Lack of collision in a simplified 1-d model for fluid-solid interaction

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November 10, 2004

Abstract

In this paper we consider a simplified model for fluid-solid interaction in one space dimension. The fluid is assumed to be governed by the viscous Burgers equation. It is coupled with a finite number of solid masses in the form of point particles, which share the velocity of the fluid and are accelerated by the pressure difference of the fluid on both sides. We prove global existence and uniqueness of solutions. This requires of proving that the solid particles never collide in finite time, a key fact that follows from suitable a priori estimates together with uniqueness results for ordinary differential equations.

We also describe the asymptotic behavior of solutions as $t \to \infty$, extending previous results established for a single solid mass.

The possible 2-D analogues of these results in the context of Navier-Stokes equations are open problems.

Keywords and Phrases. Fluid-solid interaction, lack of collision, global existence.

AMS Subject Classification. 35R35, 76R99, 35K15, 35B40.

1 Introduction

The problem of fluid-solid interaction has attracted a lot of attention in recent years. Important progress has been achieved in the context of Navier-Stokes equations, coupled with rigid or elastic bodies (see, for instance, [B], [CST], [DE], [SST], [TT] and the references therein). However, as far as we know, there is no result in the literature in the context of the Navier-Stokes equations clarifying either the possible collision of the immersed bodies in finite time, or collision with the exterior boundary when the fluid is confined. This is a relevant problem since, for instance, in two space dimensions the solution would be smooth for all $t > 0$ if finite-time collision could be excluded, according to [T], [T2].

In this paper we consider a simplified model for a 1-D fluid containing several immersed rigid bodies, whose main difference with the Navier-Stokes system lies in the fact that we ignore the pressure terms in the standard fluid equation. We have studied a similar model including a nonlinear convection term in a previous paper [VZ]. There, existence and uniqueness of solutions and their large-time behavior were established for the case of only one solid mass immersed in the fluid. The results were then extended to the multi-dimensional case in [MZ].

The problem of possible collision of two immersed point masses remained open, and this question is answered negatively in the present paper, no collision occurs in finite time. The problem of collision in finite time is open in more than one dimension. We observe that, from a mechanical point of view, it is easy to understand that collision is more likely to occur in several space dimensions, since there is more room around the particles for the fluid to escape, making the possibility of collision more plausible.

Statement of the Problem. We consider a simplified model for a 1-D fluid containing a finite number N of point particles floating in it and located at the points $x = h_i(t)$, $i = 1, \cdots, N$ moving with the fluid. Without loss of generality we may assume that, in the absence of collision,

$$h_1(t) < h_2(t) < \cdots < h_N(t), \tag{1.1}$$

holds for some time $0 < t < T$. The complete system under consideration reads as follows:

$$\begin{cases} u_t - u_{xx} + \kappa(u^2)_x = 0, & x \in I_i(t), \quad i = 0, \cdots, N, \quad t > 0 \\ h_i'(t) = u(h_i(t), t), & i = 1, \cdots, N, \quad t > 0 \\ m_i\, h_i''(t) = [u_x](h_i(t), t), & i = 1, \cdots, N, \quad t > 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \\ h_i(0) = h_{i,0}, & h_i'(0) = h_{i,1}, \quad i = 1, \cdots, N. \end{cases} \tag{1.2}$$

Here and in the sequel, $I_i(t)$ stand for the intervals occupied by the fluid, separated by the point particles

$$I_0(t) = (-\infty, h_1(t)); \quad I_i(t) = (h_i(t), h_{i+1}(t)),\ i = 1, \cdots N-1; \quad I_N(t) = (h_N(t), \infty). \tag{1.3}$$

The masses m_i, $i = 1, \cdots, N$, of the particles are positive real numbers. The coefficient κ measures the ratio between convection and diffusion, which for the heat equation it is zero.

When convection enters, and depending on the direction of convection, it can be positive or negative.

We refer to this problem as Problem (**P**). We denote by $[f](x_0)$ the jump of a space function $f(x)$ at the point x_0,

$$[f](x) = \lim_{s\to 0}(f(x+s) - f(x-s)),$$

i.e., the difference between the values taken when the point is approached above (i.e., for $x > x_0$)) and from the left (for $x < x_0$). Thus, according to the transmission conditions satisfied at the point mass locations $x = h_i(t)$, $i = 1, \cdots, N$, the velocities of the fluid and the particles coincide, and each particle is accelerated by the difference of the velocity gradient on both sides of it. Thus, the velocity gradient acts as a pressure. Note that, to simplify the presentation, we can choose the particles to be of unit mass $m_i = 1$, but this does not entail any real change.

We will construct local in time solutions with the regularity that corresponds to the usual strong solutions of fluid flow theory, we will prove that the particle trajectories of these solutions never meet in finite time, so that the strong solution can be uniquely continued for all time.

The same techniques apply in the case of the Dirichlet problem in a bounded interval or on the half line, whose boundary represents a rigid wall. In that case, the issue of the possible finite-time collision of the immersed particles with the boundary also arises. Again, we prove that no finite-time collision with the wall occurs and the strong solution exists globally in time. See Section 6.

Two sections are then devoted to study the asymptotic behaviour under the additional (and natural) assumption that $u_0 \in L^1(\mathbb{R})$. Continuing the work of [VZ] for the case of one particle, in Section 7 solutions are shown to approach a Burgers profile with explicit form and convergence rates, Theorem 7.1. And a more striking result is proved in Section 8: though particles do not collide in finite time, they do approach each other in infinite time if convection is not strong, and in fact it can be shown that not only $h_i(t) \sim ct^{1/2}$, for all $i = 1, \cdots, N$, with the same $c > 0$, but also

$$h_j(t) - h_i(t) \sim t^{-a}, \tag{1.4}$$

for all $i \neq j$, with a precise exponent $a = a(\kappa)$ that is calculated in terms of the Burgers asymptotic profile, that is positive if $\kappa < 1$, see Theorem 8.3. So there is a kind of collision in infinite time after all.

Comments. Our continuous medium is governed by the viscous Burgers equation. In fact, the basic mathematical model is the heat equation, and the convection term $(u^2)_x$ adds only minor mathematical difficulties and small corrections to the qualitative behavior.

The problem of collision for fluids governed by the Navier-Stokes equations is open in both variants: whether two rigid bodies may collide or whether one rigid body may collide with the exterior boundary. We make a comment on that issue in the final section.

2 Lack of collision. Statement of the main results

We start our study by stating the main result on existence and uniqueness of a global in time solution for the 1-D model in $\mathbb{R}$, i.e., the problem without collisions posed in the whole line.

Theorem 2.1 (Existence of strong solutions without collision) *Let $u_0 \in H^1(\mathbb{R})$, and let us for $i = 1, \cdots, N$ have quantities $h_{i,0}$, $h_{i,1}, m_i \in \mathbb{R}$ be such that*

$$-\infty < h_{1,0} < h_{2,0} < \cdots < h_{N,0} < \infty, \tag{2.1}$$

and $m_i > 0$. Under the compatibility conditions

$$u_0(h_{i,0}) = h_{i,1}, \quad i = 1, \cdots, N, \tag{2.2}$$

there exists a unique global solution $(u, h_1(t), \cdots, h_N(t))$ of Problem $(\mathbf{P})$ *such that*

$$u \in C([0, \infty); H^1(\mathbb{R})), \quad u_{xx} \in L^2(Q^i),\ i = 0, ..., N, \tag{2.3}$$

where $Q^i = \{(x, t) : x \in I_i(t), 0 < t < T\}$, while the h_i's satisfy

$$h_i(t) \in C^1([0, \infty); \mathbb{R}),\ i = 1, \cdots, N, \quad h_i'' \in L^2(0, T), \tag{2.4}$$

for all finite $T > 0$, and the initial data are taken. Moreover, the particles do not collide in finite time, i.e.,

$$h_1(t) < h_2(t) < \cdots < h_N(t), \quad \forall t > 0. \tag{2.5}$$

The extra regularity $u_{xx} \in L^2$ makes it possible to define the velocity jumps $[u_x](h_i(t), t)$ at the point particles and to make sense of the dynamic conditions $m_i h_i''(t) = [u_x](h_i(t), t)$ a.e. in time. Besides, the partial differential equation is satisfied a.e. in the fluid. We call *strong solutions* the solutions with the regularity stated in the theorem.

The absence of collision result is more precise: If a strong solution is defined without collision in the time interval $0 \le t < T$ for some $T > 0$, then the particles cannot collide at $t = T$ and the solution can be uniquely continued for all $t \ge T$ as a strong solution without collision.

In the paper we establish two improvements of these results. On the one hand, we will prove that the strong solutions are actually classical, see Theorem 3.1; u turns out to be C^∞ smooth with the exception of the derivative jumps across the particle trajectories. The trajectories are C^∞ curves for $t > 0$.

As a second improvement, we will extend the existence and uniqueness result to cover initial data $u_0 \in L^2(\mathbb{R})$, cf. Theorem 5.1. This result also covers the situation where $u_0 \in H^1$, even if the set of initial speeds of the interfaces is not consistent with the data for u, i.e., $h_{i,1} \neq u_0(h_{i,0})$ for some i.

The case of a single point mass (i.e., $N = 1$) has been studied in [VZ], cf. Theorem 2.1. Considering several point particles introduces conceptual and technical difficulties. The main one is the possibility of collision in finite time, with associated blow-up of some important estimates on the solutions. Excluding such occurrence is the main purpose of the paper.

Let us comment at this point on the mathematical approach. The model under consideration may be viewed as a *free boundary problem*, since it combines a partial differential equation to be satisfied by the fluid with the motion of the point particles that can be viewed as free boundaries; the location of these particles is part of the unknown. The paper proceeds with the proof of Theorem 2.1 step by step in a number of subsections. Assuming that the point particles are located at different points at the initial time (which is one of the main assumptions of Theorem 2.1 above), a change of variables allows to reduce it to finding a fixed point in a suitable functional setting. Methods of the theory of evolution equations and fixed point arguments allow us to prove the *local in time* existence and uniqueness. Then, solutions may be extended to the maximal existence time $[0, T_{\max})$. Global existence is then equivalent to the fact that $T_{\max} = \infty$. The study of global solvability needs uniform global estimates. The *energy*

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} u^2 dx + \frac{1}{2} \sum_{i=1}^{N} m_i \, |h_i'(t)|^2 \tag{2.6}$$

is dissipated along trajectories according to the *dissipation law*

$$\frac{dE(t)}{dt} = - \int_{\mathbb{R}} |u_x(x,t)|^2 dx. \tag{2.7}$$

This shows that the energy does not blow-up while solutions exist. The existence of strong solutions needs additional estimates which are obtained essentially by using u_{xx} as test function and extend the ones already introduced in [VZ].

However, these facts are not sufficient to guarantee *global existence*. Indeed, the classical continuation argument allowing to extend the solution for all $t > 0$ requires of knowing a priori that the particles do not collide. Thus, the key point in the proof of Theorem 2.1 is establishing the lack of collision property (2.5). Our proof of that fact is based on the simple observation that the particle dynamics is governed by the ordinary differential equation (ODE):

$$h_i'(t) = u(h_i(t), t), \; t > 0, \quad i = 1, \cdots, N. \tag{2.8}$$

Thus, if two particles collide in, say, time $t = T > 0$, (without loss of generality we may assume that the colliding particles are located at the points $h_1(t)$ and $h_2(t)$), both solve the Cauchy problem

$$\begin{cases} h_i'(t) = u(h_i(t), t), & 0 \leqslant t < T, \; i = 1, 2 \\ h_1(T) = h_2(T). \end{cases} \tag{2.9}$$

It is then sufficient to show uniqueness of solutions of (2.9) backwards in time. Uniqueness for problem (2.9) will follow from the regularity of the fluid solution u. More precisely, it is sufficient to show that $u(h,t)$ is Lipschitz continuous in the space variable h, since u enters in (2.9) as the nonlinearity of the ODE, together with some integrability property in time. This regularity is proved by means of a local H^2_x-estimate, which is a crucial step in our result. It is basically the same estimate we needed before to construct the strong solutions but with the key observation that there is a version which is valid up to the possible collision time with bounds independent of the distance between consecutive particles. Collision is then eliminated as a consequence of the obtained regularity.

3 Local existence of strong solutions

We start our practical work by constructing strong solutions in a certain time interval, establishing existence, and showing the main properties.

