol{\beta}^T \, dt,\end{aligned} \tag{1.18c}$$

where

$$\boldsymbol{\beta} := (1+t)^{\frac{N}{2} + \frac{1}{2+N}} \left(\mathbf{g} - \mathbf{M}_1 (4\pi \, t)^{-\frac{N}{2}} \right). \tag{1.18d}$$

Moreover, all the integrals involved in the definition of $[\mathbf{M}_2^2]$ *for* $N = 2$ *and* $[\mathbf{M}_2]$ *for* $N \geq 3$ *are well defined.*

Remark 1.5 *The second term in the asymptotic expansion of the solution contains some terms that may not be explicitly computed in terms of the initial data. This is the case both in dimension $N = 2$ and $N = 3$. When $N = 2$, the definition of $[\mathbf{M}_2^2]$ contains several time integrals that involve the solution $(\mathbf{v}, \mathbf{g})$ for all time $t \geq 0$. The same phenomenon occurs in [20], Theorem 3, for scalar convection-diffusion equations on the whole space $\mathbb{R}^N$.*

It is convenient to display the results of Theorem 1.3 as an asymptotic development as $t \to \infty$ in $\mathbf{L}^2(\Omega)$:

- When $N = 2$:

$$\mathbf{v}(t) = \mathbf{M}_1\, G(t) + |\log(1+t)|\, [\mathbf{M}_2^1]\, \nabla G(t) + [\mathbf{M}_2^2]\, \nabla G(t) + o\left(|\log t|\, t^{-1}\right). \quad (1.19a)$$

- When $N \geq 3$:

$$\mathbf{v}(t) = \mathbf{M}_1\, G(t) + [\mathbf{M}_2]\, \nabla G(t) + o\left(t^{-\frac{N}{4}-\frac{1}{2}}\right). \quad (1.19b)$$

For the solution $\tilde{v}$ of the heat equation on the whole space $\mathbb{R}^N$, with initial data $\tilde{v}_0$, we have the asymptotic expansion in $L^2(\mathbb{R}^N)$:

$$\tilde{v}(t) = \widetilde{M_1}\, G(t) + \widetilde{\mathbf{M}}_2 \cdot \nabla G(t) + o\left(t^{-\frac{N}{4}-\frac{1}{2}}\right), \quad (1.19c)$$

where $\widetilde{M_1} = \int_{\mathbb{R}^N} \tilde{v}_0\, d\mathbf{x}$ and $\widetilde{\mathbf{M}}_2 = -\int_{\mathbb{R}^N} \tilde{v}_0\, \mathbf{x}\, d\mathbf{x}$.

Comparing (1.19a) for $N = 2$ and (1.19b) for $N = 3$ with the known results for the heat equation (1.19c) we observe some slight differences due to the presence of the solid mass. In dimension $N = 2$ the main difference is due to the presence of a time logarithmic multiplicative factor on the second term of the asymptotic expansion involving ∇G. This was already observed to be the case in [20] for the quadratic convective nonlinearity in dimension $N = 2$. We also see the presence of this time logarithmic factor on the error term. The main difference in the case $N = 3$ comes from the definition of the factor $[\mathbf{M}_2]$ multiplying the second term ∇G. Indeed, the definition of $[\mathbf{M}_2]$ in the statement of the Theorem clearly shows the impact of the coupling between the heat equation and the solid mass.

1.4 Sketches of the proofs of Theorems 1.2 and 1.3

The first step to prove Theorem 1.2 consists in establishing the decay rate of the solution $(\mathbf{v}, \mathbf{g})$ of system (1.4) in $\mathbf{L}^p$ $(1 \leq p \leq \infty)$. We get this result componentwise by multiplying the heat equation by non-linear functions of v, integrating by parts and using Hölder, Sobolev and interpolation inequalities. The problem is then reduced to solve an ordinary differential inequation and the conclusion arises by exhibiting a suitable super-solution.

Remark 1.6 *As we already stressed it for v and g in subsection 1.2, M_1 shall stand subsequently for the component $M_{1,i}$ of the vector $\mathbf{M}_1 := \int_\Omega \mathbf{v}_0\, dx + m\, \mathbf{g}_0$.*

In a second step, we introduce:

$$\bar{\mathbf{v}}(\mathbf{x},t) := \mathbf{v}(\mathbf{x},t) - \mathbf{M}_1\, G(\mathbf{x},t), \quad \forall\, \mathbf{x} \in \Omega \quad \text{and} \quad \bar{\mathbf{g}}(t) := \mathbf{g}(t) - \mathbf{M}_1\, J(t),$$

where $J(t) := G(t,\mathbf{x})\,|_{\mathbf{x}\in\partial B} = (4\pi\, t)^{-\frac{N}{2}} e^{-\frac{1}{4t}}$. Since G is the fundamental solution of the heat equation, $\bar{\mathbf{v}}$ solves on $\Omega \times (0,\infty)$:

$$\bar{\mathbf{v}}_t - \Delta\bar{\mathbf{v}} - \mathbf{g}\cdot\nabla\bar{\mathbf{v}} = \mathbf{M}_1\, \mathbf{g}\cdot\nabla G. \tag{1.20}$$

On the other hand, simple computations yield:

$$J'(t) = (4\pi)^{-\frac{N}{2}} t^{-\frac{N}{2}-1}\, e^{-\frac{1}{4t}} \left(-\frac{N}{2} + \frac{1}{4t}\right),$$

for all $t \geq 0$ and also $\int_{\partial\Omega} \frac{\partial G}{\partial \mathbf{n}}\, d\sigma_x = \frac{1}{2}\,(4\pi)^{-\frac{N}{2}}\, \sigma_N\, t^{-\frac{N}{2}-1}\, e^{-\frac{1}{4t}}$, where σ_N stands for the measure of the sphere ∂B of $\mathbb{R}^N$. Thus, with the correcting term,

$$\varepsilon(t) := \frac{1}{2}\, m\, (4\pi)^{-\frac{N}{2}}\, t^{-\frac{N}{2}-1} e^{-\frac{1}{4t}} \left(\frac{\sigma_N}{m} - N + \frac{1}{2t}\right), \tag{1.21}$$

it follows that

$$m\, J'(t) = -\int_{\partial\Omega} \frac{\partial G}{\partial \mathbf{n}}\, d\sigma_x + \varepsilon(t).$$

Therefore, the ODE governing the evolution of $\bar{\mathbf{g}}$ reads as follows:

$$m\, \bar{\mathbf{g}}' = -\int_{\partial\Omega} \mathbf{n}\cdot\bar{\mathbf{v}}\, d\sigma_x - \mathbf{M}_1\, \varepsilon(t). \tag{1.22}$$

In order to prove that $\mathbf{M}_1\, G$ is the first term in the asymptotic development of $\mathbf{v}$, we have to prove that $\bar{\mathbf{v}}$ decreases faster than $\mathbf{v}$ and G separately do. The decay rate for $\bar{\mathbf{v}}$ is obtained by using the same arguments employed when analysing the decay rate of $\mathbf{v}$. However the proof is technically more involved due to the presence of the correcting terms on the right hand side of (1.20) and (1.22).

In a third step, we rewrite equations (1.20) and (1.22), using the so-called similarity variables and rescaled functions. Working in weighted Sobolev spaces, we determine the decay rate of $\bar{\mathbf{v}} = \mathbf{v} - \mathbf{M}_1\, G$ in these similarity variables. Expressing this result in the classical variables, we prove, in particular, the decay of the $\mathbf{L}^1$ norm of $\bar{\mathbf{v}}$.

The conclusion of Theorem 1.2 results by interpolation of the $\mathbf{L}^p$ estimate with the $\mathbf{L}^1$ decay of the solution.

The outline of the proof of Theorem 1.3 is the following: we begin by determining the expressions of $[\mathbf{M}_2]$ distinguishing the dimension $N = 2$ and $N \geq 3$, using scaling arguments and similarity variables. The most serious difficulty consists in estimating the term involving the integrals on the interface. This task requires estimates on $\Delta\mathbf{v}$ and $\mathbf{v}_t$.

1.5 Plan of the paper

This article is organised as follows: at the beginning of the following section, we give some basic estimates like, for example, the energy dissipation law. Then, we study the decay rate in $\mathbf{L}^p$ of a solution of a generalised version of system (1.4). This system is similar to (1.4), but a little more complex because it contains some additional non-linear terms. As an application of these results, we deduce the decay rate of the solution $\mathbf{v}$ of system (1.4), as well as the decay rate of $\bar{\mathbf{v}} = \mathbf{v} - \mathbf{M}_1 G$. The decay of the $\mathbf{L}^1$ norm is proved in section 3 by classical parabolic techniques, using similarity variables and scaling arguments. However, in our case, the presence of the second unknown $\mathbf{g}$ requires special care. These arguments allow us to perform the proof of Theorem 1.2, combining the decay rate of the $\mathbf{L}^1$ norm with the results of section 2.1. In section 4, we compute the decay rate of $\|\Delta\mathbf{v}\|_4$ and $\|\mathbf{u}_t\|_4$. Afterwards, in section 5, we identify the second term in the asymptotic development in similarity variables and give the proof of Theorem 1.3. As we already pointed it out, the existence and uniqueness of the solution of (1.4) as well as the proofs of some technical Lemmas will be carried out in two Appendices at the end of the paper.

2 Decay rates

From now on, we shall work with the scalar functions v and g introduced in subsection 1.2 to denote any of the components v_i, g_i of the vectors $\mathbf{v}$, $\mathbf{g}$.

2.1 Basic a priori estimates

We first state some basic estimates:

- *Energy dissipation:*
  Multiplying by v and integrating by parts the first equation of system (1.5), we find:

$$\frac{1}{2}\left[\int_\Omega v^2(t,x)\,d\mathbf{x} + m\,|g(t)|^2\right] + \int_0^t\int_\Omega |\nabla v|^2\,d\mathbf{x}\,ds = \frac{1}{2}\left[\int_\Omega v^2(0,\mathbf{x})\,d\mathbf{x} + m\,|g(0)|^2\right]. \quad (2.1)$$

- *L^p estimates:*
  In the same way as above, we multiply the equation by $j'(v)$, with j a real valued convex function and we integrate with respect to $\mathbf{x}$ to obtain:

$$\frac{d}{dt}\left[\int_\Omega j(v)\,d\mathbf{x} + m\,j(g(t))\right] = -\int_\Omega |\nabla v|^2 j''(v) d\mathbf{x}.$$

If we choose for $j(v)$ an approximation of the function $|v|^p$, we deduce that the quantity

$$\int_\Omega |v|^p \, d\mathbf{x} + m\, |g|^p, \tag{2.2}$$

decreases in time whenever $v_0 \in L^p(\Omega)$ for all $1 \leq p < \infty$.

The first stage in the analysis of the large time behaviour of (1.5) consists in establishing the decay rate of the solution. But, instead of studying directly (1.5), we prefer considering the following more general framework in which the same decay properties hold.

2.2 General decay results

We consider, in this subsection, any smooth global in time solution $(\mathbf{v}, \mathbf{g})$:

$$\mathbf{v} \in C\left([0,\infty), \mathbf{L}^2(\Omega)\right) \cap \mathbf{L}^2\left((0,\infty), \dot{\mathbf{H}}^1(\Omega)\right) \cap L^\infty((0,\infty), \mathbf{L}^1(\Omega)),$$
$$\mathbf{g} \in C\left([0,\infty), \mathbb{R}^N\right),$$

of the following non-linear system:

$$\begin{cases} \mathbf{v}_t - \Delta \mathbf{v} - [\mathbf{U}]\, \mathbf{g} - \mathbf{V}(t) \cdot \nabla \mathbf{v} = \boldsymbol{\varepsilon}_1(\mathbf{x}, t), & \text{on } \Omega \times (0,\infty), \\ \mathbf{v} = \mathbf{g}, & \text{on } \partial\Omega \times (0,\infty), \\ m\, \mathbf{g}'(t) = -\displaystyle\int_{\partial\Omega} \mathbf{n} \cdot \nabla \mathbf{v} \, d\sigma_x + \boldsymbol{\varepsilon}_2(t), & \text{on } (0,\infty), \\ \mathbf{v}(0) = \mathbf{v}_0, \quad \mathbf{g}(0) = \mathbf{g}_0, \end{cases} \tag{2.3}$$

where $[\mathbf{U}](\mathbf{x}, t)$ is a matrix valued function and $\mathbf{V}(t)$, $\boldsymbol{\varepsilon}_1(t)$ and $\boldsymbol{\varepsilon}_2(t)$ three vector valued functions which will be specified later.

In the sequel, we will apply the results obtained for the general system (2.3) in the following particular cases:

Application 1 *If we specify* $[\mathbf{U}] = [\mathbf{0}]$, $\mathbf{V} = \mathbf{g}$, $\boldsymbol{\varepsilon}_1 = \mathbf{0}$ *and* $\boldsymbol{\varepsilon}_2 = \mathbf{0}$ *we obtain system* (1.4). *This case will be considered in subsection 2.3, Proposition 2.2.*

Application 2 *In view of equations* (1.20) *and* (1.22), $(\bar{\mathbf{v}}, \bar{\mathbf{g}})$ *solves system* (2.3) *with* $\mathbf{V} = \mathbf{g}$, $[\mathbf{U}] = [\mathbf{0}]$, $\boldsymbol{\varepsilon}_1 := \mathbf{M}_1\, \mathbf{g} \cdot \nabla G$ *and* $\boldsymbol{\varepsilon}_2(t) := -\mathbf{M}_1\, \varepsilon(t)$ *where* $\varepsilon(t)$ *is defined by* (1.21). *This case will be treated in section 3, Proposition 3.1.*

Application 3 *Consider* $(\mathbf{v}, \mathbf{g})$, *the solution of system* (1.4). *Then, its time derivative* $(\mathbf{v}_t, \mathbf{g}_t)$ *solves system* (2.3) *with* $[\mathbf{U}] = \nabla \mathbf{v}$ *and* $\mathbf{V} = \mathbf{g}$. *This case will be investigated in section 4.*

In the following Proposition, we describe the decay rate in $\mathbf{L}^p$ of the solution $(\mathbf{v}, \mathbf{g})$ of the general system (2.3).

Proposition 2.1 *Let us denote:*

$$\delta_1 := 2\,N \sup_{t\in(0,\infty)} (\|\mathbf{v}\|_1 + m\,|\mathbf{g}|). \tag{2.4}$$

- *Fix $1 < p < \infty$ and assume also that there exists $C_p > 0$ and $\alpha_p > 0$ such that the functions:*

$$\vartheta_1(t) := t\,\|\,[\mathbf{U}]\,\|_p, \tag{2.5a}$$

$$\vartheta_2(t) := \epsilon_p(t)\,t^{\frac{N}{2}(1-\frac{1}{p})+1}, \tag{2.5b}$$

$$\epsilon_p(t) := \max\left(\|\boldsymbol{\varepsilon}_1\|_p,\, \frac{1}{m}|\boldsymbol{\varepsilon}_2|\right), \tag{2.5c}$$

  fulfil the estimate:

$$\vartheta_1(t) + \vartheta_2(t) \le C_p\,\left(1 + t^{-\alpha_p}\right), \quad \forall\, t > 0. \tag{2.6}$$

  Then, any smooth solution $(\mathbf{v}, \mathbf{g})$ of system (2.3) satisfies the following decay properties:

$$\|\mathbf{v}\|_p \le C(p)\,\delta_p\,t^{-\frac{N}{2}(1-\frac{1}{p})}, \tag{2.7a}$$

$$|\mathbf{g}| \le C(p)\,\delta_p\,t^{-\frac{N}{2}(1-\frac{1}{p})}, \quad \forall\, t \ge 1, \tag{2.7b}$$

  where δ_p is a positive constant defined by:

$$\delta_p := \delta_1\,\max\left\{(1+\alpha_p)^{\frac{N}{2}(1-\frac{1}{p})},\ \left[\delta_1^{-1} \sup_{t\in(1,\infty)} \vartheta_2(t)\right]^{\frac{N(p-1)}{2p+N(p-1)}},\ \sup_{t\in(1,\infty)} \vartheta_1(t)^{\frac{N}{2}(1-\frac{1}{p})}\right\}. \tag{2.8}$$

  Moreover, the constants $C(p)$ in estimates (2.7) depends on p and N only.

- *Assume furthermore that:*

$$C_p \text{ and } \alpha_p \text{ in (2.6) are uniformly bounded for all } p \text{ large enough.} \tag{2.9}$$

  In this case, estimates (2.7) remain valid for $p = \infty$ with δ_p as in (2.8) with $p = \infty$.

Remark 2.1 *The following comments are in order:*

- *In view of the definitions (2.5), it is obvious that ϑ_1 and ϑ_2 depend on p. Nevertheless, to shorten notations, we have not make this depedence explicit.*

- *We do not make any assumption on the decay properties of the potential* $\mathbf{V}$ *because the term* $\mathbf{V}\cdot\nabla\mathbf{v}$ *vanishes in all the estimates, since* $\mathbf{V}$ *depends only on* t.

- *The decay rate* (2.7a) *we obtain for* v *coincides with the one of the solution of the heat equation on* $\mathbb{R}^N$ *and with those of the 1-d model for fluid-solid interaction in [18].*

Proof of Proposition 2.1: We treat separately the cases $1 < p < \infty$ and $p = \infty$. We proceed componentwise, using the rules of notation of section 1.1: v and g stand for any component v_i and g_i of $\mathbf{v}$ and $\mathbf{g}$. The corresponding first momentum will be denoted by M_1 although it stands for the quantity $M_{1,i}$.

The case $1 < p < \infty$**:**
Multiplying the equation (2.3-i) by $v|v|^{p-2}$ and integrating by parts, the term $\int_\Omega \mathbf{V}\cdot\nabla v|v|^{p-2}v\,d\mathbf{x}$ vanishes according to Green's formula and we get:

$$\frac{1}{p}\frac{d}{dt}[\|v\|_p^p + m|g|^p] = -\frac{4(p-1)}{p^2}\|\nabla|v|^{\frac{p}{2}}\|_2^2 + \int_\Omega \mathbf{g}\cdot\mathbf{U}\,v\,|v|^{p-2}\,d\mathbf{x} + \int_\Omega \varepsilon_1 v\,|v|^{p-2}\,d\mathbf{x} + \varepsilon_2\,g\,|g|^{p-2}, \quad \forall\, t \geq 0. \quad (2.10)$$

We begin by estimating the last three terms. Let us denote:

$$\begin{aligned} I(t) :=& \left|\int_\Omega \mathbf{g}\cdot\mathbf{U}v\,|v|^{p-2}\,d\mathbf{x}\right| + \left|\int_\Omega \varepsilon_1 v\,|v|^{p-2}\,d\mathbf{x}\right| + \left|\varepsilon_2\,g\,|g|^{p-2}\right|, \\ =& I_1(t) + I_2(t) + I_3(t), \quad \forall\, t \geq 0, \end{aligned}$$

and prove the following Lemma:

Lemma 2.1 *There exists a constant* $C > 0$ *depending on* m *and* N *only, such that:*

$$I(t) \leq C\|[\mathbf{U}]\|_p \left[\|v\|_p^p + m\left(\max_{i=1,\dots,N}|g_i|\right)^p\right] + C\epsilon_p(t)\left[\|v\|_p^p + m|g|^p\right]^{1-1/p}, \quad \forall\, t \geq 0. \quad (2.11)$$

Proof of Lemma 2.1:

- Concerning I_1, we have:

$$\left|\int_\Omega \mathbf{g}\cdot\mathbf{U}v|v|^{p-2}\,d\mathbf{x}\right| \leq \int_\Omega |\mathbf{g}\cdot\mathbf{U}|\,|v|^{p-1}\,d\mathbf{x}, \quad \forall\, t \geq 0, \quad (2.12)$$

and:

$$|\mathbf{g}\cdot\mathbf{U}| \le \left(\max_{i=1,\dots,N}|g_i|\right)\left(\sum_{i=1}^{N}|U_i|\right) \le N^{1/2}\left(\max_{i=1,\dots,N}|g_i|\right)|\mathbf{U}|. \tag{2.13}$$

Since, obviously $|\mathbf{U}| \le |[\mathbf{U}]|$, applying Hölder's inequality we get with (2.12) and (2.13):

$$I_1(t) \le\ N^{1/2}\left(\max_{i=1,\dots,N}|g_i|\right)\|[\mathbf{U}]\|_p\|v\|_p^{p-1}, \quad \forall\, t\ge 0. \tag{2.14}$$

However:

$$\left(\max_{i=1,\dots,N}|g_i|\right)\|v\|_p^{p-1} \le C\left[\|v\|_p^p + m\left(\max_{i=1,\dots,N}|g_i|\right)^p\right], \quad \forall\, t\ge 0, \tag{2.15}$$

with $C=C(m)$. Combining (2.12), (2.14) and (2.15), it comes:

$$I_1(t) \le C\|[\mathbf{U}]\|_p\left[\|v\|_p^p + m\left(\max_{i=1,\dots,N}|g_i|\right)^p\right], \quad \forall\, t\ge 0. \tag{2.16}$$

- For I_2, one checks easily, by Hölder's inequality that:

$$I_2(t) \le \|\varepsilon_1\|_p\|v\|_p^{p-1} \le \|\boldsymbol{\varepsilon}_1\|_p\|v\|_p^{p-1}, \quad \forall\, t\ge 0. \tag{2.17}$$

- At last, I_3 satisfies:

$$I_3(t) \le |\varepsilon_2|\,|g|^{p-1} \le |\boldsymbol{\varepsilon}_2|\,|g|^{p-1}, \quad \forall\, t\ge 0. \tag{2.18}$$

Using the notation (2.5c) and combining (2.17) and (2.18), we obtain that:

$$I_2 + I_3 \le \epsilon_p\left[\|v\|_p^{p-1} + m|g|^{p-1}\right], \quad \forall\, t\ge 0.$$

Applying the inequality $a^\gamma + b^\gamma \le 2^{1-\gamma}(a+b)^\gamma$, valid for any $0<\gamma\le 1$ and $a,b>0$, with $\gamma = 1-1/p$, we get:

$$I_2 + I_3 \le C\epsilon_p\left[\|v\|_p^p + m|g|^p\right]^{1-1/p}, \quad \forall\, t\ge 0, \tag{2.19}$$

with $C=C(m)$. Putting together (2.16) and (2.19), we obtain (2.11). ■

Going back to equation (2.10), we give now estimates for the term involving the gradient of $|v|^{\frac{p}{2}}$.

Lemma 2.2 *For any $N\ge 2$ and $p>1$,*

$$\frac{\|v\|_p^{p\left(1+\frac{2}{N\,(p-1)}\right)}}{\left[\|v\|_1 + m|g|\right]^{\frac{2p}{N(p-1)}}} \le C\|\nabla|v|^{\frac{p}{2}}\|_2^2, \quad \forall\, t\ge 0, \tag{2.20}$$

where the constant $C>0$ depends only on N, but is independent of v.

Remark 2.2 *This Lemma is similar to Lemma 1 in [12]. However* (2.20) *contains the term* $|g|$ *and the constant* C *is here independent of* p*.*

We need also the same kind of estimate involving the second unknown $|g|$:

Lemma 2.3 *For any* $N \geq 2$ *and* $p > 1$*,*

$$\frac{(m|g|^p)^{1+\frac{2}{N(p-1)}}}{[\|v\|_1 + m|g|]^{\frac{2p}{N(p-1)}}} \leq C\|\nabla|v|^{\frac{p}{2}}\|_2^2, \quad \forall\, t \geq 0, \tag{2.21}$$

where $C > 0$ *depends on* N *and* m *only.*

The proofs of Lemma 2.2 and Lemma 2.3 will be given in Appendix B.

Observe now that:

$$\left[\|v\|_p^p + m|g|^p\right]^{1+\frac{2}{N(p-1)}} \leq 2^{\frac{2}{N(p-1)}} \left[\|v\|_p^{p(1+\frac{2}{N(p-1)})} + (m|g|^p)^{1+\frac{2}{N(p-1)}}\right], \quad \forall\, t \geq 0, \tag{2.22}$$

because of the inequality $(a+b)^\gamma \leq 2^{\gamma-1}(a^\gamma + b^\gamma)$ which is valid for any $\gamma > 1$ and $a, b > 0$. Inequalities (2.20), (2.21) and (2.22) yield:

$$\left[\|v\|_p^p + m\,|g|^p\right]^{1+\frac{2}{N(p-1)}} \leq C\, 2^{\frac{2}{N(p-1)}}\, [\|v\|_1 + m\,|g|]^{\frac{2p}{N(p-1)}}\, \|\nabla|v|^{\frac{p}{2}}\|_2^2, \tag{2.23}$$

with C uniform with respect to p. In view of the definition (2.4) of δ_1, we get:

$$\|v\|_1 + m\,|g| \leq \frac{1}{2\,N}\,\delta_1,$$

and then

$$\begin{aligned}\left[\|v\|_p^p + m\,|g|^p\right]^{1+\frac{2}{N(p-1)}} &\leq C\, \left(2^{\frac{1}{p}-1}\,\frac{\delta_1}{N}\right)^{\frac{2p}{N(p-1)}} \|\nabla|v|^{\frac{p}{2}}\|_2^2 \\ &\leq C\, \left(\frac{\delta_1}{N}\right)^{\frac{2p}{N(p-1)}} \|\nabla|v|^{\frac{p}{2}}\|_2^2, \end{aligned} \tag{2.24}$$

where C does not depend on p.

In all the sequel we will be very careful on how the constants in the estimates depend on δ_1 and p.

Introducing the functions:

$$X_p := \left[\|v\|_p^p + m\,|g|^p\right]^{1/p}, \tag{2.25a}$$

$$Y_p := \left[\|\mathbf{v}\|_p^p + m\,|\mathbf{g}|^p\right]^{1/p} = \left[\sum_{i=1}^N \|v_i\|_p^p + m\,|g_i|^p\right]^{1/p}, \tag{2.25b}$$

we can summarise (2.10), (2.11) and (2.24) by:

$$\frac{1}{p}\,(X_p^p)' + C\,\frac{(p-1)}{p^2}\,\left(\frac{\delta_1}{N}\right)^{-\frac{2p}{N(p-1)}}\;X_p^{p+\frac{2p}{N(p-1)}} \\ - C\,\|[\mathbf{U}]\|_p\,\left[\|v\|_p^p + m\left(\max_{i=1,\dots,N}|g_i|\right)^p\right] - C\,\epsilon_p\,X_p^{p-1} \leq 0, \quad \forall\, t \geq 0. \tag{2.26}$$

This last inequality holds for each component v and g of $\mathbf{v}$ and $\mathbf{g}$. Adding together these N inequalities, we get:

$$\frac{1}{p}\,(Y_p^p)' + C\,\frac{(p-1)}{p^2}\,\left(\frac{\delta_1}{N}\right)^{-\frac{2p}{N(p-1)}}\;\sum_{i=1}^{N}\left[\|v_i\|_p^p + m\,|g_i|^p\right]^{1+\frac{2}{N(p-1)}} \\ - C\,\|[\mathbf{U}]\|_p\,\left[\sum_{i=1}^{N}\|v_i\|_p^p + m\,N\left(\max_{i=1,\dots,N}|g_i|\right)^p\right] \\ - C\,\epsilon_p\,\sum_{i=1}^{N}\left[\|v_i\|_p^p + m\,|g_i|^p\right]^{1-\frac{1}{p}} \leq 0, \quad \forall\, t \geq 0. \tag{2.27}$$

A convexity argument yields:

$$Y_p^{p+\frac{2\,p}{N(p-1)}} = \left[\sum_{i=1}^{N}\|v_i\|_p^p + m\,|g_i|^p\right]^{1+\frac{2}{N(p-1)}} \leq N^{\frac{2}{N(p-1)}}\sum_{i=1}^{N}\left[\|v_i\|_p^p + m\,|g_i|^p\right]^{1+\frac{2}{N(p-1)}},$$

and then:

$$\begin{aligned} {\delta_1}^{-\frac{2p}{N(p-1)}}\,Y_p^{p+\frac{2\,p}{N(p-1)}} &= \left(\frac{\delta_1}{N}\right)^{-\frac{2p}{N(p-1)}}\;N^{-\frac{2\,p}{N(p-1)}}\;Y_p^{p+\frac{2\,p}{N(p-1)}} \\ &\leq \left(\frac{\delta_1}{N}\right)^{-\frac{2p}{N(p-1)}}\;N^{-\frac{2}{N(p-1)}}\;Y_p^{p+\frac{2\,p}{N(p-1)}} \\ &\leq \left(\frac{\delta_1}{N}\right)^{-\frac{2p}{N(p-1)}}\;\sum_{i=1}^{N}\left[\|v_i\|_p^p + m\,|g_i|^p\right]^{1+\frac{2}{N(p-1)}}. \end{aligned}$$

One proves as well, by concavity of the function $x \mapsto x^{1-\frac{1}{p}}$, that:

$$\sum_{i=1}^{N}\left[\|v_i\|_p^p + m\,|g_i|^p\right]^{1-\frac{1}{p}} \leq C\,\left[\sum_{i=1}^{N}\|v_i\|_p^p + m\,|g_i|^p\right]^{1-\frac{1}{p}} = C\,Y_p^{p-1}, \tag{2.28a}$$

with $C := N^{1/p} \leq N$. On the other hand:

$$\left(\max_{i=1,\dots,N}|g_i|\right) \leq \left(\sum_{i=1}^{N}|g_i|^p\right)^{\frac{1}{p}},$$

for all $1 \leq p < \infty$. We deduce that:

$$\sum_{i=1}^{N} \|v_i\|_p^p + m\, N \left(\max_{i=1,\dots,N} |g_i|\right)^p \leq N\, Y_p^p. \tag{2.28b}$$

Relations (2.28) together with (2.27) yield:

$$\frac{1}{p}\,(Y_p^p)' + C\,\frac{(p-1)}{p^2}\,\delta_1^{-\frac{2p}{N(p-1)}}\,Y_p^{p+\frac{2p}{N(p-1)}} - C\,\|[\mathbf{U}]\|_p\,Y_p^p - C\,\epsilon_p\,Y_p^{p-1} \leq 0, \quad \forall\, t \geq 0. \tag{2.29}$$

According to notations (2.5) of Proposition 2.1, (2.26) reads:

$$\frac{1}{p}\,(Y_p^p)' + C\,\frac{(p-1)}{p^2}\,\delta_1^{-\frac{2p}{N(p-1)}}\,Y_p^{p+\frac{2p}{N(p-1)}} - C\,t^{-1}\,\vartheta_1\,Y_p^p - C\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-1}\,\vartheta_2\,Y_p^{p-1} \leq 0,$$

for all $t \geq 0$. Multiplying both sides by Y_p^{1-p}, it comes:

$$Y_p' + C(p)\,\delta_1^{-\frac{2p}{N(p-1)}}\,Y_p^{1+\frac{2p}{N(p-1)}} - C\;t^{-1}\,\vartheta_1\,Y_p - C\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-1}\,\vartheta_2 \leq 0, \tag{2.30}$$

with $C(p) := \frac{(p-1)}{p^2}$. We introduce then the function Z_p defined on $(0,\infty)$ by:

$$Z_p(t) := \bar{C}_p\,\delta_p \left(\frac{N}{2} + \gamma_1\,t^{-\alpha_p}\right)^{\frac{N}{2}\left(1-\frac{1}{p}\right)} t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)},$$

where δ_p and α_p are the constants defined by (2.8) and in the hypothesis (2.6) respectively and γ_1 and $\bar{C}_p$ are as follows:

$$\gamma_1 := \sup_{t\in(0,1)} \vartheta_1(t)\,t^{\alpha_p} + \delta_1^{-1} \sup_{t\in(0,1)} \vartheta_2(t)\,t^{\alpha_p}, \tag{2.31}$$

and

$$\bar{C}_p := \left[3\,C(p)^{-1}\,\max\left\{\frac{N}{2}, C(p),\, C\right\}\right]^{\frac{N}{2}}, \tag{2.32}$$

$C(p)$ and C being the constants in (2.30). The assumption (2.6) of the Proposition ensures that γ_1 is well defined. Direct computations yield:

$$Z_p' + C(p)\,\delta_1^{-\frac{2p}{N(p-1)}}\,Z_p^{1+\frac{2p}{N(p-1)}} - C\,t^{-1}\,\vartheta_1\,Z_p - C\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-1}\,\vartheta_2 = F(t), \tag{2.33}$$

for all $t \geq 0$, where $F(t)$ is defined on $(0,\infty)$ by:

$$\begin{aligned} F(t) := t^{-1}\,Z_p \Bigg\{ &-\frac{N}{2}\left(1-\frac{1}{p}\right) \\ &+ \gamma_1\,t^{-\alpha_p}\left[C(p)\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - \alpha_p\,\frac{N}{2}\left(1-\frac{1}{p}\right)\left(\frac{N}{2}+\gamma_1\,t^{-\alpha_p}\right)^{-1}\right] \\ &+ C(p)\,\frac{N}{2}\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - C\,\vartheta_1 - C\left(\frac{N}{2}+\gamma_1\,t^{-\alpha_p}\right)^{-\frac{N}{2}\left(1-\frac{1}{p}\right)} (\bar{C}_p\,\delta_p)^{-1}\,\vartheta_2 \Bigg\}. \end{aligned} \tag{2.34}$$

We are going to prove that any solution Y_p of (2.30) is a sub-solution of (2.33) and hence, by Theorem 1.5.3 of [15], that:

$$Y_p(t) \leq Z_p(t), \quad \forall\, t \geq 0. \tag{2.35}$$

Observe that $t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)}$ in the definition of Z_p is the term we need in the right hand side of (2.35) in order conclude the proof of the decay rate (2.7). The term added in the expression of Z_p in which $t^{-\alpha_p}$ appears has no incidence on the asymptotic behaviour as $t \to \infty$, but it is required to get (2.35) in the neighbourhood of $t = 0$.