3.1 Some basic estimates

As a preliminary step, we derive a number of integral a priori estimates that hold for all strong solutions in their existence time.

• **Energy dissipation.** Multiplying by u and integrating by parts we get:

$$E(t) + \int_0^t \int_{\mathbb{R}} \mid u_x \mid^2 \, dxds = E(\tau), \tag{3.1}$$

for all $t > \tau \geq 0$, where the energy is as in (2.6).

• **Conservation of momentum.** Integrating the first equation in (1.2) with respect to $x \in \mathbb{R}$ and using the jump condition we get

$$\int_{\mathbb{R}} u(x,t)\, dx + \sum_i m_i\, h_i'(t) = \int_{\mathbb{R}} u_0(x) dx + \sum_i m_i\, h_{i,1}, \quad \forall t > 0, \tag{3.2}$$

whenever, in addition to the assumptions of Theorem 2.1, $u_0 \in L^1(\mathbb{R})$.

• L^p **estimates.** More generally, given a convex C^1 real function $j(\cdot)$ we can multiply in (1.2) by $\beta(u) = j'(u)$ and integrate with respect to $x \in \mathbb{R}$ to get

$$\frac{d}{dt}\left[\int_{\mathbb{R}} j(u)dx + \sum_i m_i\, j(h_i'(t))\right] \leq 0.$$

Specializing $j(u)$ to be an approximation of the function $|u|$, we deduce that the following quantity decreases in time:

$$\int_{\mathbb{R}} \mid u \mid dx + \sum_i m_i \mid h_i'(t) \mid, \tag{3.3}$$

whenever $u_0 \in L^1(\mathbb{R})$.

We draw a similar conclusion when $j(u) = |u|^p$ for some $p \geq 1$. Therefore, we get L^p, $1 \leq p < \infty$ estimates for the solution at all times $t > 0$ in terms of the initial data $u_0 \in L^p(\mathbb{R})$. For the case $p = \infty$ we use the function $j(u) = (u - k)_+$ with $k \geq \max\{\|u_0\|_\infty, \ |h_{i,1}|\}$, to get

$$\|u(t)\|_\infty, \, |h_i'(t)| \leq \max\{\|u_0\|_\infty, \, |h_{i,1}|\}, \qquad \forall t > 0 \tag{3.4}$$

where we use the short notation $u(t)$ for $u(\cdot, t)$.

• **Positivity.** Taking $j(u)$ to be an approximation of the functions $u^+ = \max\{u, 0\}$, $u^- = (-u)^+$, we deduce that the following quantities

$$\int u^+ dx + \sum_i m_i \, (h_i'(t))^+, \qquad \int u^- dx + \sum_i m_i \, (h_i'(t))^-$$

decrease in time. Consequently, when $u_0 \geq 0$ a.e. and $h_{i,1} \geq 0$ we deduce that

$$u(x,t) \geq 0 \quad \text{a.e.} \quad x \in \mathbb{R}, \quad \forall t > 0, \quad h_i'(t) \geq 0, \quad \forall t > 0. \tag{3.5}$$

3.2 Existence of local strong solutions

In order to prove local (in time) existence, we use a classical technique of free-boundary theory that reduces the problem to a fixed space domain by means of a change of coordinates. In this way, we get a problem where the particles are fixed, at the price of considering more general parabolic equations with variable coefficients and quadratic nonlinearities. The presence of the quadratic nonlinearity in the system adds some technical difficulties with respect to the linear case, but they are not essential. This was already observed in [VZ] in the case of one particle.

Step 1. *Change of variables.* We introduce the variables,

$$v(y,t) = u(x, \, t), \quad x = \varphi(y,t), \tag{3.6}$$

where function φ is strictly increasing in y and such that

$$\varphi(h_{i,0}, \, t) = h_i(t), \, i = 1, \cdots, N, \, t > 0. \tag{3.7}$$

Obviously, after the change of variables (3.6) the particles, located previously at the points $h_1(t), \cdots, h_N(t)$ for the fluid solution u, are now located at the points $h_{i,0}$, $i = 1, \cdots, N$ for the new fluid solution v, and do not move in this frame. A function φ as in (3.7) can be easily found if

$$h_1(t) < h_2(t) < \cdots < h_N(t), \tag{3.8}$$

i.e., provided that no collision occurs. Now, if we combine the fact of having assumed that the N particles are located at different points in the initial time $t = 0$, with the property of dissipation of energy, $E(t) \leq E(0)$, with $E(t)$ given (2.6), we easily conclude that the separation property (3.8) is guaranteed to be true at least in the time interval $[0, T_*)$, with

$$T_* = \frac{m}{2\sqrt{2}} \min_{1\leq i\leq N-1} \frac{|h_{i+1,0} - h_{i,0}|}{[E(0)]^{1/2}}, \tag{3.9}$$

where $m = \min_i m_i > 0$. This follows from the elementary estimate for the gaps between consecutive mass points, $l_i(t) = h_{i+1}(t) - h_i(t)$, $i = 1, \ldots, N-1$:

$$l_i(t) \geq l_i(0) - \int_0^t |h'_{i+1}(s) - h'_i(s)|\, ds\,, \tag{3.10}$$

and the energy estimate (3.1).

For simplicity, we may take φ to be continuous and linear in the y variable in each interval I_i between two point particles, so that it will be invertible for all $0 < t < T$ (under the separation assumption (3.8)), and will depend on t in the same way as the functions $h_i(t)$. Besides, if $y = \psi(x, t)$ is the inverse of $x = \varphi(y, t)$, it is also continuous and piecewise linear in x for each fixed $t > 0$ (such that (3.8) holds), and we can write

$$u(x, t) = v(\psi(x, t),\, t)\,. \tag{3.11}$$

Clearly,

$$\psi(h_i(t),\, t) = h_{i,0},\ i = 1, \cdots, N. \tag{3.12}$$

We have on the i-th interval

$$\psi_x = \frac{1}{\varphi_y} = \frac{l_{i,0}}{l_i(t)}, \tag{3.13}$$

which is piecewise constant as a function of x. We still have to define φ and ψ on the end intervals, I_0 and I_N, and in that case we can use a simple translation:

$$\begin{aligned} \varphi(y, t) &= y + h_1(t) - h_{1,0} && \text{if } \ y \leq h_{1,0}, \\ \varphi(y, t) &= y + h_N(t) - h_{N,0} && \text{if } \ y \geq h_{N,0}. \end{aligned}$$

In that case $\psi_x = \varphi_y = 1$. Summing up, the change of space variable is uniformly bi-Lipschitz continuous as long as we do not approach a collision.

Step 2. *Reformulated problems.* With this change of variables, system (1.2) can be written in the following equivalent way:

$$\begin{cases} v_t - |\psi_x|^2 v_{yy} + \psi_t\, v_y + \psi_x \kappa (v^2)_y = 0, \quad y \in I_i^*, \quad i = 0, \ldots, N,\, t > 0 \\ h'_i(t) = v(h_{i,0},\, t), \quad i = 1, \ldots, N,\, t > 0 \\ m_i\, h''_i(t) = [v_y/\varphi_y](h_{i,0}, t), \quad i = 1, \ldots, N, \quad t > 0 \\ v(y, 0) = u_0(y), \quad y \in \mathbb{R} \\ h_i(0) = h_{i,0},\, h'_i(0) = h_{i,1}, \quad i = 1, \ldots, N, \end{cases} \tag{3.14}$$

where the intervals I_i^* now are independent of time, i.e.,

$$I_0^* = (-\infty,\, h_{1,0}), \quad I_i^* = (h_{i,0},\, h_{i+1,0}), \quad i = 1, \cdots, N-1, \quad I_N^* = (h_{N,0},\, \infty). \tag{3.15}$$

The main differences between (1.2) and (3.14) are the following:

(1) The particles in the transformed system (3.14) are fixed in time; The acceleration $h_i''(t)$ of the particles is given by the jump of the quotient v_y/φ_y.

(2) The first equation in (3.14) is a parabolic PDE with variable coefficients. Note that the coefficients ψ_t and ψ_x and $|\psi_x|^2$ of this parabolic equation (as well as the one entering in the equation for the acceleration of the point particles) depend on the trajectories $h_i(t)$, $i = 1, \cdots, N$, of the particles. Note also that $\psi_{xx} = 0$ in each interval I_i^*.

Now, under the assumption of lack of collision for a time interval $0 \le t \le T$ we have uniform parabolicity, i.e., $|\psi_x|$ is bounded above and below by positive finite constants, uniformly on $x \in \mathbb{R}$ and $t \in [0, T-\delta)$, for all $\delta > 0$. Indeed, by formula (3.13) $|\psi_x|$ depends only on the distances $l_i(t) = h_{i+1}(t) - h_i(t)$, $i = 1, \ldots, N-1$, and it goes to infinity linearly with $1/l(t)$ where $l(t) = \min\{l_i(t) : \ i = 1, \ldots, N-1\}$, so that a problem with this lower bound occurs only at collision, $l(t) \to 0$.

Step 3. *Iterative scheme for local existence.* In order to prove local (in time) existence and uniqueness of strong solutions of (3.14), and, consequently, of (1.2), we will use a Fixed Point argument on the particle trajectories.

The iteration step proceeds as follows: we prescribe (freeze) the dynamics of the particles on the coefficients of the equation. More precisely, we consider functions $\widehat{h}_i(t) \in H^2(0,T)$ for some $T_* > T > 0$ and such that

$$\widehat{h}_i(0) = h_{i,0},\ \widehat{h}_i'(0) = h_{i,1},\ i = 1, \ldots, N. \tag{3.16}$$

Note that this is an affine subspace of $H^2(0,T)$ in which the semi-norm $\|h_i''\|_{L^2}$ provides a complete estimate on the norm of h_i in H^2, since h can be fully reconstructed from h_i'' and the initial data $h_{i,0}$ and $h_{i,1}$.

We also assume the non-intersection of the trajectories $x = \widehat{h}_i(t)$, as well as the compatibility condition (2.2). With this choice of functions $\widehat{h}_i = \widehat{h}_i(t)$ we now consider the problem

$$\begin{cases} v_t - |\widehat{\psi}_x|^2 v_{yy} + \widehat{\psi}_t\, v_y + \widehat{\psi}_x \kappa\, (v^2)_y = 0, & x \in I_i^*,\ i = 0, \ldots, N,\ t > 0 \\ v(h_{i,0}, t) = \widehat{h}_i'(t), \quad i = 1, \ldots, N,\ t > 0 \\ v(y, 0) = u_0(y), \quad y \in \mathbb{R}, \\ h_i(0) = h_{i,0}, \quad h_i'(0) = h_{i,1}, \quad i = 1, \ldots, N. \end{cases} \tag{3.17}$$

Here, $\widehat{\psi}_x$, $\widehat{\psi}$, and $\widehat{\psi}_t$ stand for the derivatives of the function $\widehat{\psi}$ that performs the change of variables (3.11)-(3.12) for the functions $\widehat{h}_i$. Problem (3.17) can be solved independently on each space interval I_i^*. On each of these intervals it consists of a parabolic PDE with a quadratic nonlinear term, plus nonhomogeneous initial and boundary conditions.

Local in time existence and uniqueness can be easily proved by parabolic techniques, using for instance the results of [LSU]. We obtain a unique solution v in the class

$$v \in C([0,T];\, H^1(\mathbb{R})),$$

existing in a time interval $0 < t < T$. It is constructed in each of the strips $Q(i,T) = I_i^* \times (0,T)$, $i = 0, \cdots, N$. The proof uses a priori estimates that we will review in detail because we need some information about the precise dependence for the fixed point argument. Note that due to the quadratic term in the PDE the existence time could in principle be finite and very small, but this difficulty will be eliminated by the a priori estimates.

A PRIORI ESTIMATES. This is a way to obtain a priori estimates for $v, v_y, v_{yy}, v_t \in L^2(Q_T)$: assuming that the solution $v(x,t)$ of (3.17) is defined in the strip $Q = \mathbb{R} \times (0,T)$ for some $T > 0$, we perform the calculations in each of the space intervals I_i^*.

(i) To be specific, let us consider an interval $I_i^* = \{h_{i,0} < y < h_{i+1,0}\}$ between two particles; the calculations in the end intervals will be commented upon later, they are similar and easier. Let us assume for simplicity of notation (and without loss of generality) that it is $(0,L)$, hence $L = l_{i,0}$. First, we eliminate the boundary conditions by introducing the function

$$w(y,t) = v(y,t) - \frac{y}{L}\,\widehat{h}_1'(t) - \frac{L-y}{L}\,\widehat{h}_0'(t) \tag{3.18}$$

defined in $Q_1 = \{0 < y < L,\ 0 < t < T\}$. In this way, w vanishes on the lines $y = 0$ and $y = L$ for $0 \le t \le T$. It also satisfies in Q_1 an equation of the form

$$w_t - a(y,t)w_{yy} + b(y,t)w_y + c(y,t)\,(w^2)_y + d(y,t)w = f(y,t)\,, \tag{3.19}$$

where

$$\begin{aligned} &a(y,t) = |\widehat{\psi}_x|^2, \quad b(y,t) = \widehat{\psi}_t + 2\kappa\widehat{\psi}_x\,\frac{(L-y)\widehat{h}_0'(t) + y\widehat{h}_1'(t)}{L}, \\ &c(y,t) = \widehat{\psi}_x\kappa, \qquad\quad d(y,t) = 2\kappa\widehat{\psi}_x\,\frac{(\widehat{h}_1'(t) - \widehat{h}_0'(t))}{L}. \end{aligned} \tag{3.20}$$

We see that these coefficients are regular enough and a and c do not depend on y inside each interval I_i^* (but the value changes when we change interval). Indeed, a and c are as smooth as $\widehat{h}_i$, H^2 in time, while b is continuous and piecewise linear in y and as smooth as $\widehat{h}_i'$, i.e., H^1 in time, and d is piecewise constant in y and H^1 in time. Hence, all of them are bounded in Q. As for the right-hand side, we may write it as $f(y,t) = f_{(1)}(y,t) + f_{(2)}(y,t)$ with

$$f_{(1)}(y,t) = \frac{y}{L}\widehat{h}_1''(t) + \frac{L-y}{L}\widehat{h}_0''(t),$$

which has L^2 regularity in time and is smooth in space, with norm bound by the H^2 norm of the $\widehat{h}_i$. Besides, we have

$$f_{(2)}(y,t) = \frac{\widehat{\psi}_t}{L}(\widehat{h}_1'(t) - \widehat{h}_0'(t)) + \frac{2\kappa}{L^2}\widehat{\psi}_x\,(\widehat{h}_1'(t) - \widehat{h}_0'(t))((L-y)\widehat{h}_0'(t) + y\widehat{h}_1'(t)),$$

which is more regular and easier to treat. More precisely, it is H^1 in time and for each $t > 0$, piecewise linear in space.

(ii) Next, we obtain classical energy estimates by multiplying the equation by w and w_{yy} and integrating in y and t in the rectangle $R = I \times (t_1, t_2)$, with $I = (0, L)$, $0 \le t_1 < t_2 < T$. In the first case we get

$$\begin{aligned} \frac{1}{2}\int_I w^2(t_2)\,dy + \iint_R a(t) w_y^2\,dydt &= \frac{1}{2}\int_I w^2(t_1)\,dy + \iint_R d(y,t) w^2\,dydt \\ + \iint_R f(y,t)\,w\,dydt - \iint_R b(y,t) w_y\,w\,dydt &- \iint_R 2c(t) w^2 w_y\,dydt\,. \end{aligned} \tag{3.21}$$

Note that, due to the boundary conditions, the last term disappears upon space integration. This is important since it is the only term with a power of w higher than quadratic. We use this estimate as follows: since the trajectories $\widehat{h}_i$ do not collide, $a(t)$ is bounded below by a constant, say $\lambda > 0$. There is possibly bad term $\iint dw^2\,dydt$, but d is bounded since $T < T_*$, T_* being the a priori lower bound on the collision time (3.9). Therefore, we may use Hölder's inequality plus a Gronwall argument in the usual way, to get from (3.21) the bounds

$$\int_I w^2(t)\,dy \le C_1, \qquad \iint_R a(t) w_y^2\,dydt \le C_2, \tag{3.22}$$

These estimates depend continuously on the norm of $\widehat{h}_i$ in $H^2(0,T)$ since $T < T_*$. It follows from standard embeddings[1] that

$$\int_0^{T_*} \|w(t)\|^2_{L^\infty(I)}\,dt \le \sup_t \|w(t)\|_2 \int_0^T \|w_y\|_2\,dt \le C_3\sqrt{T}\sup_t \|w(t)\|_2 \left[\int_0^T \|w_y\|_2^2\,dt\right]^{1/2}, \tag{3.23}$$

so that $w \in L^2(0, T : L^\infty(I))$. The constants $C_1, \cdots, C_3$ depend on $\int_I w_0^2\,dy$, the norm of $\widehat{h}$ in $H^2(0,T)$ and T, but not on any information about $u_{0,x}$ if $t \in (0,T)$.

(iii) We now multiply equation (3.19) by w_{yy} and integrate in the rectangle R and get after some manipulations,

$$\begin{aligned} \frac{1}{2}\int_I w_y^2(t_2)\,dy + \iint_R a(t) w_{yy}^2\,dydt &= \frac{1}{2}\int_I w_y^2(t_1)\,dy + \iint_R d\,w_{yy} w\,dydt \\ + \iint_R b(y,t) w_y\,w_{yy}\,dydt + \iint_R c(t)\,(w^2)_y\,w_{yy}\,dydt &- \iint_R f(y,t)\,w_{yy}\,dydt\,. \end{aligned} \tag{3.24}$$

Note that $d_y = 0$ so that $\iint_R d\,w_{yy} w\,dydt = 0$. By arguing as before we get from (3.24) an inequality of the form

$$X(t_2) + c\int_{t_1}^{t_2} Y(t)\,dt \le X(t_1) + \int_{t_1}^{t_2} (F(t) + B(t)X(t))\,dt\,, \tag{3.25}$$

[1] The embedding we are using is the one-dimensional interpolation inequality $\|f\|_\infty \le C\|f\|_2^{1/2}\|f'\|_2^{1/2}$, valid for all functions $f \in H^1(\mathbb{R})$ or $f \in H_0^1(I)$). We use it with $f = w(t)$.

involving the nonnegative integrals $X(t)=\int_I w_y^2(t)\,dx$ and $Y=\int_I w_{yy}^2(t)\,dx$. It has coefficients $F(t)$ and $B(t)$ which are functions obtained from bounds of a,b,c. In particular, we may take

$$F(t)=K\int_I f(y,t)^2\,dy,\quad B(t)=K\left(|b|_\infty^2+|c|_\infty^2\|w(t)\|_\infty^2\right), \tag{3.26}$$

with a constant $K>0$. The norm $\|w(t)\|_\infty$ is taken in $L^\infty(I)$. Note that F and B are not necessarily bounded, but they are at least integrable (the latter according to (3.23)). We apply the Gronwall technique to formula (3.25) to get for $0\le t_1<t\le T$

$$X(t)\le X(t_1)\exp\left(\int_{t_1}^t B(r)\,dr\right)+\int_{t_1}^t F(s)\exp\big(\int_s^t B(r)\,dr\big)\,ds. \tag{3.27}$$

It follows that there are bounds for $0<t<T$

$$\int_I w_y^2(t)\,dx\le C_4, \tag{3.28}$$

$$\int_0^{T_*}\int_I w_{yy}^2\,dxdt\le C_5. \tag{3.29}$$

When restricted to $0\le t\le T<T_*$, the constants depend continuously on the norms of $w_{0,x}$ in $L^2(I)$, which can be traced back to the initial information (the norm $\|u_0\|_{H^1}$ in $H^1(\mathbb{R})$ and the distances l_{i0}) and also on the norms of $\widehat{h}_i$ in $H^2(0,T_*)$, $i=1,\dots,N$, which enter f and the coefficients a,b,c.

We remark at this point that estimates (3.22) and (3.28) imply that $w(y,t)$ is uniformly bounded in each strip $I\times(0,T)$. Moreover, using (3.29) it follows that w_t is uniformly bounded in $L^2(I\times(0,T))$. From known regularity results for evolution equations, it follows from the bounds for w_y, w_{yy} and w_t that $w\in C([0,T]:H^1(I))\subset L^\infty(I\times(0,T))$ with an estimate that depends on the same arguments.

Indeed, the above proof can be performed on all intervals I_i^* with $i=1,\cdots,N-1$, and we obtain the above uniform estimates for $0<t<T<T_*$. The end intervals I_0^* and I_N^* are semi-infinite, so that the proof needs slight modifications. In particular, $w(y,t)=v(y,t)-\widehat{h}_1'(t)$ for $y\le\widehat{h}_1(t)$, $w(y,t)=v(y,t)-\widehat{h}_N'(t)$ for $y\ge\widehat{h}_N(t)$. We leave the details to the reader.

Summing up, we find a finite time $T_*>0$ as in (3.9) such that estimates for w,w_x,w_{yy},w_t obtained above hold for $t\in(0,T)$, for all $0<T<T_*$, with constants that depend on the data in terms of bounds of these norms of $w_{0,y}$, $\widehat{h}_i(t)$ stated after (3.29) and upper and lower bounds for the l_{i0}.

Using formula (3.18) we see that similar estimates hold for the function $v(x,t)$ in the time interval $(0,T)$ and the norms depend on the data in the same way.

Step 4. *Fixed point argument.* Now comes an important consequence that allows us to close the iteration process and prove the following existence result.

Proposition 3.1 *Under the assumptions of Theorem* 2.1 *there exists a time interval* $0 \le t \le T'$ *in which Problem* (**P**) *admits a strong solution.*

Proof: Indeed, let $v^{(i)}$ be the restriction of v to the strip $R_i = I_i^* \times (0,T)$. The regularity of $v^{(i)}$ in R_i allows us to define the traces of $v_y^{(i)}$ on the bounding lines $y = h_{i,0}$, $y = h_{i+1,0}$ for $0 \le t \le T$, as $L^2(0,T)$ functions. Indeed, we know that in each subinterval I_i^* the solution v^i belongs to $L^2(0,T;H^2(I_i^*))$, and also that $\partial_t v^{(i)} \in L^2_{x,t}$. By interpolation, we deduce that

$$v^{(i)} \in H^s(0,T_* : H^{2-2s}(I_i^*)).$$

Since the boundary trace of functions in the space $H^\sigma(I)$ of an interval I, is well defined if $\sigma > 1/2$, we deduce that the trace of $v_y^{(i)}$ on the bounding sides of I_i^* belongs to $H^s(0,T)$ for all $s < 1/4$.

Next, we notice that on any line $\gamma_i := \{y = h_{i,0},\ 0 \le t \le T\}$, there are two such traces, one coming from $v^{(i)}$, and one from $v^{(i-1)}$, so that we can define the jump,

$$[v_y](h_{i,0},t) = v_y^{(i+1)}(h_{i,0},t) - v_y^{(i)}(h_{i,0},t).$$

A similar jump appears in φ_y. Actually, the dynamics of the particles is controlled by the jump of v_y/φ_y. Note also that for all intermediate intervals, $v_y^{(i)} = w_y^{(i)} + l_i'(t)/l_{i,0}$, and $l_i'(t) = \widehat{h}_{i+1}'(t) - \widehat{h}_i'(t)$. On the end intervals, $u_x = v_y = w_y$.

We now use this trace to update the value of the functions $h_i(t)$ by means of the dynamic condition

$$m_i\, h_i''(t) = [v_y/\varphi_y](h_{i,0},t), \quad i = 1,\ldots,N, \quad t > 0, \tag{3.30}$$

plus initial conditions $h_i(0) = h_{i,0}$, $h_i'(0) = h_{i,1}$, so that each new h_i may be seen as determined by v, hence by the $\widehat{h}_i$, and belongs to the class $H^2[0,T)$, more precisely, to $H^{2+s}(0,T)$. Note that, due to the construction, $\widehat{\psi}_i$ is as regular as $\widehat{h}_i$ was. Consequently, solving (3.17) yields a nonlinear map $\mathcal{N}$ given by

$$\mathcal{N} : (\widehat{h}_1,\ldots,\widehat{h}_N) \to w \to v \to (h_1,\ldots,h_N)\,. \tag{3.31}$$

Acting on the space $\mathcal{X} = \left(H^2(0,T))\right)^N$ into itself, $\mathcal{N}$ is a bounded and compact map, since the image is bounded in $H^{2+s}(0,T)$ with $s < 1/4$. Moreover, it is easy to see from the use of Gronwall's inequality on formula (3.25) that $\mathcal{N}$ maps a ball of H^2 into itself on two conditions: the ball must be taken large enough and the time variable must be restricted to a finite interval $[0,T']$ with $T' > 0$ small enough, depending on the norms of the data u_0 in $H^1(\mathbb{R})$ and the data of the particles $h_{i,0}, h_{i,1}$. This implies the existence of a fixed point, hence existence of a local in time solution for our problem. At this time we need the compatibility condition (2.2).