Thus, it is sufficient to prove that:

$$F(t) \geq 0, \quad \forall\, t \geq 0. \tag{2.36}$$

Note that $t^{-1}\, Z_p$, in the definition of $F(t)$, is positive. Since

$$\frac{N}{2}\left(1-\frac{1}{p}\right)\left(\frac{N}{2}+\gamma_1\, t^{-\alpha_p}\right)^{-1} \leq 1,$$

we obtain also that:

$$C(p)\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - \alpha_p\,\frac{N}{2}\left(1-\frac{1}{p}\right)\left(\frac{N}{2}+\gamma_1\, t^{-\alpha_p}\right)^{-1} \geq C(p)\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - \alpha_p.$$

On the other hand $\left(\frac{N}{2}+\gamma_1\, t^{-\alpha_p}\right)^{-\frac{N}{2}\left(1-\frac{1}{p}\right)} \leq 1$, because $N \geq 2$, and $\bar{C}_p^{-1} \leq 1$ (obvious with the definition (2.32)) and $N/2 \geq 1$. Hence, to get (2.36), it is sufficient to prove that:

$$-\frac{N}{2}+\gamma_1\, t^{-\alpha_p}\left[C(p)\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - \alpha_p\right] + C(p)\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - C\,\vartheta_1 - C\,\delta_p^{-1}\,\vartheta_2 \geq 0, \quad \forall\, t \geq 0. \tag{2.37}$$

Dividing in (2.37) by $\max\left\{\frac{N}{2}, C(p), C\right\} \geq 1$, the problem is reduced to prove that:

$$\gamma_1\, t^{-\alpha_p}\left[\tilde{C}\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - \alpha_p\right] + \tilde{C}\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - 1 - \vartheta_1 - \delta_p^{-1}\,\vartheta_2 \geq 0, \tag{2.38}$$

for all $t \geq 0$, where $0 < \tilde{C} < 1$ is defined by

$$\tilde{C} := C(p)\,\max\left\{\frac{N}{2}, C(p), C\right\}^{-1}. \tag{2.39}$$

We proceed in two steps proving first that (2.38) holds on $(1, \infty)$ and then on $(0, 1)$.

- When $t \in (1,\infty)$:
  According to the definition (2.8), we have

$$\delta_p \geq \delta_1\, \delta_1^{-\frac{N(p-1)}{2p+N(p-1)}} \sup_{t\in(1,\infty)} \vartheta_2(t)^{\frac{N(p-1)}{2p+N(p-1)}}. \tag{2.40}$$

  Basic computations yield:

$$\tilde{C}\,(\bar{C}_p\,\delta_1^{-1}\delta_p)^{\frac{2p}{N(p-1)}} \geq \tilde{C}\,\bar{C}_p^{\frac{2p}{N(p-1)}}\,\delta_1^{-\frac{2p}{2p+N(p-1)}} \sup_{t\in(1,\infty)} \vartheta_2(t)^{\frac{2p}{2p+N(p-1)}}. \tag{2.41}$$

  From (2.40), we deduce also that, for all $t \geq 1$:

$$\delta_p^{-1}\vartheta_2 \leq \delta_p^{-1} \sup_{t\in(1,\infty)} \vartheta_2(t) \leq \delta_1^{-\frac{2p}{2p+N(p-1)}} \sup_{t\in(1,\infty)} \vartheta_2(t)^{\frac{2p}{2p+N(p-1)}}. \tag{2.42}$$

  Combining (2.41) and (2.42), we get, for all $t \geq 1$:

$$\tilde{C}\,(\bar{C}_p\,\delta_1^{-1}\,\delta_p)^{\frac{2p}{N(p-1)}} \geq \tilde{C}\,\bar{C}_p^{\frac{2p}{N(p-1)}}\,\delta_p^{-1}\,\vartheta_2. \tag{2.43}$$

  Comparing the definitions (2.32) and (2.39) of $\bar{C}_p$ and $\tilde{C}$, one remarks that $\bar{C}_p = (3\,\tilde{C}^{-1})^{\frac{N}{2}}$ and therefore that $\tilde{C}\,\bar{C}_p^{\frac{2p}{N(p-1)}} = 3\,(3\,\tilde{C}^{-1})^{\frac{1}{p-1}}$. Since $0 < \tilde{C} < 1$, this implies:

$$\tilde{C}\,\bar{C}_p^{\frac{2p}{N(p-1)}} \geq 3, \tag{2.44}$$

  and (2.43) becomes:

$$\frac{1}{3}\,\tilde{C}\,(\bar{C}_p\,\delta_1^{-1}\,\delta_p)^{\frac{2p}{N(p-1)}} \geq \delta_p^{-1}\,\vartheta_2, \quad \forall t \geq 1. \tag{2.45a}$$

  On the other hand since, by the definition (2.8),

$$\delta_p \geq \delta_1 \sup_{t\in(1,\infty)} \vartheta_1(t)^{\frac{N}{2}\left(1-\frac{1}{p}\right)},$$

  we have also:

$$\tilde{C}\,(\bar{C}_p\,\delta_1^{-1}\,\delta_p)^{\frac{2p}{N(p-1)}} \geq \tilde{C}\,\bar{C}_p^{\frac{2p}{N(p-1)}}\,\vartheta_1, \quad \forall t \geq 1,$$

  and as a consequence of (2.44) we get:

$$\frac{1}{3}\tilde{C}\,(\bar{C}_p\,\delta_1^{-1}\,\delta_p)^{\frac{2p}{N(p-1)}} \geq \vartheta_1, \quad \forall t \geq 1. \tag{2.45b}$$

  Finally, once again, according to the definition (2.8):

$$\delta_p \geq (1+\alpha_p)^{\frac{N}{2}\left(1-\frac{1}{p}\right)}\,\delta_1,$$

what leads to:

$$\tilde{C}\,(\bar{C}_p\,\delta_1^{-1}\,\delta_p)^{\frac{2p}{N(p-1)}} \geq \tilde{C}\,\bar{C}_p^{\frac{2p}{N(p-1)}}(1+\alpha_p),$$

and hence, with (2.44):

$$\frac{1}{3}\tilde{C}\,(\bar{C}_p\,\delta_1^{-1}\,\delta_p)^{\frac{2p}{N(p-1)}} \geq (1+\alpha_p) \geq 1. \tag{2.45c}$$

Summing together the three relations (2.45), we get:

$$\tilde{C}\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - 1 - \vartheta_1 - \delta_p^{-1}\,\vartheta_2 \geq 0, \quad \forall t \geq 1.$$

We deduce also, from (2.45c) that $\tilde{C}\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - \alpha_p \geq 0$ for all $t > 0$, what allows us to conclude that (2.38) is true for all $t \geq 1$.

- When $t \in (0,1)$:
  We must now establish the estimate (2.38) on the interval $(0,1)$. Since $t^{\alpha_p} > 0$, (2.38) is equivalent to:

$$\gamma_1\,\left[\tilde{C}\,(\bar{C}_p\,\delta_1^{-1}\,\delta_p)^{\frac{2p}{N(p-1)}} - \alpha_p\right] + \tilde{C}\,t^{\alpha_p}\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - t^{\alpha_p} - \vartheta_1\,t^{\alpha_p} \\ - \delta_p^{-1}\,\vartheta_2\,t^{\alpha_p} \geq 0.$$

  According to (2.45c) we have $\tilde{C}\,t^{\alpha_p}\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - t^{\alpha_p} \geq 0$, as well as $\tilde{C}\,(\delta_1^{-1}\,\bar{C}_p\,\delta_p)^{\frac{2p}{N(p-1)}} - \alpha_p \geq 1$. From the definition (2.8) of δ_p, we deduce straightforwardly that $\delta_p \geq \delta_1$. Hence, it remains only to check that:

$$\gamma_1 - \vartheta_1\,t^{\alpha_p} - \delta_1^{-1}\,\vartheta_2\,t^{\alpha_p} \geq 0, \quad \forall\, 0 < t \leq 1, \tag{2.46}$$

  what is obvious in view of the definition (2.31) of γ_1.

  The proof is then completed for $p < \infty$.

The case $p = \infty$:
Applying Lemma 2.2 and Lemma 2.3 with $p = 2$ and replacing v by $|v|^q$ and g by $|g|^q$ respectively ($q > 1$), we obtain:

$$\|v\|_{2q}^{2q\left(1+\frac{2}{N}\right)} \leq C\,\left[\|v\|_q^q + m\,|g|^q\right]^{\frac{4}{N}}\,\|\nabla|v|^q\|_2^2,$$

and

$$\left(m\,|g|^{2q}\right)^{1+\frac{2}{N}} \leq C\,\left[\|v\|_q^q + m\,|g|^q\right]^{\frac{4}{N}}\,\|\nabla|v|^q\|_2^2.$$

Arguing as for (2.23), these estimates provide:

$$\left[\|v\|_{2q}^{2q} + m\,|g|^{2q}\right]^{1+\frac{2}{N}} \leq C\,2^{\frac{2}{N}}\,\left[\|v\|_q^q + m\,|g|^q\right]^{\frac{4}{N}}\,\|\nabla|v|^q\|_2^2$$
$$\leq C\,\left[\|\mathbf{v}\|_q^q + m\,|\mathbf{g}|^q\right]^{\frac{4}{N}}\,\|\nabla|v|^q\|_2^2. \qquad (2.47)$$

The relation (2.10) together with the definitions (2.5), Lemma 2.1 and the notations (2.25) yields:

$$\frac{1}{p}\frac{d}{dt}X_p^p - C\,t^{-1}\,\vartheta_1\,\left[\|v\|_p^p + m\,\left(\max_{i=1,\dots,N}|g_i|\right)^p\right] - C\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-1}\,\vartheta_2\,X_p^{p-1}$$
$$\leq -4\frac{p-1}{p^2}\,\|\nabla|v|^{\frac{p}{2}}\|_2^2,$$

for all $1 < p < \infty$. Taking $p = 2\,q$ and combining with (2.47), one gets:

$$\frac{1}{2q}\frac{d}{dt}X_{2q}^{2q} + C(q)\,Y_q^{-\frac{4q}{N}}\,X_{2q}^{2q\left(1+\frac{2}{N}\right)} - C\,t^{-1}\,\vartheta_1\,\left[\|v\|_{2q}^{2q} + m\,\left(\max_{i=1,\dots,N}|g_i|\right)^{2q}\right]$$
$$-\,C\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-1}\,\vartheta_2\,X_{2q}^{2q-1} \leq 0, \quad (2.48)$$

with $C(q) = C\,(2q-1)/q^2$. Summing together the N inequalities (2.48) corresponding to each component of $\mathbf{v}$ and $\mathbf{g}$, we obtain:

$$\frac{1}{2q}\frac{d}{dt}Y_{2q}^{2q} + C(q)\,Y_q^{-\frac{4q}{N}}\,\sum_{i=1}^{N}\left[\|v_i\|_{2q}^{2q} + m\,|g_i|^{2q}\right]^{\left(1+\frac{2}{N}\right)}$$
$$-\,C\,t^{-1}\,\vartheta_1\,\left[\sum_{i=1}^{N}\|v_i\|_{2q}^{2q} + m\,N\,\left(\max_{i=1,\dots,N}|g_i|\right)^{2q}\right]$$
$$-\,C\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-1}\,\vartheta_2\,\sum_{i=1}^{N}\left[\|v_i\|_{2q}^{2q} + m\,|g_i|^{2q}\right]^{1-\frac{1}{2q}} \leq 0. \quad (2.49)$$

But, by convexity of the function $x \mapsto x^{1+\frac{2}{N}}$:

$$Y_{2q}^{2q\left(1+\frac{2}{N}\right)} \leq N^{\frac{2}{N}}\sum_{i=1}^{N}\left[\|v_i\|_{2q}^{2q} + m\,|g_i|^{2q}\right]^{\left(1+\frac{2}{N}\right)}, \qquad (2.50a)$$

and by concavity of $x \mapsto x^{1-\frac{1}{2q}}$ we get:

$$\sum_{i=1}^{N}\left[\|v_i\|_{2q}^{2q} + m\,|g_i|^{2q}\right]^{1-\frac{1}{2q}} \leq C\,Y_{2q}^{2q\left(1-\frac{1}{2q}\right)}, \qquad (2.50b)$$

with $C := N^{1/2q} \leq N$ for all $1 \leq q < \infty$. On the other hand,

$$\sum_{i=1}^{N} \|v_i\|_{2q}^{2q} + m\,N\,\left(\max_{i=1,\dots,N} |g_i|\right)^{2q} \leq N\,Y_{2q}^{2q}. \tag{2.50c}$$

The relation (2.49) combined with estimates (2.50) provides:

$$\frac{1}{2q}\frac{d}{dt}Y_{2q}^{2q} + C(q)\,Y_q^{-\frac{4q}{N}}\,Y_{2q}^{2q\left(1+\frac{2}{N}\right)} - C\,t^{-1}\,\vartheta_1\,Y_{2q}^{2q} - C\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-1}\,\vartheta_2\,Y_{2q}^{2q-1} \leq 0.$$

Dividing all the terms by Y_{2q}^{2q-1}, we get:

$$Y_{2q}' + C(q)\,Y_q^{-\frac{4q}{N}}\,Y_{2q}^{1+\frac{4q}{N}} - C\,t^{-1}\,\vartheta_1\,Y_{2q} - C\,t^{-\frac{N}{2}\left(1-\frac{1}{2q}\right)-1}\vartheta_2 \leq 0. \tag{2.51}$$

Let us introduce the new functions:

$$Z_q(s) = Y_q(e^s)\,e^{s\frac{N}{2}\left(1-\frac{1}{q}\right)}, \quad \forall s \geq 0, \tag{2.52a}$$

$$Y_q(t) = Z_q(\log t)\,t^{-\frac{N}{2}\left(1-\frac{1}{q}\right)}, \quad \forall t \geq 1. \tag{2.52b}$$

The function Z_{2q} satisfies, according to (2.51):

$$Z_{2q}' - \frac{N}{2}\left(1 - \frac{1}{2q}\right) Z_{2q} + C(q)\,Z_q^{-\frac{4q}{N}}\,Z_{2q}^{1+\frac{4q}{N}} - C\,\vartheta_1\,Z_{2q} - C\,\vartheta_2 \leq 0. \tag{2.53}$$

For all $q > 1$, $\frac{N}{2}\left(1 - \frac{1}{2q}\right)$ is bounded above by $N/2$. The hypothesis (2.6) and (2.9) allow us to rewrite (2.53) as follows:

$$Z_{2q}' + C(q)\,Z_q^{-\frac{4q}{N}}\,Z_{2q}^{1+\frac{4q}{N}} - C\,Z_{2q} - C \leq 0. \tag{2.54}$$

Set then $\widehat{Z}_q = \max(Z_q, 1)$. Since $\widehat{Z}_q \geq Z_q$, we have $\widehat{Z}_q^{-\frac{4q}{N}} \leq Z_q^{-\frac{4q}{N}}$ and hence:

$$Z_{2q}' + C(q)\,\widehat{Z}_q^{-\frac{4q}{N}}\,Z_{2q}^{1+\frac{4q}{N}} - C\,Z_{2q} - C \leq 0. \tag{2.55}$$

For q large enough, note that the constant $C = 1$ satisfies (2.55). Indeed, since $\widehat{Z}_q \geq 1$, then $\widehat{Z}_q^{-\frac{4q}{N}} \leq 1$ and therefore:

$$C(q)\,\widehat{Z}_q^{-\frac{4q}{N}} - C \leq C(q) - C \leq 0,$$

for q large enough because $C(q) := C\,\left(\frac{2q-1}{q^2}\right) \to 0$ as $q \to \infty$.

Since 1 and Z_{2q} both satisfy (2.55), we can draw the same conclusion for $\widehat{Z}_{2q}$:

$$\widehat{Z}_{2q}' + C(q)\,\widehat{Z}_q^{-\frac{4q}{N}}\,\widehat{Z}_{2q}^{1+\frac{4q}{N}} - C\,\widehat{Z}_{2q} - C \leq 0.$$

Moreover, since $\widehat{Z}_{2q} \geq 1$, $C\,\widehat{Z}_{2q} \geq C$ and we obtain also that $\widehat{Z}_{2q}$ solves:

$$\widehat{Z}_{2q}' + C(q)\,\widehat{Z}_q^{-\frac{4q}{N}}\,\widehat{Z}_{2q}^{1+\frac{4q}{N}} - C\,\widehat{Z}_{2q} \leq 0. \tag{2.56}$$

Above in the proof, in the case $p < \infty$, in (2.35), we proved that there exists a constant $C_q > 0$ such that

$$Y_q(t) \leq C_q\,\delta_q\,t^{-\frac{N}{2}\left(1-\frac{1}{q}\right)}, \quad \forall\, t \geq 1. \tag{2.57}$$

This relation, in view of the definition (2.52) of Z_{2q}, ensures that Z_{2q} is bounded on $(0,\infty)$. We can introduce then:

$$\widehat{Z}_q^*(s_0) = \sup_{s \geq s_0} \widehat{Z}_q(s) < \infty, \tag{2.58}$$

and deduce from (2.56) that:

$$\widehat{Z}_{2q}' + C(q)\,\widehat{Z}_q^*(0)^{-\frac{4q}{N}}\,\widehat{Z}_{2q}^{1+\frac{4q}{N}} - C\,\widehat{Z}_{2q} \leq 0, \tag{2.59}$$

because $\widehat{Z}_q^*(0) \geq \widehat{Z}_q(s)$ and hence $\widehat{Z}_q^*(0)^{-\frac{4q}{N}} \leq \widehat{Z}_q(s)^{-\frac{4q}{N}}$ for all $s \geq 0$. We apply the following Lemma, whose proof is given in Appendix B, to equation (2.59):

Lemma 2.4 *Any positive solution z of:*

$$z' + C_1\,z^{1+\gamma} - C_2\,z \leq 0, \tag{2.60}$$

where C_1, C_2 and γ are given positive constants, satisfies the estimate:

$$z(t) \leq \left(\frac{C_1}{C_2}\right)^{-1/\gamma} \left(1 - e^{-C_2\,\gamma\,t}\right)^{-1/\gamma}. \tag{2.61}$$

We obtain for $\widehat{Z}_{2q}$ the estimate:

$$\widehat{Z}_{2q}(s) \leq \widehat{Z}_q^*(0) \left(\frac{C\,q^2}{2q-1}\right)^{\frac{N}{4q}} \left(1 - e^{-C\,\frac{4q}{N}\,s}\right)^{-\frac{N}{4q}},$$

and therefore:

$$\begin{aligned}
\widehat{Z}_{2q}^*(1/q) = \sup_{s \geq 1/q} \widehat{Z}_{2q}(s) &\leq \sup_{s \geq 1/q} \left[\widehat{Z}_q^*(0) \left(\frac{C\,q^2}{2q-1}\right)^{\frac{N}{4q}} \left(1 - e^{-C\,\frac{4q}{N}\,s}\right)^{-\frac{N}{4q}}\right] \\
&\leq \widehat{Z}_q^*(0) \left(\frac{C\,q^2}{2q-1}\right)^{\frac{N}{4q}} \left(1 - e^{-C\,\frac{4}{N}}\right)^{-\frac{N}{4q}} \\
&\leq \widehat{Z}_q^*(0) \left(\frac{C\,q^2}{2q-1}\right)^{\frac{N}{4q}},
\end{aligned}$$

where $C > 0$ does not depend on q. Taking $q = 2^n$ and iterating this argument, we obtain:

$$\widehat{Z}^*_{2^{n+1}}(2) \leq \widehat{Z}^*_{2^{n+1}}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}\right) \leq \prod_{k=0}^{n} \left(\frac{C\, 2^{2k}}{2^{k+1}-1}\right)^{\frac{N}{4}\frac{1}{2^k}} \widehat{Z}^*_1(0).$$

Taking the logarithm of the right hand side, we get

$$\log\left(\prod_{k=0}^{n} \left(\frac{C\, 2^{2k}}{2^{k+1}-1}\right)^{\frac{N}{4}\frac{1}{2^k}}\right) = \frac{N}{4} \sum_{k=0}^{n} \frac{1}{2^k} \log\left(\frac{C\, 2^{2k}}{2^{k+1}-1}\right). \tag{2.62}$$

Moreover:

$$\left|\frac{1}{2^k} \log\left(\frac{C\, 2^{2k}}{2^{k+1}-1}\right)\right| \leq C\, \frac{k}{2^k}.$$

Therefore, the series in the right hand side of (2.62) converges as $n \to \infty$ and we deduce that:

$$\limsup_{n\to\infty} \widehat{Z}^*_{2^n}(2) < \infty.$$

According to the definition (2.52) of Z_q and (2.58) of $\widehat{Z}^*_{2q}$, this result leads to:

$$C_\infty := \limsup_{n\to\infty} \left\{ \sup_{t \geq e^2} Y_{2^n}(t)\, t^{\frac{N}{2}\left(1-\frac{1}{2^n}\right)} \right\} < \infty.$$

Taking into account the hypothesis (2.9) and in view of the definition (2.8) of δ_p, we can define:

$$\delta_\infty := \lim_{p\nearrow\infty} \sup \delta_p.$$

Taking $q = 2^n$ and passing to the limit as $n \nearrow \infty$ in (2.57), one deduces that there exists a constant $C(\infty) := C_\infty\, \delta_\infty^{-1} > 0$ such that

$$\|v\|_\infty \leq \limsup_{n\to\infty} Y_{2^n}(t) \leq C(\infty)\, \delta_\infty\, t^{-\frac{N}{2}},$$

for all t large enough. ■

2.3 Decay rates

We are now in a position to prove the following Proposition, as announced in Application 2:

Proposition 2.2 *Assume that the initial data* $(\mathbf{v}_0, \mathbf{g}_0) \in \mathbf{L}^2(\Omega) \times \mathbb{R}^N$ *of system (1.4) are such that* $\mathbf{v}_0 \in \mathbf{L}^1(\Omega)$. *Then:*

$$\begin{aligned} \|\mathbf{v}(t)\|_p \leq & C\,[\|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|]\,t^{-\frac{N}{2}(1-\frac{1}{p})}, \\ |\mathbf{g}(t)| \leq & C\,[\|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|]\,t^{-\frac{N}{2}}, \quad \forall\, t > 0, \end{aligned}$$

for all $1 \leq p \leq \infty$. *The constant* $C > 0$ *in these estimates depends on* p, m *and* N *but is independent of the initial data.*

Remark 2.3 *The complete asymptotic analysis will show that these decay estimates are sharp. The decay rate of* $\mathbf{g}$ *is a consequence of the* $\mathbf{L}^\infty$ *estimate of* $\mathbf{v}$, *because of the transmission condition* $\mathbf{v} = \mathbf{g}$ *on the interface* $\partial\Omega$.

Proof : As explained in Application 1, we only have to apply Proposition 2.1, setting $\mathbf{V} = \mathbf{0}$, $[\mathbf{U}] = [\mathbf{0}]$. Condition (2.6) is trivially satisfied. The decay property (2.2) ensures that, since $\mathbf{v}_0$ is in $\mathbf{L}^1(\Omega)$:

$$\|\mathbf{v}\|_1 + m\,|\mathbf{g}| \leq \|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|, \quad \forall\, t \geq 0.$$

Therefore, in this case, we can set: $\delta_1 = 2\,N\,(\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|)$, and the proof of the Proposition is then completed. ■

3 The first term in the asymptotic expansion

This section is devoted to the proof of Theorem 1.2.

As we have pointed out in section 1.3, the first momentum $\mathbf{M}_1$, defined by $\mathbf{M}_1 := \int_\Omega \mathbf{v}(t)\,d\mathbf{x} + m\,\mathbf{g}(t)$, is constant in time. The role played by this quantity in the description of the large time behaviour of $\mathbf{v}$ is made precise in Theorem 1.2. Actually, $\mathbf{M}_1\,G(t)$ is the first term in the asymptotic development of $\mathbf{v}$.

In application 2 we have defined $\bar{\mathbf{v}} := \mathbf{v} - \mathbf{M}_1\,G$ and $\bar{\mathbf{g}}$, the trace of $\bar{\mathbf{v}}$ on the boundary of Ω. The pair, $(\bar{\mathbf{v}}, \bar{\mathbf{g}})$ solves:

$$\begin{cases} \bar{\mathbf{v}}_t - \Delta\bar{\mathbf{v}} - \mathbf{g}\cdot\nabla\bar{\mathbf{v}} = \boldsymbol{\varepsilon}_1, & \mathbf{x} \in \Omega, \quad t > 0, \\ \bar{\mathbf{v}}(\mathbf{x}, t) = \bar{\mathbf{g}}(t), & \mathbf{x} \in \partial B, \quad t > 0, \\ m\bar{\mathbf{g}}'(t) = -\displaystyle\int_{\partial\Omega} \mathbf{n}\cdot\nabla\bar{\mathbf{v}}\,d\sigma_x + \boldsymbol{\varepsilon}_2(t), & t > 0, \\ \bar{v}(\mathbf{x}, 0) := \bar{v}_0(\mathbf{x}) = v_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, & \bar{g}(0) := \bar{g}_0 = h_1, \end{cases} \tag{3.1}$$

where $\boldsymbol{\varepsilon}_1 := \mathbf{M}_1\,\mathbf{g}\cdot\nabla G$ and $\boldsymbol{\varepsilon}_2 = -\mathbf{M}_1\,\varepsilon$ (see (1.21)). Namely:

$$\boldsymbol{\varepsilon}_2(t) := \frac{1}{2}\,m\,\mathbf{M}_1\,(4\pi)^{-\frac{N}{2}}\,t^{-\frac{N}{2}-1}e^{-\frac{1}{4t}}\left(-\frac{\sigma_N}{m} + N - \frac{1}{2t}\right). \tag{3.2}$$

The following Proposition concerns the decay rate of $\bar{\mathbf{v}}$ and $\bar{\mathbf{g}}$:

Proposition 3.1 *Assume that the initial data* $(\mathbf{v}_0, \mathbf{g}_0) \in \mathbf{L}^2(\Omega) \times \mathbb{R}^N$ *of* (1.4) *are such that* $\mathbf{v}_0 \in \mathbf{L}^1(\Omega)$. *Then define, for all* $t \geq 0$*:*

$$\delta_1(t) := 2\, N \sup_{s \geq t} \left[\|\bar{\mathbf{v}}(s)\|_1 + m\, |\bar{\mathbf{g}}(s)| \right], \tag{3.3}$$

and also set, for all $t \geq 1$ *and all* $1 < p \leq \infty$*, distinguishing the values of the mass* m *of the ball:*

- *When* $m = \dfrac{\sigma_N}{N}$*:*

$$\delta_p(t) = \delta_1(t) \max \left\{ \left(1 + \frac{N}{2}\right)^{\frac{N}{2}(1-\frac{1}{p})}, \right. \\ \left. \left(\delta_1(t)^{-1} \left[\|\mathbf{v}(t)\|_1 + m |\mathbf{g}(t)| \right] t^{-\frac{N}{2} \min\{\frac{1}{p} + \frac{2}{N}, 1 - \frac{1}{N}\}} \right)^{\frac{N(p-1)}{2\,p + N\,(p-1)}} \right\}. \tag{3.4a}$$

- *When* $m \neq \dfrac{\sigma_N}{N}$*:*

$$\delta_p(t) = \delta_1(t) \max \left\{ \left(1 + \frac{N}{2}\right)^{\frac{N}{2}(1-\frac{1}{p})}, \right. \\ \left. \left(\delta_1(t)^{-1} \left[\|\mathbf{v}(t)\|_1 + m |\mathbf{g}(t)| \right] t^{-\frac{N}{2} \min\{\frac{1}{p}, 1 - \frac{1}{N}\}} \right)^{\frac{N(p-1)}{2\,p + N\,(p-1)}} \right\}. \tag{3.4b}$$

Then, $\delta_1(t)$ *is bounded on* $[0, \infty)$ *and for any* $t_0 \geq 1$*, the solution* $(\bar{\mathbf{v}}, \bar{\mathbf{g}})$ *of* (3.1) *satisfies the following decay properties:*

$$\|\bar{\mathbf{v}}(t)\|_p \leq C\, \delta_p(t_0)\, t^{-\frac{N}{2}(1-\frac{1}{p})}, \tag{3.5a}$$

$$|\bar{\mathbf{g}}(t)| \leq C\, \delta_\infty(t_0)\, t^{-\frac{N}{2}}, \quad \forall\, t \geq t_0 + 1. \tag{3.5b}$$

Remark 3.1 *We shall prove later that* $\|\bar{\mathbf{v}}(t)\|_1 + m\, |\bar{\mathbf{g}}(t)|$ *goes to 0 as* $t \to \infty$ *and hence that also* $\delta_1(t)$ *and* $\delta_p(t)$ *go to* 0 *as* $t \to \infty$*. Then, choosing* $t_0 = t/2$ *we will be able to improve the decay rate of* $\|\bar{\mathbf{v}}(t)\|_p$ *and* $|\bar{\mathbf{g}}(t)|$*.*

Let us define

$$\mathcal{K}(t, \mathbf{x}) := \exp\left(\frac{\mathbf{x}^2}{4(t+1)} \right), \tag{3.6}$$

and recall that $K(\mathbf{x}) := \mathcal{K}(0, \mathbf{x})$. Since K is radially symmetric, we will sometimes use the notation $K(r)$ with $r = |\mathbf{x}| \in \mathbb{R}_+$ instead of $K(\mathbf{x})$.

In the sequel, we will perform the proof of the following Proposition, improving the decay rate of $\bar{v}$ given in Proposition 3.1 for particular values of p.

Proposition 3.2 *Let* $\mathbf{v}_0$ *be in* $\mathbf{L}^2(K,\Omega)$. *Then the solution* $(\mathbf{v},\mathbf{g})$ *of system* (1.4) *satisfies the estimate:*

$$\text{When } N=2: \quad \|\mathbf{v}-\mathbf{M}_1\,G\|_p \le C\,|\log(1+t)|\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}, \quad \forall t\ge 1, \tag{3.7a}$$

$$\text{When } N\ge 3: \quad \|\mathbf{v}-\mathbf{M}_1\,G\|_p \le C\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}, \quad t\ge 1, \tag{3.7b}$$

for all $1\le p\le 2$. *The constant* C *in these estimates depends on* p, N *and* m.

Remark 3.2 *Estimates* (3.7) *fit exactly those of the heat equation on the whole space* $\mathbb{R}^N$, *the case* $N=2$ *being excepted, where a logarithmic term appears in the decay rate. We shall show that this logarithmic term is due to the contribution of the solid mass in the system and that estimates* (3.7) *are sharp when* $p=2$.

Proof of Theorem 1.2: Assuming that Proposition 3.2 holds, let us proceed to complete the proof of Theorem 1.2.

Relation (3.7) with $p=1$ provides the estimates:

$$\text{When } N=2: \quad \|\bar{\mathbf{v}}\|_1 \le C\,|\log(1+t)|\,t^{-\frac{1}{2}}, \quad \forall\,t\ge 1,$$
$$\text{When } N\ge 3: \quad \|\bar{\mathbf{v}}\|_1 \le C\,t^{-\frac{1}{2}}, \quad \forall t\ge 1.$$

From Proposition 2.2 we deduce that:

$$|\bar{\mathbf{g}}(t)| \le C\,t^{-\frac{N}{2}}, \quad \forall\,t>0.$$

Therefore the positive constant $\delta_1(t)$ of Proposition 3.1 can be estimated as follows:

$$\text{When } N=2: \quad \delta_1(t) \le C\,|\log(1+t)|\,t^{-\frac{1}{2}}, \quad \forall\,t\ge 1, \tag{3.8a}$$

$$\text{When } N\ge 3: \quad \delta_1(t) \le C\,t^{-\frac{1}{2}}, \quad \forall t\ge 1. \tag{3.8b}$$

On the other hand, (3.5) ensures that, for all $t_0\ge 1$:

$$\|\bar{\mathbf{v}}(t)\|_p \le C\,\delta_p(t_0)\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)}, \quad \forall\,t\ge t_0+1, \tag{3.9}$$

the constant $\delta_p(t_0)$ being defined by (3.4).