From the fixed point $h_i(t) \in H^2(0,T')$, we may recover the function $w(x,t)$ solution of equation (3.19), and from them the solution v of problem (3.14). We easily check that we have uniform bounds for v in $L^\infty(0,T':H^1(\mathbb{R}))$ and in $L^2(0,T':H^2(\mathbb{R}^*))$, where $\mathbb{R}^*$ is $\mathbb{R}$ minus the set of particle locations $\{h_{i,0}\}$. Moreover, we have $v_t \in L^2(0,T':L^2(\mathbb{R}))$. It follows that $v \in C([0,T'):L^2(\mathbb{R}))$.

We also need to establish continuity into $H^1(\mathbb{R})$. We only need to prove it for w. Observe that $(w_y^2)_t = 2w_y\, w_{yt}$. Now, on every interval $I = I_i^*$ we have

$$\int_0^t \int_I w_y w_{yt}\, dx = -\int_0^t \int_I w_{yy} w_t\, dx,$$

and the later quantity is controlled. Therefore, $w_y \in C([0,T'):L^2(I_i^*))$ for every i, hence, $w_y, v_y \in C([0,T'):L^2(\mathbb{R}))$.

The conclusions for v can be translated into similar conclusions for u by undoing the coordinate transformation (3.6). We have thus obtained a strong solution in a small time interval. ■

The time T' of existence is in principle less than the first collision time T_* and possibly small. Actually, for small time T' operator $\mathcal{N}$ is a contraction in $H^2(0,T')$. This is the way to uniqueness that we study next.

3.3 Uniqueness of local strong solutions

In this section we consider two strong solutions without collision as constructed above, defined for a time $T>0$ and having the same initial data. We will prove that they coincide by arguing on the difference $w = w_1 - w_2$. It satisfies an equation of the form

$$w_t - a_1(y,t)w_{yy} + B(y,t)\, w_y + C(y,t)w = g(y,t)\,, \tag{3.32}$$

on every space interval $x \in I_i^*$, and as long as both solutions exist. Here,

$$B(y,t) = b_1(y,t) + 2c_1(y,t)w_1, \quad C = 2c_1(y,t)\, w_{2,y} + d_1(y,t),$$

with a_i, b_i, c_i , and d_i the coefficients given by (3.20) for the solution w_i, $i=1,2$. Finally, g is given by

$$\begin{array}{l} g(y,t) = f_1(y,t) - f_2(y,t) + (a_1-a_2)w_{2,yy} \\ -(b_1-b_2)w_{2,y} - (c_1-c_2)(w_2)_y^2 - (d_1-d_2)w_2, \end{array} \tag{3.33}$$

f_i being now the second member of equation (3.19) for solution w_i. We put $f = f_1 - f_2$. Initial and boundary data for w on every such domain are zero.

We will be using the norms for vector functions $h(t) = (h_1(t),\cdots,h_N(t))$ in the spaces $(L^\infty(0,t))^N$,$(W^{1,\infty}(0,t))^N$ and $(H^2(0,t))^N$ that we will represent by $\|h\|_{0,t}$, $\|h\|_{1,t}$, $\|h\|_{2,t}$ respectively. Next, we observe that the coefficients a_i, b_i, and c_i entering the equations of the two solutions are bounded in the following form:

- a_i, b_i and c_i are uniformly bounded above. Moreover, a_i, c_i, d_i and $b_{i,y}$ are independent of y and there exists a constant such that

$$0 \le a_i(t), c_i(t), d_i(t) \le C, \quad |b_i(t)|, |b_{i,y}(t)| \le C\|h\|_{1,t}.$$

- a_i is bounded below away from zero with a similar bound. For two solutions we have

$$|a_1 - a_2|, |b_1 - b_2|, |c_1 - c_2|, |d_1 - d_2| \le C\|h_1 - h_2\|_{1,t}.$$

with a universal constant $C > 0$.

- The right-hand side f_i depends in the same way on the norm $\|h_i\|_{2,t}$. Moreover,

$$\|f_1 - f_2\|_{L^2(0,T)} \le C\|h_1 - h_2\|_{2,t}.$$

Taking also into account the proved regularity of the solutions, it follows from standard regularity theory that $w = w_1 - w_2$ is bounded in each strip of the from $Q_{i,t} = I_i^* \times (0,t)$ with an estimate of the form

$$\|w\|_{2,1;Q_{i,t}} \le C(t)\|h\|_{2,t},$$

where the norm in the left-hand side controls up to two space derivatives and one time derivative in L^2 in $Q_{i,t}$.

Calculation of the estimates. Multiplying equation (3.32) by the factor w and integrating in an interval $I = I_i$ we have the first estimate

Lemma 3.1 *For $t > 0$ small enough and $R = I \times (0,t)$ we have*

$$\sup_{0\le s\le t} \int_I w^2(s)\, dy + \iint_R w_y^2\, dydt \le C\|h\|_{2,t}^2 \tag{3.34}$$

with a constant that depends only on the solutions u_1 and u_2 and time t. It is uniform for all small times and depends on the known bounds of the solutions.

Proof: We start from the formula

$$\begin{aligned}\frac{1}{2}\int_I w^2(t_2)\, dy + \iint_R a_1(t)w_y^2\, dydt = -\iint_R B(y,t)w_y\, w\, dydt - \iint_R C(y,t)w^2\, dydt \\ + \iint_R f(y,t)\, w\, dydt + \iint_R \{(a_1 - a_2)w_{2,yy} - (b_1 - b_2)w_{2,y} - (c_1 - c_2)(w_2)_y^2\}w\, dydt\,,\end{aligned} \tag{3.35}$$

with $f = f_1 - f_2$. Let us examine the terms in the right-hand side. We have six integrals. Firstly,

$$I_1 = \iint_R C(t)w^2\, dydt = \iint_R 2c_1 w_{2,y}\, w^2\, dydt = -4\iint_R c_1 w_2\, ww_y\, dydt,$$

which can be bounded as

$$|I_1| \le 4\|c_1\|_\infty \|w_2\|_\infty \iint_R |w\, w_y|\, dydt.$$

This can be absorbed by the left-hand side for small t. In the same way, using $B = 2c_1 w_1 + b_1$ with $b_{1,y} = K(t)$, we get

$$I_2 = \iint_R B(y,t) w_y\, w\, dydt = 2\iint_R c_1 w_1 w w_y\, dydt - \frac{1}{2}\iint_R K(t) w^2\, dydt.$$

The last integral is no problem since $K(t)$ is bounded as far as there is no collision. As for the previous one, since w_1 is bounded, it can be estimated from above by

$$\frac{\|w_1\|_\infty^2}{\varepsilon} \iint_R c_1^2 w^2\, dydt + \varepsilon \iint_R w_y^2\, dydt.$$

Next, we have

$$I_3 = \iint_R f(y,t)\, w\, dydt \le C\|h\|_{2,t} (\iint_R w^2\, dydt)^{1/2} \le C\|h\|_{2,t}^2 + C\iint_R w^2\, dydt.$$

This originates the main term in the right-hand side of the estimate.

The last three terms have a similar treatment for all of them, as follows:

$$|\iint_R (a_1 - a_2) w_{2,yy}\, w\, dydt| = |\iint_R (a_1 - a_2) w_{2,y}\, w_y\, dydt| \le C\, \|h\|_{0,t} \sup_t \|w_{2,y}\|_2 (\iint_R |w_y|^2\, dydt) t^{1/2},$$

$$|\iint_R (b_1 - b_2) w_{2,y}\, w\, dydt| \le C\, \|h\|_{1,t} \left(\iint_R |w_{2,y}(t)|^2\right)^{1/2} \left(\iint_R w^2\right)^{1/2},$$

$$|\iint_R (c_1 - c_2)(w_2)_y^2\, w\, dydt| \le C\, \|h\|_{0,t} \sup_t \|w_2(t)\|_\infty \left(\iint_R |w_{2,y}(t)|^2\right)^{1/2} \left(\iint_R w^2\right)^{1/2}.$$

Summing up and working a bit, we get the result, (3.34). ■

Multiplying equation (3.32) by the factor w_{yy} and integrating in an interval $I = I_i$ we have the second estimate

Lemma 3.2 *For $t > 0$ small enough and $R = I \times (0,t)$ we have*

$$\sup_{0 \le s \le t} \int_I w_y^2(s)\, dy + \iint_R w_{yy}^2\, dydt \le C\|h\|_{2,t}^2 \tag{3.36}$$

with a constant that depends only on the solutions u_1 and u_2 and time t. It is uniform for all small times and depends on the known bounds of the solutions.

Proof: We start from the formula

$$
\begin{aligned}
&-\iint_R w_t w_{yy}\,dydt + \iint_R a_1(t) w_{yy}^2\,dydt = \iint_R B(y,t) w_y\, w_{yy}\,dydt + \iint_R C(y,t) w w_{yy}\,dydt \\
&-\iint_R f(y,t)\, w_{yy}\,dydt - \iint_R \{(a_1-a_2) w_{2,yy} - (b_1-b_2) w_{2,y} - (c_1-c_2)(w_2)_y^2\} w_{yy}\,dydt\,.
\end{aligned}
$$

The left-hand side can be transformed into an expression that is bounded below by

$$
I = I(t) = \sup_{0\le s\le t} \frac{1}{2}\int_I w_y^2(s)\,dy + c\iint_R w_{yy}^2\,dydt.
$$

On the right-hand side we again have six integrals. The third contributes the main term. The rest are as follows:

$$
I_1 = \iint_R 2c_1 w_{2,y}\, w w_{yy}\,dydt \le \varepsilon \iint_R w_{yy}^2\,dydt + \frac{4}{\varepsilon} c_1^2 \iint_R w_{2,y}^2\, w^2 dydt
$$

and the last integral, let us call it $I_{1,2}$, is calculated as follows: $w(t)$ is bounded for every $t > 0$ and the L^∞ norm is L^2 integrable in time (in fact, it is even bounded in time). The integral is then bounded by

$$
c_1^2 \int_0^t \|w^2(t)\|_\infty^2 (\int_I w_{2,y}^2\,dy) dt = c_1^2 \sup_t \|w_{2,y}(t)\|_2^2 \int_0^t \|w(t)\|_\infty^2 dt.
$$

Arguing as in (3.23), the last integral can be bounded by

$$
t^{1/2} \sup_t \|w(t)\|_2 (\iint_R w_y^2\,dydt)^{1/2},
$$

and is small for small t. Therefore, $I_{1,2}$ can also be absorbed by the left-hand side. This is a quite interesting technical trick to take care of the quadratic convective term of the equation, and will be used again below. Next,

$$
I_2 = \iint_R B(y,t) w_y\, w_{yy}\,dydt = -\iint_R c_1 w_{1,yy}(w_y)^2\,dydt - \frac{K}{2}\iint_R w_y^2\,dydt.
$$

The last integral does not create problems since it will be controlled by I by an application of the Gronwall technique, as seen before. As for the previous one, we have the upper bound

$$
C(\iint_R w_{1,yy}^2 dydt)^{1/2} (\iint_R w_y^4\,dydt)^{1/2}.
$$

The last factor can be bounded using the interpolation inequality

$$
\int_I w_y^4\,dy \le \|w(t)\|_\infty^2 ||w_{yy}||_2^2 \le \|w(t)\|_2 ||w_y(t)||_2 ||w_{yy}||_2^2.
$$

Using the previous lemma we arrive at a bound of the form

$$\iint_R w_y^4\, dy \le C\|h\|_{2,t} \sup_t \|w_y(t)\|_2 \iint_R w_{yy}^2\, dy.$$

which is good enough for our purposes. The last three terms are easy to control as follows:

$$|\iint_R (a_1 - a_2) w_{2,yy}\, w_{yy}\, dydt| \le C\, \|h\|_{0,t} \|w_2\|_{2,t} \|w_{yy}\|_2,$$

$$|\iint_R (b_1 - b_2) w_{2,y}\, w_{yy}\, dydt| \le C\, \|h\|_{1,t} \sup_t \|w_{2,y}(t)\|_2 \|w_{yy}(t)\|_2,$$

$$|\iint_R (c_1 - c_2)((w_2)^2)_y\, w_{yy}\, dydt| \le C\, \|h\|_{0,t} \|w_2(t)\|_\infty \|w_{2,y}(t)\|_2 \|w_{yy}(t)\|_2.$$

Summing up and working a bit, we get the result. ∎

Once the estimates for w in terms of h are obtained, we continue with the proof of uniqueness. The next step is to use the trace theorem and then the dynamic condition (3.30) to transfer the information to h'' and get a contraction result.