- When $m\ne \dfrac{\sigma_N}{N}$:
  Since the quantity $\|\mathbf{v}\|_1 + m\,|\mathbf{g}|$ decreases in time (see (2.2)) and

  $$\left(1+\frac{N}{2}\right)^{\frac{N}{2}\left(1-\frac{1}{p}\right)} \le C_N := \left(1+\frac{N}{2}\right)^{\frac{N}{2}},$$

  from (3.4) we deduce that, for all $1<p\le\infty$:

$$\delta_p(t) \le \delta_1(t)\,\max\left\{C_N,\ \left(C\,\delta_1(t)^{-1}\,t^{-\frac{N}{2}\,\min\{\frac{1}{p},1-\frac{1}{N}\}}\right)^{\frac{N(p-1)}{2\,p+N\,(p-1)}}\right\}. \tag{3.10}$$

Since $0 \leq \frac{N(p-1)}{2\,p+N\,(p-1)} \leq \frac{N}{N+2} \leq 1$, we can assume that the constant $C^{\frac{N(p-1)}{2\,p+N\,(p-1)}}$ is independent of p and rewrite the inequality (3.10):

$$\delta_p(t) \leq C_N\, \delta_1(t)^{\frac{2p}{2p+N(p-1)}} \left[\max\left\{\delta_1(t),\, t^{-\frac{N}{2}\,\min\{\frac{1}{p},1-\frac{1}{N}\}}\right\}\right]^{\frac{N(p-1)}{2\,p+N\,(p-1)}}. \quad (3.11)$$

According to (3.8),

– When $N=2$:

$$\max\left\{\delta_1(t),\, t^{-\frac{N}{2}\,\min\{\frac{1}{p},1-\frac{1}{N}\}}\right\} \leq \max\left\{|\log(1+t)|\, t^{-\frac{1}{2}},\; t^{-\frac{N}{2}\,\min\{\frac{1}{p},1-\frac{1}{N}\}}\right\}$$
$$\leq t^{-\frac{1}{2}} \max\left\{|\log(1+t)|,\, t^{\frac{1}{2p}[p-\min\{N,p(N-1)\}]}\right\},$$

and, because $N \geq 2$, basic computations yield

$$p - \min\{N,\, p(N-1)\} \geq 0 \;\Leftrightarrow\; p \geq N.$$

Therefore, from (3.11) and (3.8), we deduce for all t large enough:

∗ For all $1 < p \leq N$:

$$\delta_p(t) \leq C\, \left(|\log(1+t)|\, t^{-\frac{1}{2}}\right)^{\frac{2p}{2p+N(p-1)}} \left(|\log t|\, t^{-\frac{1}{2}}\right)^{\frac{N(p-1)}{2p+N(p-1)}}$$
$$\leq C\,|\log(1+t)|\, t^{-\frac{1}{2}}. \quad (3.12a)$$

∗ For all $N < p \leq \infty$:

$$\delta_p(t) \leq C\, \left(|\log(1+t)|\, t^{-\frac{1}{2}}\right)^{\frac{2p}{2p+N(p-1)}} \left(t^{-\frac{N}{2p}}\right)^{\frac{N(p-1)}{2p+N(p-1)}}$$
$$\leq C\,|\log(1+t)|^{\frac{2p}{2p+N(p-1)}}\, t^{-\frac{1}{2}+\theta(N,p)}, \quad (3.12b)$$

with $\theta(N,p) := \frac{N}{2}\,\frac{(p-1)(p-N)}{p\,(2\,p+N(p-1))}$.

– When $N \geq 3$: the only difference with the case $N=2$ comes from (3.8), that is to say from the absence of logarithmic term. Consequently, we get the following estimates for $\delta_p(t)$, for all t large enough:

∗ For all $1 < p \leq N$:

$$\delta_p(t) \leq C\, t^{-\frac{1}{2}}. \quad (3.12c)$$

∗ For all $N < p \leq \infty$:

$$\delta_p(t) \leq C\, t^{-\frac{1}{2}+\theta(N,p)}. \quad (3.12d)$$

• When $m = \dfrac{\sigma_N}{N}$: the definition of $\delta_p(t)$ is distinct and according to (3.4), we must turn (3.10) into:

$$\delta_p(t) \leq \delta_1(t)\, \max\left\{C_N,\, \left(C\,\delta_1(t)^{-1}\, t^{-\frac{N}{2}\,\min\{\frac{1}{p}+\frac{2}{N},1-\frac{1}{N}\}}\right)^{\frac{N(p-1)}{2\,p+N\,(p-1)}}\right\}.$$

The same kind of computations as above leads to:

– When $N = 2$ and for all $1 < p \leq \infty$ and all t large enough:

$$\delta_p(t) \leq C\,|\log(1+t)|\,t^{-\frac{1}{2}}. \tag{3.13a}$$

– When $N \geq 3$ and for all $1 < p \leq \infty$ and all t large enough:

$$\delta_p(t) \leq C\,t^{-\frac{1}{2}}. \tag{3.13b}$$

The estimates of Theorem 1.2 arise straightforwardly when combining (3.9) with (3.12) and (3.13) and specifying $t_0 = t/2$. ■

We perform now the proof of Proposition 3.1.

Proof of Proposition 3.1: It is quite easy to check that $\delta_1(t)$ is bounded for all $t \geq 0$. Indeed, according to the definition of $\bar{\mathbf{v}}$ and $\bar{\mathbf{g}}$ in Application 2, we have:

$$\|\bar{\mathbf{v}}\|_1 + m\,|\bar{\mathbf{g}}| \leq \|\mathbf{v}\|_1 + m\,|\mathbf{g}| + |\mathbf{M}_1|\,\|G\|_1 + |\mathbf{M}_1|\,m\,|J|.$$

Explicit computations give $\|G\|_1 \leq 1$ and $|J| \leq (4\pi\,t)^{-\frac{N}{2}} e^{-\frac{1}{4t}}$ and:

$$|\mathbf{M}_1| \leq \|\mathbf{v}(t)\|_1 + m\,|\mathbf{g}(t)|.$$

On the other hand, relation (2.2) ensures that $\|\mathbf{v}(t)\|_1 + m\,|\mathbf{g}(t)| \leq \|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|$ so that:

$$\|\bar{\mathbf{v}}\|_1 + m\,|\bar{\mathbf{g}}| \leq C\,\|\mathbf{v}_0\| + m\,|\mathbf{g}_0|, \quad \forall\, t \geq 0.$$

The proof of estimates (3.5) derives from Proposition 2.1. We have:

$$\|\nabla G\|_p \leq C\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}, \quad \forall\, t \geq 0,$$

for all $1 \leq p \leq \infty$, where the constant C does not depend on p. On the other hand, Proposition 2.2 ensures that:

$$|\mathbf{g}| \leq C\,[\|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|]\,t^{-\frac{N}{2}}, \quad \forall\, t \geq 0. \tag{3.14}$$

Therefore, since $\boldsymbol{\varepsilon}_1 = \mathbf{M}_1\,\mathbf{g}\cdot\nabla G$, we deduce that:

$$\|\varepsilon_1\|_p \leq C\,|\mathbf{g}|\,\|\nabla G\|_p \leq C\,[\|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|]\,t^{-\frac{N}{2}\left(2-\frac{1}{p}\right)-\frac{1}{2}}, \quad \forall\, t \geq 0. \tag{3.15}$$

The definition (3.2) of $\boldsymbol{\varepsilon}_2$ leads to the estimates:

- When $m = \dfrac{\sigma_N}{N}$:

$$|\boldsymbol{\varepsilon}_2| \leq C\,|\mathbf{M}_1|\,t^{-\frac{N}{2}-2} \leq C\,[\|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|]\,t^{-\frac{N}{2}-2}, \quad \forall\, t \geq 0. \tag{3.16a}$$

- When $m \neq \dfrac{\sigma_N}{N}$:

$$|\boldsymbol{\varepsilon}_2| \leq C\,|\mathbf{M}_1|\,t^{-\frac{N}{2}-1} \leq C\,[\|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|]\,t^{-\frac{N}{2}-1}, \quad \forall\, t \geq 0. \tag{3.16b}$$

Note that de decay rate of the correctiong term $\boldsymbol{\epsilon}_2$ is of order $t^{-N/2-2}$ when $m = \sigma_N/N$ and only of order $t^{-N/2-1}$ when $m \neq \sigma_N/N$. That leads to distinguish these two cases in Theorem 1.2.

From (3.15) and (3.16) and according to the definition (2.5) of ϑ_2, we deduce that:

- When $m = \dfrac{\sigma_N}{N}$:

$$\vartheta_2 \leq C\,[\|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|]\,t^{-\frac{N}{2}\max\left\{\frac{1}{p}+\frac{2}{N},1-\frac{1}{N}\right\}}, \quad \forall\, 0 < t \leq 1, \tag{3.17a}$$

$$\vartheta_2 \leq C\,[\|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|]\,t^{-\frac{N}{2}\min\left\{\frac{1}{p}+\frac{2}{N},1-\frac{1}{N}\right\}}, \quad \forall\, t \geq 1. \tag{3.17b}$$

- When $m \neq \dfrac{\sigma_N}{N}$:

$$\vartheta_2 \leq C\,[\|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|]\,t^{-\frac{N}{2}\max\left\{\frac{1}{p},1-\frac{1}{N}\right\}}, \quad \forall\, 0 < t \leq 1, \tag{3.17c}$$

$$\vartheta_2 \leq C\,[\|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|]\,t^{-\frac{N}{2}\min\left\{\frac{1}{p},1-\frac{1}{N}\right\}}, \quad \forall\, t \geq 1. \tag{3.17d}$$

In model (3.1), $[\mathbf{U}] = [\mathbf{0}]$ and $\mathbf{V} = \mathbf{g}$ with the notations of system (2.3). Thus, $\vartheta_1 = 0$ and according to (3.17), the hypotheses (2.6) and (2.9) are fulfilled with $\alpha_p = \frac{N}{2} + 1$. Note in particular that α_p is independent of p. Therefore, Proposition 2.1 applies and relation (3.5) holds with $t_0 = 1$ for all $1 \leq p \leq \infty$. The constant δ_p is defined by (2.8) and δ_∞ by (2.8) specifying $p = \infty$.

To get estimates (3.5) for any $t_0 \geq 1$, remark that the proof above applies for the functions $\bar{v}(t + t_0)$ and $\bar{g}(t + t_0)$ and the initial conditions $\bar{\mathbf{v}}(t_0)$ and $\bar{\mathbf{g}}(t_0)$. Indeed, all the estimates (3.14), (3.15), (3.16) and (3.17) remain valid replacing $[\|\mathbf{v}_0\|_1 + m\,|\mathbf{g}_0|]$ by $[\|\mathbf{v}(t_0)\|_1 + m\,|\mathbf{g}(t_0)|]$, because this quantity decreases in time (see (2.2)). According to (3.17), we can simplify the expression of $\delta_p(t_0)$ and turn (2.8) into (3.4). ■

Proof of Proposition 3.2: We use the so-called similarity variables (we refer to [10], [11], [12] and [20] for details):

$$\mathbf{y} := \frac{\mathbf{x}}{\sqrt{1+t}}, \quad s := \log(1+t), \tag{3.18a}$$

or equivalently:

$$\mathbf{x} := e^{\frac{s}{2}}\,\mathbf{y}, \quad t := e^s - 1, \tag{3.18b}$$

together with the rescaled functions:

$$\boldsymbol{\xi}(\mathbf{y},s) := e^{\frac{s}{2}N}\,\mathbf{v}(\mathbf{y}e^{\frac{s}{2}}, e^s-1) \quad \text{and} \quad \boldsymbol{\zeta}(s) := e^{\frac{s}{2}N}\,\mathbf{g}(e^s-1). \tag{3.19}$$

Equivalently, we can express $\mathbf{v}$ and $\mathbf{g}$ with respect to $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$:

$$\mathbf{v}(\mathbf{x},t) = (t+1)^{-\frac{N}{2}}\,\boldsymbol{\xi}\left(\frac{\mathbf{x}}{\sqrt{1+t}}, \log(1+t)\right), \quad \mathbf{g}(t) = (t+1)^{-\frac{N}{2}}\,\boldsymbol{\zeta}(\log(1+t)).$$

This change of variables maps both fixed domains B and Ω on the time dependent ones B_s and $\Omega_s := \mathbb{R}^N \setminus B_s$, where B_s stands for the ball of radius $r_s := e^{-\frac{s}{2}}$ centred at the origin. In these new variables, the law of conservation of momentum reads as follows:

$$\mathbf{M}_1 = \int_{\Omega_s} \boldsymbol{\xi}(\mathbf{y},s)\,d\mathbf{y} + m\,e^{-s\frac{N}{2}}\boldsymbol{\zeta}(s), \quad \forall s \geq 0. \tag{3.20}$$

The vector valued functions $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ solve the following system:

$$\begin{cases} \boldsymbol{\xi}_s + L_s\boldsymbol{\xi} - \dfrac{N}{2}\boldsymbol{\xi} - e^{-s\,\frac{N-1}{2}}\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi} = \mathbf{0}, & \mathbf{y} \in \Omega_s, \quad s > 0, \\ \boldsymbol{\xi}(\mathbf{y},s) = \boldsymbol{\zeta}(s), \quad \mathbf{y} \in \partial B_s, & \qquad\qquad s > 0, \\ \boldsymbol{\zeta}'(s) - \frac{N}{2}\boldsymbol{\zeta}(s) = -\dfrac{e^{\frac{Ns}{2}}}{m}\displaystyle\int_{\partial\Omega_s} \mathbf{n}\cdot\nabla\boldsymbol{\xi}\,d\sigma_y, & \qquad\qquad s > 0, \\ \boldsymbol{\xi}(\mathbf{y},0) = \mathbf{v}_0(\mathbf{y}), \quad \mathbf{y} \in \Omega, \quad \boldsymbol{\zeta}(0) = \mathbf{h}_1, & \end{cases} \tag{3.21}$$

where the operator L_s is defined componentwise by:

$$L_s\xi := -\Delta\xi - \frac{\mathbf{y}}{2}\cdot\nabla\xi.$$

Note that the domain Ω_s where (3.21) holds, evolves in the new time variable s. Thus, L_s (which is, apparently, time independent) has to be viewed as a time dependent unbounded operator in $L^2(K,\Omega_s)$ with domain $H^2(K,\Omega_s) \cap H_0^1(K,\Omega_s)$. We will denote merely by L this unbounded operator in $L^2(K,\mathbb{R}^N)$ with domain $H^2(K,\mathbb{R}^N)$ (see [10]). We introduce also:

$$\theta_1(\mathbf{y}) := (4\pi)^{-\frac{N}{2}}\exp\left(\frac{-|\mathbf{y}|^2}{4}\right). \tag{3.22}$$

This function θ_1 corresponds to the heat kernel in similarity variables. In addition, the function θ_1 solves:

$$L\,\theta_1 - \frac{N}{2}\theta_1 = 0 \quad \text{on } \mathbb{R}^N, \tag{3.23}$$

i.e. it is an eigenfunction associated with the eigenvalue $\lambda_1 := \frac{N}{2}$ of L. In fact, λ_1 is simple and it is the first eigenvalue of L, which has a discrete spectrum that can be computed explicitly (see [10]).

We are mainly interested in the large time behaviour of

$$\bar{\boldsymbol{\xi}}(\mathbf{y},s) := \boldsymbol{\xi}(\mathbf{y},s) - \mathbf{M}_1\,\theta_1(\mathbf{y}),\ \ \mathbf{y}\in\Omega_s\ \text{ and }\ \bar{\boldsymbol{\zeta}}(s) := \boldsymbol{\zeta}(s) - \mathbf{M}_1\,\theta_1(\mathbf{y}),\ \ \mathbf{y}\in\partial B_s.$$

These functions play the role of $\bar{\mathbf{v}}$ and $\bar{\mathbf{g}}$ in similarity variables. They are bounded: this is a consequence of the decay properties of $|\mathbf{g}|$ and $\|\mathbf{v}\|_\infty$ in Proposition 2.2. Since θ_1 is also bounded on $\mathbb{R}^N$, one deduces that:

$$|\bar{\boldsymbol{\zeta}}(s)| \le C \quad \text{and} \quad \|\bar{\boldsymbol{\xi}}\|_\infty \le C, \quad \forall s>0. \tag{3.24}$$

Combining (3.21) and (3.23), one deduces that the pair $(\bar{\boldsymbol{\xi}},\bar{\boldsymbol{\zeta}})$ solves:

$$\begin{cases} \bar{\boldsymbol{\xi}}_s + L_s\bar{\boldsymbol{\xi}} - \dfrac{N}{2}\bar{\boldsymbol{\xi}} - e^{-s\,\frac{N-1}{2}}\boldsymbol{\zeta}\cdot\nabla\bar{\boldsymbol{\xi}} = e^{-s\,\frac{N-1}{2}}\mathbf{M}_1\boldsymbol{\zeta}\cdot\nabla\theta_1, \quad \mathbf{y}\in\Omega_s, & s>0,\\ \bar{\boldsymbol{\xi}}(\mathbf{y},s) = \bar{\boldsymbol{\zeta}}(s), \quad \mathbf{y}\in\partial B_s, & s>0,\\ m\left(\bar{\boldsymbol{\zeta}}(s)e^{-s\frac{N}{2}}\right)' = -\displaystyle\int_{\partial\Omega_s}\mathbf{n}\cdot\nabla\bar{\boldsymbol{\xi}}\,d\sigma_y + e^{-s\frac{N}{2}}\,\boldsymbol{\rho}(s), & s>0,\\ \bar{\boldsymbol{\xi}}(\mathbf{y},0) = \mathbf{v}_0(\mathbf{y}), \quad \mathbf{y}\in\Omega, \quad \bar{\boldsymbol{\zeta}}(0) = \mathbf{h}_1, \end{cases} \tag{3.25}$$

with

$$\boldsymbol{\rho}(s) := \frac{1}{2}\,m\,\mathbf{M}_1\,(4\pi)^{-\frac{N}{2}}\exp\left(-\frac{r_s^2}{4}\right)\left(-\frac{\sigma_N}{m} + N - \frac{e^{-s}}{2}\right), \tag{3.26}$$

and $r_s = e^{-\frac{s}{2}}$ is the radius of the ball B_s. In (3.25-iii), the quantity $e^{-s\frac{N}{2}}\,\boldsymbol{\rho}(s)$ is a correcting term due to the contribution of θ_1. Remark that this system can also be derived from (3.1) in a straightforward way by making the change of variables (3.18).

From now on, we will work componentwise, using the rules of notation of section 1.1. We shall use in the sequel: $(\cdot,\cdot)_s$, the scalar product of $L^2(K,\Omega_s)$ and $\|\cdot\|_s$ the associated norm. Moreover, $\chi(s)$ stands for $K(r_s)$ and hence:

$$\chi(s) := \exp\left(\frac{e^{-s}}{4}\right) = 1 + C(s)\,e^{-s}, \quad \forall s>0, \tag{3.27}$$

where $C(s)$ is a positive function such that $0 < C_1 \le C(s) \le C_2 < \infty$, for all $s>0$.

Multiplying componentwise the first equation of system (3.25) in the weighted Sobolev space $L^2(K,\Omega_s)$ by $\bar{\xi}$ we obtain:

$$(\bar{\xi}_s,\bar{\xi})_s + (L_s\,\bar{\xi},\bar{\xi})_s - \frac{N}{2}(\bar{\xi},\bar{\xi})_s - e^{-s\,\frac{N-1}{2}}(\boldsymbol{\zeta}\cdot\nabla\bar{\xi},\bar{\xi})_s \\ - e^{-s\,\frac{N-1}{2}}M_1(\boldsymbol{\zeta}\cdot\nabla\theta_1,\bar{\xi})_s = 0, \quad \forall s>0. \tag{3.28}$$

Integrating by parts, it comes:

$$(L_s\,\bar\xi,\bar\xi)_s = \|\nabla\bar\xi\|_s^2 - \int_{\partial\Omega_s} \frac{\partial\bar\xi}{\partial\mathbf{n}}\bar\xi\,K\,d\sigma_y.$$

Then, according to the coupling condition on the interface $\partial\Omega_s$ we can rewrite (3.28) as follows:

$$(\bar\xi_s,\bar\xi)_s + \|\nabla\bar\xi\|_s^2 - \frac{N}{2}\,\|\bar\xi\|_s^2 - e^{-s\,\frac{N-1}{2}}\,(\boldsymbol{\zeta}\cdot\nabla\bar\xi,\bar\xi)_s - e^{-s\,\frac{N-1}{2}}M_1\,(\boldsymbol{\zeta}\cdot\nabla\theta_1,\bar\xi)_s$$
$$+\,m\,\chi\,e^{-\frac{s}{2}N}\,\bar\zeta'\bar\zeta - \frac{N}{2}\,m\,\chi\,e^{-\frac{s}{2}N}\,\bar\zeta^2 - e^{-s\frac{N}{2}}\,\rho\,\chi\,\bar\zeta = 0. \quad (3.29)$$

In order to analyse the first term in (3.29) involving the time derivative, we need the following identity:

Lemma 3.1 *For all function $f\in C^1((0,+\infty),W^{1,1}(\mathbb{R}^N))$,*

$$\frac{d}{ds}\left[\int_{\Omega_s} f(\mathbf{z},s)d\mathbf{z}\right]\bigg|_{s=s_0} = \int_{\Omega_{s_0}} f_s(\mathbf{y},s_0)\,d\mathbf{y} + \frac{e^{-\frac{s_0}{2}}}{2}\int_{\partial\Omega_{s_0}} f(\mathbf{y},s_0)\,d\mathbf{y}. \quad (3.30)$$

The proof of this Lemma will be given at the end of this paper, in Appendix B.

Applying the above Lemma to the function $\bar\xi^2\,K$ in the domain Ω_s, we deduce that:

$$\frac{1}{2}\frac{d}{ds}\|\bar\xi\|_s^2 = (\bar\xi_s,\bar\xi)_s + \frac{e^{-\frac{s}{2}}}{4}\int_{\partial B_s}\bar\zeta^2\,K\,d\sigma_y = (\bar\xi_s,\bar\xi)_s + \frac{e^{-\frac{s}{2}N}}{4}\,\chi\,\bar\zeta^2\,\sigma_N. \quad (3.31)$$

On the other hand, a simple computation gives the following identity for the term of (3.29) involving the time derivative of $\bar\zeta$:

$$\frac{1}{2}\frac{d}{ds}\left[m\,\chi\,e^{-\frac{s}{2}N}\,\bar\zeta^2\right] = m\,\chi\,e^{-\frac{s}{2}N}\,\bar\zeta\,\bar\zeta' - \frac{1}{4}\,m\,(\frac{e^{-s}}{2}+N)\,\chi\,e^{-\frac{s}{2}N}\,\bar\zeta^2. \quad (3.32)$$

Combining together the relations (3.29), (3.31) and (3.32), we get:

$$\frac{1}{2}\frac{d}{ds}\|\bar\xi\|_s^2 + \frac{1}{2}\frac{d}{ds}\left[m\,\chi\,e^{-\frac{s}{2}N}\,\bar\zeta^2\right] - \frac{N}{2}\,\|\bar\xi\|_s^2 - e^{-s\,\frac{N-1}{2}}\,(\boldsymbol{\zeta}\cdot\nabla\bar\xi,\bar\xi)_s$$
$$- e^{-s\,\frac{N-1}{2}}M_1\,(\boldsymbol{\zeta}\cdot\nabla\theta_1,\bar\xi)_s - \frac{1}{4}\,m\,\chi\,e^{-\frac{s}{2}N}\,\bar\zeta^2\left(\frac{\sigma_N}{m}+N-\frac{e^{-s}}{2}\right)$$
$$- e^{-s\frac{N}{2}}\,\rho\,\chi\,\bar\zeta = 0. \quad (3.33)$$

Taking into account that $\nabla\theta_1 = -\frac{\mathbf{y}}{2}\,\theta_1 = -\frac{\mathbf{y}}{2}\,(4\pi)^{-\frac{N}{2}}\,K^{-1}$, we deduce that:

$$M_1\,(\boldsymbol{\zeta}\cdot\nabla\theta_1,\bar\xi)_s = -M_1\,(4\pi)^{-\frac{N}{2}}\,(\boldsymbol{\zeta}\cdot\frac{\mathbf{y}}{2}\,K^{-1},\bar\xi)_s.$$

Keeping in mind that $|\boldsymbol{\zeta}|$ is bounded, it comes

$$
\begin{aligned}
|(\boldsymbol{\zeta} \cdot \frac{\mathbf{y}}{2} K^{-1}, \bar{\xi})_s| &\le \int_{\Omega_s} |\boldsymbol{\zeta} \cdot \frac{\mathbf{y}}{2} K^{-1} \bar{\xi} K| \, d\mathbf{y} \\
&\le \|\boldsymbol{\zeta} \cdot \frac{\mathbf{y}}{2} K^{-1} \|_s \, \|\bar{\xi}\|_s \le C \, \|\bar{\xi}\|_s, \quad \forall s > 0. \qquad (3.34)
\end{aligned}
$$

On the other hand, we have the obvious inequalities

$$
|(\boldsymbol{\zeta} \cdot \nabla \bar{\xi}, \bar{\xi})_s| \le C \, \|\nabla \bar{\xi}\|_s \, \|\bar{\xi}\|_s \le C \, \|\nabla \bar{\xi}\|_s^2 + C \, \|\bar{\xi}\|_s^2, \quad \forall s > 0. \qquad (3.35)
$$

Combining (3.33), (3.34) and (3.35), we obtain:

$$
\begin{aligned}
\frac{1}{2} \frac{d}{ds} \|\bar{\xi}\|_s^2 + \frac{1}{2} \frac{d}{ds} \left[m \, \chi \, e^{-s \frac{N}{2}} \, \bar{\zeta}^2 \right] &\le -(1 - C \, e^{-s \, \frac{N-1}{2}}) \, \|\nabla \bar{\xi}\|_s^2 \\
&+ \left(\frac{N}{2} + C \, e^{-s \, \frac{N-1}{2}} \right) \|\bar{\xi}\|_s^2 + C \, e^{-s \, \frac{N-1}{2}} \, \|\bar{\xi}\|_s \\
&- \frac{1}{4} m \, e^{-s \frac{N}{2}} \, \chi \, \bar{\zeta}^2 \left(\frac{e^{-s}}{2} - N - \frac{\sigma_N}{m} \right) + e^{-s \frac{N}{2}} \, \rho \, \chi \, \bar{\zeta}, \quad \forall s > 0. \qquad (3.36)
\end{aligned}
$$

Taking into account once again the fact that $\bar{\zeta}$ (see (3.24)), ρ and χ are bounded, we can simplify the above estimate as follows:

$$
\begin{aligned}
\frac{1}{2} \frac{d}{ds} \left[\|\bar{\xi}\|_s^2 + m \, \chi \, e^{-s \frac{N}{2}} \, \bar{\zeta}^2 \right] &\le -(1 - C \, e^{-s \, \frac{N-1}{2}}) \|\nabla \bar{\xi}\|_s^2 \\
&+ \left(\frac{N}{2} + C \, e^{-s \, \frac{N-1}{2}} \right) \|\bar{\xi}\|_s^2 + C \, e^{-s \, \frac{N-1}{2}} \|\bar{\xi}\|_s + C \, e^{-s \frac{N}{2}}, \quad \forall s > 0. \qquad (3.37)
\end{aligned}
$$

In order to obtain an ordinary differential inequation for $\|\bar{\xi}\|_s^2 + m \, \chi \, e^{-s \frac{N}{2}} \, \bar{\zeta}^2$, one needs an estimate for the term $\|\nabla \bar{\xi}\|_s^2$.

First of all, let us recall some classical results about the operator L: this is a self-adjoint unbounded operator in $L^2(K)$ with domain $D(L) := H^2(K)$. Its eigenvalues are

$$
\lambda_k := \frac{N + k - 1}{2}, \quad k \in \mathbb{N}^*,
$$

and the first eigenvalue is simple. Its eigenspace, denoted E_1, is spanned by θ_1 (we refer to [10] for details). Moreover, we can express the eigenvalues by means of the Rayleigh principle. That reads, for λ_1 and λ_2:

$$
\inf_{\varphi \in L^2(K)} \frac{\|\nabla \varphi\|^2}{\|\varphi\|^2} = \lambda_1, \quad \text{and} \quad \inf_{\varphi \in E_1^{\perp}} \frac{\|\nabla \varphi\|^2}{\|\varphi\|^2} = \lambda_2. \qquad (3.38)
$$

Note that the condition $\varphi \in E_1^{\perp}$ means precisely that $\int_{\Omega} \varphi \, d\mathbf{y} = 0$. Thus, λ_1 and λ_2 are the minima of the Reyleigh quotient on $H^1(K)$ and on the subspace of $H^1(K)$ of functions of null mass respectively.

However, we are dealing with L_s on Ω_s and not with L on $\mathbb{R}^N$. But because of the coupling condition (3.21-ii) on the interface $\partial\Omega_s$, any function of $H^1(\Omega_s)$ can be extended to be in $H^1(\mathbb{R}^N)$ by setting

$$\xi(\mathbf{y}, s) := \zeta(s) \text{ on } B_s. \tag{3.39}$$

When s is large, we are going to show that $\bar{\xi} = \xi - M_1\,\theta_1$ is "almost" in $E_1^{\perp}$ since it tends to zero as $t \to \infty$ in $L^1(\mathbb{R}^N)$. This together with the definition of λ_2 shows that $\|\nabla\bar{\xi}\|_s^2 \geq \frac{N+1}{2}\|\bar{\xi}\|_s^2$ up to a small correcting term. The task consists in evaluating sharply this correcting term. The ideas we shall apply are quite close of those of [9, Lemma 2]. We state the Lemma in a more general framework, in order to apply it in other cases as well:

Lemma 3.2 *Let Ψ be a function of $H^1(\Omega_s, K)$ and $\psi(s)$ a real valued bounded function on $(0,\infty)$ such that $\Psi|_{\Omega_s} = \psi$. Suppose furthermore that*

$$\mathcal{M}_1 := \int_{\Omega_s} \Psi\, d\mathbf{y} + m\, e^{-s\frac{N}{2}}\,\psi, \tag{3.40}$$

is a constant. Then $\overline{\Psi} := \Psi - \mathcal{M}_1\,\theta_1$ satisfies the estimate:

$$\|\nabla\overline{\Psi}\|_s^2 \geq \frac{N+1}{2}\|\overline{\Psi}\|_s^2 - Ce^{-s\frac{N}{2}}, \quad \forall s > 0. \tag{3.41}$$

Proof : We extend $\overline{\Psi}$ to be a function defined in the whole space $\mathbb{R}^N$ by setting:

$$\overline{\Psi}(\mathbf{y}, s) := \psi(s) - \mathcal{M}_1\theta_1(\mathbf{y}), \quad \mathbf{y} \in B_s. \tag{3.42}$$

We introduce then:

$$r_1(s) := \mathcal{M}_1 - \frac{(\Psi(s), \theta_1)}{\|\theta_1\|^2}.$$

Remark that $r_1(s) = 0$ if and only if $\overline{\Psi} \in E_1^{\perp}$. According to the expression (3.40) of $\mathcal{M}_1$ and since $\|\theta_1\|^2 = (4\pi)^{-\frac{N}{2}}$, it comes:

$$\begin{aligned} r_1(s) =& \int_{\Omega_s} \Psi d\mathbf{y} + m\, e^{-s\frac{N}{2}}\psi(s) - \int_{\mathbb{R}^N} \Psi\, d\mathbf{y} \\ =& m\, e^{-s\frac{N}{2}}\psi(s) - \int_{B_s} \psi\, d\mathbf{y} = e^{-s\frac{N}{2}}\psi\left(m - \frac{\sigma_N}{N}\right). \end{aligned}$$

Since $|\psi|$ is bounded, it follows that:

$$|r_1(s)| \leq C\, e^{-s\frac{N}{2}}, \quad \forall s > 0. \tag{3.43}$$

If we set now

$$\Psi_1 := \overline{\Psi} + r_1\theta_1, \tag{3.44}$$

then $\Psi_1 \in E_1^\perp$ and according to (3.38):

$$\|\nabla \Psi_1\|^2 \geq \frac{N+1}{2}\|\Psi_1\|^2, \quad \forall s > 0. \tag{3.45}$$

Therefore, combining (3.44) and (3.45) we obtain the inequality:

$$\|\nabla \overline{\Psi}\|^2 + r_1^2\,\|\nabla \theta_1\|^2 + 2\,r_1\,(\nabla \overline{\Psi}, \nabla \theta_1) \geq \frac{N+1}{2}\,(\|\overline{\Psi}\|^2 + r_1^2\,\|\theta_1\|^2 + 2\,r_1\,(\overline{\Psi}, \theta_1)),$$

that is to say

$$\|\nabla \overline{\Psi}\|^2 \geq \frac{N+1}{2}\,\|\overline{\Psi}\|^2 - r_1^2\left(\|\nabla \theta_1\|^2 - \frac{N+1}{2}\,\|\theta_1\|^2\right) \\ - 2\,r_1\left((\nabla \overline{\Psi}, \nabla \theta_1) - \frac{N+1}{2}\,(\overline{\Psi}, \theta_1)\right). \tag{3.46}$$

The function θ_1, being an eigenfunction of L associated with the eigenvalue $\lambda_1 = \frac{N}{2}$, satisfies the following classical relations:

$$(\nabla \theta_1, \nabla \varphi) = \lambda_1(\theta_1, \varphi), \quad \forall \varphi \in H^1(K), \qquad \|\nabla \theta_1\|^2 = \lambda_1 \|\theta_1\|^2. \tag{3.47}$$

Consequently, we can turn (3.46) into:

$$\|\nabla \overline{\Psi}\|^2 \geq \frac{N+1}{2}\,\|\overline{\Psi}\|^2 + \frac{1}{2}\,r_1^2\,\|\theta_1\|^2 + r_1\,(\overline{\Psi}, \theta_1).$$

Observe that $\overline{\Psi} = \Psi_1 - r_1\,\theta_1$ and $\Psi_1 \perp \theta_1$. Thus, the inequality above can be rewritten as follows

$$\|\nabla \overline{\Psi}\|^2 \geq \frac{N+1}{2}\,\|\overline{\Psi}\|^2 - \frac{1}{2}\,r_1^2\,\|\theta_1\|^2.$$

We denote $\|\cdot\|_{B_s}$ the scalar product in $L^2(K, B_s)$ and $(\cdot,\cdot)_{B_s}$ the associated norm. We get then

$$\|\nabla \overline{\Psi}\|_s^2 \geq \frac{N+1}{2}\,\|\overline{\Psi}\|_s^2 + R(s), \tag{3.48}$$

where

$$R(s) := \frac{N+1}{2}\,\|\overline{\Psi}\|_{B_s}^2 - \|\nabla \overline{\Psi}\|_{B_s}^2 - \frac{1}{2}\,r_1^2\,\|\theta_1\|^2.$$

Let us now estimate the reminder $R(s)$:

- Because of the definition (3.42) and since $|\psi|$ is bounded, we obtain:

  $$\frac{N+1}{2}\,\|\overline{\Psi}\|_{B_s}^2 = \frac{N+1}{2}\int_{B_s} (\psi - \mathcal{M}_1\theta_1)^2\,d\mathbf{y} \leq C\,|B_s| \leq C\,e^{-s\frac{N}{2}}, \tag{3.49}$$

  where $|B_s| = \frac{\sigma_N}{N}\,e^{-s\frac{N}{2}}$ is the measure of the ball B_s.

- The same argument shows that:

$$\|\nabla\overline{\Psi}\|_{B_s}^2 = \|\nabla\theta_1\|_{B_s}^2 = \mathcal{M}_1^2\,\|\frac{\mathbf{y}}{2}\theta_1\|_{B_s}^2 \le C\,e^{-s}\int_{B_s}\theta_1^2\,d\mathbf{y} \le C\,e^{-s\frac{N+2}{2}}. \quad (3.50)$$

- From inequality (3.43), it follows that:

$$\frac{1}{2}\,r_1^2\|\theta_1\|^2 \le C\,e^{-s\,N}, \quad \forall s>0. \quad (3.51)$$

Summarising now (3.49), (3.50) and (3.51), we obtain:

$$|R(s)| \le C\,e^{-s\frac{N}{2}}, \quad \forall s>0.$$

This last estimate together with (3.48) yields the conclusion of the Lemma. ∎

One plugs now the estimate (3.41) into (3.37):

$$\frac{1}{2}\frac{d}{ds}\left[\|\bar\xi\|_s^2 + m\,\chi e^{-s\frac{N}{2}}\,\bar\zeta^2\right] \le -(1-C\,e^{-s\,\frac{N-1}{2}})\left(\frac{N+1}{2}\|\bar\xi\|_s^2 - C\,e^{-s\frac{N}{2}}\right) + \left(\frac{N}{2} + C\,e^{-s\,\frac{N-1}{2}}\right)\|\bar\xi\|_s^2 + C\,e^{-s\,\frac{N-1}{2}}\|\bar\xi\|_s + C\,e^{-s\frac{N}{2}}, \quad \forall s>0,$$

and after grouping together the terms involving $\|\bar\xi\|_s^2$ we obtain that:

$$\frac{d}{ds}\left[\|\bar\xi\|_s^2 + m\,\chi\,e^{-\frac{s}{2}N}\,\bar\zeta^2\right] + (1-C\,e^{-s\,\frac{N-1}{2}})\,\|\bar\xi\|_s^2 \le C\,e^{-s\,\frac{N-1}{2}}\|\bar\xi(s)\|_s + C\,e^{-s\frac{N}{2}}, \quad \forall s>0. \quad (3.52)$$

We now rewrite this equality in terms of the quantity:

$$X(s) := \|\bar\xi(s)\|_s^2 + m\,\chi(s)\,e^{-s\frac{N}{2}}\,\bar\zeta^2(s). \quad (3.53)$$

It comes:

$$\frac{d}{ds}X(s) + (1-C\,e^{-s\,\frac{N-1}{2}})\,X(s) - C\,e^{-s\,\frac{N-1}{2}}\|\bar\xi\|_s \le (1-C\,e^{-s\,\frac{N-1}{2}})\,m\,\chi e^{-\frac{s}{2}N}\,\bar\zeta^2(s) + C\,e^{-s\frac{N}{2}}, \quad \forall s>0. \quad (3.54)$$

Recalling that $|\bar\zeta(s)|$ as well as $|\chi(s)|$ are bounded, the inequality (3.54) can be turned into:

$$X'(s) + (1-C\,e^{-s\,\frac{N-1}{2}})\,X(s) - C\,e^{-s\,\frac{N-1}{2}}\,\|\bar\xi\|_s \le C\,e^{-s\frac{N}{2}}.$$

But, since $\sqrt{X} \ge \|\bar\xi\|_s$, it comes

$$X'(s) + (1-Ce^{-s\,\frac{N-1}{2}})X(s) - Ce^{-s\,\frac{N-1}{2}}\sqrt{X} \le Ce^{-s\frac{N}{2}}. \quad (3.55)$$

This differential inequality yields the desired result by applying a suitable Gronwall-type inequality.

Lemma 3.3 *Let X be a non-negative function on $(0,\infty)$ which satisfies:*

$$X'(s) + (1 - C_1\, e^{-s\,\frac{N-1}{2}})\, X(s) - C_2\, e^{-s\,\frac{N-1}{2}}\,(1+s)^\gamma \,\sqrt{X(s)} \le C_3\, s^\alpha\, e^{-s\,\beta}, \tag{3.56}$$

where $\alpha \ge 0$, $\beta > 0$, $\gamma \ge 0$, C_1, C_2 and C_3 are given constants. Then, $X(s)$ satisfies the following decay properties:

- *The case $N = 2$:*
  - *When $0 < \beta < 1$:*

$$X(s) \le C(1+s)^{\alpha+1}\, e^{-\beta\, s}, \quad \forall\, s \ge 0. \tag{3.57a}$$

  - *When $\beta = 1$:*

$$X(s) \le C\,(1+s)^{\frac{1}{2}\max\{2(\alpha+1),\alpha+2\gamma+3,4\}}\, e^{-s}, \quad \forall\, s \ge 0. \tag{3.57b}$$

  - *When $\beta > 1$:*

$$X(s) \le C\,(1+s)^{2(\gamma+1)}\, e^{-s}, \quad \forall\, s \ge 0. \tag{3.57c}$$

- *The case $N \ge 3$:*
  - *When $0 < \beta \le 1$:*

$$X(s) \le C\,(1+s)^{\alpha+1}\, e^{-\beta\, s}, \quad \forall\, s \ge 0. \tag{3.57d}$$

  - *When $\beta > 1$:*

$$X(s) \le C\, e^{-s}, \quad \forall\, s \ge 0. \tag{3.57e}$$

We refer to Appendix B for the proof.

With the definition (3.53) of X and the estimates of Lemma 3.3 applied to (3.55), we obtain

- When $N = 2$:

$$\|\bar{\xi}\|_s^2 + m\,\chi\, e^{-s\frac{N}{2}}\,\bar{\zeta}^2 \le C\, s^2 e^{-s}, \quad \forall s \ge 1. \tag{3.58}$$

  In particular:

$$\|\xi - M_1\theta_1\|_s^2 \le C\, s^2 e^{-s}, \quad \forall s \ge 1,$$

  and, according to the definition (3.19) of ξ:

$$\int_{\Omega_s} e^{s\frac{N}{2}}\, |v(e^s - 1, e^{\frac{s}{2}}\mathbf{y}) - M_1\, e^{-s\frac{N}{2}}\theta_1\,(e^{\frac{s}{2}}\mathbf{y})|^2 K(\mathbf{y})\, e^{s\frac{N}{2}}\, d\mathbf{y} \le C\, s^2 e^{-s},$$

for all $s \geq 1$. Getting back to the variables $\mathbf{x}$ and t and since $d\mathbf{x} = e^{s\frac{N}{2}} d\mathbf{y}$, it follows that:

$$\int_\Omega |v(t,\mathbf{x}) - M_1\, G(t,\mathbf{x})|^2\, \mathcal{K}(\mathbf{x},t)\, d\mathbf{x} \leq C \log^2(1+t)\, t^{-\frac{N}{2}-1}, \quad \forall t \geq 1,$$

where $\mathcal{K}(\mathbf{x},t)$ is as in (3.6). Hence:

$$\|v(t) - M_1 G(t)\|_{L^2(\Omega,\mathcal{K}(t))} \leq C\, |\log(1+t)|\, t^{-\frac{N}{4}-\frac{1}{2}}. \tag{3.59a}$$

- When $N \geq 3$:

$$\|\bar{\xi}\|_s^2 + m\,\chi\, e^{-s\frac{N}{2}}\, \bar{\zeta}^2 \leq C\; e^{-s}, \quad \forall s \geq 1. \tag{3.59b}$$

In this case, the logarithmic term does not appear and we find simply:

$$\|v(t) - M_1 G(t)\|_{L^2(\Omega,\mathcal{K}(t))} \leq C\, t^{-\frac{N}{4}-\frac{1}{2}}, \quad \forall t \geq 1. \tag{3.59c}$$

Remark 3.3 *Comparing* (3.59) *with the estimates of Proposition 3.1, namely with* $\|\bar{v}\|_2 \leq C\, t^{-\frac{N}{4}}$*, we have gained a decay rate of the order of* $|\log(1+t)|\, t^{-\frac{1}{2}}$ *in dimension* $N = 2$ *and* $t^{-\frac{1}{2}}$ *in dimension* $N \geq 3$*. Moreover, the result is also improved by the presence of the weight* $\mathcal{K}(\mathbf{x},t)$ *into the norm in* (3.59)*. Note that similar results are true for the pure heat equation. In that case, when subtracting the fundamental solution of an appropriate mass, solutions gain a decay rate of the order of* $t^{-1/2}$ *in any space dimension.*

Let us finally prove that, from the estimates (3.59), one can deduce the relations (3.7) of Proposition 3.2.

Fix p in $[1,2]$ and $t > 0$. Then, by Hölder's inequality, we obtain:

$$\|v\|_p^p \leq \left(\int_\Omega |v|^2 \mathcal{K}(t)\, d\mathbf{x}\right)^{\frac{p}{2}} \left(\int_\Omega \mathcal{K}(t)^{-p/(2-p)}\, d\mathbf{x}\right)^{1-\frac{p}{2}}. \tag{3.60a}$$

A straight computation gives

$$\left(\int_\Omega \mathcal{K}(t)^{-\frac{p}{2-p}}\, d\mathbf{x}\right)^{1-\frac{p}{2}} \leq \left(\int_{\mathbb{R}^N} \mathcal{K}(t)^{-\frac{p}{2-p}}\, d\mathbf{x}\right)^{1-\frac{p}{2}} = \left(\frac{4\pi(2-p)}{p}(1+t)\right)^{\frac{N}{2}(1-\frac{p}{2})} \leq C\, t^{\frac{N}{2}(1-\frac{p}{2})}, \quad \forall t > 0. \tag{3.60b}$$

So, (3.60a) provides $\|v\|_p^p \leq C\, \|v\|_{L^2(\Omega,\mathcal{K}(t))}^p\, t^{\frac{N}{2}(1-\frac{p}{2})}$, for all $t > 0$. This relation, together with the estimates (3.59), yields the conclusion of the Proposition 3.2. ■

4 Higher order estimates

In this section we shall study the decay rate of $\Delta \mathbf{v}$ in $\mathbf{L}^4(\Omega)$, $\mathbf{v}$ being the solution of the main system under consideration (1.4). This result will be used in the sequel in order to obtain the second term in the large time asymptotic development of $\mathbf{v}$. We proceed in several steps. We compute first the decay rate of $\nabla \mathbf{v}$. Then, we deduce the decay rate of $\mathbf{v}_t$ in $\mathbf{L}^4(\Omega)$ that will yield that of $\Delta \mathbf{v}$ in a straightforward way.

Lemma 4.1 *Let the initial data* $(\mathbf{v}_0, \mathbf{g}_0)$ *of* (1.4) *be in* $\mathbf{L}^2(\Omega) \times \mathbb{R}^N$ *then the solution* $(\mathbf{v}, \mathbf{g})$ *satisfies:*

$$\|\nabla \mathbf{v}\|_2 \leq C\,(1+t)^{-\frac{N}{4}-\frac{1}{2}}, \quad \forall\, t > 0. \tag{4.1}$$

The proof of this Lemma will be given in the Appendix B.

Remark 4.1 *Note that the decay rate in* (4.1) *coincides with that of the solution of the heat equation in* $\mathbb{R}^N$ *with initial data in* $L^1(\mathbb{R}^N)$*.*

We are interested now in finding estimates for the time derivatives $\mathbf{v}_t$ and $\mathbf{g}_t$. We denote $\mathbf{w} := \mathbf{v}_t$ and $\mathbf{k} := \mathbf{g}_t$. Taking time derivative in system (1.4), we get:

$$\begin{cases} \mathbf{w}_t - \Delta \mathbf{w} - \mathbf{g} \cdot \nabla \mathbf{w} - \mathbf{k} \cdot \nabla \mathbf{v} = 0, & \mathbf{x} \in \Omega, \quad t > 0, \\ \mathbf{w}(\mathbf{x}, t) = \mathbf{k}(t), & \mathbf{x} \in \partial B, \quad t > 0, \\ m\, \mathbf{k}'(t) = -\displaystyle\int_{\partial\Omega} \mathbf{n} \cdot \nabla \mathbf{w}\, d\sigma_x, & \qquad\quad t > 0, \\ \mathbf{w}(\mathbf{x}, 0) = \mathbf{v}_0'(\mathbf{x}) = \Delta \mathbf{v}_0(\mathbf{x}) + \mathbf{g}_0 \cdot \nabla \mathbf{v}_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathbf{k}(0) = \mathbf{g}'(0) = -\dfrac{1}{m} \displaystyle\int_{\partial\Omega} \mathbf{n} \cdot \nabla \mathbf{v}_0\, d\sigma_x. & \end{cases} \tag{4.2}$$

This system fits the general form of system (2.3), specifying $[\mathbf{U}] = \nabla \mathbf{v}$, $\mathbf{V} = \mathbf{g}$, $\boldsymbol{\varepsilon}_1 = \mathbf{0}$ and $\boldsymbol{\varepsilon}_2 = \mathbf{0}$. We note that the associated first momentum vanishes here. Indeed:

$$\int_\Omega \mathbf{w}\, d\mathbf{x} + m\, \mathbf{k} = \frac{d}{dt} \left(\int_\Omega \mathbf{v}\, d\mathbf{x} + m\, \mathbf{g} \right) = \mathbf{0}, \tag{4.3}$$

because $\int_\Omega \mathbf{v}\, d\mathbf{x} + m\, \mathbf{g} = \mathbf{M}_1$ is a constant vector, as proved in subsection 1.3. In order to apply Proposition 2.1 to system (4.2), we note that Lemma 4.1 provides an estimate for $[\mathbf{U}] = \nabla \mathbf{v}$ in $\mathbf{L}^2(\Omega)$. However, we also need a uniform estimate on $\|\mathbf{w}\|_1 + m\,|\mathbf{k}|$. This is obtained in the following Lemma:

Lemma 4.2 *The solution* $(\mathbf{w}, \mathbf{k})$ *of system* (4.2)*, assuming that the initial data are such that* $(\mathbf{v}_0, \mathbf{g}_0) \in \mathbf{H}^2(\Omega, K) \times \mathbb{R}^N$ *and* $\mathbf{v}_0|_{\partial\Omega} = \mathbf{g}_0$*, satisfies the following decay properties:*

$$\|\mathbf{w}\|_p \leq C\, t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}, \qquad |\mathbf{k}| \leq C\, t^{-\frac{N}{4}-\frac{1}{2}}, \tag{4.4}$$

for all $1 \leq p \leq 2$ and for t large enough. In particular, when $p = 1$:

$$\|\mathbf{w}\|_1 + m\,|\mathbf{k}| \leq C\,t^{-\frac{1}{2}}. \tag{4.5}$$

Remark 4.2 *The time derivative of the solution of the heat equation decays in L^p with a rate of the order of $t^{-\frac{N}{2}(1-\frac{1}{p})-1}$. The decay rate in (4.4) is weaker by a factor $t^{1/2}$ but it suffices for the purpose of applying Proposition 2.1 to system (4.2). This decay rate coincides with that of solutions of the heat equation of zero mass and initial data in $L^1(\mathbb{R}^N, |1+\mathbf{x}|)$. Note also that the decay rate we obtain for $|k|$ in (4.4) is of the order of the L^2-estimate on w. One could expect a better decay rate for k too, according to the continuity condition of the interface.*

Proof : We introduce similarity variables and rescaled functions:

$$\boldsymbol{\nu}(\mathbf{y}, s) := e^{s\frac{N}{2}}\mathbf{w}(e^{\frac{s}{2}}\mathbf{y}, e^s - 1), \qquad \boldsymbol{\varsigma}(s) := e^{s\frac{N}{2}}\mathbf{k}(e^s - 1). \tag{4.6}$$

In the same way as for system (3.21), we obtain, in view of (4.2) and the change of variables (4.6), that $(\boldsymbol{\nu}, \boldsymbol{\varsigma})$ solves:

$$\begin{cases} \boldsymbol{\nu}_s + L_s\boldsymbol{\nu} - \dfrac{N}{2}\boldsymbol{\nu} - e^{-s\,\frac{N-1}{2}}\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\nu} - e^{-s\,\frac{N-1}{2}}\boldsymbol{\varsigma}\cdot\nabla\boldsymbol{\xi} = \mathbf{0}, \quad \mathbf{y} \in \Omega_s, & s > 0, \\ \boldsymbol{\nu}(\mathbf{y}, s) = \boldsymbol{\varsigma}(s), \quad \mathbf{y} \in \partial B_s, & s > 0, \\ \boldsymbol{\varsigma}'(s) - \frac{N}{2}\boldsymbol{\varsigma}(s) = -\dfrac{e^{\frac{Ns}{2}}}{m}\displaystyle\int_{\partial\Omega_s} \mathbf{n}\cdot\nabla\boldsymbol{\nu}\,d\sigma_y, & s > 0, \\ \boldsymbol{\nu}(\mathbf{y}, 0) = \mathbf{v}_0'(\mathbf{y}) = \Delta\mathbf{v}_0(\mathbf{y}) + \mathbf{g}_0\cdot\nabla\mathbf{v}_0(\mathbf{y}), \quad \mathbf{y} \in \Omega, & \\ \boldsymbol{\varsigma}(0) = \mathbf{g}'(0) = -\dfrac{1}{m}\displaystyle\int_{\partial\Omega} \mathbf{n}\cdot\nabla\mathbf{v}_0\,d\sigma_x. & \end{cases} \tag{4.7}$$

where $(\boldsymbol{\xi}, \boldsymbol{\zeta})$ solves system (3.21). Working componentwise and multiplying the main equation in the weighted Sobolev space $L^2(\Omega_s, K)$ by ν, integrating by parts and using the transmission condition on $\partial\Omega_s$ (we refer to the proof of Proposition 3.2, identity (3.33) for details), it comes:

$$\frac{1}{2}\frac{d}{ds}\|\nu\|_s^2 + \frac{1}{2}\frac{d}{ds}\left[m\,e^{-s\frac{N}{2}}\chi\,\varsigma^2\right] + \|\nabla\nu\|_s^2 - \frac{N}{2}\|\nu\|_s^2 - e^{-s\frac{N-1}{2}}(\boldsymbol{\zeta}\cdot\nabla\nu, \nu)_s$$
$$- e^{-s\frac{N-1}{2}}(\boldsymbol{\varsigma}\cdot\nabla\xi, \nu)_s - \frac{1}{4}\left(N + \frac{\sigma_N}{m} - \frac{e^{-s}}{2}\right)m\,e^{-\frac{sN}{2}}\chi\,\varsigma^2 = 0. \tag{4.8}$$

Moreover, the following estimates hold:

- Since $|\boldsymbol{\zeta}| \leq C$, the first non-linear term satisfies, applying Cauchy-Schwarz's inequality:

$$\begin{aligned} \left|e^{-s\frac{N-1}{2}}(\boldsymbol{\zeta}\cdot\nabla\nu, \nu)_s\right| &\leq C\,e^{-s\frac{N-1}{2}}\|\nabla\nu\|_s\,\|\nu\|_s \\ &\leq C\,e^{-s\frac{N-1}{2}}(\|\nabla\nu\|_s^2 + \|\nu\|_s^2), \end{aligned}$$

and, by Poincaré's inequality in the weighted Sobolev space $H^1(\Omega_s, K)$:

$$\left| e^{-s\frac{N-1}{2}} (\boldsymbol{\zeta} \cdot \nabla \nu, \nu)_s \right| \leq C\, e^{-s\frac{N-1}{2}}\, \|\nabla \nu\|_s^2. \tag{4.9a}$$

- For the second one, the same arguments yield:

$$\begin{aligned} \left| e^{-s\frac{N-1}{2}} (\boldsymbol{\varsigma} \cdot \nabla \xi, \nu)_s \right| &\leq C\, e^{-s\frac{N-1}{2}} (|\boldsymbol{\varsigma}|^2 + \|\nabla \xi\|_s^2\, \|\nu\|_s^2) \\ &\leq C\, e^{-s\frac{N-1}{2}} (|\boldsymbol{\varsigma}|^2 + \|\nabla \xi\|_s^2\, \|\nabla \nu\|_s^2). \end{aligned} \tag{4.9b}$$

Therefore, combining relations (4.8) and (4.9a) and summing the N relations corresponding to each component, we deduce that:

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\|\boldsymbol{\nu}\|_s^2 + \frac{1}{2}\frac{d}{ds}\left[m\, e^{-s\frac{N}{2}} \chi\, \boldsymbol{\varsigma}^2\right] &\leq -\sum_{i=1}^{N} \left(1 - C\, e^{-s\frac{N-1}{2}} (1 + \|\nabla \xi_i\|_s^2)\right) \|\nabla \nu_i\|_s^2 \\ &+ \frac{N}{2}\, \|\boldsymbol{\nu}\|_s^2 + \frac{1}{4}\left(N + \frac{\sigma_N}{m} - \frac{e^{-s}}{2}\right) m\, e^{-s\frac{N}{2}} \chi\, |\boldsymbol{\varsigma}|^2 + C\, e^{-s\frac{N-1}{2}} |\boldsymbol{\varsigma}|^2. \end{aligned} \tag{4.10}$$

Since,

$$\sum_{i=1}^{N} \|\nabla \xi_i\|_s^2\, \|\nabla \nu_i\|_s^2 \leq \|\nabla \boldsymbol{\xi}\|_s^2\, \|\nabla \boldsymbol{\nu}\|_s^2$$

and

$$\frac{1}{4}\left(N + \frac{\sigma_N}{m} - \frac{e^{-s}}{2}\right) m\, e^{-s\frac{N}{2}} \chi\, |\boldsymbol{\varsigma}|^2 \leq C\, e^{-s\frac{N-1}{2}} |\boldsymbol{\varsigma}|^2,$$

we turn (4.10) into:

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\|\boldsymbol{\nu}\|_s^2 + \frac{1}{2}\frac{d}{ds}\left[m\, e^{-s\frac{N}{2}} \chi\, \boldsymbol{\varsigma}^2\right] &\leq -\left(1 - C\, e^{-s\frac{N-1}{2}} (1 + \|\nabla \boldsymbol{\xi}\|_s^2)\right) \|\nabla \boldsymbol{\nu}\|_s^2 \\ &+ \frac{N}{2}\, \|\boldsymbol{\nu}\|_s^2 + C\, e^{-s\frac{N-1}{2}} |\boldsymbol{\varsigma}|^2. \end{aligned} \tag{4.11}$$

On the other hand, taking into account that $\varsigma = \nu|_{\Omega_s}$ and using the trace inequality (B.5), we deduce that:

$$\begin{aligned} |\varsigma|^2 &\leq C \left(\int_{\Omega_s} |\nabla \nu|^2\, d\mathbf{x}\right)^{1/2} \left(\int_{\Omega_s} \nu^2\, d\mathbf{x}\right)^{1/2} \\ &\leq C \left(\int_{\Omega_s} |\nabla \nu|^2\, K\, d\mathbf{x}\right)^{1/2} \left(\int_{\Omega_s} \nu^2\, K\, d\mathbf{x}\right)^{1/2} = C\, (\|\nabla \nu\|_s\, \|\nu\|_s), \end{aligned}$$

and, by Poincaré's inequality, we deduce that:

$$|\boldsymbol{\varsigma}|^2 \le C\,\|\nabla\boldsymbol{\nu}\|_s^2. \tag{4.12}$$

Plugging this estimate, (4.11) reduces to:

$$\frac{1}{2}\frac{d}{ds}\left[\|\boldsymbol{\nu}\|_s^2 + m\,e^{-s\frac{N}{2}}\chi\,\boldsymbol{\varsigma}^2\right] \le -\left(1 - C\,e^{-s\frac{N-1}{2}}(1+\|\nabla\boldsymbol{\xi}\|_s^2)\right)\|\nabla\boldsymbol{\nu}\|_s^2 + \frac{N}{2}\,\|\boldsymbol{\nu}\|_s^2 + C\,e^{-s\frac{N-1}{2}}|\boldsymbol{\varsigma}|^2. \tag{4.13}$$

The first momentum of the solution $(\boldsymbol{\nu},\boldsymbol{\varsigma})$ of system (4.7) being zero (see (4.3)), the conclusion of Lemma 3.2 applies to $\boldsymbol{\nu}$ and hence we get:

$$\|\nabla\boldsymbol{\nu}\|_s^2 \ge \frac{N+1}{2}\,\|\boldsymbol{\nu}\|_s^2 - C\,e^{-s\frac{N}{2}},$$

Plugging this estimate into (4.13) and using (4.12), the following inequality holds for $X(s) := \|\boldsymbol{\nu}\|_s^2 + m\,e^{-s\frac{N}{2}}\chi\,|\boldsymbol{\varsigma}|^2$:

$$X'(s) + \left(1 - C\,e^{-s\frac{N-1}{2}}(1+\|\nabla\boldsymbol{\xi}\|_s^2)\right)\,X(s) \le C\,e^{-s\frac{N}{2}}, \quad \forall\, s \ge 0. \tag{4.14}$$

A computation similar to (3.33), yields:

$$\|\nabla\xi\|_s^2 = -\frac{1}{2}\frac{d}{ds}\left[\|\xi\|_s^2 + m\,\chi\,e^{-s\frac{N}{2}}\zeta^2\right] + \frac{N}{2}\|\xi\|_s^2 + e^{-s\frac{N-1}{2}}(\boldsymbol{\zeta}\cdot\nabla\xi,\xi)_s - \frac{1}{4}\,m\,e^{-s\frac{N}{2}}\chi\,\zeta^2\left(\frac{e^{-s}}{2} - \frac{\sigma_N}{m} - N\right).$$

Multiplying by $e^{-s\frac{N-1}{2}}$ and integrating with respect to s from 0 to ∞, we get:

$$\int_0^\infty e^{-s\frac{N-1}{2}}\,\|\nabla\xi\|_s^2\,ds = \frac{1}{2}\left[\|\xi(0)\|_0^2 + m\,\chi(0)\,\zeta^2(0)\right] + \frac{N+1}{4}\int_0^\infty e^{-s\frac{N-1}{2}}\,\|\xi\|_s^2\,ds - \frac{1}{4}\,m\int_0^\infty e^{-s\frac{2N-1}{2}}\chi\,\zeta^2\,\left(\frac{e^{-s}}{2} - \frac{\sigma_N}{m} - 1\right)\,ds + \int_0^\infty e^{-s(N-1)}(\boldsymbol{\zeta}\cdot\nabla\xi,\xi)_s\,ds. \tag{4.15}$$

Applying twice the Cauchy-Schwarz's inequality to the last term, we get:

$$\left|\int_0^\infty e^{-s(N-1)}(\boldsymbol{\zeta}\cdot\nabla\xi,\xi)_s\,ds\right| \le \left(\int_0^\infty e^{-s\frac{3(N-1)}{2}}\,|\boldsymbol{\zeta}|^2\|\xi\|_s^2\,ds\right)^{\frac{1}{2}}\left(\int_0^\infty e^{-s\frac{N-1}{2}}\,\|\nabla\xi\|_s^2\,ds\right)^{\frac{1}{2}}. \tag{4.16}$$

According to (3.58) and (3.59b), $|\boldsymbol{\zeta}|$ and $\|\xi\|_s^2$ are both bounded. Thus, (4.16) can be rewritten as:

$$\left|\int_0^\infty e^{-s(N-1)}(\boldsymbol{\zeta}\cdot\nabla\xi,\xi)_s\,ds\right| \leq C_1\left(\int_0^\infty e^{-s\frac{N-1}{2}}\|\nabla\xi\|_s^2\,ds\right)^{\frac{1}{2}}. \tag{4.17}$$

The same arguments ensure that:

$$\left|\frac{1}{2}\left[\|\xi(0)\|_0^2 + m\,\chi(0)\,\zeta^2(0)\right] + \frac{N+1}{4}\int_0^\infty e^{-s\frac{N-1}{2}}\|\xi\|_s^2\,ds \right.$$
$$\left. -\frac{1}{4}\,m\int_0^\infty e^{-s\frac{2N-1}{2}}\chi\,\zeta^2\left(\frac{N-1}{m}+\frac{e^{-s}}{2}-\frac{\sigma_N}{m}-N\right)ds\right| \leq C_2. \tag{4.18}$$

Therefore, with the notation $Y := \left(\int_0^\infty e^{-s\frac{N-1}{2}}\|\nabla\xi\|_s^2\,ds\right)^{\frac{1}{2}}$, we deduce from relations (4.15), (4.17) and (4.18) that $Y^2 - C_1\,Y - C_2 \leq 0$, what implies, in particular, that $Y < \infty$ and also:

$$\left(\int_0^\infty e^{-s\frac{N-1}{2}}\|\nabla\boldsymbol{\xi}\|_s^2\,ds\right)^{\frac{1}{2}} < \infty.$$

Setting then

$$\phi(s) := e^{-s\frac{N-1}{2}}(1+\|\nabla\boldsymbol{\xi}\|_s^2),$$

the differential inequality (4.14) can be rewritten:

$$X'(s) + (1 - C\,\phi(s))\,X(s) \leq C\,e^{-s\frac{N}{2}}.$$

This yields, after integration the estimate $X(s) \leq C\,e^{-s}$, since $\int_0^\infty \phi(s)\,ds < \infty$, where C depends on the initial data $X(0)$. That is to say:

$$\|\boldsymbol{\nu}\|_s^2 + m\,e^{-s\frac{N}{2}}\chi\,\boldsymbol{\varsigma}^2 \leq C\,e^{-s}, \quad \forall\, s \geq 0, \tag{4.19}$$

for all $N \geq 2$. According to the definition (4.6) of $\boldsymbol{\nu}$, we get:

$$\int_{\Omega_s} e^{s\frac{N}{2}}|\mathbf{w}(e^s-1, e^{\frac{s}{2}}\mathbf{y})|^2\,K(\mathbf{y})\,e^{s\frac{N}{2}}\,d\mathbf{y} \leq C\,e^{-s},$$

for all $s > 0$. Coming back to classical variables $\mathbf{x}$ and t and since $d\mathbf{x} = e^{s\frac{N}{2}}d\mathbf{y}$, it follows that:

$$\|\mathbf{w}\|_{\mathbf{L}^2(\Omega,\mathcal{K}(\mathbf{x},t))} \leq C\,t^{-\frac{N}{4}-\frac{1}{2}}, \quad \forall\, t > 0,$$

where $\mathcal{K}(\mathbf{x},t)$ is as in (3.6). We may also proceed exactly as in (3.60) to get the expected estimates for $\|\mathbf{w}\|_p$ for all $1 \leq p \leq 2$. In the same way, (4.19) together with the definition (4.6) of $\boldsymbol{\varsigma}$ leads to

$$e^{s\frac{N}{2}}\,\mathbf{k}^2(e^s-1) \leq C\,e^{-s} \;\Leftrightarrow\; |\mathbf{k}(t)| \leq C\,t^{-\frac{N}{4}-\frac{1}{2}}, \quad \forall\, t > 0.$$

■

For any function $f \in H^2(\Omega)$, we denote $[D_2 f]$ the $N \times N$ Hessian matrix of f. The following holds:

Lemma 4.3 *Let f be a function of $H^1(\Omega)$ such that $\Delta f \in L^2(\Omega)$ and $f|_\Omega$ is constant. Then:*

$$\|[D_2 f]\|_2 \leq C_N\,(\|\Delta f\|_2 + \|\nabla f\|_2),$$

for all $N \geq 2$, where C_N stands for a positive constant depending on the dimension N only.

The proof of this technical Lemma will be given in the Appendix B, at the end of the paper.