The technical tool consists of using Rellich's multiplier, of the form $\zeta(y,t) = (y-c)w_y$, in each interval I_i^*. We get the expression for the trace. Let us write the interval as (h_1, h_2) for ease of notation. Integration of $\partial_y((y-h_1)\, w_y^2)$ in the rectangle $\mathcal{R} = (h_1, h_2) \times (0,t)$ gives

$$(3.37)\quad \begin{aligned} &(h_2 - h_1) \int_0^t (w_y(h_2-, t))^2\, dt = 2 \iint (y-h_1) w_y w_{yy}\, dydt + \iint (w_y)^2\, dydt \\ &= \iint (2(y-h_1)w_y - w))\, w_{yy}\, dydt \le C \left(\iint w_{yy}^2\, dydt \right)^{1/2} \left(\iint (w_y^2 + w^2)\, dydt \right)^{1/2}, \end{aligned}$$

with double integrals in $\mathcal{R}$. An analogous result is obtained for the trace on the boundary $y = h_1$. In view of the previous lemmas, this implies an improved estimate for the trace at the lateral boundaries of the form

$$(3.38)\quad \int_0^t (w_x)^2 (h_i(t)+, t)\, dt,\ \int_0^t (w_x)^2 (h_{i+1}(t)-, t)\, dt \le K \|h''\|_{L^2(0,t)} \|w\|_{L^2(0,t:H^1(I))}\, .$$

Moreover, we have $w \in L^\infty(0,T : H^1(R))$. Hence, $\|w\|_{L^2(0,t:H^1(I))} \le t^{1/2}\, \|w\|_{L^\infty(0,t:H^1(I))}$, which puts an extra factor $t^{1/2}$ into the desired constant. Using the dynamical condition together with (3.38), we get

$$(3.39)\quad \|h''(t)\|_{L^2(0,t)} \le C t^{1/2} \|h\|_{2,t}.$$

Uniqueness in the small follows. By a standard continuation argument we get uniqueness as far as both solutions are defined.

Moreover, we can prove along the same lines a stronger result, i.e., continuous dependence of the solution with respect to H^1-perturbations of the initial data u_0. ∎

3.4 Continuation

We have found existence and uniqueness of a strong solution in a time interval $(0, T)$ where the time T depends on the bounds for the norms of the data u_0 in $H^1(\mathbb{R})$, and on the data $h_{i,1}$, $1 \le i \le N$ and also on lower bounds for the distances $l_i = h_{i+1,0} - h_{i,0}$, $1 \le i \le N-1$. We see that the only obstruction to the continuation in time of a solution is the possibility of two particles meeting. The continuation of the solution as long as the curves $h_i(t)$ do not meet is a standard argument. ■

As a conclusion of the study of this subsection we have obtained the following result, that we formulate in terms of u without any new difficulty.

Proposition 3.2 *Under the assumptions of Theorem* 2.1 *there exists a time interval* $0 \le t \le T$ *in which Problem* (**P**) *admits a unique strong solution. The solution can be continued (in a unique way) until two particles collide, or for all time if they never do.*

We call $T = T_{max}(u_0, h_1, \cdot, h_N)$ the maximal time of existence. It will be finite only if mass collision occurs.

The estimates of Subsection 3.1 are now justified. We can in particular conclude that u is bounded in $Q = \mathbb{R} \times (0, T)$ and so are the interface speeds $h_i'(t)$, $0 \le t \le T$, by virtue of estimate (3.4). Actually, as time increases we will get decay estimates that say that both u and $h_i'(t)$ go to zero with suitable rates. A detailed analysis will be performed in Section 7.

3.5 Further regularity

Once we have the basic regularity $h_i''(t) \in H^s$ for every $s < 1/4$, we may use a bootstrap argument to get further regularity of the interfaces $h_i(t)$.

Indeed, $h_i''(t) \in H^s$ with $s < 1/4$. We go to the equation satisfied by w to get better regularity in a H^σ space, and $H^{\sigma-2}$ regularity for w_t. Hence better w_y traces. We now use the dynamics of the trajectories, $m_i\, h_i'' = [u_x(h_i(t), t)]$ to get an improved set of functions $h_i(t)$. The bootstrap argument allows one to iterate until we get $h_i \in C^\infty(0, T)$. Summing up, we get

Theorem 3.1 *The trajectories* $h_i(t)$ *are* C^∞ *curves in the interval* $(0, T)$*, and* C^1 *down to* $t = 0$*. The solution* u *is a smooth function of* x *and* t *in each region between trajectories; more precisely, it is* C^∞ *smooth up to the lateral boundaries of those domains.*

4 Global strong solutions without collision

We now address the problem of eliminating the possibility of collision, so that the existence, uniqueness and regularity of strong solutions proved so far can be extended to the interval $0 < t < \infty$.

4.1 Improved H^2 estimates

In order to perform the analysis near a possible collision point we need a better understanding of the derivative estimates, that we work out in this section in terms of the original variable u. We will prove that the constants in some important estimates do not blow up as we approach a possible collision. While the beginning of the proof follows more or less standard ideas, already used in [VZ], the end of the proof implies a quite sophisticated use of differential inequalities on a priori bounds to eliminate the danger of blow-up caused by a cubic term of the form $\int u_x^3\,dx$. This term is originated by the particles and does not appear in free flow.

Proposition 4.1 *Every strong solution of problem* (1.2) *with* $u_0 \in H^1(\mathbb{R})$*, defined in a time interval* $(0,T)$ *without collision, satisfies the estimates:*

$$\int_{\mathbb{R}} u_x(x,t)^2\,dx \le C(u,T)\int_{\mathbb{R}} u_{0,x}^2\,dx\,, \tag{4.1}$$

for all $t \in (0,T)$*, and*

$$\sum_{i=0}^{N}\int_0^T \|u_{xx}\|^2_{L^2(I_i(t))}dt + \sum_{i=1}^{N}\int_0^T m_i\,|h_i''(t)|^2 \leqslant C(u,T). \tag{4.2}$$

Moreover, C *depends on the particular solution only through the norms of the initial data, specifically, the norm of* u_0 *in* $H^1(\mathbb{R})$ *and* $\sum_i m_i h_{i,1}^2$.

Proof. The proof has delicate arguments, so we will divide the proof into several steps. To better keep track of the constants that appear in the proof, whenever they are not explicit, they are denoted by C if they depend on the stated norms of the data, and by small type, c, if they do not (i.e., if they are universal).

(i) We begin by multiplying the parabolic equation satisfied by u by u_{xx} and integrating on $h_i(t) < x < h_{i+1}(t)$, which gives

$$\int_{h_i(t)}^{h_{i+1}(t)} |u_{xx}|^2 dx = \int_{h_i(t)}^{h_{i+1}(t)} u_t u_{xx} dx + 2\kappa \int_{h_i(t)}^{h_{i+1}(t)} u u_x u_{xx} dx. \tag{4.3}$$

We now analyze the two integrals on the right hand side of (4.3). First, we have

$$\int_{h_i(t)}^{h_{i+1}(t)} u_t u_{xx} dx = -\frac{1}{2}\int_{h_i(t)}^{h_{i+1}(t)} \partial_t\left(|u_x|^2\right) + u_t\,u_x\Big|_{h_i(t)}^{h_{i+1}(t)}. \tag{4.4}$$

(ii) Let us examine the last term. As a consequence of the interface conditions, after differentiating we have

$$u_t(h_i(t)\pm 0,t) = h_i''(t) - h_i'(t)u_x(h_i(t)\pm 0,t),$$

where we have used the lateral values for the derivatives, since the solutions are supposed to be continuous but they may have derivative jumps across the lines $x = h_i(t)$. The notation $u_x(h_i(t)+0,t)$ (resp. $u_x(h_i(t)-0,t)$) means of course the limit of $u_x(x,t)$ as x tends to $h_i(t)$ from above (resp. from below). We abbreviate $h_i(t) \pm 0$ into the simpler notation $h_i(t)^{\pm}$. We get

$$u_t\, u_x\Big|_{h_i(t)}^{h_{i+1}(t)} = \{h''(t)u_x(h(t),t) - h'(t)(u_x)^2(h(t),t)\}\Big|_{h(t)=h_i(t)+}^{h(t)=h_{i+1}(t)^-},$$

where the lateral limits are only important in the argument of u_x.

(iii) In the same way, the integral in the right-hand side of (4.4) can be computed as

$$-\frac{1}{2}\int_{h_i(t)}^{h_{i+1}(t)} \partial_t\left(|u_x|^2\right) = -\frac{1}{2}\frac{d}{dt}\int_{h_i(t)}^{h_{i+1}(t)} |u_x|^2 dx + \frac{1}{2}(|u_x(x,t)|^2 h')\Big|_{h_i(t)^+}^{h_{i+1}(t)^-}.$$

In this way we get

$$\int_{h_i(t)}^{h_{i+1}(t)} u_t\, u_{xx}\, dx = -\frac{1}{2}\frac{d}{dt}\int_{h_i(t)}^{h_{i+1}(t)} |u_x|^2 dx + \Big(u_x(h'' - \frac{u_x h'}{2})\Big)\Big|_{h_i(t)^+}^{h_{i+1}(t)^-}. \tag{4.5}$$

We remark that, in view of the ODE governing the acceleration of the particles in (1.1),

$$m_i\,(h_i''(t))^2 = h_i''(t)\,[u_x(h_i,\, t)]\,,$$

with $[u_x(h_i(t),\, t)] = u_x(h_i(t)^+,\, t) - u_x(h_i(t)^-,\, t)$.

The second integral in (4.3) can be estimated as follows:

$$2\kappa\int_{h_i(t)}^{h_{i+1}(t)} |u u_x u_{xx}| dx \leqslant 2\kappa^2 \int_{h_i(t)}^{h_{i+1}(t)} u^2 u_x^2 dx + \frac{1}{2}\int_{h_i(t)}^{h_{i+1}(t)} |u_{xx}|^2 dx. \tag{4.6}$$

(iv) Let us now put together the results. Making the same calculation in each interval $I_i(t)$, $i = 0, \ldots, N$ and adding all the above expressions, we deduce that

$$\left\{\begin{aligned} &\sum_{i=0}^{N}\int_{I_i(t)} |u_{xx}|^2 dx + \frac{d}{dt}\int_{\mathbb{R}} |u_x|^2 dx + 2\sum_{i=1}^{N} m_i\, \left|h_i''(t)\right|^2 \\ &\qquad \leqslant \sum_{i=1}^{N} h_i'(t)\left[(u_x)^2(h_i,\, t)\right]_{h_i^-}^{h_i^+} + 4\kappa^2\int_{\mathbb{R}} u^2 u_x^2 dx. \end{aligned}\right. \tag{4.7}$$

(v) The last term on the right-hand side of (4.7) poses no problem since u is bounded. The first term however is more difficult to treat and poses a severe problem. In order to estimate it, we take into account the fact that $h_i'(t) = u(h_i(t),t)$, so that we can write

$$J = \sum_{i=N}^{1}\left[(u\, u_x^2)(h_i,\, t)\right]_{h_i^-}^{h_i^+} = \sum_{i=0}^{N}\int_{I_i} \partial_x\left(u u_x^2\right)\, dx = \sum_{i=0}^{N}\int_{I_i}\left(u_x^3 + 2u\, u_x\, u_{xx}\right)\, dx\,. \tag{4.8}$$

The right-hand member of (4.7) is therefore bounded by

$$\frac{1}{2}\sum_{i=0}^{N}\int_{I_i(t)}|u_{xx}|^2+4(1+\kappa^2)\int_{\mathbb{R}}u^2u_x^2dx+\int_{\mathbb{R}}|u_x|^3\,dx\,. \tag{4.9}$$

We can now go back to (4.7) and get for all $0<t<T$:

$$\begin{aligned}\frac{1}{2}\sum_{i=0}^{N}\int_{I_i(t)}|u_{xx}|^2dx+\frac{d}{dt}\int_{\mathbb{R}}|u_x|^2dx+2\sum_{i=1}^{N}m_i\,|h_i''(t)|^2\\ \leqslant C_1(t)\int_{\mathbb{R}}|u_x|^2\,dx+\int_{\mathbb{R}}|u_x|^3\,dx,\end{aligned} \tag{4.10}$$

where $C_1(t)=4(1+\kappa^2)\,\|u(t)\|^2_{L^\infty(\mathbb{R})}$. This is the basic integral inequality from which the estimates are derived.