Remark 4.3 *Although the result of the above Lemma suffices for our purpose, it can be improve as follows, when $N \leq 3$ only, using an argument of elliptic regularity:*

$$\|[D_2 f]\|_2 \leq C_N\,\|\Delta f\|_2.$$

Combining together Lemma 4.1, Lemma 4.2 and Lemma 4.3, we can now deduce:

Proposition 4.1 *Let $\mathbf{v}$ be the solution of system (1.4) and assume that $(\mathbf{v}_0, \mathbf{g}_0) \in \mathbf{H}^2(\Omega, K) \times \mathbb{R}^N$ and $\mathbf{v}_0|_{\partial\Omega} = \mathbf{g}_0$. Then:*

$$\|\Delta \mathbf{v}\|_p \leq C\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}, \tag{4.20}$$

for all $1 \leq p \leq 4$, all $N \geq 2$ and all t large enough.

Remark 4.4 *Note that the Laplacian of solutions of the heat equation with initial data in $L^1(\mathbb{R}^N)$ decays in L^p with a rate $t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-1}$. We do not know whether the decay rate in (4.20) is sharp for the system under consideration.*

Proof : We proceed in two steps:

- When $1 \leq p \leq 2$:
  Taking into account that $|\mathbf{g}| \leq C\,t^{-\frac{N}{2}}$ (see Proposition 2.2) and $\|\nabla \mathbf{v}\|_p \leq C\,t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}$ according to Lemma 4.1 and in view of the main equation in (1.5), we deduce that:

  $$\|\mathbf{v}_t - \Delta \mathbf{v}\|_p \leq C\,t^{-\frac{N}{2}} t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}. \tag{4.21}$$

  This, together with the decay rate of $\mathbf{w} = \mathbf{v}_t$ in Lemma 4.2, allows us to deduce (4.20) for $1 \leq p \leq 2$.

- When $2 < p \le 4$:
  According to relation (B.1), we have:

$$\|\nabla v\|_4^4 \le \|\nabla v\|_2^2 \, \|[D_2 v]\|_2^2,$$

which, combined with Lemma 4.3 gives:

$$\|\nabla \mathbf{v}\|_4^4 \le C_N \, \|\nabla \mathbf{v}\|_2^2 \, (\|\Delta \mathbf{v}\|_2 + \|\nabla \mathbf{v}\|_2)^2.$$

Lemma 4.1 provides the decay rates of $\|\nabla \mathbf{v}\|_2$ and we have $\|\Delta \mathbf{v}\|_2 \le C\, t^{-\frac{N}{4}-\frac{1}{2}}$ as proved in the case $1 \le p \le 2$. Therefore, we obtain:

$$\|\nabla \mathbf{v}\|_4 \le C\, t^{-\frac{N}{4}-\frac{1}{2}}.$$

We now apply Proposition 2.1 to system (4.2) to obtain the decay rate of $\mathbf{v}_t = \mathbf{w}$ in $\mathbf{L}^4(\Omega)$. We set $[\mathbf{U}] := [\nabla \mathbf{v}]$, $\mathbf{V} = \mathbf{g}$, $\boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_2 = \mathbf{0}$. Then $\vartheta_1 = t \, \| \, [\mathbf{U}] \, \|_4$ and $\vartheta_2 = 0$. Thus, assumption (2.6) of Proposition 2.1 holds for $p = 4$ with $\alpha_4 = N/4 - 1/2$. Proposition 2.1 then yields:

$$\|\mathbf{w}\|_4 \le C\, \delta_4 \, t^{-\frac{N}{2}\left(1-\frac{1}{4}\right)}, \tag{4.22}$$

for all $t \ge 1$, where δ_4 is defined by (2.8). Moreover, according to definition (2.8), we have $\delta_4 \le C\, \delta_1$ with $\delta_1 = 2 \sup_{t\in(0,\infty)}(\|\mathbf{w}\|_1 + m\,|\mathbf{k}|)$. We proceed then as in the proof of Proposition 3.1: we split the time interval $(0,t)$ into two subintervals $(0, t/2)$ and $(t/2, t)$. On $(0, t/2)$ we use the decay property (4.5) of $\|\mathbf{w}\|_1 + m\,|\mathbf{k}|$, and then we apply (4.22) on $(t/2, t)$. This provides:

$$\|\mathbf{w}\|_4 \le C\, t^{-\frac{N}{2}\left(1-\frac{1}{4}\right)-\frac{1}{2}}.$$

As we said, relation (4.21) clearly holds for all $1 \le p < \infty$ and then provides the conclusion of Proposition 4.1 for $p = 4$. An interpolation inequality yields (4.20) for all $2 < p \le 4$.

■

5 Second term in the asymptotic development

In this section, we will make more precise the conclusions of Theorem 1.2 for $p = 2$, analysing the large time behaviour of:

$$\mathbf{v}_1 := t^{\frac{1}{2}}\,\bar{\mathbf{v}}(t) = t^{\frac{1}{2}}(\mathbf{v} - \mathbf{M}_1\, G) \quad \text{and} \quad \boldsymbol{\xi}_1 := e^{\frac{s}{2}}\,\bar{\boldsymbol{\xi}} = e^{\frac{s}{2}}(\boldsymbol{\xi} - \mathbf{M}_1\, \boldsymbol{\theta}_1),$$

in $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^2(K, \Omega_s)$ respectively.

5.1 Identification of the second term

In this subsection, we identify the quantities entering in the second term of the asymptotic expansion and we show that they are well-defined.

Observe that the decay rate of $\bar{\boldsymbol{\xi}}$ when $N = 2$, in (3.58), namely the fact that $\|\bar{\boldsymbol{\xi}}\|_s \leq C\, s\, e^{-\frac{s}{2}}$, does not guarantee that $\boldsymbol{\xi}_1$ is bounded in $\mathbf{L}^2(\Omega_s, K)$. Nevertheless, $s^{-1}\boldsymbol{\xi}_1$ is bounded. However, for $N \geq 3$, $\boldsymbol{\xi}_1$ itself is bounded in $\mathbf{L}^2(K, \Omega_s)$. This fact will be relevant all along this section.

To begin, define $\boldsymbol{\zeta}_1$ as the trace of $\boldsymbol{\xi}_1$ on $\partial\Omega_s$. Since $(\boldsymbol{\xi}_1, \boldsymbol{\zeta}_1) = e^{\frac{s}{2}}(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\zeta}})$ and $(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\zeta}})$ solves system (3.25), we deduce that $(\boldsymbol{\xi}_1, \boldsymbol{\zeta}_1)$ solves:

$$\begin{cases} \boldsymbol{\xi}_{1,s} + L_s\boldsymbol{\xi}_1 - \dfrac{N+1}{2}\boldsymbol{\xi}_1 = e^{-s\frac{N-2}{2}}\boldsymbol{\zeta} \cdot \nabla\boldsymbol{\xi}, \quad \mathbf{y} \in \Omega_s, & s > 0, \\ \boldsymbol{\xi}_1(\mathbf{y}, s) = \boldsymbol{\zeta}_1(s), \quad \mathbf{y} \in \partial B_s, & s > 0, \\ m\,(\boldsymbol{\zeta}_1'(s) - \frac{N+1}{2}\boldsymbol{\zeta}_1(s))\, e^{-s\frac{N}{2}} = -\displaystyle\int_{\partial\Omega_s} \mathbf{n} \cdot \nabla\boldsymbol{\xi}_1\, d\sigma_y + e^{-s\frac{N}{2}}\boldsymbol{\rho}_1(s), & s > 0, \\ \boldsymbol{\xi}_1(\mathbf{y}, 0) = \mathbf{v}_0(\mathbf{y}) - \mathbf{M}_1\theta_1(\mathbf{y}) \quad \mathbf{y} \in \Omega, \quad \boldsymbol{\zeta}_1(0) = \mathbf{h}_1 - (4\pi)^{\frac{N}{2}}\mathbf{M}_1, \end{cases} \tag{5.1}$$

where $\boldsymbol{\rho}_1(s) := e^{\frac{s}{2}}\,\boldsymbol{\rho}(s)$ and $\boldsymbol{\rho}(s)$ as in (3.26), i.e.

$$\boldsymbol{\rho}(s) := \frac{1}{2}\, m\, \mathbf{M}_1\, (4\pi)^{-\frac{N}{2}} e^{-r_s^2/4} \left(-\frac{\sigma_N}{m} + N - \frac{e^{-s}}{2} \right).$$

We denote $\theta_{2,1}, \theta_{2,2}, \ldots, \theta_{2,N}$ the N eigenfunctions of L associated to $\lambda_2 = \frac{N+1}{2}$ that span the eigenspace E_2:

$$\theta_{2,i}(\mathbf{y}) := \frac{\partial\theta_1}{\partial y_i}(\mathbf{y}) = -\frac{y_i}{2}\theta_1(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^N, \quad \boldsymbol{\theta}_2 := \nabla\theta_1. \tag{5.2}$$

As we shall see, $\boldsymbol{\xi}_1$ behaves for large s as $[\mathbf{M}_2(s)]\,\boldsymbol{\theta}_2$ where $[\mathbf{M}_2(s)]$ is as follows:

- When $N = 2$, $[\mathbf{M}_2(s)] := s\,[\mathbf{M}_2^1] + [\mathbf{M}_2^2]$, is an affine function with $[\mathbf{M}_2^1]$ and $[\mathbf{M}_2^2]$ two constant matrices that we shall identify.

- When $N \geq 3$, $[\mathbf{M}_2]$ is a constant matrix to be determined.

To shorten notations and avoid distinguishing dimensions $N = 2$ and $N \geq 3$, we will sometimes use the notation:

$$\eta_N(s) := \begin{cases} \sqrt{1+s}, & \text{when } N = 2, \\ 1, & \text{when } N \geq 3. \end{cases}$$

The fact that $\boldsymbol{\theta}_2$ enters in the large time behaviour of the solution $\boldsymbol{\xi}_1$ of (5.1) can easily be motivated. For instance, when dealing with the solutions of

$$w_s + L\, w - \frac{N+1}{2}\, w = 0, \quad \text{in } \mathbb{R}^N,\ s \geq 0,$$

by the Fourier expansion of the solution w on the basis of eigenfunctions of L it can be easily seen that, when w is of zero mass, the leading term is the projection onto E_2. System (5.1) can be viewed as a perturbation of this ideal situation. Its dynamics, although it is essentially of the same nature, is more complex.

When $N = 2$, the projection of $\boldsymbol{\xi}_1$ over $\boldsymbol{\theta}_2$ grows linearly with $s > 0$ and therefore this case needs a distinguished treatment.

As we shall see, the matrices $[\mathbf{M}_2^1]$, $[\mathbf{M}_2^2]$ $(N = 2)$ and $[\mathbf{M}_2]$ $(N \geq 3)$ entering in the second term of the asymptotic expansion are as follows:

- When $N = 2$:

$$[\mathbf{M}_2^1] := \lim_{s\to\infty} 2\, s^{-1}\, (4\pi)^{N/2} \int_{\Omega_s} \boldsymbol{\xi}_1\, \boldsymbol{\theta}_2^T\, K\, d\mathbf{y}, \tag{5.3a}$$

$$[\mathbf{M}_2^2] := \lim_{s\to\infty} \left(2\, (4\pi)^{N/2} \int_{\Omega_s} \boldsymbol{\xi}_1\, \boldsymbol{\theta}_2^T\, K\, d\mathbf{y} - s\, [\mathbf{M}_2^1] \right). \tag{5.3b}$$

- When $N \geq 3$:

$$[\mathbf{M}_2] := \lim_{s\to\infty} 2\, (4\pi)^{N/2} \int_{\Omega_s} \boldsymbol{\xi}_1\, \boldsymbol{\theta}_2^T\, K\, d\mathbf{y}. \tag{5.3c}$$

The following Proposition guarantees that the limits above are well-defined.

Proposition 5.1 *The matrices* $[\mathbf{M}_2^1]$, $[\mathbf{M}_2^2]$ *(*$N = 2$*) and* $[\mathbf{M}_2]$ *(*$N \geq 3$*) are well defined by relations* (5.3)*, i.e. the limits always exist in* $\mathbb{R}$ *and the identities* (1.18) *of Theorem 1.3 hold.*

Moreover, $\eta_N^{-1}|\boldsymbol{\beta}|$ *is bounded and all the generalised integrals in the definitions* (1.18) *are finite.*

Proof : Fix i between 1 and N and multiply the main equation of (5.1) in $L^2(\Omega_s, K)$ by $\theta_{2,i}$. It comes, for each component ξ_1 of $\boldsymbol{\xi}_1$:

$$(\xi_{1,s}, \theta_{2,i})_s + (L_s\xi_1, \theta_{2,i})_s - \frac{N+1}{2}(\xi_1, \theta_{2,i})_s = e^{-s\frac{N-2}{2}}(\boldsymbol{\zeta} \cdot \nabla\xi, \theta_{2,i})_s. \tag{5.4}$$

By Green's formula, it follows that:

$$(L_s\xi_1, \theta_{2,i})_s - (\xi_1, L_s\theta_{2,i})_s = \int_{\partial\Omega_s} \frac{\partial\theta_{2,i}}{\partial\mathbf{n}}\, \xi_1\, \chi\, d\sigma_y - \int_{\partial\Omega_s} \frac{\partial\xi_1}{\partial\mathbf{n}}\, \theta_{2,i}\, \chi\, d\sigma_y. \tag{5.5}$$

By direct computation, and since $\theta_{2,i} = -y_i\, \theta_1/2$ and $\nabla\theta_1 = -\mathbf{y}\, \theta_1/2$, we get on $\partial\Omega_s$:

$$\frac{\partial\theta_{2,i}}{\partial\mathbf{n}} = \nabla\, \theta_{2,i} \cdot \mathbf{n} = -\frac{n_i}{2}\, \theta_1 - \frac{y_i}{2}\, \nabla\theta_1 \cdot \mathbf{n} = \frac{y_i}{2}\, \theta_1 \left(e^{s/2} - \frac{e^{-s/2}}{2} \right), \tag{5.6}$$

because $\mathbf{n} = -e^{s/2}\,\mathbf{y}$ on $\partial\Omega_s$. Since θ_1 is radially symmetric, we get with (5.6):

$$\int_{\partial\Omega_s} \frac{\partial\theta_{2,i}}{\partial\mathbf{n}}\, d\sigma_y = 0.$$

On the other hand, taking into account the fact that $L_s\,\theta_{2,i} = \frac{N+1}{2}\,\theta_{2,i}$ and that $\xi_1 = \zeta_1$ is constant on the boundary $\partial\Omega_s$, we deduce that

$$\int_{\partial\Omega_s} \frac{\partial\theta_{2,i}}{\partial\mathbf{n}}\, \xi_1\,\chi\, d\sigma_y = 0.$$

Therefore (5.5) can be turned into:

$$(L_s\xi_1, \theta_{2,i})_s - \frac{N+1}{2}(\xi_1, \theta_{2,i})_s = -\int_{\partial\Omega_s} \frac{\partial\xi_1}{\partial\mathbf{n}}\,\theta_{2,i}\,\chi\, d\sigma_y. \tag{5.7}$$

But θ_1 is a radially symmetric function and hence

$$\int_{\partial\Omega_s} \frac{\partial\theta_1}{\partial\mathbf{n}}\,\theta_{2,i}\,\chi\, d\sigma_y = -\int_{\partial\Omega_s} \frac{\partial\theta_1}{\partial\mathbf{n}}\,\frac{y_i}{2}\theta_1\,\chi\, d\sigma_y = 0,$$

for all $s > 0$. According to the definition of $\xi_1\,(= e^{\frac{s}{2}}(\xi - M_1\,\theta_1))$, it follows that the term of the right hand side in (5.7) can be reduced to

$$\int_{\partial\Omega_s} \frac{\partial\xi_1}{\partial\mathbf{n}}\,\theta_{2,i}\,\chi\, d\sigma_y = -\frac{1}{2}\,e^{\frac{s}{2}}\,(4\pi)^{-\frac{N}{2}}\int_{\partial\Omega_s} \frac{\partial\xi}{\partial\mathbf{n}}\,y_i\, d\sigma_y. \tag{5.8}$$

Plugging (5.7) and (5.8) into (5.4), we obtain

$$(\xi_{1,s}, \theta_{2,i})_s = -\frac{1}{2}\,e^{\frac{s}{2}}(4\pi)^{-\frac{N}{2}}\int_{\partial\Omega_s} \frac{\partial\xi}{\partial\mathbf{n}} y_i\, d\sigma_y + e^{-s\frac{N-2}{2}}(\boldsymbol{\zeta}\cdot\nabla\xi, \theta_{2,i})_s. \tag{5.9}$$

The use of Lemma 3.1 yields

$$\frac{d}{ds}(\xi_1, \theta_{2,i})_s = (\xi_{1,s}, \theta_{2,i})_s + \frac{e^{-\frac{s}{2}}}{2}\int_{\partial\Omega_s} \xi_1\,\theta_{2,i}\,\chi\, d\sigma_y,$$

but since $\xi_1|_{\partial\Omega_s} = \zeta_1$ is constant for all $s \geq 0$, the last term vanishes and we deduce that:

$$\frac{d}{ds}(\xi_1, \theta_{2,i})_s = (\xi_{1,s}, \theta_{2,i})_s.$$

We rewrite (5.9) as:

$$\frac{d}{ds}(\xi_1, \theta_{2,i})_s = -\frac{1}{2}\,e^{\frac{s}{2}}\,(4\pi)^{-\frac{N}{2}}\int_{\partial\Omega_s} \frac{\partial\xi}{\partial\mathbf{n}}\,y_i\, d\sigma_y + e^{-s\frac{N-2}{2}}(\boldsymbol{\zeta}\cdot\nabla\xi, \theta_{2,i})_s. \tag{5.10}$$

The main difficulty consists now in estimating the term involving the integral on the boundary of Ω_s. This will be done using the results of section 4.

Lemma 5.1 *Let $\boldsymbol{\xi}$ be solution of* (3.21) *with initial data* $(\boldsymbol{\xi}_0, \boldsymbol{\zeta}_0) \in \mathbf{H}^2(\Omega, K) \times \mathbb{R}^N$ *such that* $\boldsymbol{\xi}_0|_{\partial\Omega} = \boldsymbol{\zeta}_0$*, then the following inequality holds:*

$$\left| \int_{\partial\Omega_s} \frac{\partial \xi}{\partial \mathbf{n}} \, y_i \, d\sigma_y \right| \leq C \, e^{-s \frac{3N}{8}}, \tag{5.11}$$

for all $N \geq 2$ and s large enough.

Proof : Going back to the variables $\mathbf{x} := \mathbf{y} e^{\frac{s}{2}}$ and $t := e^s - 1$, we have:

$$\int_{\partial\Omega_s} \frac{\partial \xi}{\partial \mathbf{n}} \, y_i \, d\sigma_y = (1+t)^{\frac{1}{2}} \int_{\partial\Omega} \frac{\partial v}{\partial \mathbf{n}} \, x_i \, d\sigma_x, \tag{5.12}$$

where $v(\mathbf{x}, t) = (1+t)^{-N/2} \, \xi\left(\mathbf{x}/\sqrt{1+t}, \log(1+t)\right)$. Applying Green's formula, we get

$$\int_\Omega \Delta v \, \rho_i(\mathbf{x}) \, d\mathbf{x} - \int_\Omega v \, \Delta \rho_i(\mathbf{x}) \, d\mathbf{x} = \int_{\partial\Omega} \frac{\partial v}{\partial \mathbf{n}} \, x_i \, d\sigma_x, \tag{5.13}$$

where we specify $\rho_i(\mathbf{x}) := x_i \, e^{1-|\mathbf{x}|^2}$. Note that, on $\partial\Omega$:

$$\frac{\partial \rho_i}{\partial \mathbf{n}} = x_i \, (2 \, |\mathbf{x}| - 1) \, e^{1-|\mathbf{x}|^2},$$

because $\mathbf{n} = -\mathbf{x}$ on $\partial\Omega$ and hence:

$$\int_{\partial\Omega} \frac{\partial \rho_i}{\partial \mathbf{n}} \, d\sigma_x = 0, \quad \int_{\partial\Omega} v \, \frac{\partial \rho_i}{\partial \mathbf{n}} \, d\sigma_x = 0,$$

because $v = g$ is constant on $\partial\Omega$. Applying Hölder's inequality in L^4 and $L^{\frac{4}{3}}$ for the first term in (5.12), and according to Proposition 4.1, we obtain that:

$$\left| \int_\Omega \Delta v \, \rho_i \, (\mathbf{x}) \, d\mathbf{x} \right| \leq C \, t^{-\frac{3N}{8} - \frac{1}{2}}, \tag{5.14}$$

for all $N \geq 2$. Note that, since $\boldsymbol{\xi} = e^{s\frac{N}{2}} \mathbf{v}(\mathbf{y} \, e^{\frac{s}{2}}, e^s - 1)$, $\boldsymbol{\xi}_0$ and $\mathbf{v}_0$ coincide and the condition $\boldsymbol{\xi}_0 \in \mathbf{H}^2(\Omega, K)$ implies $\mathbf{v}_0 \in \mathbf{H}^2(\Omega)$. Hence, Proposition 4.1 applies. On the other hand, by direct computations, we get:

$$\Delta \rho_i = 2 \, x_i \, (2 \, |\mathbf{x}|^2 - N - 1) \, e^{1-|\mathbf{x}|^2},$$

then:

$$\int_\Omega G \, \Delta \rho_i \, d\mathbf{x} = 0,$$

for all $t > 0$, because G is radially symmetric. Therefore

$$\left| \int_\Omega v \, \Delta \rho_i \, d\mathbf{x} \right| = \left| \int_\Omega (v - M_1 \, G) \, \Delta \rho_i \, d\mathbf{x} \right| \leq \|v - M_1 \, G\|_\infty \int_\Omega |\Delta \rho_i(\mathbf{x})| d\mathbf{x},$$

that is to say, with Theorem 1.2, when $m \neq \frac{\sigma_N}{N}$:

$$\left| \int_\Omega v \, \Delta\rho_i(\mathbf{x}) \, d\mathbf{x} \right| \leq C \begin{cases} |\log(1+t)|^{\frac{2}{2+N}} \, t^{-\frac{N}{2}-\frac{1}{N+2}}, & \text{when } N = 2, \\ t^{-\frac{N}{2}-\frac{1}{N+2}}, & \text{when } N \geq 3. \end{cases} \tag{5.15}$$

When $m = \frac{\sigma_N}{N}$, we can obtain a slightly better estimate but (5.15) is enough for our purpose. Summarising, (5.13) with (5.14) and (5.15), it comes

$$\left| \int_{\partial\Omega} \frac{\partial v}{\partial \mathbf{n}} x_i \, d\sigma_x \right| \leq C \, t^{-\frac{3N}{8}-\frac{1}{2}},$$

for all t large enough and all $N \geq 2$ because $\frac{N}{2} + \frac{1}{2+N} \geq \frac{3N}{8} + \frac{1}{2}$ for all $N \geq 2$. According to (3.18) and (5.12), estimate (5.11) holds. ■

Let us address now the second term of the right hand side of equality (5.10). In view of the explicit form (5.2) of $\theta_{2,i}$, we get $\theta_{2,i} \, K = -\frac{1}{2}(4\,\pi)^{-\frac{N}{2}} y_i$ and then, integrating by parts, we obtain that:

$$(\boldsymbol{\zeta} \cdot \nabla \boldsymbol{\xi}, \theta_{2,i})_s = \frac{1}{2} \, \zeta_i \, (4\pi)^{-\frac{N}{2}} \int_{\Omega_s} \boldsymbol{\xi} \, d\mathbf{y}, \tag{5.16}$$

where ζ_i stands for the $i-$th component of the vector $\boldsymbol{\zeta}$. But, in similarity variables, according to (3.20), $\mathbf{M}_1 = \int_{\Omega_s} \boldsymbol{\xi} \, d\mathbf{y} + m \, e^{-s\frac{N}{2}} \boldsymbol{\zeta}$, for all $s \geq 0$, so (5.16) can be rewritten as:

$$(\boldsymbol{\zeta} \cdot \nabla \boldsymbol{\xi}, \theta_{2,i})_s = \frac{1}{2} \mathbf{M}_1 \, (4\pi)^{-\frac{N}{2}} \zeta_i - \frac{1}{2} \, m \, (4\pi)^{-\frac{N}{2}} \, e^{-s\frac{N}{2}} \boldsymbol{\zeta} \, \zeta_i.$$

Finally, integrating (5.10) in time from 0 to s, we obtain that:

$$\begin{aligned} (\boldsymbol{\xi}_1, \theta_{2,i})_s = -\frac{1}{2} \, (4\pi)^{-\frac{N}{2}} (\mathbf{v}_0, y_i)_{\mathbf{L}^2(\Omega)} - \frac{1}{2} \, (4\pi)^{-\frac{N}{2}} \int_0^s e^{\frac{\alpha}{2}} \left(\int_{\partial\Omega_\alpha} \mathbf{n} \cdot \nabla \boldsymbol{\xi} \, y_i \, d\sigma_y \right) d\alpha \\ - \frac{1}{2} (4\pi)^{-\frac{N}{2}} \int_0^s e^{-\alpha\frac{N-2}{2}} (m \, e^{-\alpha\frac{N}{2}} \boldsymbol{\zeta} \, \zeta_i - \mathbf{M}_1 \, \zeta_i) \, d\alpha. \end{aligned} \tag{5.17}$$

Lemma 5.1 ensures that

$$\int_0^s \left| e^{\frac{\alpha}{2}} \int_{\partial\Omega_\alpha} \mathbf{n} \cdot \nabla \boldsymbol{\xi} \, y_i \, d\sigma_y \right| d\alpha \leq C \int_0^\infty e^{-\alpha\frac{3N-4}{8}} \, d\alpha < \infty, \tag{5.18}$$

for all $N \geq 2$. Hence, this integral is convergent.

On the other hand, according to Theorem 1.2 (once again, the estimate below can be slightly improved when $m = \frac{\sigma_N}{N}$ but it is sufficient for our purpose):

$$\left| \mathbf{g}(t) - \mathbf{M}_1 \, (4\pi t)^{-N/2} \right| \leq C \begin{cases} |\log(1+t)|^{\frac{1}{2}} \, t^{-\frac{5}{4}}, & \text{when } N = 2, \\ t^{-\frac{N}{2}-\frac{1}{2+N}}, & \text{when } N \geq 3, \end{cases} \tag{5.19}$$

which in similarity variables can be rewritten as:

$$\left|\boldsymbol{\zeta}(s) - \mathbf{M}_1\,(4\pi)^{-\frac{N}{2}}\right| \leq C\,\eta_N(s)\,e^{-s\frac{1}{2+N}}. \tag{5.20}$$

Then

$$\boldsymbol{\zeta} = e^{-s\frac{1}{2+N}}\boldsymbol{\beta}(s) + \mathbf{M}_1\,(4\pi)^{-\frac{N}{2}},$$

with $\boldsymbol{\beta} = \boldsymbol{\beta}(s)$ as in (1.18d). Relation (5.20) means that $\eta_N^{-1}|\boldsymbol{\beta}(s)|$ is bounded for all $s \geq 0$. We obtain then:

$$\begin{aligned} e^{-\alpha\frac{N-2}{2}}\left(m\,e^{-\alpha\frac{N}{2}}\boldsymbol{\zeta}\zeta_i - \mathbf{M}_1\,\zeta_i\right) = \\ e^{-\alpha\frac{N-2}{2}}\,(4\pi)^{-\frac{N}{2}}\Big[-\mathbf{M}_1\,M_{1,i} + m\,(4\pi)^{-\frac{N}{2}}\,e^{-\alpha\frac{N}{2}}\,\mathbf{M}_1\,M_{1,i} \\ -(4\pi)^{\frac{N}{2}}\,e^{-\alpha\frac{1}{2+N}}\,\mathbf{M}_1\,\beta_i + m\,e^{-\alpha\frac{N}{2}-\alpha\frac{1}{2+N}}(\boldsymbol{\beta}\,M_{1,i} + \mathbf{M}_1\,\beta_i) \\ +m\,(4\pi)^{\frac{N}{2}}\,e^{-\alpha\frac{N}{2}-\alpha\frac{2}{N+2}}\boldsymbol{\beta}\,\beta_i\Big], \quad \forall\alpha > 0. \end{aligned}$$

Integrating now from 0 to s, we obtain:

- When $N = 2$:

$$\begin{aligned} \int_0^s \left(m\,e^{-\alpha}\boldsymbol{\zeta}\,\zeta_i - \mathbf{M}_1\,\zeta_i\right)\,d\alpha = -s\,\mathbf{M}_1\,M_{1,i}\,(4\pi)^{-1} \\ + m\,(4\pi)^{-2}\,\mathbf{M}_1\,M_{1,i}(1-e^{-s}) - \mathbf{M}_1\int_0^s e^{-\frac{\alpha}{4}}\beta_i\,d\alpha \\ + \int_0^s (4\pi)^{-1}\,m\,e^{-\alpha\frac{5}{4}}(\boldsymbol{\beta}\,M_{1,i} + \mathbf{M}_1\,\beta_i)\,d\alpha + \int_0^s m\,e^{-\alpha\frac{3}{2}}\boldsymbol{\beta}\,\beta_i\,d\alpha. \end{aligned} \tag{5.21}$$

Then, combining (5.17) and (5.21), it comes:

$$\begin{aligned} 2\,(4\pi)^{N/2}\int_{\Omega_s}\boldsymbol{\xi}_1\,\theta_{2,i}\,d\mathbf{y} = s\,M_{1,i}\,\mathbf{M}_1\,(4\pi)^{-1} - (\mathbf{v}_0, y_i)_{\mathbf{L}^2(\Omega)} \\ - \int_0^s e^{\frac{\alpha}{2}}\left(\int_{\partial\Omega_\alpha}\mathbf{n}\cdot\nabla\boldsymbol{\xi}\,y_i\,d\sigma_y\right)d\alpha - m\,(4\pi)^{-2}\,\mathbf{M}_1\,M_{1,i}(1-e^{-s}) \\ + \mathbf{M}_1\int_0^s e^{-\frac{\alpha}{4}}\beta_i\,d\alpha - \int_0^s (4\pi)^{-1}\,m\,e^{-\alpha\frac{5}{4}}(M_{1,i}\,\boldsymbol{\beta} + \mathbf{M}_1\,\beta_i)\,d\alpha \\ - \int_0^s m\,e^{-\alpha\frac{3}{2}}\beta_i\,\boldsymbol{\beta}\,d\alpha. \end{aligned} \tag{5.22}$$

Then, dividing by s, taking into account (5.18) and letting s go to ∞, we get the equality between (5.3a) and (1.18a). We get also:

$$\Big[2\,(4\pi)^{N/2}\int_{\Omega_s}\boldsymbol{\xi}_1\,\theta_2^T\,K\,d\mathbf{y} - s\,(4\pi)^{-1}\mathbf{M}_1\,\mathbf{M}_1^T\Big] = -\int_\Omega \mathbf{v}_0\,\mathbf{y}^T\,d\mathbf{y}$$
$$-\int_0^s e^{\frac{\alpha}{2}}\left(\int_{\partial\Omega_\alpha}(\mathbf{n}\cdot\nabla\boldsymbol{\xi})\,\mathbf{y}^T\,d\sigma_y\right)d\alpha - (4\pi)^{-2}\,m\,\mathbf{M}_1\,\mathbf{M}_1^T\,\left(1-e^{-s}\right)$$
$$+\int_0^s e^{-\alpha\frac{1}{4}}\mathbf{M}_1\,\boldsymbol{\beta}^T\,d\alpha - (4\pi)^{-1}\int_0^s m\,e^{-\alpha\frac{5}{4}}(\boldsymbol{\beta}\,\mathbf{M}_1^T + \mathbf{M}_1\,\boldsymbol{\beta}^T)\,d\alpha$$
$$-\int_0^s m\,e^{-\alpha\frac{3}{2}}\boldsymbol{\beta}\,\boldsymbol{\beta}^T\,d\alpha. \quad (5.23)$$

- When $N \geq 3$:
  We find the following expression:

$$2\,(4\pi)^{N/2}\int_{\Omega_s}\boldsymbol{\xi}_1\,\theta_2^T\,K\,d\mathbf{y} =$$
$$-\int_\Omega \mathbf{v}_0\,\mathbf{y}^T\,d\mathbf{y} - \int_0^s e^{\frac{\alpha}{2}}\left(\int_{\partial\Omega_\alpha}(\mathbf{n}\cdot\nabla\boldsymbol{\xi})\,\mathbf{y}^T\,d\sigma_y\right)d\alpha$$
$$-\ \mathbf{M}_1\,\mathbf{M}_1^T\,(4\pi)^{-N}\left[\frac{m}{N-1}\left(1-e^{-s(N-1)}\right) - (4\pi)^{\frac{N}{2}}\,\frac{2}{N-2}\left(1-e^{-s\frac{N-2}{2}}\right)\right]$$
$$+\ \mathbf{M}_1\int_0^\infty e^{-\alpha\frac{N-2}{2}-\alpha\frac{1}{2+N}}\boldsymbol{\beta}^T\,d\alpha - m\int_0^s e^{-\alpha(N-1)-\alpha\frac{2}{2+N}}\boldsymbol{\beta}\,\boldsymbol{\beta}^T\,d\alpha.$$
$$-\ (4\pi)^{-\frac{N}{2}}\,m\int_0^s e^{-\alpha(N-1)-\alpha\frac{1}{2+N}}(\boldsymbol{\beta}\,\mathbf{M}_1^T + \mathbf{M}_1\,\boldsymbol{\beta}^T)\,d\alpha. \quad (5.24)$$

Letting $s \to \infty$ in (5.23) and (5.24), we find that the expressions (1.18b) and (5.3b) and (1.18c) and (5.3c) coincide respectively. That concludes the proof of Proposition 5.1. ■

5.2 Proof of Theorem 1.3

We shall proceed in several steps, establishing a sequence of preliminary Lemmas. Theorem 1.3 will then hold immediately.

We recall the definition of $[\mathbf{M}_2(s)]$ (see section 5.1):

$$[\mathbf{M}_2(s)] := s\,[\mathbf{M}_2^1] + [\mathbf{M}_2^2],\ \text{ when } N = 2, \quad (5.25a)$$
$$[\mathbf{M}_2(s)] := [\mathbf{M}_2],\ \text{ when } N \geq 3, \quad (5.25b)$$

$[\mathbf{M}_2^1]$, $[\mathbf{M}_2^2]$ and $[\mathbf{M}_2]$ being three constant matrices defined equivalently by (5.3) or (1.18). Furthermore, one introduces:

$$\bar{\boldsymbol{\xi}}_1 := e^{\frac{s}{2}}(\boldsymbol{\xi} - \mathbf{M}_1\,\theta_1) - [\mathbf{M}_2(s)]\,\boldsymbol{\theta}_2, \quad \bar{\boldsymbol{\zeta}}_1 := \bar{\boldsymbol{\xi}}_1, \quad \text{on } \partial B_s, \quad (5.26a)$$

what reads also, according to the rules of notation of section (1.1):

$$\bar{\xi}_1 := e^{\frac{s}{2}}(\xi - M_1\,\theta_1) - \mathbf{M}_2(s)\cdot\boldsymbol{\theta}_2, \quad \bar{\zeta}_1 := \bar{\xi}_1, \quad \text{on } \partial B_s. \tag{5.26b}$$

Remark that $\bar{\boldsymbol{\zeta}}_1$, because of the contribution of $\boldsymbol{\theta}_2$, is non-constant along the boundary $\partial\Omega_s$ whenever $[\mathbf{M}_2(s)] \neq [\mathbf{0}]$. The function $\bar{\boldsymbol{\xi}}_1(\mathbf{y}, s)$ solves for all $s > 0$ and all $\mathbf{y} \in \Omega_s$:

$$\bar{\boldsymbol{\xi}}_{1,s} + L_s\bar{\boldsymbol{\xi}}_1 - \frac{N+1}{2}\bar{\boldsymbol{\xi}}_1 - e^{-s\frac{N-2}{2}}\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi} = \begin{cases} -[\mathbf{M}_2^1]\,\boldsymbol{\theta}_2, & \text{when } N = 2,\\ \mathbf{0}, & \text{when } N \geq 3.\end{cases}$$

Multiplying componentwise by $\bar{\xi}_1$ in $L^2(K, \Omega_s)$ and according to the rules of notations of section 1.1, it comes:

$$(\bar{\xi}_{1,s}, \bar{\xi}_1)_s + (L_s\bar{\xi}_1, \bar{\xi}_1)_s - \frac{N+1}{2}\|\bar{\xi}_1\|_s^2 - e^{-s\frac{N-2}{2}}(\boldsymbol{\zeta}\cdot\nabla\xi, \bar{\xi}_1)_s = \begin{cases} -\displaystyle\sum_{i=1}^{N} M_{2,i}^1(\theta_{2,i}, \bar{\xi}_1)_s, & \text{when } N = 2,\\ 0, & \text{when } N \geq 3.\end{cases} \tag{5.27}$$

As we did for $\bar{\xi}$ in the proof of Proposition 3.2, we are going to show that $\|\bar{\xi}_1\|_s$ solves an ordinary differential inequality and then apply a Gronwall-type inequality.

For the first term in (5.27), we have the estimate below:

Lemma 5.2 *Let $\bar{\xi}_1$ be defined by* (5.26)*. Then there exists $C > 0$ such that*

$$\frac{1}{2}\frac{d}{ds}\|\bar{\xi}_1\|_s^2 \leq (\bar{\xi}_{1,s}, \bar{\xi}_1)_s + C\,\eta_N^2\,e^{-s\frac{N-2}{2} - s\frac{2}{N+2}}. \tag{5.28}$$

Proof : According to Lemma 3.1, we can write:

$$(\bar{\xi}_{1,s}, \bar{\xi}_1)_s = \frac{1}{2}\frac{d}{ds}\|\bar{\xi}_1\|_s^2 - \frac{e^{-\frac{s}{2}}}{4}\int_{\partial\Omega_s}\bar{\zeta}_1^2\,\chi\,d\sigma_y. \tag{5.29}$$

Since $\xi|_{\partial\Omega_s} = \zeta$ and $\theta_1|_{\partial\Omega_s} = (4\pi)^{-\frac{N}{2}}\chi^{-1}$ are constant and $\theta_{2,i} = -y_i\,\theta_1/2$, we have:

$$\int_{\partial\Omega_s}\zeta\,\theta_{2,i}\,\chi\,d\sigma_y = -\int_{\partial\Omega_s}\frac{y_i}{2}\,\zeta\,\theta_1\,\chi\,d\sigma_y = 0,$$

and

$$\int_{\partial\Omega_s}\theta_1\,\theta_{2,i}\,\chi\,d\sigma_y = -\int_{\partial\Omega_s}\frac{y_i}{2}\theta_1^2\,\chi\,d\sigma_y = 0,$$

for all $i = 1, \ldots, N$. We have also:

$$\int_{\partial\Omega_s}\theta_{2,i}\,\theta_{2,j}\,\chi\,d\sigma_y = \frac{1}{4}\int_{\partial\Omega_s}y_i\,y_j\,\theta_1^2\,\chi\,d\sigma_y = 0,$$

for $i \neq j$. Therefore, since θ_1 is equal to $(4\pi)^{-\frac{N}{2}}\,\chi^{-1}$ on $\partial\Omega_s$ and $\theta_{2,i}$ to $-\frac{y_i}{2}\,(4\pi)^{-\frac{N}{2}}\,\chi^{-1}$, the following equality holds:

$$\int_{\partial\Omega_s} \bar{\zeta}_1^2\,\chi\,d\sigma_y = e^s\,\left(\zeta - M_1\,(4\pi)^{-\frac{N}{2}}\,\chi^{-1}\right)^2 \chi\,\sigma_N\,e^{-s\frac{N-1}{2}} + (4\pi)^{-N}\,\chi^{-1}\,e^{-s\frac{N+1}{2}}\,\sigma_{N,1}\,|\mathbf{M}_2(s)|^2, \quad (5.30)$$

where

$$\sigma_{N,1} := \int_{\partial B} x_1^2\,d\sigma_x = \int_{\partial B} x_i^2\,d\sigma_x,$$

for any $i = 1,\ldots,N$ due to the symmetry of the ball. Direct computations provide $\chi^{-1}(s) = 1 + C(s)\,e^{-s}$. The first term in the right hand side of (5.30) satisfies:

$$\left| e^s\,\left(\zeta - M_1\,(4\pi)^{-\frac{N}{2}}\,\chi^{-1}\right)^2 \chi\,\sigma_N\,e^{-s\frac{N-1}{2}}\right| \leq C\,\left(\zeta - M_1\,(4\pi)^{-\frac{N}{2}}\right)^2\,e^{-s\frac{N-3}{2}} + C\,e^{-s\frac{N+1}{2}},$$

which, combined with the estimate (5.20), yields:

$$\left| e^s\,(\zeta - M_1\,(4\pi)^{-\frac{N}{2}}\,\chi^{-1})^2 \chi\,\sigma_N\,e^{-s\frac{N-1}{2}}\right| \leq C\,\eta_N^2\,e^{-s\frac{2}{N+2}}\,e^{-s\frac{N-3}{2}}, \quad (5.31)$$

because $\frac{N+1}{2} \geq \frac{N-3}{2} + \frac{2}{N+2}$. The second term in (5.30) can be estimated as follows:

$$\left|(4\pi)^{-N}\,\chi^{-1}\,e^{-s\frac{N+1}{2}}\,\sigma_{N,1}\,|\mathbf{M}_2(s)|^2\right| \leq C\,e^{-s\frac{N+1}{2}} \begin{cases} s^2, & \text{when } N = 2, \\ 0, & \text{when } N \geq 3. \end{cases} \quad (5.32)$$

Relations (5.29) and (5.30) yield (5.28) with estimates (5.31) and (5.32). ■

Let us now address the second term of (5.27). Integrating by parts, we get:

$$(L_s\bar{\xi}_1, \bar{\xi}_1)_s = \|\nabla\bar{\xi}_1\|_s^2 - \int_{\partial\Omega_s} \frac{\partial\bar{\xi}_1}{\partial\mathbf{n}}\,\bar{\xi}_1\,\chi\,d\sigma_y. \quad (5.33)$$

This term is the one involving the most important technical difficulties. We shall treat separately the two terms on the right hand side.

Lemma 5.3 *Let $\bar{\xi}_1$ be defined by (5.26), then*

$$\|\nabla\bar{\xi}_1\|_s^2 \geq \frac{N+2}{2}\|\bar{\xi}_1\|_s^2 - C \begin{cases} (1+s)\,e^{-\frac{s}{2}}, & \textit{when } N = 2, \\ e^{-s\frac{N-2}{2} - s\frac{2}{N+2}}, & \textit{when } N \geq 3. \end{cases}$$

Proof : The proof follows the same ideas as in the proof of Lemma 3.2, but this time is technically more involved. All along the proof, we extend ξ and $\bar{\xi}_1$ on B_s by setting $\xi := \zeta$ and

$$\bar{\xi}_1 = e^{\frac{s}{2}}(\zeta - M_1\theta_1) - \mathbf{M}_2(s)\cdot\boldsymbol{\theta}_2. \quad (5.34)$$

Denoting by $\mathbb{P}$ the orthogonal projection from $L^2(K)$ onto the subspace $(E_1 \cup E_2)^\perp$, we have:

$$\bar{\xi}_1 = \frac{(\theta_1, \bar{\xi}_1)}{\|\theta_1\|^2}\theta_1 + \sum_{i=1}^{N} \frac{(\theta_{2,i}, \bar{\xi}_1)}{\|\theta_{2,i}\|^2}\theta_{2,i} + \mathbb{P}(\bar{\xi}_1) = \mathbb{P}(\bar{\xi}_1) - e^{\frac{s}{2}}\, r_1(s)\, \theta_1 - \mathbf{r}_2(s) \cdot \boldsymbol{\theta}_2,$$

where

$$r_1(s) = M_1 - \frac{(\xi, \theta_1)}{\|\theta_1\|^2} \tag{5.35a}$$

and

$$r_{2,i}(s) = M_{2,i}(s) - e^{s/2}\frac{(\xi, \theta_{2,i})}{\|\theta_{2,i}\|^2}, \quad \forall\, i = 1, \ldots, N. \tag{5.35b}$$

Therefore, we get:

$$\mathbb{P}(\bar{\xi}_1) = \bar{\xi}_1 + e^{\frac{s}{2}}\, r_1(s)\, \theta_1 + \mathbf{r}_2 \cdot \boldsymbol{\theta}_2, \tag{5.36}$$

Since the third eigenvalue of L is $\lambda_3 = \frac{N+2}{2}$ and $\mathbb{P}(\bar{\xi}_1) \in (E_1 \cup E_2)^\perp$, we have:

$$\|\nabla \mathbb{P}(\bar{\xi}_1)\|^2 \geq \frac{N+2}{2}\, \|\mathbb{P}(\bar{\xi}_1)\|^2. \tag{5.37}$$

The following relations of orthogonality in $L^2(K)$: $(\theta_1, \theta_{2,i}) = 0$, $(\mathbb{P}(\bar{\xi}_1), \theta_1) = 0$ and $(\mathbb{P}(\bar{\xi}_1), \theta_{2,i}) = 0$ for all $i = 1, \ldots, N$, together with the identities below:

$$(\nabla f, \nabla \theta_1) = \frac{N}{2}(f, \theta_1) \quad \text{and} \quad (\nabla f, \nabla \theta_{2,i}) = \frac{N+1}{2}(f, \theta_{2,i}),$$

for all $i = 1, \ldots, N$ and for all $f \in H^1(K)$, resulting from the fact that θ_1 and $\theta_{2,i}$ are eigenfunctions of L, allow us to expand and simplify (5.37):

$$\begin{aligned}
&\|\nabla \bar{\xi}_1\|^2 + e^s\, r_1^2(s) \|\nabla \theta_1\|^2 + \sum_{i=1}^{N} r_{2,i}(s)^2 \|\nabla \theta_{2,i}\|^2 + 2\, e^{\frac{s}{2}}\, r_1(s)(\nabla \bar{\xi}_1, \nabla \theta_1) \\
&\qquad + 2 \sum_{i=1}^{N} r_{2,i}(s)(\nabla \bar{\xi}_1, \nabla \theta_{2,i}) \geq \\
&\frac{N+2}{2}\Bigg[\|\bar{\xi}_1\|^2 + e^s\, r_1^2(s)\, \|\theta_1\|^2 + \sum_{i=1}^{N} r_{2,i}^2(s) \|\theta_{2,i}\|^2 + 2\, e^{\frac{s}{2}}\, r_1(s)\, (\bar{\xi}_1, \theta_1) \\
&\qquad + 2 \sum_{i=1}^{N} r_{2,i}(s)(\bar{\xi}_1, \theta_{2,i})\Bigg].
\end{aligned}$$

Since $\|\nabla\theta_1\|^2 = \frac{N}{2}\|\theta_1\|^2$ and $\|\nabla\theta_{2,i}\|^2 = \frac{N+1}{2}\|\theta_{2,i}\|^2$, we obtain that:

$$\|\nabla\bar{\xi}_1\|^2 + e^s\, r_1^2(s)\frac{N}{2}\|\theta_1\|^2 + \frac{N+1}{2}\sum_{i=1}^{N} r_{2,i}(s)^2\|\theta_{2,i}\|^2 + N\, e^{\frac{s}{2}}\, r_1(s)(\bar{\xi}_1,\theta_1)$$
$$+ (N+1)\sum_{i=1}^{N} r_{2,i}(s)(\bar{\xi}_1,\theta_{2,i}) \geq$$
$$\frac{N+2}{2}\left[\|\bar{\xi}_1\|^2 + e^s\, r_1^2(s)\,\|\theta_1\|^2 + \sum_{i=1}^{N} r_{2,i}^2(s)\|\theta_{2,i}\|^2 + 2\, e^{\frac{s}{2}}\, r_1(s)\,(\bar{\xi}_1,\theta_1)\right.$$
$$\left. +2\sum_{i=1}^{N} r_{2,i}(s)(\bar{\xi}_1,\theta_{2,i})\right],$$

that is to say:

$$\|\nabla\bar{\xi}_1\|^2 \geq \frac{N+2}{2}\|\bar{\xi}_1\|^2 + e^s\, r_1^2(s)\,\|\theta_1\|^2 + 2\, e^{\frac{s}{2}}\, r_1(s)\,(\bar{\xi}_1,\theta_1)$$
$$+ \frac{1}{2}\sum_{i=1}^{N} r_{2,i}^2(s)\,\|\theta_{2,i}\|^2 + \sum_{i=1}^{N} r_{2,i}(s)\,(\bar{\xi}_1,\theta_{2,i}).$$

Plugging the relation (5.36) of $\bar{\xi}_1$ into the relation above, we get:

$$\|\nabla\bar{\xi}_1\|^2 \geq \frac{N+2}{2}\|\bar{\xi}_1\|^2 + e^s\, r_1^2(s)\,\|\theta_1\|^2 - 2\, e^s\, r_1^2(s)\,\|\theta_1\|^2$$
$$+ \frac{1}{2}\sum_{i=1}^{N} r_{2,i}^2(s)\,\|\theta_{2,i}\|^2 - \sum_{i=1}^{N} r_{2,i}^2(s)\,\|\theta_{2,i}\|^2.$$

We simplify the inequality above as follows:

$$\|\nabla\bar{\xi}_1\|^2 \geq \frac{N+2}{2}\|\bar{\xi}_1\|^2 - e^s r_1^2(s)\|\theta_1\|^2 - \frac{1}{2}\sum_{i=1}^{N} r_{2,i}(s)^2\|\theta_{2,i}\|^2,$$

that is to say:

$$\|\nabla\bar{\xi}_1\|_s^2 - \frac{N+2}{2}\|\bar{\xi}_1\|_s^2 \geq -\|\nabla\bar{\xi}_1\|_{B_s}^2 + \frac{N+2}{2}\|\bar{\xi}_1\|_{B_s}^2 - e^s r_1^2(s)\|\theta_1\|^2$$
$$- \frac{1}{2}\sum_{i=1}^{N} r_{2,i}^2(s)\|\theta_{2,i}\|^2, \quad (5.38)$$

since $\|f\|^2 = \|f\|_s^2 + \|f\|_{B_s}^2$ for all function $f \in L^2(K)$ and all $s > 0$. It remains to estimate each term of the right hand side of (5.38).

First term: From the definition of $\bar{\xi}_1$ on B_s (see (5.26)), we deduce that $\nabla\bar{\xi}_1 = -e^{\frac{s}{2}} M_1 \nabla\theta_1 - \sum_{i=1}^N M_{2,i}(s)\nabla\theta_{2,i}$ on B_s and hence:

$$\|\nabla\bar{\xi}_1\|^2_{B_s} = e^s M_1^2 \|\nabla\theta_1\|^2_{B_s} + \sum_{i=1}^N M_{2,i}(s)^2 \|\nabla\theta_{2,i}\|^2_{B_s}. \tag{5.39}$$

Indeed, straight computations yield $\nabla\theta_1 = -\frac{\mathbf{y}}{2}\theta_1 = \boldsymbol{\theta}_2$ and $\nabla\theta_{2,i} = -\frac{\mathbf{e}_i}{2}\theta_1 + \frac{y_i}{4}\mathbf{y}\,\theta_1$, where $\mathbf{e}_i$ is the vector whose components are $e_{i,j} = \delta_{ij}$, $j = 1,\ldots,N$. Thus

$$(\nabla\theta_1, \nabla\theta_{2,i})_{B_s} = \int_{B_s} \frac{y_i}{2}\left(1 - \frac{|\mathbf{y}|^2}{2}\right)\theta_1^2\, K\, d\mathbf{y} = 0,$$

because θ_1 and K are radially symmetric functions. We also have:

$$\begin{aligned}(\nabla\theta_{2,i}, \nabla\theta_{2,j})_{B_s} =& \frac{1}{4}\int_{B_s}\left(-\mathbf{e}_i + y_i\,\frac{\mathbf{y}}{2}\right)\cdot\left(-\mathbf{e}_j + y_j\,\frac{\mathbf{y}}{2}\right)\theta_1^2\, K\, d\mathbf{y}\\ =& -\frac{1}{4}\int_{B_s} y_i\, y_j\left(1 - \frac{|\mathbf{y}|^2}{4}\right)\theta_1^2\, K\, d\mathbf{y},\end{aligned}$$

and hence $(\nabla\theta_{2,i}, \nabla\theta_{2,j})_{B_s} = 0$ when $i \neq j$.

- From $\nabla\theta_1 = -\frac{\mathbf{y}}{2}\theta_1$, we deduce that:

$$\int_{B_s} |\nabla\theta_1|^2\, K\, d\mathbf{y} = \frac{1}{4}(4\pi)^{-\frac{N}{2}}\int_{B_s}\mathbf{y}^2\,\theta_1\, d\mathbf{y}.$$

It comes:

$$\int_{B_s} |\nabla\theta_1|^2\, K\, d\mathbf{y} = \frac{1}{4}(4\pi)^{-\frac{N}{2}}\int_{B_s}\mathbf{y}^2\,\theta_1\, d\mathbf{y} = \int_{B_s} |\nabla\theta_1|^2\, K\, d\mathbf{y} \leq C\, e^{-s\frac{N+2}{2}}. \tag{5.40}$$

- In the same way, we have:

$$\|\nabla\theta_{2,i}\|^2_{B_s} \leq C e^{-s\frac{N}{2}}, \quad \forall i = 1,\ldots,N. \tag{5.41}$$

Combining (5.39), (5.40) and (5.41), we obtain:

$$\|\nabla\bar{\xi}_1\|^2_{B_s} \leq C \begin{cases}(1+s)^2\, e^{-s} & \text{when } N = 2,\\ e^{-s\frac{N}{2}} & \text{when } N \geq 3,\end{cases} \tag{5.42}$$

taking into account the definition (5.25) of $\mathbf{M}_2(s)$.

Second term: According to the definition (5.34) of $\bar{\xi}_1$ on B_s, we get

$$\|\bar{\xi}_1\|^2_{B_s} = e^s\, \|\zeta - M_1\,\theta_1\|^2_{B_s} + \sum_{i=1}^N M_{2,i}(s)^2\, \|\theta_{2,i}\|^2_{B_s}, \tag{5.43}$$

because $((\zeta - M_1\,\theta_1), \theta_{2,i})_{B_s} = 0$ for all $i = 1,\ldots,N$ and all $s > 0$.

- Remark that

$$\|\zeta - M_1\,\theta_1\|^2_{B_s} \leq 3\,\Big(\,\|\zeta - M_1\,(4\pi)^{-\frac{N}{2}}\|^2_{B_s} + \,M_1^2\,\|(4\pi)^{-\frac{N}{2}} -\,\theta_1(r_s)\|^2_{B_s} \\ +M_1^2\,\|\theta_1(r_s) - \theta_1\|^2_{B_s}\Big)\,.$$

We know, according to (5.20), that $|\zeta - M_1\,(4\pi)^{-\frac{N}{2}}| \leq C\eta_N\,e^{-s\frac{1}{2+N}}$ and hence,

$$\|\zeta - M_1\,(4\pi)^{-\frac{N}{2}}\|^2_{B_s} \leq C\,\eta_N^2\,e^{-s\frac{N}{2}-s\frac{2}{N+2}}.$$

By definition, $\theta_1(r_s) = (4\,\pi)^{-N/2}\exp(-e^{-s}/4)$ and $\exp(-e^{-s}/4) = 1 - e^{-s}/4 + o(e^{-s})$. Therefore, $|\theta_1(r_s) - (4\,\pi)^{-N/2}| \leq C\,e^{-s}$, and

$$\|(4\pi)^{-\frac{N}{2}} -\,\theta_1(r_s)\|^2_{B_s} \leq C\,e^{-s\frac{N+4}{2}}.$$

On the other hand, $\theta_1(\mathbf{y}) = 1 + O(|\mathbf{y}|^2)$ and then $|\theta_1(\mathbf{y}) - \theta_1(r_s)| \leq C\,r_s^2 \leq C\,e^{-s}$ for all $\mathbf{y} \in B_s$. Therefore:

$$\|\theta_1 - \theta_1(r_s)\|^2_{B_s} \leq C\,e^{-s\frac{N+4}{2}},$$

and finally:

$$\|\zeta - M_1\,\theta_1\|^2_{B_s} \leq C\,\eta_N^2\,e^{-s\frac{N}{2}-s\frac{2}{N+2}}, \tag{5.44}$$

because $\frac{N}{2} + \frac{2}{N+2} \leq \frac{N+4}{2}$ for all $N \geq 2$.

- Moreover $\|\theta_{2,i}\|_{B_s} \leq C\,e^{-s\frac{N+2}{2}}$.

This last estimate, together with (5.43) and (5.44) yields:

$$\|\bar{\xi}_1\|^2_{B_s} \leq C\,\eta_N^2\,e^{-s\frac{N-2}{2}-s\frac{2}{N+2}} + C\,\begin{cases}(1+s)^2\,e^{-s\frac{N+2}{2}}, & \text{when } N = 2,\\ e^{-s\frac{N+2}{2}}, & \text{when } N \geq 3,\end{cases} \tag{5.45}$$

that is to say:

$$\|\bar{\xi}_1\|^2_{B_s} \leq C\,\eta_N^2\,e^{-s\frac{N-2}{2}-s\frac{2}{N+2}}, \tag{5.46}$$

for all $N \geq 2$, because $\frac{N+2}{2} > \frac{N-2}{2} + \frac{2}{N+2}$.

Third term: Let us recall (see (5.35a)) that $r_1(s) := M_1 - \frac{(\xi,\theta_1)}{\|\theta_1\|^2}$ which, combined with (3.20), yields:

$$r_1(s) = \int_{\Omega_s} \xi(\mathbf{y}, s)\,d\mathbf{y} + m\,e^{-s\frac{N}{2}}\zeta(s) - \frac{(\xi,\theta_1)}{\|\theta_1\|^2},$$

But:

$$\frac{(\xi,\theta_1)}{\|\theta_1\|^2} = \frac{1}{(4\,\pi)^{-\frac{N}{2}}}\int_{\Omega_s} \xi\,\theta_1\,K\,d\mathbf{y} = \frac{\sigma_N}{N}\,e^{-s\frac{N}{2}}\,\zeta,$$

and hence:

$$r_1(s) = e^{-s\frac{N}{2}}\,\zeta\,(m - \frac{\sigma_N}{N}).$$

Since $|\zeta|$ is bounded:

$$e^s\, r_1^2\, \|\theta_1\|^2 \le C e^{-s(N-1)}. \tag{5.47}$$

Forth term: According to the definition (5.35b) of $r_{2,i}(s)$, and (5.3) of $M_{2,i}(s)$, we have:

- When $N = 2$:

$$\begin{aligned} r_{2,i}(s) &= M_{2,i}(s) - e^{\frac{s}{2}}\,\frac{(\xi, \theta_{2,i})}{\|\theta_{2,i}\|^2} \\ &= M_{2,i}^1\, s + M_{2,i}^2 - e^{\frac{s}{2}}\,\frac{(\xi, \theta_{2,i})}{\|\theta_{2,i}\|^2} \\ &= M_{2,i}^1\, s + \lim_{\alpha\to\infty}\left(\frac{(\xi_1, \theta_{2,i})_\alpha}{\|\theta_{2,i}\|^2} - \alpha\, M_{2,i}^1\right) - e^{\frac{s}{2}}\,\frac{(\xi, \theta_{2,i})}{\|\theta_{2,i}\|^2}. \end{aligned} \tag{5.48}$$

By definition $\xi_1 = e^{\frac{s}{2}}(\xi - M_1\,\theta_1)$, hence

$$(\xi_1, \theta_{2,i})_s = e^{\frac{s}{2}}(\xi, \theta_{2,i})_s - M_1\, e^{\frac{s}{2}}\,(\theta_1, \theta_{2,i})_s = e^{\frac{s}{2}}(\xi, \theta_{2,i})_s, \tag{5.49}$$

because $(\theta_1, \theta_{2,i})_s = 0$ for all $s \ge 0$. On the other hand, since $\xi = \zeta$ on B_s, we can rewrite (5.48) as follows:

$$\begin{aligned} r_{2,i}(s) = \Bigg[\lim_{\alpha\to\infty}\left(e^{\alpha/2}\frac{(\xi, \theta_{2,i})_\alpha}{\|\theta_{2,i}\|^2} - \alpha\, M_{2,i}^1\right) - \left(e^{\frac{s}{2}}\frac{(\xi, \theta_{2,i})_s}{\|\theta_{2,i}\|^2} - s\, M_{2,i}^1\right) \\ - e^{\frac{s}{2}}\frac{(\zeta, \theta_{2,i})_{B_s}}{\|\theta_{2,i}\|^2}\Bigg]. \end{aligned} \tag{5.50}$$

The last term vanishes:

$$(\zeta, \theta_{2,i})_{B_s} = \zeta \int_{B_s} \theta_{2,i}\, K\, d\mathbf{y} = -\zeta \int_{B_s} \frac{y_i}{2}\,\theta_1\, K\, d\mathbf{y} = 0,$$

because θ_1 and K are radially symmetric. Identity (5.50) reads equivalently, since $\|\theta_{2,i}\|^2 = (1/2)\,(4\pi)^{-N/2}$ for all $i = 1, \ldots, N$:

$$\begin{aligned} \mathbf{r}_2(s) = \Bigg[\lim_{\alpha\to\infty}\left(2\,(4\pi)^{N/2}\, e^{\alpha/2} \int_{\Omega_\alpha} \boldsymbol{\xi}_1\, \boldsymbol{\theta}_2^T\, d\mathbf{y} - \alpha\, \mathbf{M}_2^1\right) \\ - \left(2\,(4\pi)^{N/2}\, e^{s/2} \int_{\Omega_s} \boldsymbol{\xi}_1\, \boldsymbol{\theta}_2^T\, d\mathbf{y} - s\, \mathbf{M}_2^1\right)\Bigg]. \end{aligned}$$

According to (5.22) and (5.23), the identity above reads:

$$\mathbf{r}_2(s) = -\int_s^\infty \left(e^{\frac{\alpha}{2}} \int_{\partial\Omega_\alpha} \frac{\partial \xi}{\partial \mathbf{n}} \, \mathbf{y} \, d\sigma_y \right) d\alpha - \mathbf{M}_1 \, M_1 \, (4\pi)^{-2} \, m \, e^{-s} + M_1 \int_s^\infty e^{-\alpha \frac{1}{4}} \boldsymbol{\beta} \, d\alpha - m \, (4\pi)^{-1} \int_s^\infty e^{-\alpha \frac{5}{4}} (\mathbf{M}_1 \, \beta + M_1 \, \boldsymbol{\beta}) \, d\alpha - m \int_s^\infty e^{-\alpha \frac{3}{2}} \, \boldsymbol{\beta} \, \beta \, d\alpha.$$

On the other hand, Lemma 5.1 provides us the estimate

$$\left| \int_s^\infty \left(e^{\frac{\alpha}{2}} \int_{\partial\Omega_\alpha} \frac{\partial \xi}{\partial \mathbf{n}} \, y_i \, d\sigma_y \right) d\alpha \right| \le C \, e^{-s \frac{3N-4}{8}} = C \, e^{-\frac{s}{4}}.$$

The definition (1.18d) of $\boldsymbol{\beta}$, combined with the estimate (5.20), ensures that $\eta_N^{-1} |\boldsymbol{\beta}|$ is bounded. The other terms are then easy to estimate. The one having the weakest decay rate is $\int_s^\infty e^{-\alpha \frac{1}{4}} \boldsymbol{\beta} \, d\alpha$. Indeed, by l'Hospital's rule:

$$\left| \int_s^\infty e^{-\alpha \frac{1}{4}} \boldsymbol{\beta} \, d\alpha \right| \le C \int_s^\infty \sqrt{1+\alpha} \, e^{-\frac{\alpha}{4}} \, d\alpha \le C \, (1+s) \, e^{-\frac{s}{4}}.$$

Finally, we obtain that:

$$|r_{2,i}(s)|^2 \le C \, (1+s) \, e^{-\frac{s}{2}}. \tag{5.51a}$$

- When $N \ge 3$:
  The definitions (5.35b) of $r_{2,i}(s)$ and (5.3) of $M_{2,i}(s)$ lead to:

$$r_{2,i}(s) = M_{2,i} - e^{\frac{s}{2}} \frac{(\xi, \theta_{2,i})}{\|\theta_{2,i}\|^2} = \lim_{\alpha \to 0} \frac{(\xi_1, \theta_{2,i})_\alpha}{\|\theta_{2,i}\|^2} - e^{\frac{s}{2}} \frac{(\xi, \theta_{2,i})}{\|\theta_{2,i}\|^2}.$$

According to (5.49), the identity above reads:

$$r_{2,i}(s) = \left[\lim_{\alpha \to \infty} \left(e^{\alpha/2} \frac{(\xi, \theta_{2,i})_\alpha}{\|\theta_{2,i}\|^2} - e^{\frac{s}{2}} \frac{(\xi, \theta_{2,i})_s}{\|\theta_{2,i}\|^2} \right) - e^{\frac{s}{2}} \frac{(\zeta, \theta_{2,i})_{B_s}}{\|\theta_{2,i}\|^2} \right].$$

The last term vanishes and we can rewrite equivalently:

$$\mathbf{r}_2(s) = \lim_{\alpha \to \infty} 2 \, (4\pi)^{N/2} \left(e^{\alpha/2} \int_{\Omega_\alpha} \boldsymbol{\xi}_1 \, \boldsymbol{\theta}_2^T \, d\mathbf{y} - e^{s/2} \int_{\Omega_s} \boldsymbol{\xi}_1 \, \boldsymbol{\theta}_2^T \, d\mathbf{y} \right).$$