(vi) Now, $u_0\in H^1(\mathbb{R})$ implies that $u_0\in L^\infty(\mathbb{R})$ and we know that the L^∞ norm of a solution, $\|u(t)\|_\infty$, is controlled uniformly for all the existence time, see formula (3.4). Therefore, $C_1(t)$ is a priori bounded by C, a function of $\|u_0\|_{H^1(\mathbb{R})}$ and $E(0)$. In the absence of the last term which involves a cubic integral we would use a Gronwall argument to get

$$\int_{\mathbb{R}}u_x(x,t)^2\,dx\le e^{Ct}\int_{\mathbb{R}}u_x(x,0)^2\,dx, \tag{4.11}$$

for all $0\le t\le T$.

(vii) But we still have to control the effect of the cubic nonlinearity. In principle, this could lead to blow-up in the estimate for $\int u_x^2\,dx$. Blow-up will be avoided by using, in particular, the fact that $\iint u_x^2\,dxdt$ is bounded (by the first energy estimate (3.1)). The argument goes in two steps as follows.

To begin with, we estimate $\int u_x^3\,dx$ by interpolating between the norm of u_x in L^2 and the norm of u_{xx} in L^2 plus the amplitude of the possible jumps of u_x of value $|h''(t)|$. To do this we use the following interpolation inequality: Let us consider the class of L^2 functions f such that its derivatives f_x may be decomposed into a regular part, belonging to L^∞ (that we shall denote by $f_{x,reg}$ and a finite number N of Dirac deltas with finite amplitude a_i located at points x_i, $i=1,...,N$, then

$$||f||_\infty\le c\{||f||_2^{1/2}||f_{x,reg}||_2^{1/2}+\sum_{i=1}^{N}|a_i|\}. \tag{4.12}$$

We will prove the estimate below for the reader's convenience. Applying this inequality with $f=u_x$ to estimate the cubic term, we have for fixed $t>0$

$$\int|u_x|^3\,dx\le c||u_x||_2^{5/2}||u_{xx,reg}||_2^{1/2}+c||u_x||_2^2\sum|h_i''(t)|. \tag{4.13}$$

Using Young's inequality we get for every $\epsilon > 0$ a constant $c > 0$ such that

$$\int |u_x|^3 \, dx \leq \epsilon \sum |h_i''(t)|^2 + \epsilon \int u_{xx,reg}^2 \, dx + c \left(\int u_x^2 \, dx \right)^{5/3} + c \left(\int u_x^2 \, dx \right)^2 . \tag{4.14}$$

According to this estimate, the inequality in (4.10) now reads as follows

$$\begin{aligned} &\frac{1}{4} \sum_{i=0}^{N} \int_{I_i(t)} |u_{xx}|^2 dx + \frac{d}{dt} \int_{\mathbb{R}} |u_x|^2 dx + \sum_{i=1}^{N} m_i \, |h_i''(t)|^2 \\ &\leqslant C \left[\int_{\mathbb{R}} |u_x|^2 \, dx + \left(\int u_x^2 \, dx \right)^{5/3} + \left(\int u_x^2 \, dx \right)^2 \right] \leq C \left[\int u_x^2 \, dx + \left(\int u_x^2 \, dx \right)^2 \right]. \end{aligned} \tag{4.15}$$

In view of this, for $Y(t) = 1 + \int u_x(x,t)^2 \, dx$ we get a differential inequality of the form

$$\frac{dY}{dt} \leq C \, Y^2. \tag{4.16}$$

This inequality can in principle lead to blow-up in finite time. However, we know that for every $0 \leq t \leq T$

$$\int_t^T Y(t) \, dt = T + \int_t^T \int_{\mathbb{R}} u_x^2 \, dx dt \leq T + E(0) \tag{4.17}$$

where $E(0)$ is the energy of the initial data. This implies that blow-up does not occur [2]. We can write inequality (4.16) in the form

$$\frac{dY}{dt} \leq A(t) \, Y, \quad A(t) = CY(t), \tag{4.18}$$

and apply Gronwall to get

$$Y(t) \leq Y_0 \, e^B, \quad B(t) = \int_0^t A(t) \, dt \leq C(t + E(0)), \tag{4.19}$$

where C is the constant of formula (4.11). Therefore, $Y(t)$ does not blow up. We conclude that $\int_{\mathbb{R}} u_x^2 \, dx$ is uniformly bounded in time in the form (4.1) and

$$C(u,T) \leq c \|u_0\|_\infty^2 (T + E(0)), \tag{4.20}$$

with c a universal constant that depends only on κ. Moreover, $\int_{\mathbb{R}} u_x^3 \, dx$ is also bounded in time by the interpolation inequality (4.14).

[2] We thank Alessio Porretta for the present simple version of the argument to avoid blow-up.

(viii) Once $\int_{\mathbb{R}} u_x^2\,dx$ is uniformly bounded, we integrate inequality (4.15) with respect to time in the interval $0 \leqslant t \leqslant T$ to get

$$\sum_{i=0}^{N} \int_{0,T} \|u_{xx}\|^2_{L^2(I_i(t))} dt + \sum_{i=1}^{N} \int_0^T m_i\, |h_i''(t)|^2$$

$$\leqslant 4 \int u_{0,x}^2\,dx + C \int_0^T \left[\int_{\mathbb{R}} u_x^2\,dx + (\int_{\mathbb{R}} u_x^2\,dx)^2 \right] dt \leq C(u,T).$$

This concludes the proof of the Proposition. ■

Remarks. (1) The important fact in these estimates is that they are controlled in terms of the norms of the data, uniformly in any finite time interval. In particular, they are independent of the distances between particles before collision. Therefore, blow up of these quantities is avoided for $T < \infty$. As we have announced, this will be basic tool in the proof that collision cannot occur.

(2) *Proof of Estimate* (4.12): We have

$$f^2(x) = -\int_{-\infty}^{x} 2 f f_{x,reg} dx + \sum [f^2](x_i),$$

where the sum runs over the indexes i such that the points of discontinuity x_i lie in the interval of integration $(-\infty, x)$. We now observe that

$$|[f^2](x_i)| \leq 2||f||_\infty |a_i|.$$

Consequently, it follows that

$$||f||_\infty^2 \leq 2||f||_2 ||f_{x,reg}||_2 + 2||f||_\infty \sum_{i=1}^{N} |a_i|.$$

Resolving this quadratic inequality for $||f||_\infty$ we get the claimed interpolation inequality (4.12). ■

4.2 Lack of collision. End of proof of Theorem 2.1

As described above, once the local (in time) existence and uniqueness of the solutions of (1.1) has been proved, the solution $(u,\, h_1, \ldots,\, h_N)$ can be extended to a maximal time interval $[0,\, T)$ with $T = T_{max}$ by a standard procedure. In view of the above estimates, finite-time blow-up is excluded as long as no collision occurs. The following alternative holds for the maximal time of existence $T_{\max}$:

- Either $T_{\max} = \infty$, and the solution $(u,\, h_1, \ldots\, h_N)$ is globally defined in time; or

- $T_{\max} < \infty$ and, in this case we have finite-time collision:

$$\exists i \neq j;\ h_i(T_{\max}) = h_j(T_{\max}). \tag{4.21}$$

Consequently, it is sufficient to exclude this possibility. Indeed, a main contribution of the paper is precisely proving that collision does not occur. This fact will follow from a uniqueness argument for the ODE (2.9), i.e.,

$$h_i'(t) = u(h_i(t),\, t)$$

which governs the dynamics of the particles, since collision means non-uniqueness backwards in time for this equation, starting at a collision time.

In view of the equation, uniqueness holds if u remains Lipschitz continuous in space with a Lipschitz constant that is integrable in time in each space interval limited by consecutive particles. Indeed, from (2.9) we deduce that

$$|h_1'(t) - h_2'(t)| = |u(h_1(t), t) - u(h_2(t), t)| \leq M(t)|h_1(t) - h_2(t)| \tag{4.22}$$

where $M(t) = ||u_x(\cdot, t)||_{L^\infty(h_1(t), h_2(t))}$. If $M(t) \in L^1(\delta, T)$, then an application of Gronwall's Lemma implies that the data at time $t = T$ determine uniquely the solution of equation (2.8). Now, the fact that $M(t) \in L^2(\delta, T)$ is a consequence of Proposition 4.2 below. ■

Proposition 4.2 *Under the assumptions of Theorem* 2.1*, we have* $u_x \in L^2((0, T) : L^\infty(\mathbb{R}))$.

Proof. This is an immediate consequence of the interpolation inequality (4.12) and the estimates (4.1) and (4.2). ■

5 Solutions with L^2 data

We now extend the class of initial data to consider $u_0 \in L^2(\mathbb{R})$. As we will see, existence of a solution is easy and the solution is strong for all times $t \geq \tau > 0$. More precisely, we have

Theorem 5.1 *Given a function $u_0 \in L^2(\mathbb{R})$, mass locations $h_{i,0} \in \mathbb{R}$ as in* (2.1)*, and speeds $h_{i,1} \in \mathbb{R}$, there exists a unique function $u \in C([0, \infty) : L^2(\mathbb{R}))$ and functions $h_i(t) \in C^1([0, \infty))$ such that for all $t \geq \tau > 0$ they form a strong solution of the problem posed for $t \geq \tau$, and besides u and h_i take the prescribed initial data at $t = 0$. Moreover, there exists a constant $C > 0$, depending only on the L^2 norm of u_0 and the initial data of the particles, such that*

$$\|u_x(\cdot, t)\|_{L^2(\mathbb{R})} \leq C\, t^{-1/2}, \quad \|u(t)\|_\infty \leq C\, t^{-1/4}, \quad |h_i'(t)| \leq C\, t^{-1/4}. \tag{5.1}$$

Besides, the function tu_{xx} is uniformly bounded in $L^2(Q_T)$ for all finite T. Concerning the interfaces, the expressions $t^{1/2}h_i''(t)$ are uniformly bounded in $L^2(0, T)$. Finally, the initial data of $h_i'(t)$ are taken with the rate $|h_i'(t) - h_{1,i}| \leq Ct^{1/4}$.

NOTE. In the previous statement, we use the notation u_{xx} for the second derivative function defined a.e. away from the particle trajectories, forgetting for ease of notation the derivative jumps at the curves $x = h_i(t)$. We should have written $u_{xx,reg}$.

Proof. EXISTENCE. The construction of the solution proceeds by approximation of u_0 by a sequence $u_{0n} \in H^1(\mathbb{R})$, so that compatibility holds at $t = 0$ between u_{0n} and the initial particle speeds. We obtain a sequence of strong solutions u_n. We then pass to the limit $n \to \infty$. The existence of the limit is guaranteed once we have the uniform energy estimates (3.1).