Taking into account (5.24), one obtains:

$$
\begin{aligned}
\mathbf{r}_2(s) = -\int_s^\infty e^{\frac{\alpha}{2}} \left(\int_{\partial\Omega_\alpha} \frac{\partial \xi}{\partial \mathbf{n}} \, \mathbf{y} \, d\sigma_y \right) d\alpha \\
+ \mathbf{M}_1 M_1 (4\pi)^{-N} \left[(4\pi)^{\frac{N}{2}} \frac{1}{N-2} e^{-s\frac{N-2}{2}} - \frac{m}{N-1} e^{-s(N-1)} \right] \\
+ M_1 \int_s^\infty e^{-\alpha\frac{N-2}{2} - \alpha\frac{1}{N+2}} \boldsymbol{\beta} \, d\alpha \\
- m \, (4\pi)^{-\frac{N}{2}} \int_s^\infty e^{-\alpha(N-1) - \alpha\frac{1}{2+N}} (\mathbf{M}_1 \beta + M_1 \boldsymbol{\beta}) \, d\alpha \\
- m \int_s^\infty e^{-\alpha(N-1) - \alpha\frac{2}{N+2}} \boldsymbol{\beta} \beta \, d\alpha.
\end{aligned}
$$

The term having the weakest decay rate in this case is $M_1 \, \mathbf{M}_1 (4\pi)^{-N/2}/(N-2) \, e^{-s(N-2)/2}$ when $N = 3$ and $\int_s^\infty e^{\frac{\alpha}{2}} \left(\int_{\partial\Omega_\alpha} \frac{\partial \xi}{\partial \mathbf{n}} \, \mathbf{y} \, d\sigma_y \right) d\alpha$ when $N \geq 4$, because $\frac{N-2}{2} + \frac{1}{N+2} \geq \frac{3N-4}{8}$, for all $N \geq 3$. According to Lemma 5.1, this yields:

$$
|r_{2,i}(s)|^2 \leq C \, e^{-s} \quad \text{when } N = 3, \tag{5.51b}
$$

$$
|r_{2,i}(s)|^2 \leq C \, e^{-s\frac{3N-4}{4}} \quad \text{when } N \geq 4. \tag{5.51c}
$$

To complete the proof of the Lemma 5.3, it suffices to plug the estimates (5.42), (5.46), (5.47) and (5.51) into (5.38). ■

We address now the second term of equality (5.33):

Lemma 5.4 *The function $\bar{\xi}_1$ in (5.26) satisfies for all $N \geq 2$:*

$$
\int_{\partial\Omega_s} \frac{\partial \bar{\xi}_1}{\partial \mathbf{n}} \bar{\xi}_1 \, \chi \, d\sigma_y \leq -\frac{1}{2} \frac{d}{ds} \left(m \, \chi \, \bar{\zeta}^2 e^{-s\frac{N-2}{2}} \right) - \frac{1}{2} \, m \, \chi \, \bar{\zeta}^2 e^{-s\frac{N-2}{2}} + C \, \eta_N^2 \, e^{-s\frac{3N-4}{8}}.
$$

The proof of this Lemma will be performed in the Appendix B, as well as the proof of the following one:

Lemma 5.5 *The following inequalities hold:*

$$
|(\boldsymbol{\zeta} \cdot \nabla \xi, \bar{\xi}_1)_s| \leq C \, e^{-\frac{s}{2}} \|\nabla \bar{\xi}_1\|_s^2 + C \, e^{-\frac{s}{2}} \|\bar{\xi}_1\|_s^2 + \begin{cases} C \, (1+s) \, e^{-\frac{s}{2}} \|\bar{\xi}_1\|_s + C \, (1+s)^{\frac{3}{2}} \, e^{-\frac{s}{4}}, & \textit{when } N = 2, \\ C \, e^{-\frac{s}{2}} \|\bar{\xi}_1\|_s + C \, e^{-s\frac{3N-4}{8}}, & \textit{when } N \geq 3. \end{cases}
$$

We dispose now of all the tools to deduce the decay rate of $\bar{\xi}_1$. We rewrite (5.27) using (5.33), and it comes:

$$(\bar{\xi}_{1,s},\bar{\xi}_1)_s + \|\nabla\bar{\xi}_1\|_s^2 - \int_{\partial\Omega_s}\frac{\partial\bar{\xi}_1}{\partial\mathbf{n}}\bar{\xi}_1\,\chi\,d\sigma_y - \frac{N+1}{2}\|\bar{\xi}_1\|_s^2$$
$$- e^{-s\frac{N-2}{2}}(\boldsymbol{\zeta}\cdot\nabla\xi,\bar{\xi}_1)_s = \begin{cases} -\sum_{i=1}^{N} M_{i,2}^1\,(\theta_{2,i},\bar{\xi}_1)_s, & \text{when } N=2,\\ 0, & \text{when } N\geq 3.\end{cases} \tag{5.52}$$

On the other hand, Lemmas 5.2 and 5.4 ensure that:

$$\frac{1}{2}\frac{d}{ds}\left(\|\bar{\xi}_1\|_s^2 + m\,\chi\,\bar{\zeta}^2\,e^{-s\frac{N-2}{2}}\right) + \frac{1}{2}\,m\,\chi\,\bar{\zeta}^2\,e^{-s\frac{N-2}{2}} \leq (\bar{\xi}_{1,s},\bar{\xi}_1)_s$$
$$- \int_{\partial\Omega_s}\frac{\partial\bar{\xi}_1}{\partial\mathbf{n}}\,\bar{\xi}_1\,\chi\,d\sigma_y + C\,\eta_N^2\,e^{-s\frac{3N-4}{8}},$$

because $\frac{3N-4}{8}\leq\frac{N-2}{2}+\frac{2}{N+2}$, for all $N\geq 2$. That yields, together with (5.52):

$$\frac{1}{2}\frac{d}{ds}\left(\|\bar{\xi}_1\|_s^2 + m\,\chi\,\bar{\zeta}^2\,e^{-s\frac{N-2}{2}}\right) + \frac{1}{2}\,m\,\chi\,\bar{\zeta}^2\,e^{-s\frac{N-2}{2}} \leq -\|\nabla\bar{\xi}_1\|_s^2$$
$$+ \frac{N+1}{2}\|\bar{\xi}_1\|_s^2 + e^{-s\frac{N-2}{2}}(\boldsymbol{\zeta}\cdot\nabla\xi,\bar{\xi}_1)_s$$
$$+ \begin{cases} -\sum_{i=1}^{N} M_{i,2}^1(\theta_{2,i},\bar{\xi}_1)_s + C\,\eta_N^2\,e^{-s\frac{3N-4}{8}}, & \text{when } N=2,\\ C\,e^{-s\frac{3N-4}{8}}, & \text{when } N\geq 3.\end{cases}$$

The expression (5.36) of $\bar{\xi}_1$ leads to $|(\theta_{2,i},\bar{\xi}_1)_s| \leq C\,|r_{2,i}| \leq C\,(1+s)^{3/2}\,e^{-s/4}$, according to (5.51). Applying Lemma 5.5, we get

$$\frac{1}{2}\frac{d}{ds}\left(\|\bar{\xi}_1\|_s^2 + m\chi(s)\bar{\zeta}^2 e^{-s\frac{N-2}{2}}\right) + \frac{1}{2}\,m\,\chi\,\bar{\zeta}^2 e^{-s\frac{N-2}{2}} \leq$$
$$-(1 - C\,e^{-s\frac{N-1}{2}})\|\nabla\bar{\xi}_1\|_s^2 + \left(\frac{N+1}{2} + C\,e^{-s\frac{N-1}{2}}\right)\|\bar{\xi}_1\|_s^2$$
$$+ C\begin{cases}(1+s)\,e^{-s\frac{N-1}{2}}\|\bar{\xi}_1\|_s + C\,(1+s)^{\frac{3}{2}}\,e^{-\frac{s}{4}}, & \text{when } N=2,\\ e^{-s\frac{N-1}{2}}\|\bar{\xi}_1\|_s + C\,e^{-s\frac{3N-4}{8}}, & \text{when } N\geq 3.\end{cases} \tag{5.53}$$

According to Lemma 5.3, we deduce that:

$$-\left(1 - C\,e^{-s\frac{N-1}{2}}\right)\|\nabla\bar{\xi}_1\|_s^2 + \left(\frac{N+1}{2} + C\,e^{-s\frac{N-1}{2}}\right)\|\bar{\xi}_1\|_s^2$$
$$\leq -\frac{1}{2}\|\bar{\xi}_1\|_s^2 + C\,e^{-s\frac{N-1}{2}}\,\|\bar{\xi}_1\|_s^2 + C\begin{cases}(1+s)^3\,e^{-s/2} & \text{when } N=2,\\ e^{-s\frac{N-2}{2}-s\frac{2}{N+2}} & \text{when } N\geq 3.\end{cases}$$

Combining this last estimate with (5.53), we get:

$$X(s)' + (1 - Ce^{-s\frac{N-1}{2}})X(s)$$
$$\leq \begin{cases} C\,(1+s)\,e^{-s\frac{N-1}{2}}\sqrt{X(s)} + C\,(1+s)^{\frac{3}{2}}\,e^{-\frac{s}{4}}, & \text{when } N=2,\\ C\,e^{-s\frac{N-1}{2}}\sqrt{X(s)} + C\,e^{-s\frac{3N-4}{8}}, & \text{when } N\geq 3, \end{cases} \tag{5.54}$$

where

$$X(s) := \|\bar{\xi}_1\|_s^2 + m\,\chi\,\bar{\zeta}^2\,e^{-s\frac{N-2}{2}}.$$

This differential inequality fits with the general form of Lemma 3.3, which yields:

$$X(s) \leq C \begin{cases} (1+s)^{\frac{5}{2}}\,e^{-\frac{s}{4}}, & \text{when } N=2,\\ (1+s)\,e^{-\frac{5s}{8}}, & \text{when } N=3,\\ (1+s)\,e^{-s}, & \text{when } N=4,\\ e^{-s}, & \text{when } N\geq 5. \end{cases}$$

To conclude the proof of Theorem 1.3, one has only to rewrite the estimates above in the classical variables $\mathbf{x}$ and t, using the formulas (3.18b).

A Existence and uniqueness of solutions

This section is devoted to prove Theorem 1.1 on the existence and uniqueness of solutions of (1.4).

A.1 The underlying semigroup

We begin by building the underlying semigroup of system (1.4). Thus, let us introduce the Hilbert space

$$\mathcal{H} := \mathbf{L}^2(\Omega)\times\mathbb{R}^N = \{V = \begin{pmatrix}\mathbf{v}\\ \mathbf{g}\end{pmatrix} \in \mathbf{L}^2(\Omega)\times\mathbb{R}^N\},$$

endowed with the scalar product

$$\langle V_1, V_2\rangle := \int_\Omega \mathbf{v}_1\cdot\mathbf{v}_2\,dx + m\mathbf{g}_1\cdot\mathbf{g}_2.$$

Let

$$D(A) := \{\begin{pmatrix}\mathbf{v}\\ \mathbf{g}\end{pmatrix} \in \mathcal{H} : \mathbf{v}\in\mathbf{H}^2(\Omega),\ \mathbf{v}|_{\partial B} = \mathbf{g}\},$$

be the domain of the operator A which is defined by:

$$AV := \begin{pmatrix} \Delta\mathbf{v}\\ -\dfrac{1}{m}\displaystyle\int_{\partial\Omega}\mathbf{n}\cdot\nabla\mathbf{v}\,d\sigma_x \end{pmatrix}.$$

The linearised (around the trivial $(\mathbf{0}, \mathbf{0})$ solution) version of system (1.4) is as follows:

$$\begin{cases} \mathbf{v}_t - \Delta \mathbf{v} = 0, & \mathbf{x} \in \Omega, \quad t > 0, \\ \mathbf{v}(\mathbf{x}, t) = \mathbf{g}(t), & \mathbf{x} \in \partial B, \quad t > 0, \\ m\mathbf{g}'(t) = -\displaystyle\int_{\partial\Omega} \mathbf{n} \cdot \nabla \mathbf{v} \, d\sigma_x, & \qquad\quad t > 0, \\ \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, & \mathbf{g}(0) = \mathbf{h}_1. \end{cases}$$

It can be rewritten in an abstract form as:

$$\begin{cases} V_t = A\,V & \text{on } \Omega \times (0, \infty), \\ V(0) = V_0, \end{cases} \tag{A.1}$$

with $V_0 = \begin{pmatrix} \mathbf{v}_0 \\ \mathbf{h}_1 \end{pmatrix}$. One can easily check that

- A is m-dissipative. Indeed

$$\langle AV, V \rangle = \int_\Omega \Delta \mathbf{v} \cdot \mathbf{v} \, d\mathbf{x} - \mathbf{g} \cdot \int_{\partial\Omega} \mathbf{n} \cdot \nabla \mathbf{v} \, d\sigma_x \tag{A.2a}$$

$$= -\int_\Omega |\nabla \mathbf{v}|^2 \, d\mathbf{x} \leq 0, \quad \forall\, V \in D(A). \tag{A.2b}$$

- For any $F \in \mathcal{H}$, the equation

$$-AV + V = F, \tag{A.3}$$

  admits a unique solution in $D(A)$. To prove this, we classically set

$$\mathcal{V} := \{V = \begin{pmatrix} \mathbf{v} \\ \mathbf{g} \end{pmatrix} \in \mathcal{H}, \text{ such that } \mathbf{v} \in \mathbf{H}^1(\Omega), \mathbf{v}|_{\partial B} = \mathbf{g}\}.$$

  The weak form of (A.3) is

$$\begin{cases} V \in \mathcal{V}, \\ a(V, W) = \langle F, W \rangle, \quad \forall\, W \in \mathcal{V}, \end{cases} \tag{A.4}$$

  where

$$\begin{aligned} a(V, W) :=& \langle -A\,V + V, W \rangle \\ &= \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{w} \, d\mathbf{x} + \int_\Omega \mathbf{v} \cdot \mathbf{w} \, d\mathbf{x} + m\,\mathbf{g} \cdot \mathbf{k}, \end{aligned}$$

  is a bilinear, continuous, coercive form on $\mathcal{V} \times \mathcal{V}$, and $W \mapsto \langle F, W \rangle \in \mathbb{R}$ is a linear continuous operator on $\mathcal{V}$. Lax-Milgram's Theorem ensures the existence of a unique solution in $\mathcal{V}$ of the equation (A.4). Classical considerations of regularity for elliptic equations allow us to prove that the solution is in fact in $D(A)$.

- A is self-adjoint.

Therefore, A is the generator of an analytic semigroup of contractions $S(t) : \mathcal{H} \mapsto \mathcal{H}$. This means, in particular, that for any $V_0 := (\mathbf{v}_0, \mathbf{g}_0) \in \mathcal{H}$, there exists a unique solution $V(t) = S(t)V_0$ in $C([0,\infty), \mathcal{H})$ of equation (A.1). This implies also some basic decay estimates, namely:

$$\begin{aligned} |\langle AS(t)V_0, S(t)V_0\rangle| &\leq \frac{1}{t}\|V_0\|_{\mathcal{H}}^2, \quad \forall V_0 \in \mathcal{H}, \\ \int_0^\infty |\langle AS(s)V_0, S(s)V_0\rangle| ds &\leq \frac{1}{2}\|V_0\|_{\mathcal{H}}^2, \quad \forall V_0 \in \mathcal{H}. \end{aligned} \tag{A.5}$$

A.2 Proof of Theorem 1.1

We deal now with the non-linear term. Fix $T > 0$ and $V_0 \in \mathcal{H}$ and let us introduce the space

$$X := \{V \in \mathcal{C}([0,T], \mathcal{H}), \langle AV, V\rangle \in L^1(0,T)\} = \mathcal{C}([0,T], \mathcal{H}) \cap L^2([0,T], \mathcal{V}),$$

endowed with the norm

$$\|V\|_X := \|V\|_{L^\infty(0,T,\mathcal{H})} + \left(\int_0^T |\langle AV, V\rangle| ds\right)^{\frac{1}{2}}.$$

Then consider the mapping

$$[\phi](V)(t) := S(t)V_0 + \int_0^t S(t-s)\mathcal{N}(V(s))ds, \quad 0 \leq t \leq T,$$

where

$$\mathcal{N}(V(s)) := \begin{pmatrix} \mathbf{g} \cdot \nabla \mathbf{v} \\ 0 \end{pmatrix}.$$

To shorten notations, we shall write

$$W(t) := \int_0^t S(t-s)\mathcal{N}(V(s))ds.$$

Denoting $R := \|V_0\|_{\mathcal{H}}$ and $B := B(0, 3R) \subset X$, we are going to prove that ϕ is a strict contraction from B to B provided T is small enough. We shall then apply the Banach fixed point Theorem. As a consequence, we will obtain existence and uniqueness of a local in time solution of system (1.4).

Fix V in B. We have,

$$\|\phi(V)(t)\|_{\mathcal{H}} \leq \|V_0\|_{\mathcal{H}} + \|W(t)\|_{\mathcal{H}}, \quad 0 \leq t \leq T,$$

and also ,

$$\|W(t)\|_{\mathcal{H}} \leq \int_0^T \|\mathcal{N}(V(s))\|_{\mathcal{H}} ds \leq T^{\frac{1}{2}} \left(\int_0^T \|\mathcal{N}(V(s))\|_{\mathcal{H}}^2 ds \right)^{\frac{1}{2}} \tag{A.6a}$$

$$\leq T^{\frac{1}{2}} \|\mathbf{g}\|_{L^\infty(0,T)} \left(\int_0^T |\langle AV(s), V(s) \rangle| ds \right)^{\frac{1}{2}}, \quad 0 \leq t \leq T, \tag{A.6b}$$

because of the expression (A.2) of $|\langle AV, V \rangle|$. On the other hand, according to relation (A.5), in order to prove that $\phi(B) \subset B$, we only need to estimate

$$\int_0^T |\langle AW(t), W(t) \rangle| dt.$$

By Duhamel's principle, W solves the following abstract system:

$$\begin{cases} W_t(t) = AW(t) + \mathcal{N}(V(t)), \quad 0 \leq t \leq T, \\ W(0) = 0. \end{cases}$$

Multiplying in $\mathcal{H}$ this equation by W, we obtain

$$\frac{1}{2} \frac{d}{dt} \|W\|_{\mathcal{H}}^2 = \langle AW(t), W(t) \rangle + \langle \mathcal{N}(V(t)), W(t) \rangle, \quad 0 < t \leq T,$$

so that, integrating from 0 to T, it comes:

$$\frac{1}{2} \|W(T)\|_{\mathcal{H}}^2 - \int_0^T \langle AW(s), W(s) \rangle \, ds = \int_0^T \langle \mathcal{N}(V(s)), W(s) \rangle \, ds.$$

As pointed it out in (A.2), $\langle AW(s), W(s) \rangle \leq 0$ for all $s > 0$ and so we get the inequality:

$$\begin{aligned} \int_0^T |\langle AW(s), W(s) \rangle| ds &\leq \int_0^T \langle \mathcal{N}(V(s)), W(s) \rangle \, ds \\ &\leq \left(\int_0^T \|\mathcal{N}(V(s))\|_{\mathcal{H}}^2 ds \right)^{\frac{1}{2}} \left(\int_0^T \|W(s)\|_{\mathcal{H}}^2 \, ds \right)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\int_0^T |\langle AW(s), W(s) \rangle| \, ds \leq T^{\frac{1}{2}} \|W\|_{L^\infty(0,T,\mathcal{H})} \left(\int_0^T \|\mathcal{N}(V(s))\|_{\mathcal{H}}^2 ds \right)^{\frac{1}{2}}. \tag{A.7}$$

The relations (A.6) provide estimates for both $\|W\|_{L^\infty(0,T,\mathcal{H})}$ and $\left(\int_0^T \|\mathcal{N}(V(s))\|_{\mathcal{H}}^2 ds \right)^{\frac{1}{2}}$. We obtain:

$$\int_0^T |\langle AW(s), W(s) \rangle| \, ds \leq T \|\mathbf{g}\|_{L^\infty(0,T)}^2 \left(\int_0^T |\langle AV(s), V(s) \rangle| ds \right). \tag{A.8}$$

Combining now the relations (A.5), (A.6) and (A.8), we obtain

$$\|\phi(V)\|_X \leq (1+2^{-\frac{1}{2}})\|V_0\|_{\mathcal{H}} + 2\,T^{\frac{1}{2}}\|\mathbf{g}\|_{L^\infty(0,T)} \left(\int_0^T |\langle AV(s), V(s)\rangle| ds\right)^{\frac{1}{2}},$$

that is to say, since $V \in B$,

$$\|\phi(V)\|_X \leq (1+2^{-\frac{1}{2}})R + 18T^{\frac{1}{2}}R^2.$$

Obviously, $(1+2^{-\frac{1}{2}})R+18T^{\frac{1}{2}}R^2 < 3R$ if T is small enough. Therefore, it follows that $\phi(B) \subset B$, whenever T is small with respect to R.

Let us now consider V_1 and V_2 in B and estimate the difference

$$\phi(V_1)(t) - \phi(V_2)(t) = \int_0^t S(t-s)[\mathcal{N}(V_1(s)) - \mathcal{N}(V_2(s))]\, ds, \quad 0 \leq t \leq T.$$

Taking the norm of this quantity in $\mathcal{H}$, we obtain the inequality

$$\|\phi(V_1)(t) - \phi(V_2)(t)\|_{\mathcal{H}} \leq \int_0^T \|\mathcal{N}(V_1(s)) - \mathcal{N}(V_2(s))\|_{\mathcal{H}} ds,$$

and by Jensen's inequality:

$$\|\phi(V_1)(t) - \phi(V_2)(t)\|_{\mathcal{H}} \leq T^{\frac{1}{2}} \left(\int_0^T \|\mathcal{N}(V_1(s)) - \mathcal{N}(V_2(s))\|_{\mathcal{H}}^2 ds\right)^{\frac{1}{2}}. \qquad \text{(A.9)}$$

But observe that:

$$\begin{aligned}\|\mathcal{N}(V_1(t)) - \mathcal{N}(V_2(t))\|_{\mathcal{H}}^2 &= \int_\Omega (\mathbf{g}_1 \cdot \nabla v_1 - \mathbf{g}_2 \cdot \nabla v_2)^2 d\mathbf{x} \\ &= \int_\Omega [(\mathbf{g}_1 - \mathbf{g}_2)\cdot \nabla v_1 + \mathbf{g}_2 \cdot (\nabla v_1 - \nabla v_2)]^2 d\mathbf{x},\end{aligned}$$

and hence

$$\begin{aligned}\|\mathcal{N}(V_1(t)) - \mathcal{N}(V_2(t))\|_{\mathcal{H}}^2 &\leq 2\int_\Omega |\mathbf{g}_1 - \mathbf{g}_2|^2 |\nabla v_1|^2 d\mathbf{x} + 2\int_\Omega |\mathbf{g}_2|^2 |\nabla v_1 - \nabla v_2|^2 d\mathbf{x} \\ &\leq 2\|\mathbf{g}_1 - \mathbf{g}_2\|_{L^\infty(0,T)}^2 \int_\Omega |\nabla v_1|^2 d\mathbf{x} + 2\|\mathbf{g}_2\|_{L^\infty(0,T)}^2 \int_\Omega |\nabla v_1 - \nabla v_2|^2 d\mathbf{x}.\end{aligned}$$

We deduce that

$$\begin{aligned}&\left(\int_0^T \|\mathcal{N}(V_1(s)) - \mathcal{N}(V_2(s))\|_{\mathcal{H}}^2 ds\right)^{\frac{1}{2}} \\ &\qquad \leq \sqrt{2}\|\mathbf{g}_1 - \mathbf{g}_2\|_{L^\infty(0,T)} \left(\int_0^T |\langle AV_1(s), V_1(s)\rangle|\, ds\right)^{\frac{1}{2}} \\ &\qquad + \sqrt{2}\|\mathbf{g}_2\|_{L^\infty(0,T)} \left(\int_0^T |\langle A(V_1(s) - V_2(s)), V_1(s) - V_2(s)\rangle|\, ds\right)^{\frac{1}{2}},\end{aligned}$$

that is to say:

$$\left(\int_0^T \|\mathcal{N}(V_1(s)) - \mathcal{N}(V_2(s))\|_{\mathcal{H}}^2 ds\right)^{\frac{1}{2}} \\ \leq \sqrt{2}\|V_1 - V_2\|_X \|V_1\|_X + \sqrt{2}\|V_2\|_X \|V_1 - V_2\|_X.$$

Since V_1 and V_2 are in $B(0,3R)$, we obtain that:

$$\left(\int_0^T \|\mathcal{N}(V_1(s)) - \mathcal{N}(V_2(s))\|_{\mathcal{H}}^2 ds\right)^{\frac{1}{2}} \leq 6R\sqrt{2}\|V_1 - V_2\|_X. \tag{A.10}$$

This relation, together with (A.9) provides:

$$\|\phi(V_1)(t) - \phi(V_2)(t)\|_{\mathcal{H}} \leq 6R\sqrt{2}T^{\frac{1}{2}}\|V_1 - V_2\|_X, \quad \forall\, 0 \leq t \leq T. \tag{A.11}$$

Let $\widetilde{W}(t)$ be defined by $\widetilde{W}(t) := \phi(V_1)(t) - \phi(V_2)(t)$. So $\widetilde{W}(t)$ solves:

$$\begin{cases} \widetilde{W}_t(t) = A\widetilde{W}(t) + \mathcal{N}(V_1)(t) - \mathcal{N}(V_2)(t), & 0 \leq t \leq T, \\ \widetilde{W}(0) = 0. \end{cases}$$

The same arguments we used to prove (A.7) yield here:

$$\int_0^T |\langle A\widetilde{W}(s), \widetilde{W}(s)\rangle|\, ds \\ \leq T^{\frac{1}{2}}\|\widetilde{W}\|_{L^\infty(0,T,\mathcal{H})}\left(\int_0^T \|\mathcal{N}(V_1)(s) - \mathcal{N}(V_2)(s)\|_{\mathcal{H}}^2\, ds\right)^{\frac{1}{2}},$$

and with (A.10) and (A.11), it comes:

$$\left(\int_0^T |\langle A\widetilde{W}(s), \widetilde{W}(s)\rangle|\, ds\right)^{\frac{1}{2}} \leq 6R\sqrt{2}T^{\frac{1}{2}}\|V_1 - V_2\|_X. \tag{A.12}$$

Combining (A.11) and (A.12), we conclude that:

$$\|\phi(V_1) - \phi(V_2)\|_X \leq 12R\sqrt{2}T^{\frac{1}{2}}\|V_1 - V_2\|_X.$$

Thus, assuming T is small enough with respect to R, ϕ is a strict contraction from B to B.

Using the Banach fixed point Theorem, we prove local in time existence and uniqueness of a solution on $(0,T)$.

This solution V can be continued up to a maximal time T^*, $0 < T^* \leq \infty$ and the following alternative holds: Either $T^* < \infty$ and $\|V(t)\|_X \to \infty$ as $t \to T^*$ (finite time blow up) or $T^* = \infty$.

The energy estimate (2.1) ensures that $T^* = \infty$ and the solution is global in time. This completes the proof of Theorem 1.1.

B Proofs of Lemmas

Proof of Lemma 2.2: We distinguish the cases $N = 2$ and $N \geq 3$.

The case $N = 2$*:*
Applying the inequality:

$$\int_\Omega |u|^4 d\mathbf{x} \leq C \left(\int_\Omega |u|^2 d\mathbf{x} \right) \left(\int_\Omega |\nabla u|^2 d\mathbf{x} \right), \tag{B.1}$$

with $|v|^{\frac{p}{2}}$ instead of u, we get:

$$\int_\Omega |v|^{2p} d\mathbf{x} \leq C \left(\int_\Omega |v|^p d\mathbf{x} \right) \left(\int_\Omega (\nabla |v|^{\frac{p}{2}})^2 d\mathbf{x} \right). \tag{B.2}$$

On the other hand, Hölder's inequality provides:

$$\int_\Omega |v|^p \, d\mathbf{x} \leq \left(\int_\Omega |v|^{q\,(p-1)+1} \, d\mathbf{x} \right)^{1/q} \left(\int_\Omega |v| \, d\mathbf{x} \right)^{(q-1)/q}, \quad \forall\, q \geq 1. \tag{B.3}$$

Choosing q such that $q\,(p-1) + 1 = 2\,p$, i.e. $q = 1 + p/(p-1)$, inequality (B.3) yields:

$$\int_\Omega |v|^p \, d\mathbf{x} \leq \left(\int_\Omega |v|^{2\,p} \, d\mathbf{x} \right)^{1/q} \left(\int_\Omega |v| \, d\mathbf{x} \right)^{(q-1)/q},$$

that is to say:

$$\frac{\|v\|_p^p}{\|v\|_1^{1-1/q}} \leq \left(\int_\Omega |v|^{2\,p} \, d\mathbf{x} \right)^{\frac{1}{q}}.$$

Combining this relation with (B.2), it comes:

$$\frac{\|v\|_p^{\frac{p^2}{p-1}}}{\|v\|_1^{\frac{p}{p-1}}} \leq C \int_\Omega (\nabla |v|^{\frac{p}{2}})^2 \, d\mathbf{x},$$

and this yields (2.20).

The case $N \geq 3$*:* We apply Sobolev's inequality below:

$$\|u\|_{2^*} \leq C \|\nabla u\|_2, \quad \text{where } \frac{1}{2^*} = \frac{1}{2} - \frac{1}{N},$$

with $2^* = 2N/(N-2)$ and $u = |v|^{\frac{p}{2}}$. It comes:

$$\|v\|^p_{Np/(N-2)} \leq C \|\nabla(|v|^{\frac{p}{2}})\|_2^2. \tag{B.4}$$

On the other hand, choosing $q = \frac{N(p-1)+2}{(N-2)(p-1)}$ in Hölder's inequality (B.3), we have $q\,(p-1)+1 = N\,p/(N-2)$ and (B.3) reads:

$$\|v\|_p^p \leq \left(\int_\Omega |v|^{\frac{N\,p}{N-2}}\,d\mathbf{x}\right)^{1-\frac{2\,p}{N(p-1)+2}} \left(\int_\Omega |v|\,d\mathbf{x}\right)^{\frac{2\,p}{N(p-1)+2}}.$$

We rewrite the above relation as:

$$\frac{\|v\|_p^{p\left(1+\frac{2}{N(p-1)}\right)}}{\|v\|_1^{\frac{2\,p}{N(p-1)}}} \leq C\,\|v\|_{Np/(N-2)}^p.$$

Combining this last relation with (B.4), we finally obtain:

$$\|v\|_p^{p(1+\frac{2}{N(p-1)})} \leq C\|\nabla(|v|^{\frac{p}{2}})\|_2^2\|v\|_1^{\frac{2p}{N(p-1)}}.$$

Thus

$$\frac{\|v\|_p^{p(1+\frac{2}{N(p-1)})}}{[\|v\|_1+m|g|]^{\frac{2p}{N(p-1)}}} \leq \frac{\|v\|_p^{p(1+\frac{2}{N(p-1)})}}{\|v\|_1^{\frac{2p}{N(p-1)}}} \leq C\|\nabla|v|^{\frac{p}{2}}\|_2^2,$$

and the proof of (2.20) is complete. ■

Proof of Lemma 2.3: Let $(\sigma_x, r) \in \partial\Omega \times \mathbb{R}_+$ denote the spherical coordinates. For any point $\sigma_x \in \partial\Omega$ and any smooth function $u(\sigma_x, r)$, compactly supported:

$$\begin{aligned} u^2(\sigma_x, 1) &= -2\int_1^\infty u(\sigma_x, r)\,\frac{\partial u}{\partial r}(\sigma_x, r)\,dr \\ &\leq 2\int_1^\infty |u(\sigma_x, r)|\,\left|\frac{\partial u}{\partial r}(\sigma_x, r)\right|\,r^{N-1}\,dr. \end{aligned}$$

Integrating on $\partial\Omega$, we get:

$$\int_{\partial\Omega} |u|^2\,d\sigma_x \leq C\int_{\partial\Omega}\int_1^\infty |\nabla u|\,|u|\,r^{N-1}\,dr\,d\sigma_x \leq C\,\|\nabla u\|_2\,\|u\|_2. \tag{B.5}$$

Specifying u to be $|v|^{\frac{p}{2}}$, we obtain

$$m|g|^p \leq C\|\nabla|v|^{\frac{p}{2}}\|_2\|v\|_p^{\frac{p}{2}},$$

where C depends on N and m only. On the other hand, taking (2.20) into account, we deduce that:

$$\|v\|_p^{\frac{p}{2}} \leq C\|\nabla|v|^{\frac{p}{2}}\|_2^{\frac{N(p-1)}{N(p-1)+2}}[\|v\|_1+m|g|]^{\frac{p}{N(p-1)+2}},$$

so that,

$$m|g|^p \leq C\|\nabla |v|^{\frac{p}{2}}\|_2^{1+\frac{N(p-1)}{N(p-1)+2}} [\|v\|_1 + m|g|]^{\frac{p}{N(p-1)+2}},$$

and hence

$$\frac{(m|g|^p)^{1+\frac{1}{N(p-1)+1}}}{[\|v\|_1 + m|g|]^{\frac{p}{N(p-1)+1}}} \leq C^{1+\frac{1}{N(p-1)+1}} \|\nabla |v|^{\frac{p}{2}}\|_2^2, \quad \forall t \geq 0. \tag{B.6}$$

Up to now the constant C has not been depending on p, therefore $C^{1+\frac{1}{N(p-1)+1}}$ remains also uniformly bounded with respect to p.
We now define the exponent q as follows:

$$\frac{1}{N(p-1)+1} + q = \frac{2}{N(p-1)} \quad \Leftrightarrow \quad q = \frac{N(p-1)+2}{(N(p-1)+1)N(p-1)}.$$

From (B.6), it follows that:

$$\frac{(m|g|^p)^{1+\frac{2}{N(p-1)}}}{[\|v\|_1 + m|g|]^{\frac{p}{N(p-1)+1}} (m|g|^p)^q} \leq C\|\nabla |v|^{\frac{p}{2}}\|_2^2. \tag{B.7}$$

Moreover:

$$[\|v\|_1 + m|g|]^{\frac{p}{N(p-1)+1}} (m|g|^p)^q \leq [\|v\|_1 + m|g|]^{\frac{2p}{N(p-1)}}.$$

Therefore, equality (B.7), becomes

$$\frac{(m|g|^p)^{1+\frac{2}{N(p-1)}}}{[\|v\|_1 + m|g|]^{\frac{2p}{N(p-1)}}} \leq C\|\nabla |v|^{\frac{p}{2}}\|_2^2, \quad \forall t \geq 0,$$

and the proof is completed. ■

Proof of Lemma 2.4: Let z be any positive solution of:

$$z' + C_1\, z^{1+\gamma} - C_2\, z \leq 0.$$

Multiplying both sides of the inequality by $z^{-1-\gamma}$, we get:

$$z'\, z^{-1-\gamma} - C_2\, z^{-\gamma} + C_1 \leq 0 \quad \Leftrightarrow \quad -\frac{1}{\gamma}\left(z^{-\gamma}\right)' - C_2\, z^{-\gamma} + C_1 \leq 0.$$

This relation reads:

$$\frac{d}{dt}\left[e^{\gamma\, C_2\, t} z^{-\gamma}\right] \geq e^{\gamma\, C_2\, t} \gamma\, C_1 = \frac{d}{dt}\left[\frac{C_1}{C_2}\, e^{\gamma\, C_2\, t}\right].$$

Integrating from 0 to $T > 0$, we get:

$$e^{\gamma C_2 T} z(T)^{-\gamma} \geq z(0)^{-\gamma} + \frac{C_1}{C_2}\left(e^{\gamma C_2 T} - 1\right) \geq \frac{C_1}{C_2}\left(e^{\gamma C_2 T} - 1\right),$$

since $z(0)$ is assumed to be positive. This implies:

$$z(T) \leq \left(\frac{C_1}{C_2}\right)^{-\frac{1}{\gamma}} \left(1 - e^{-\gamma C_2 T}\right)^{-\frac{1}{\gamma}},$$

and the proof is completed. ■

Proof of Lemma 3.1: This formula is linked with a classical result, the Reynolds formula, in fluids mechanic (see for example [1, Lemme 1, page 69]). Since, in our case, the domain of integration but also f are time dependent, we give the details of the proof. Let $s \mapsto \boldsymbol{\phi}(\mathbf{y}, s)$ be any $C^1((0,\infty), C^1(\mathbb{R}^N, \mathbb{R}^N))$ mapping such that $\boldsymbol{\phi}(\cdot, s_0)$ is the identity and let ω be a C^1 open subset of $\mathbb{R}^N$. Then,

$$\frac{d}{ds}\left[\int_{\omega_s} f(\mathbf{z}, s)\, d\mathbf{z}\right]\bigg|_{s=s_0} = \int_{\omega_{s_0}} f_s(\mathbf{y}, s_0) d\mathbf{y} + \int_{\partial\omega_{s_0}} f(\mathbf{y}, s_0) \boldsymbol{\phi}_s(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) d\sigma_y,$$

where $\omega_s := \boldsymbol{\phi}(\omega, s)$ and the vector $\mathbf{n}(\mathbf{y})$ is the unit normal to $\partial\omega_{s_0}$ at the point $\mathbf{y}$ directed to the exterior of ω_{s_0}. Indeed:

$$\int_{\omega_s} f(\mathbf{z}, s) d\mathbf{z} = \int_{\omega_{s_0}} f(\boldsymbol{\phi}(\mathbf{y}, s), s) |J\boldsymbol{\phi}(s)| d\mathbf{y},$$

where $|J\boldsymbol{\phi}(s)|$ is the determinant of the Jacobian matrix of $\boldsymbol{\phi}(\cdot, s)$. Calculating the derivative of the above equality with respect to s, we get:

$$\begin{aligned}\frac{d}{ds}\left[\int_{\omega_s} f(\mathbf{z}, s) d\mathbf{z}\right]\bigg|_{s=s_0} &= \int_{\omega_{s_0}} f_s(\boldsymbol{\phi}(\mathbf{y}, s_0), s_0) |J\boldsymbol{\phi}(s_0)| d\mathbf{y} \\ &+ \int_{\omega_{s_0}} \nabla f(\boldsymbol{\phi}(\mathbf{y}, s_0), s_0) \cdot \boldsymbol{\phi}_s(s_0) |J\boldsymbol{\phi}(s_0)| d\mathbf{y} \\ &+ \int_{\omega_{s_0}} f(\boldsymbol{\phi}(\mathbf{y}, s_0), s_0)\, \frac{d}{ds}|J\boldsymbol{\phi}(s)|\bigg|_{s=s_0} d\mathbf{y}. \quad \text{(B.8)}\end{aligned}$$

Since we have assumed $\boldsymbol{\phi}(\cdot, s_0)$ to be the identity, $|J\boldsymbol{\phi}(s_0)| = 1$ and we simplify (B.8) as follows:

$$\begin{aligned}\frac{d}{ds}\left[\int_{\omega_s} f(\mathbf{z}, s) d\mathbf{z}\right]\bigg|_{s=s_0} &= \int_{\omega_{s_0}} f_s(\mathbf{y}, s_0) d\mathbf{y} \\ &+ \int_{\omega_{s_0}} \left(\nabla f(\mathbf{y}, s_0) \cdot \boldsymbol{\phi}_s(s_0) + f(\mathbf{y}, s_0)\, \frac{d}{ds}|J\boldsymbol{\phi}(s)|\bigg|_{s=s_0}\right) d\mathbf{y}. \quad \text{(B.9)}\end{aligned}$$

We refer to [1, Lemme 1, page 69], for the proof of:

$$\frac{d}{ds}|J\boldsymbol{\phi}(s)|\Big|_{s=s_0} = \mathrm{div}[\boldsymbol{\phi}_s(\mathbf{y}, s_0)],$$

and, applying the Stokes formula, we get:

$$\int_{\omega_{s_0}} \left(\nabla f(\mathbf{y}, s_0) \cdot \boldsymbol{\phi}_s(s_0) + f(\mathbf{y}, s_0)\, \frac{d}{ds}|J\boldsymbol{\phi}(s)|\Big|_{s=s_0} \right) d\mathbf{y}$$
$$= \int_{\partial\omega_{s_0}} f(\mathbf{y}, s_0)\boldsymbol{\phi}_s(s_0) \cdot \mathbf{n}(\mathbf{y}) d\sigma_y. \quad \text{(B.10)}$$

When we specialise $\boldsymbol{\phi}$ to be $\boldsymbol{\phi}(\mathbf{y}, s) = \mathbf{y}e^{-(s-s_0)/2}$ and $\omega = \Omega$, we obtain:

$$\boldsymbol{\phi}_s(s_0) \cdot \mathbf{n}(\mathbf{y}) = -\frac{\mathbf{y}}{2} \cdot \mathbf{n}(\mathbf{y}) = \frac{e^{-s_0/2}}{2}, \quad \mathbf{y} \in \partial\Omega_{s_0}. \quad \text{(B.11)}$$

The proof is then completed after combining (B.9), (B.10) and (B.11). ■

Proof of Lemma 3.3: First, we define the function

$$\vartheta(s) := \exp\left(\int_0^s \left(1 - C_1\, e^{-\tau \frac{N-1}{2}}\right) d\tau \right), \quad \forall s > 0.$$

It is obvious that there exists two constants $\widetilde{C}_1$ and $\widetilde{C}_2$, and a positive bounded function $C(s)$ such that

$$\vartheta(s) = C(s)\, e^s, \quad \text{with } 0 < \widetilde{C}_1 \le C(s) \le \widetilde{C}_2, \quad \forall s > 0. \quad \text{(B.12)}$$

We set then $Y(s) := \vartheta(s)\, X(s)$, for all $s > 0$. Note that $Y(0) = 0$ according to the definition of $\vartheta(s)$. The inequality (3.56) becomes

$$Y'(s) - C_2\, e^{-s\, \frac{N-1}{2}}\, (1+s)^\gamma\, \vartheta^{\frac{1}{2}}(s)\, \sqrt{Y(s)} \le C_3\, s^\alpha\, e^{-\beta\, s}\, \vartheta(s). \quad \text{(B.13)}$$

Let us now set:

$$\mu_1(s) := C_2\, e^{-s\, \frac{N-1}{2}} \vartheta^{\frac{1}{2}}(s)\, (1+s)^\gamma \quad \text{and} \quad \mu_2(s) := C_3\, s^\alpha\, e^{-\beta\, s}\, \vartheta(s).$$

According to (B.12):

$$\mu_1(s) \le C\, e^{-s \frac{N-2}{2}}\, (1+s)^\gamma \quad \text{and} \quad \mu_2(s) \le C\, s^\alpha\, e^{s\,(1-\beta)}, \quad \forall s > 0. \quad \text{(B.14)}$$

We rewrite the equation (B.13) as:

$$Y'(s) \le \mu_1(s) \sqrt{Y(s)} + \mu_2(s),$$

and integrating from 0 to s, we get

$$Y(s) \leq Y(0) + \int_0^s \mu_2(\tau) d\tau + \int_0^s \mu_1(\tau) \sqrt{Y(\tau)} d\tau.$$

Since $Y(0) = 0$ and if $\rho(s)$ stands for

$$\rho(s) := \int_0^s \mu_2(\tau) d\tau, \tag{B.15}$$

we obtain a so-called Bihari-type inequality [15, section 1.3]:

$$Y(s) \leq \rho(s) + \int_0^s \mu_1(\tau) \sqrt{Y(\tau)} d\tau.$$

Applying [15, Theorem 1.3.2, page 19], it comes that:

$$Y(s) \leq \rho(s) + \left[\left(\int_0^s \mu_1(\tau) \sqrt{\rho(\tau)} d\tau \right)^{\frac{1}{2}} + \int_0^s \mu_1(\tau) d\tau \right]^2. \tag{B.16}$$

That is to say,

- The case $N = 2$:
  From (B.14) we deduce that:

  $$\int_0^s \mu_1(\tau)\, d\tau \leq C\, (1+s)^{\gamma+1}, \quad \forall\, s \geq 0. \tag{B.17}$$

  - When $0 < \beta < 1$: We can estimate $\rho(s)$ as follows:

    $$\rho(s) \leq C\, (1+s)^{\alpha+1}\, e^{(1-\beta)\, s}, \quad \forall\, s \geq 0, \tag{B.18}$$

    because, obviously:

    $$\int_0^s \tau^\alpha\, e^{(1-\beta)\, \tau}\, d\tau \leq C\, s^{\alpha+1} e^{(1-\beta)\, s}, \quad \forall\, s \geq 0. \tag{B.19}$$

    Combining the definition (B.15) with (B.14) and (B.19), we get (B.18). On the other hand, (B.14) and (B.18) lead to:

    $$\int_0^s \mu_1(\tau) \sqrt{\rho(\tau)}\, d\tau \leq C \int_0^s (1+\tau)^{\frac{\alpha+1}{2}+\gamma}\, e^{\frac{\tau}{2}(3-N-\beta)}\, d\tau. \tag{B.20}$$

    Since $N = 2$, and $\beta < 1$, we have $3 - N - \beta = 1 - \beta > 0$. We get:

    $$\int_0^s \mu_1(\tau)\, \sqrt{\rho(\tau)}\, d\tau \leq C\, (1+s)^{\frac{\alpha+3}{2}+\gamma} e^{(1-\beta)\frac{s}{2}}, \quad \forall\, s \geq 0. \tag{B.21}$$

Plugging (B.17), (B.18) and (B.21) into (B.16), we obtain:

$$Y(s) \leq C\,(1+s)^{1+\alpha} e^{(1-\beta)\,s} + C\left[(1+s)^{\frac{\alpha+3}{4}+\frac{\gamma}{2}} e^{(1-\beta)\frac{s}{4}} + (1+s)^{\gamma+1}\right]^2,$$

and then:

$$Y(s) \leq C\,(1+s)^{1+\alpha} e^{(1-\beta)\,s} + C\,(1+s)^{\frac{\alpha+3}{2}+\gamma} e^{(1-\beta)\frac{s}{2}} + C\,(1+s)^{2\gamma+2}. \tag{B.22}$$

Since $1-\beta > 0$, we can simplify (B.22) as:

$$Y(s) \leq C\,(1+s)^{1+\alpha} e^{(1-\beta)\,s}, \tag{B.23}$$

i.e

$$\vartheta(s)\,X(s) \leq C\,(1+s)^{1+\alpha} e^{(1-\beta)\,s},$$

what yields (3.57a), taking (B.12) into account.

- When $\beta = 1$: All the estimates of the case $0 < \beta < 1$ remain valid and (B.22) leads to (3.57b), taking (B.12) into account.
- When $\beta > 1$: relations (B.14) and (B.15) ensure that ρ is bounded. This, together with (B.14) yields:

$$\int_0^s \mu_1(\tau)\sqrt{\rho(\tau)}\,d\tau \leq C\,(1+s)^{\gamma+1} \quad \forall\, s \geq 0. \tag{B.24}$$

Then, (B.17) and (B.24) with (B.16) give:

$$Y(s) \leq C + C\left[(1+s)^{\frac{\gamma+1}{2}} + (1+s)^{\gamma+1}\right]^2,$$

i.e.

$$Y(s) \leq C\,(1+s)^{2(\gamma+1)},$$

or, equivalently, (3.57c).

The case $N \geq 3$:
In this case, in view of (B.14), it follows that:

$$\int_0^s \mu_1(\tau)\,d\tau \leq C, \quad \forall\, s \geq 0. \tag{B.25}$$

- When $0 < \beta \leq 1$, estimate (B.18) remains valid as well as (B.20). Since $N \geq 3$, we have $3 - N - \beta < 0$ in (B.20) and hence:

$$\int_0^s \mu_1(\tau)\,\sqrt{\rho(\tau)}\,d\tau \leq C, \quad \forall\, s \geq 0. \tag{B.26}$$

Plugging (B.25), (B.18) and (B.26) into (B.16), it comes:

$$Y(s) \leq C\,(1+s)^{1+\alpha} e^{(1-\beta)\,s} + C,$$

what yields (3.57d), according to (B.12).

– When $\beta > 1$, ρ is bounded. This, together with (B.14) yields:

$$\int_0^s \mu_1(\tau)\sqrt{\rho(\tau)}\,d\tau \le C, \quad \forall\, s \ge 0. \tag{B.27}$$

Plugging (B.25) and (B.27) into (B.16), it comes:

$$Y(s) \le C,$$

and also (3.57e).

The proof is then completed. ■

Proof of Lemma 4.1: We shall perform the proof using the notations of subsection A.1. We have then to prove that the solution $V(t)$ of

$$\begin{cases} V_t = AV + \mathcal{N}(V), \quad t > 0, \\ V(0) = V_0 \in \mathcal{H}, \end{cases} \tag{B.28}$$

satisfies the decay rate estimate:

$$|\langle AV(t), V(t)\rangle| \le C(1+t)^{-\frac{N}{2}-1}, \quad \forall t > 0. \tag{B.29}$$

Multiplying the equation (B.28) by $AV(t)$, it comes:

$$\langle V_t, AV\rangle = \|AV\|^2_{\mathcal{H}} + \langle \mathcal{N}(V), AV\rangle,$$

that is to say, since A is self-adjoint:

$$\frac{1}{2}\frac{d}{dt}|\langle AV, V\rangle| = -\|AV\|^2_{\mathcal{H}} - \langle \mathcal{N}(V), AV\rangle. \tag{B.30}$$

Let us recall that:

$$\mathcal{N}(V) := \begin{pmatrix} \mathbf{g}\cdot\nabla v \\ 0 \end{pmatrix} \quad \text{and} \quad \langle AV, V\rangle = -\int_\Omega |\nabla v|^2\,d\mathbf{x}.$$

Then:

$$\|\mathcal{N}(V)\|_{\mathcal{H}} \le |\mathbf{g}| \left(\int_\Omega |\nabla v|^2 d\mathbf{x}\right)^{1/2} = |\mathbf{g}||\langle AV, V\rangle|^{1/2}. \tag{B.31}$$

Applying Cauchy-Schwarz's inequality to (B.30), we get:

$$\frac{1}{2}\frac{d}{dt}|\langle AV, V\rangle| \le -\|AV\|^2_{\mathcal{H}} + \|\mathcal{N}(V)\|_{\mathcal{H}}\|AV\|_{\mathcal{H}}.$$

This last relation, together with the estimate (B.31) above yields:

$$\frac{1}{2}\frac{d}{dt}|\langle AV, V\rangle| \leq -\|AV\|_{\mathcal{H}}(\|AV\|_{\mathcal{H}} + |\mathbf{g}|_{L^\infty(0,\infty)}|\langle AV, V\rangle|^{1/2}).$$

We see that the quantity $X := |\langle AV, V\rangle|$ solves the differential inequality:

$$X' \leq -2\|AV\|_{\mathcal{H}}(\|AV\|_{\mathcal{H}} + |\mathbf{g}|\, X^{1/2}). \tag{B.32}$$

Theorem 1.1 ensures that $g \in C([0,\infty),\mathbb{R})$ and, according to the decay property of Proposition 2.2, we get:

$$|\mathbf{g}| \leq C\,(1+t)^{-\frac{N}{2}}, \quad \forall t > 0. \tag{B.33}$$

Moreover $X \leq \|AV\|_{\mathcal{H}}\|V\|_{\mathcal{H}}$. Introducing then $\mu(t) := \|AV\|_{\mathcal{H}}\|V\|_{\mathcal{H}}X^{-1}$, we have

$$\|AV\|_{\mathcal{H}} = \mu(t)\|V\|^{-1}X, \tag{B.34}$$

with:

$$\mu(t) \geq 1, \quad \forall t > 0. \tag{B.35}$$

We plug (B.33) and (B.34) into (B.32) in order to obtain:

$$X' + 2\,\mu^2\,\|V\|_{\mathcal{H}}^{-2}\,X^2 - C_1\,\mu\,\|V\|_{\mathcal{H}}^{-1}\,(1+t)^{-\frac{N}{2}}X^{\frac{3}{2}} \leq 0. \tag{B.36}$$

The task consists in proving that there exists a constant $C > 0$ such that the function

$$Y(t) := C\,(1+t)^{-\frac{N}{2}-1},$$

is a supersolution of (B.36). After simple computations we obtain:

$$\begin{aligned} Y' + 2\,\mu^2\,\|V\|^{-2}\,Y^2 - C_1\,\mu\,\|V\|_{\mathcal{H}}^{-1}\,(1+t)^{-\frac{N}{2}}\,Y^{\frac{3}{2}} \\ = C\,(1+t)^{-\frac{N}{2}-2}\left(-\frac{N}{2} - 1 + 2\,(C^{\frac{1}{2}}\,\mu\,\|V\|_{\mathcal{H}}^{-1}\,(1+t)^{-\frac{N}{4}})^2\right) \\ - C_1\,C\,(C^{\frac{1}{2}}\,\|V\|^{-1}\,(1+t)^{-\frac{N}{4}}\,\mu)\,(1+t)^{-N-\frac{3}{2}}. \end{aligned}$$

Setting $Q := C^{\frac{1}{2}}\,\mu\,\|V\|_{\mathcal{H}}^{-1}\,(1+t)^{-\frac{N}{4}}$, it is then sufficient to prove that:

$$C_1\,Q\,(1+t)^{-N-\frac{3}{2}} \leq C\,(1+t)^{-\frac{N}{2}-2}\left(-\frac{N}{2} - 1 + 2\,Q^2\right), \tag{B.37}$$

for some $C > 0$ to be determined. Inequality (B.37) can be rewritten as follows:

$$2\,Q^2 - C_1\,Q\,t^{-\frac{N}{2}+\frac{1}{2}} - \left(\frac{N}{2} + 1\right) \geq 0. \tag{B.38}$$

We proved in Theorem 1.1 that $V \in C([0,\infty),\mathcal{H})$ and according to the decay rate properties in Proposition 2.2, we know that there exists a constant $C_2 > 0$ such that $\|V\|_{\mathcal{H}} \leq C_2\,(1+t)^{-\frac{N}{4}}$ for all $t > 0$, i.e $\|V\|_{\mathcal{H}}^{-1} t^{-\frac{N}{4}} \geq C_2^{-1}$. This fact, together with (B.35) implies that:

$$Q \geq C^{\frac{1}{2}}\, C_2^{-1}. \tag{B.39}$$

In view of (B.39), it is clear that, for C large enough, (B.38) holds. This concludes the proof of Lemma 4.1 ■

Proof of Lemma 4.3: We denote v_{x_i} the derivative of the function v with respect to x_i. Integrating by parts, we get:

$$\begin{aligned}\int_\Omega |\nabla v_{x_i}|^2\, d\mathbf{x} &= -\int_\Omega v_{x_i} \Delta v_{x_i}\, d\mathbf{x} + \int_{\partial\Omega} v_{x_i} \frac{\partial v_{x_i}}{\partial \mathbf{n}}\, d\sigma_x \\ &= -\int_\Omega v_{x_i} (\Delta v)_{x_i}\, d\mathbf{x} + \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial \mathbf{n}} v_{x_i}^2\, d\sigma_x.\end{aligned}$$

Summing over i from 1 to N, we obtain that:

$$\int_\Omega |[D_2 v]|^2\, d\mathbf{x} = -\int_\Omega \nabla v \cdot \nabla \Delta v\, d\mathbf{x} + \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial \mathbf{n}} |\nabla v|^2\, d\sigma_x.$$

Since v is constant on $\partial\Omega$, so $|\nabla v|^2 = \left|\frac{\partial v}{\partial \mathbf{n}}\right|^2$ on $\partial\Omega$. Integrating by parts again, the above relation becomes:

$$\int_\Omega |[D_2 v]|^2\, d\mathbf{x} = \int_\Omega (\Delta v)^2\, d\mathbf{x} - \int_{\partial\Omega} \left[\Delta v \frac{\partial v}{\partial \mathbf{n}} - \frac{1}{2}\frac{\partial}{\partial \mathbf{n}}\left|\frac{\partial v}{\partial \mathbf{n}}\right|^2\right] d\sigma_x. \tag{B.40}$$

We denote by Δ_S the Laplace Beltrami operator on $\partial\Omega$ and $r := |\mathbf{x}|$. We can express Δv in spherical coordinates as: $\Delta v = \frac{\partial^2 v}{\partial r^2} + \frac{N-1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\Delta_s v$. On $\partial\Omega$, the relation $\frac{\partial v}{\partial \mathbf{n}} = -\frac{\partial v}{\partial r}$ holds and (B.40) can be rewritten as:

$$\begin{aligned}\int_\Omega |[D_2 v]|^2\, d\mathbf{x} &= \int_\Omega (\Delta v)^2\, d\mathbf{x} + \int_{\partial\Omega} \frac{\partial v}{\partial \mathbf{n}}\left(-\Delta v + \frac{\partial^2 v}{\partial \mathbf{n}^2}\right) d\sigma_x \\ &= \int_\Omega (\Delta v)^2\, d\mathbf{x} + (N-1)\int_{\partial\Omega} \left(\frac{\partial v}{\partial \mathbf{n}}\right)^2 d\mathbf{x},\end{aligned} \tag{B.41}$$

since, v being constant on $\partial\Omega$, $\Delta_S v = 0$. We apply now a trace type inequality (see (B.5)) to estimate the term on $\partial\Omega$:

$$\int_{\partial\Omega} \left(\frac{\partial v}{\partial r}\right)^2 d\mathbf{x} \leq 2\left(\int_\Omega \left(\frac{\partial v}{\partial r}\right)^2 d\mathbf{x}\right)^{\frac{1}{2}} \left(\int_\Omega \left|\nabla \frac{\partial v}{\partial r}\right|^2 d\mathbf{x}\right)^{\frac{1}{2}}.$$

This last relation implies that:

$$\int_{\partial\Omega} \left(\frac{\partial v}{\partial r}\right)^2 d\mathbf{x} \leq C_N \|\nabla v\|_2 \, \|[D_2 v]\|_2. \tag{B.42}$$

Combining together (B.41) and (B.42), we finally obtain:

$$\|[D_2 v]\|_2^2 - C_N \, \|\nabla v\|_2 \, \|[D_2 v]\|_2 - \|\Delta v\|_2^2 \leq 0, \tag{B.43}$$

what implies in particular that $\|[D_2 v]\|_2 \leq C_N \left(\|\nabla v\|_2 + \|\Delta v\|_2\right)$. ■

Proof of Lemma 5.4: According the definition (5.26) of $\bar{\xi}_1$ and remarking that $\int_{\partial\Omega_s} \theta_{2,i} \, d\sigma_y = 0$ and $\int_{\partial\Omega_s} \frac{\partial\theta_{2,i}}{\partial\mathbf{n}} \, d\sigma_y = 0$, we get, after simplification:

$$\begin{aligned}\int_{\partial\Omega_s} \frac{\partial\bar{\xi}_1}{\partial\mathbf{n}} \, \bar{\xi}_1 \, \chi \, d\sigma_y &= e^s \int_{\partial\Omega_s} \frac{\partial\bar{\xi}}{\partial\mathbf{n}} \, \bar{\xi} \, \chi \, d\sigma_y \\ &- (4\pi)^{-\frac{N}{2}} \frac{1}{2} \sum_{i=1}^{N} \left[M_{2,i}(s)^2 \int_{\partial\Omega_s} \frac{\partial\theta_{2,i}}{\partial\mathbf{n}} \, y_i \, d\sigma_y - e^{\frac{s}{2}} M_{2,i}(s) \int_{\partial\Omega_s} \frac{\partial\xi}{\partial\mathbf{n}} \, y_i \, d\sigma_y \right].\end{aligned}$$

- The first term of the right hand side, according to the transmission condition of (3.25), can be estimated as follows:

$$e^s \int_{\partial\Omega_s} \frac{\partial\bar{\xi}}{\partial\mathbf{n}} \, \bar{\xi} \, \chi \, d\sigma_y = e^s \, \bar{\zeta} \, \chi \, \left[e^{-s\frac{N}{2}} \rho(s) - m \, \bar{\zeta}' \, e^{-s\frac{N}{2}} + m \, \frac{N}{2} \, \bar{\zeta} \, e^{-s\frac{N}{2}} \right].$$

  On the other hand, the equality:

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}\left(m \, \chi \, \bar{\zeta}^2 e^{-s\frac{N-2}{2}} \right) = \\ &\qquad m \, \bar{\zeta} \, \bar{\zeta}' \, \chi \, e^{-s\frac{N-2}{2}} + \frac{m}{4}\left(\frac{e^{-\frac{s}{2}}}{2} - N + 2 \right) \chi \, e^{-s\frac{N-2}{2}} \, \bar{\zeta}^2,\end{aligned}$$

  allows us to write that:

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\left(m \, \chi \, \bar{\zeta}^2 \, e^{-s\frac{N-2}{2}} \right) &= -e^s \int_{\partial\Omega_s} \frac{\partial\bar{\xi}}{\partial\mathbf{n}} \bar{\xi} \, \chi \, d\sigma_y + e^s \, \bar{\zeta} \, \chi \, e^{-s\frac{N}{2}} \rho(s) \\ &+ m \, \frac{N}{2} \, \chi \, \bar{\zeta}^2 \, e^{-s\frac{N-2}{2}} + \frac{m}{4}\left(\frac{e^{-\frac{s}{2}}}{2} - N + 2 \right) \chi \, e^{-s\frac{N-2}{2}} \, \bar{\zeta}^2,\end{aligned}$$

  then:

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}\left(m \, \chi \, \bar{\zeta}^2 \, e^{-s\frac{N-2}{2}} \right) + \frac{1}{2} \, m\chi \, \bar{\zeta}^2 \, e^{-s\frac{N-2}{2}} \\ &\qquad = -e^s \int_{\partial\Omega_s} \frac{\partial\bar{\xi}}{\partial\mathbf{n}} \bar{\xi} \, \chi \, d\sigma_y + e^s \, \bar{\zeta} \, \chi \, e^{-s\frac{N}{2}} \rho(s) \\ &\qquad\qquad + \frac{1}{2} \, m \, \chi \, \bar{\zeta}^2 \, e^{-s\frac{N-2}{2}} \left(\frac{N}{2} + 2 + \frac{e^{-s/2}}{4} \right).\end{aligned}$$

According to the decay rate of $\bar{\zeta}$ ($|\bar{\zeta}(s)| \leq C\,\eta_N(s)\,e^{-s\frac{1}{2+N}}$, see (5.20)) and the fact that $\rho(s)$ is bounded, we obtain that:

$$\frac{1}{2}\frac{d}{ds}\left(m\,\chi\,\bar{\zeta}^2\,e^{-s\frac{N-2}{2}}\right) + \frac{1}{2}\,m\chi\,\bar{\zeta}^2\,e^{-s\frac{N-2}{2}} \leq -e^s\int_{\partial\Omega_s}\frac{\partial\bar{\xi}}{\partial\mathbf{n}}\,\bar{\xi}\chi\,d\sigma_y + C\,\eta_N(s)\,e^{-s\frac{N-2}{2}-s\frac{1}{2+N}}. \quad \text{(B.44)}$$

- On the other hand:

$$\left|\int_{\partial\Omega_s}\frac{\partial\theta_{2,i}}{\partial\mathbf{n}}\,y_i\,d\sigma_y\right| \leq C\,e^{-s\frac{N}{2}}. \quad \text{(B.45)}$$

The combination of the results (B.44), (B.45) and Lemma 5.1 yields the inequality of Lemma 5.4 after checking that $\frac{3N-4}{8} \leq \frac{N-2}{2} + \frac{1}{N+2}$, for all $N \geq 2$. ■

Proof of Lemma 5.5: All along this proof, ξ is extended by $\xi = \zeta$ to B_s and $\bar{\xi}_1$ is as in (5.26). Since $\nabla\xi = 0$ on B_s, we can evaluate $(\boldsymbol{\zeta}\cdot\nabla\xi, \bar{\xi}_1)$ instead of $(\boldsymbol{\zeta}\cdot\nabla\xi, \bar{\xi}_1)_s$.

By definition (5.26) of $\bar{\xi}_1$, we get

$$(\boldsymbol{\zeta}\cdot\nabla\xi, \bar{\xi}_1) = e^{-\frac{s}{2}}(\boldsymbol{\zeta}\cdot\nabla\bar{\xi}_1, \bar{\xi}_1) + M_1(\boldsymbol{\zeta}\cdot\nabla\theta_1, \bar{\xi}_1) + e^{-\frac{s}{2}}\sum_{i=1}^{N} M_{2,i}(s)(\boldsymbol{\zeta}\cdot\nabla\theta_{2,i}, \bar{\xi}_1). \quad \text{(B.46)}$$

- The first term on the right hand side satisfies

$$\begin{aligned}|(\boldsymbol{\zeta}\cdot\nabla\bar{\xi}_1, \bar{\xi}_1)| &\leq C\,|\boldsymbol{\zeta}|\,(\|\nabla\bar{\xi}_1\|^2 + \|\bar{\xi}_1\|^2),\\ &\leq C\,(\|\nabla\bar{\xi}_1\|_s^2 + \|\bar{\xi}_1\|_s^2) + C\,(\|\nabla\bar{\xi}_1\|_{B_s}^2 + \|\bar{\xi}_1\|_{B_s}^2)\\ &\leq C(\|\nabla\bar{\xi}_1\|_s^2 + \|\bar{\xi}_1\|_s^2) + C\,\eta_N^2\,e^{-s\frac{N-2}{2}-s\frac{2}{2+N}}, \end{aligned} \quad \text{(B.47)}$$

(see relations (5.42) and (5.46) and remark that $\frac{N-2}{2} + \frac{2}{N+2} \leq \frac{N}{2}$, for all $N \geq 2$).

- The second term of the right hand side of (B.46) is such that, according to (5.2) and (5.36):

$$(\boldsymbol{\zeta}\cdot\nabla\theta_1, \bar{\xi}_1) = \sum_{i=1}^{N}\zeta_i\,(\theta_{2,i}, \bar{\xi}_1) = \sum_{i=1}^{N}\zeta_i\,r_{2,i}(s)\,M_{2,i}(s)\|\theta_{2,i}\|^2,$$

and with (5.51), it comes

$$|(\boldsymbol{\zeta}\cdot\nabla\theta_1, \bar{\xi}_1)| \leq C\begin{cases}(1+s)^{\frac{3}{2}}\,e^{-\frac{s}{4}}, & \text{when } N = 2,\\ e^{-s\frac{3N-4}{8}}, & \text{when } N \geq 3.\end{cases} \quad \text{(B.48)}$$

- Arguing as in (3.34), we have:

$$|(\boldsymbol{\zeta} \cdot \nabla \theta_{2,i}, \bar{\xi}_1)| \leq C \, \|\bar{\xi}_1\|_s. \tag{B.49}$$

The proof is completed after combining (B.47), (B.48) and (B.49). ■

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