We want to prove that the limit is a strong solution for all times $t \geq \tau > 0$, and for that we need further uniform estimates, improving the estimates of Proposition 4.1 to make them depend on the initial data only through $E(0)$ (and possibly T), but not on $\|u_{0,x}\|_2$. Let us review the end stage of the derivation of these estimates. This is how to proceed. Thanks to estimate (3.1), we know that $u(\cdot, t) \in L^2(0, T : H^1(\mathbb{R})) \cap L^\infty(0, T : L^2(\mathbb{R}))$ with bounds that depend only on $\int u_0^2\, dx + \sum_i m_i h_{i,1}^2 = 2E(0)$. It follows that $u \in L^2(0, T : L^\infty(\mathbb{R}))$ with a similar bound, see (3.23). It follows that the constant $C_1 = C_1(t)$ entering in (4.10) is such that

$$C_1(t) \leq 2E(0) + \|u(t)\|_2 \|u_x(t)\|_2 \in L^2(0, T),$$

and

$$\int_0^t C_1(t)\, dt \leq 2E(0)t + \|u\|_{L^\infty(0,t:L^2(\mathbb{R}))} \|u_x\|_{L^2(Q)} t^{1/2}.$$

We now consider in greater detail the inequality satisfied by $Y = 1 + \int u_x^2\, dx$. Instead of (4.16) we may write

$$\frac{dY}{dt} \leq C_1(t)\, Y + c\, Y^2, \tag{5.2}$$

where c is universal number. This inequality can be integrated explicitly (put $Z = 1/Y$) and we get an inequality of the form

$$Y(t) \geq Y(t_2) \left(e^{\int_t^{t_2} C_1(s)\, ds} + cY(t_2) \int_t^{t_2} e^{\int_r^{t_2} C_1(s)\, ds} dr \right)^{-1}.$$

valid for $0 \leq t \leq t_2 \leq T$. We combine this estimate with the a priori bound

$$\int_0^T Y(t)\, dt = \int_0^T \int_{\mathbb{R}} u_x^2\, dxdt \leq E(0)$$

to get an estimate for $Y(t_2)$ of the form $Y(t_2) \leq F/t_2$, with F only depending on $E(0)$ and T.

Corollary 5.1 *For every strong solution and every time interval* $0 \leq t \leq T$ *we have*

$$\int_{\mathbb{R}} u_x(x, t)^2\, dx \leq F(E(0), T)\, t^{-1}. \tag{5.3}$$

Moreover,

$$|u(x,t)| \leq F_1(E(0),T)\, t^{-1/4}. \tag{5.4}$$

Here, F and F_1 are bounded, positive and increasing functions of both arguments.

Proof. For the last inequality we use the interpolation formula $\|u(t)\|_\infty \leq C\|u(t)\|_2^{1/2}\|u_x(t)\|_2^{1/2}$, already used in (3.23). ∎

This result yields the second and third estimate of (5.1), that u satisfies as a limit of the estimates for the u_n.

In order to see that that u is a strong solution for $t \geq \tau > 0$ we need to check the conditions on the particle, and we would like to have estimates on h_i'', since the estimates on h_i' are immediate from the energy estimate (3.1). In order to obtain uniform second-order estimates, we go back to (4.10), that we multiply by t to avoid singularities at $t = 0$. We integrate in time from $t = \tau$ to $t = T$ to get

$$\begin{aligned} &\frac{1}{2}\sum_{i=0}^{N}\int_\tau^T\int_{I_i(t)} t\,|u_{xx}|^2 dxdt + 2\sum_{i=1}^{N}\int_\tau^T tm_i\,|h_i''(t)|^2\,dt + \left[t\int_{\mathbb{R}}|u_x|^2 dx\right]_\tau^T \\ &\quad\leqslant \int_\tau^T tC(t)\int_{\mathbb{R}}|u_x|^2\,dxdt + \int_\tau^T\int_{\mathbb{R}}|u_x|^2\,dxdt + \int_\tau^T\int_{\mathbb{R}} t|u_x|^3\,dxdt. \end{aligned} \tag{5.5}$$

where $C(t) = C\|u(t)\|^2_{L^\infty(\mathbb{R})} = O(t^{-1/2})$, according to (5.4). In view of the energy estimate, the right-hand is bounded if we control the last term. Arguing as in (4.13) it is estimated as follows using (4.12):

$$\int_\tau^T\int_{\mathbb{R}} tu_x^3\,dxdt \leq \int_\tau^T\int_{\mathbb{R}} ct||u_x||_2^{5/2}||u_{xx,reg}||_2^{1/2}\,dxdt + \int_\tau^T t\|h''(t)\|(\int_{\mathbb{R}} u_x^2\,dx)\,dt, \tag{5.6}$$

where we write $\|h''(t)\| = \sum_i |h_i''(t)|$ for short. We write the right-hand side as $I_1 + I_2$. Let us concentrate on the last term. Using the estimate $\|u_x(t)\|_2 \leq Ct^{-1/2}$ in (5.3), we get

$$I_2 \leq C\int_\tau^T t^{1/2}|h''(t)|\|u_x\|_2 \leq \epsilon\int_\tau^T t\,(\sum_i |h_i''(t)|^2)\,dt + \int_\tau^T\int_{\mathbb{R}} u_x^2\,dxdt. \tag{5.7}$$

The first term in the right-hand side is absorbed by the left-hand side of (5.5) and the last term is bounded, uniformly as $\tau \to 0$, by the energy estimate. On the other hand, in view of (5.3),

$$\begin{aligned} I_1 &\leq c\int_\tau^T\int_{\mathbb{R}} t||u_x||_2^{5/2}||u_{xx,reg}||_2^{1/2}\,dxdt \leq \varepsilon\int_\tau^T\int_{\mathbb{R}} t||u_{xx,reg}||_2^2\,dxdt \\ &+C_\varepsilon\int_\tau^T\int_{\mathbb{R}} t||u_x||_2^{10/3}\,dxdt \leq \varepsilon\int_\tau^T\int_{\mathbb{R}} t||u_{xx,reg}||_2^2\,dxdt + C_\varepsilon\int_\tau^T\int_{\mathbb{R}} t^{1/3}||u_x||_2^2\,dxdt, \end{aligned} \tag{5.8}$$

which can be handled in a similar way.

Uniqueness. We will use the above estimates of the smoothing effect at the initial time also at the level of second derivatives. Then, we use the Fixed Point Argument as in the uniqueness of strong solutions, but now we use a space with time weight. Repeating the calculation of (3.37) for $w = w_1 - w_2$ and using time t as a weight we get

$$(h_{i+1} - h_i) \int_0^t t(w_y(h_{i+1}^-, t))^2 dt \le C \left(\iint s w_{yy}^2 \, dyds \right)^{1/2} \left(\iint_0^t s\,(w_y^2 + w^2)\, dyds \right)^{1/2} .$$

We can obtain estimates for $w = w_1 - w_2$ that are completely similar to the estimates just done for u, using also the technical details of Lemmas 3.1 and 3.2. From them we deduce that

$$\iint t w_{yy}^2 dydt \le C \|t^{1/2} h''\|_2^2, \qquad \int (w_y^2 + w^2)\, dy \le C \|t^{1/2} h''\|_2^2 / t,$$

so that the last integral is estimated as

$$\int_0^t t\, dt \int (w_y^2 + w^2)\, dy \le Ct \|t^{1/2} h''\|_2^2.$$

With these bounds we arrive at

$$\int_0^t t\,(w_y(h_{i+1}^-, t))^2 dt \le Ct^{1/2} \|th''\|^2,$$

and by the dynamical condition at the interface we get

$$\|th''\|_{L^2} \le Ct^{1/4} \|th''\|_{L^2}$$

which implies uniqueness.

Initial speeds. We now check the initial behavior more closely. Indeed, the calculation done a few lines ago can be done with a lower power of t

$$\int_0^t t^s (w_y(h_{i+1}-, t))^2 dt \le C \left(\iint t w_{yy}^2 \, dydt \right)^{1/2} \left(\iint_0^t t^{2s-1}\,(w_y^2 + w^2)\, dydt \right)^{1/2},$$

and the last integral is convergent as long as $2s - 1 \ge 0$, i.e., $s \ge 1/2$. Using the dynamical condition, we conclude that $t^{s/2} h'' \in L^2(0, T)$. Integrating h'' and using Hölder's inequality, we get $h' \in C^\alpha([0, \infty)$ for every $\alpha = (1 - s)/2 \le 1/4$. This gives a detailed proof of how fast the initial speeds are taken.

Decay for large times. We still need to control the behavior of the constants in the decay estimates for large times, since the analysis performed in Subsection 4.1, on which we rely, and the improvement in this section, leaves a possibility of bad behaviour with large T that we want to eliminate. Indeed, it has been proved in [VZ] in the case of one single particle that when the initial datum belongs to $L^2(\mathbb{R})$ the solution decays like

$$\| u(t) \|_{L^p(\mathbb{R})} \leqslant C t^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}, \quad \forall t \geqslant 0 \tag{5.9}$$

for all $2 \leqslant p \leqslant \infty$ with C depending on p and $E(0)$ (which coincides with the decay rate of solutions of the linear heat equation). Taking $p = \infty$ we see that the constant $C(t)$ appearing in estimate (5.5) is bounded by $C/\sqrt{t}$ uniformly in time and C depends only on $E(0)$. It follows that

$$t \int_{\mathbb{R}} |u_x|^2 dx \leq C,$$

for a similar constant. The estimates (5.1) of the Theorem hold now uniformly in time. ■

6 Lack of collision with exterior boundaries.

In this section we sketch the proof of the global existence of solutions of the above fluid-solid system in the presence of exterior boundaries, i.e., side walls. As in previous sections, the main difficulty consists on proving that collision does not occur. But, in this case, two different collision phenomena need to be analyzed:

- The possibility that two particles collide in finite time.
- The collision of a solid mass with the exterior boundary.

To simplify the presentation, we analyze the Dirichlet problem on the half-line $x > 0$. But the same result can be obtained, with similar techniques, for the Dirichlet problem in a bounded interval. We also put $m_i = 1$, $\kappa = 1$. The system we consider reads as follows:

$$\begin{cases} u_t - u_{xx} + (u^2)_x = 0, & x \in I_i(t), \quad i = 0, \dots, N, \quad t > 0 \\ h_i'(t) = u(h_i(t), t), & i = 1, \dots, N, \quad t > 0 \\ h_i''(t) = [u_x](h_i(t), t), & i = 1, \dots, N, \quad t > 0 \\ u(0,t) = 0, & t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R} \\ h_i(0) = h_{0,i}, & h_i'(0) = h_{1,i}, \quad i = 1, \dots, N. \end{cases} \tag{6.1}$$

The masses m_i have been taken to be $m_i = 1$ for simplicity.

The main difference between this new system (6.1) and the one analyzed in the previous section is the fact that, now, we consider a Dirichlet boundary condition at $x = 0$. Accordingly, the definition of the first interval $I_0(t)$ needs to be modified. This time we have

$$I_0(t) = (0,\, h_1(t)),\ t > 0. \tag{6.2}$$

The definition of the other intervals remains unchanged (see (1.3)).

As in the previous section, we assume that the N particles occupy distinct points at the initial time away from the boundary point $x = 0$, i.e.

$$0 < h_{1,0} < h_{1,0} < \cdots < h_{N,0}. \tag{6.3}$$

We have the following result on the global existence and uniqueness of solutions:

Theorem 6.1 *For any*

$$(u_0,\, h_{1,0},\, h_{1,1}, \ldots,\, h_{N,0},\, h_{N,1}) \in L^2(\mathbb{R}^+) \times [\mathbb{R} \times \mathbb{R}]^N, \tag{6.4}$$

such that (6.3) *holds, there exists a unique global solution* $u \in C\left([0,\infty); L^2(\mathbb{R}^+)\right)$, $h_i(t) \in C^1\left([0,\infty); \mathbb{R}^+\right)$, $i = 1, \ldots, N$.

In particular, particles do not collide in finite time and neither they do with the exterior boundary $x = 0$. *In other words,*

$$0 < h_1(t) < h_2(t) < \ldots < h_N(t), \quad \forall t > 0. \tag{6.5}$$

When the initial data u_0 *belong to the space* $H^1(\mathbb{R}_+)$ *and the compatibility conditions (2.2) are satisfied, the solution is strong, and, in particular,* $u \in C([0,\infty) : H^1(\mathbb{R}_+))$.

Sketch of the proof. It is very similar to the proof of the problem posed in the whole line. We jump on the derivation of the energy identity, the L^p-decay, and the local existence of strong and weak solutions and concentrate on reviewing the argument for the lack of collision.

Lack of collision.

The local (in time) solutions can be continued up to a maximal time of existence $0 < T_{\max}$ both for weak and strong solutions. We concentrate on the latter, and once we have solved that problem, the first is then solved as in the previous case.

The goal is now proving that $T_{\max} = \infty$.

We argue as in Step 2 of the proof of Theorem 2.1. Either $T_{\max} = \infty$ or $T_{\max} < \infty$. In the later case two possibilities arise. Either there is finite-time blow-up or collision occurs. Moreover, in the later case, there are also different types of collision that may occur

$$h_i(T) = h_j(T) \text{ for some } i \neq j$$

or

$$h_1(T) = 0.$$

Finite-time blow-up of the energy can be easily excluded by the energy dissipation law (2.6) -(2.7). The fact that two different point particles collide can also be excluded, arguing as in the proof of Theorem 2.1. Consequently, the proof of global existence is reduced to showing that the point mass $h_1(t)$ may not collide with the boundary point at $x = 0$. To see this, we argue as in the proof of Theorem 2.1. Assume that

$$h_1(T) = 0. \tag{6.6}$$

Then, $h_1 = h_1(t)$ is a solution of the following Cauchy problem:

$$\begin{cases} h_1'(t) = u(h_1(t),\, t), \quad 0 < t < T \\ h_1(T) = 0. \end{cases} \tag{6.7}$$

Obviously $h \equiv 0$ is also a solution of (6.6) because of the boundary condition $u(0,t) = 0$ that u satisfies.

Therefore, it is sufficient to show the uniqueness of solutions of (6.7). Indeed, if uniqueness holds, then $h_1 \equiv 0$ and consequently $h_{1,0} = 0$, which is in contradiction with the assumptions we have made on the initial data.

To prove uniqueness of solutions of (6.7) it is sufficient to show that

$$\int_{\delta}^{T_{\max}} \int_0^{h_1(t)} \Big| u_{xx} \Big|^2 dxdt < \infty, \tag{6.8}$$

for all $0 < \delta < T_{\max}$.

To prove that (6.8) holds we argue as in the proof of Theorem 2.1, multiplying the equation in (6.1) and integrating by parts. In this case we have

$$\int_0^{h_1(t)} \Big| u_{xx} \Big|^2 dx = \int_0^{h_1(t)} u_t u_{xx} dx + 2 \int_0^{h_1(t)} u u_x dx \tag{6.9}$$

Moreover

$$\int_0^{h_1(t)} u_t u_{xx} dx = \frac{-1}{2} \int_0^{h_1(t)} \partial_t(|\, u_x \,|^2) dx + u_t u_x(h_1(t),\, t). \tag{6.10}$$

Indeed, at this point we observe that the boundary term corresponding to $x = 0$ vanishes since $u_t(0,t) = 0$, which holds because of the Dirichlet boundary condition on u.

Thus

$$\begin{aligned} \int_0^{h_1(t)} u_t u_{xx} dx = -\frac{1}{2} \frac{d}{dt} \int_0^{h_1(t)} |\, u_x \,|^2 \, dx \quad &+ |\, u_x(h_1(t),\, t) \,|^2 \, h_1'(t) \\ &+ \Big(h_1''(t) - u_x(h_1(t),\, t) h_1'(t) \Big) u_x(h_1(t),\, t). \end{aligned}$$

Once this identity has been proved (which replaces the integral over the infinite interval $(-\infty, 0)$ in the context of the Cauchy problem we analyzed in the previous section) the proof of (6.8) can be carried following the arguments in the proof of Theorem 1.1, without any further change. This concludes the proof of Theorem 6.1. ■

7 Asymptotic behavior

Once global existence is known to hold, the scaling and compactness results in [VZ] can be easily adapted to the present situation to obtain the asymptotic description of the dynamics as $t \to \infty$. We show that the effect of the balls is a small perturbation of the convective-diffusive flow for large times. We also calculate the movement of the solid points in that situation.

Theorem 7.1 (Asymptotic behavior) *Under the assumptions of Theorem* 2.1 *and assuming further that* $u_0 \in L^1(\mathbb{R})$ *it follows that*

$$t^{(1-1/p)/2} \parallel u(t) - U(t) \parallel_{L^p} \to 0, \ \ \textit{as } t \to \infty \tag{7.1}$$

for all $1 \leqslant p \leqslant \infty$, *where*

$$U(x,t) = \frac{1}{t^{1/2}} f_M(x/\sqrt{t}) \tag{7.2}$$

is the self-similar solution of Burger's equation, satisfying

$$\begin{cases} u_t - u_{xx} + \kappa\,(u^2)_x = 0, & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = M\delta_0, \end{cases} \tag{7.3}$$

where δ_0 *denotes the Dirac delta at the origin and the asymptotic momentum* M *is given by*

$$M = \int_{\mathbb{R}} u_0(x)dx + \sum_{i=1}^{N} m_i h_{i,1}. \tag{7.4}$$

If $M > 0$, *we also have*

$$t^{-1/2} \Big| h_i(t) - c\sqrt{t} \Big| \to 0, \ \ \textit{as } t \to \infty,\ i = 1, \cdots, N \tag{7.5}$$

where $c > 0$ *is uniquely determined by the equation*

$$f_M(c) = c/2. \tag{7.6}$$

Moreover,

$$t^{1/2} \Big| h_i'(t) - \frac{c}{2\sqrt{t}} \Big| \to 0,\ t \to \infty,\ i = 1, \cdots, N. \tag{7.7}$$

According to this Theorem, the dynamics of the system decouples as $t \to \infty$. Indeed, the fluid solution u behaves like a solution of Burgers viscous equation. Thus, asymptotically, the effect of the point particles vanishes in first approximation. Note, however, that the asymptotic mass M of the limit solution $\bar{u}$, according to the law (7.4), absorbs the initial momenta of the particles.

Once the asymptotic profile $\bar{u}$ of the fluid solution u is known, (7.5)-(7.7) determine the asymptotic dynamics of the particles. According to (7.5), in particular, all the particles behave, in a first approximation, as a parabola $c\sqrt{t}$, the constant c being the same for all the particles (it is characterized by the equation).

Equation (7.6) has a unique solution. This can be easily seen from the explicit formula of the asymptotic profile f_M with $\kappa = 1$:

$$f_M(y) = \frac{(e^M - 1)G(y)}{(e^M - 1)\int_{-\infty}^{-y} G(s)ds + 1}; \; G(y) = (4\pi)^{-1/2} \exp(-y^2/4). \tag{7.8}$$

For $\kappa \neq 1$ we may use rescaling. But this property can also be seen from the differential equation satisfied by f_M and this will be interesting for the next section, so we will revise this topic there.

Let us briefly sketch the proof of Theorem 7.1 on the asymptotic behavior of solutions that follows closely the developments in [VZ], now with several particles. Proceeding as in [VZ], by the method described in the end of Section 3.1, the following properties can be derived: for $1 \leqslant p < \infty$ it holds,

$$\frac{d}{dt}\left[\int_{\mathbb{R}} |u(x,t)|^p dx + \sum_{i=1}^{N} m_i |h_i'(t)|^p\right] \leqslant 0. \tag{7.9}$$

A similar conclusion holds in L^∞. Namely,

$$\| u(t) \|_\infty, |h_1'(t)|, \ldots, |h_N'(t)| \leqslant \max\left\{\| u_0 \|_\infty, |h_{1,1}|, \ldots, |h_{1,N}|\right\}. \tag{7.10}$$

Obviously, these inequalities make sense when the initial data u_0 belong to the corresponding L^p space.

Lemma 7.1 (Decay for L^1-data) *The following decay estimate is also satisfied by a solution with data $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.*

$$\| u(t) \|_{L^p(\mathbb{R})} \leqslant C_p t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}, \quad \forall t > 0 \tag{7.11}$$

for all $1 \leqslant p \leqslant \infty$, where the constant C_p depends on p and on the L^1-norm of u_0. Moreover,

$$|h_i'(t)| \leqslant Ct^{-1/2}, \quad \forall t > 0, \quad \forall i = 1, \ldots, N. \tag{7.12}$$

The proof of (7.11) is a bit more complicated but still standard and it can be carried out as in [VZ]. At this level the fact of having one single solid mass as in [VZ] or several ones, as in the present paper, does not make any difference once the global existence of solutions is guaranteed, as it is the case here.

Once the decay properties (7.11)-(7.12) have been proved, we introduce the scaled solutions

$$u_\lambda(x,t) = \lambda u(\lambda x,\, \lambda^2 t), \quad h_{i,\lambda}(t) = \frac{1}{\lambda} h_i(\lambda^2 t),\, i = 1, \dots, N, \tag{7.13}$$

for any $\lambda > 0$. These scaled solutions solve the perturbed system:

$$\begin{cases} u_{\lambda,t} - u_{\lambda,xx} + \kappa\,(u_\lambda^2)_x = 0,\, x \in I_{\lambda,i}(t),\, t > 0,\, i = 0, \dots, N \\ u_\lambda(h_{i,\lambda}(t),\, t) = h'_{i,\lambda}(t),\, t > 0,\, i = 1, \dots, N \\ \dfrac{m_i}{\lambda} h''_{i,\lambda}(t) = [u_{\lambda,x}(h_{\lambda,i}(t),\, t)]\,,\, t > 0,\, i = 1, \dots, N \\ u_\lambda(x,0) = u_{0,\lambda}(x),\, x \in \mathbb{R} \\ h_{\lambda,i}(0) = h_{i,0,\lambda},\, h'_\lambda(0) = h_{i,1,\lambda},\, i = 1, \dots, N, \end{cases} \tag{7.14}$$

with

$$u_{0,\lambda}(x) = \lambda u_0(\lambda x),\, h_{i,0,\lambda} = \frac{1}{\lambda} h_{i,0},\, h_{i,1,\lambda} = \lambda h_{i,1},\, i = 1, \dots, N, \tag{7.15}$$

and $I_{\lambda,i}(t)$ being defined as in (1.3) but for the scaled functions $h_{i,\lambda}$.

System (7.14) is very close to (1.2) except for the fact that a scale factor $1/\lambda$ appears multiplying the masses m_i. The "new masses" m_i/λ go to zero as $\lambda \to \infty$. Consequently, as $\lambda \to \infty$, formally, (7.14) approaches the Burgers equation on the whole line without jumps for u_x at the interfaces. This was rigorously proved in [VZ] for the case of one single solid mass. The same proof, with minor changes, applies in the present case.

It is important to note that, in view of the decay properties (7.11)-(7.12), $(u_\lambda,\, h_{i,\lambda},\, h'_{i,\lambda})$ are bounded in $L^p(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ for all $1 \leqslant p \leqslant \infty$ for any time $t > 0$. On the other hand, in order to prove the asymptotic result in Theorem 1.2, it is sufficient to compute the limit in $L^p(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ of $(u_\lambda(t),\, h_{i,\lambda}(t),\, h'_{i,\lambda}(t))$ for $t = 1$. This can be done as in [VZ], but in order to get compactness in that space one needs:

- To get uniform bounds on $u_{\lambda,x}$ and $u_{\lambda,xx}$ in $L^2(\mathbb{R})$, for $t > 0$ fixed. This can be done, roughly speaking, by means of the energy dissipation law and the argument employed to prove the key estimate for the lack of collision, cf. Subsections 3.1 and 4.1 respectively.
- To get uniform bounds for $|h''_\lambda(t)|$. This can also be done as in the proof of Subsection 4.1.
- To get uniform bounds on the integrability of the scaled solutions u_λ as $|x| \to \infty$. This can be done using suitable cut-off functions and integration by parts.

Once this is done, the compactness of $u_\lambda\Big|_{t=1}$ in $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, holds by standard arguments.

It is then easy to see in the variational formulation of (7.14) that the limit of u_λ as $\lambda \to \infty$ solves the Burgers equation (7.3) with the initial mass (7.4). Once this is done, it is also easy to check that, in a first approximation, all the particles behave similarly as indicated in (7.5)-(7.7).

We refer to [VZ] for the complete details of the proof when $N = 1$.

8 Asymptotic convergence of trajectories

We have shown that the particles of our model do not collide in finite time. However, they approach each other as time goes to infinity in relative size. The rate depends on the ratio κ between convection and diffusion, hence we will review this topic. In this study we assume that $M > 0$. A similar asymptotic analysis will apply to the case $M < 0$ but the case where u_0 has changing sign and $M = 0$ is not covered[3]. The differential equation satisfied by the selfsimilar profiles f_m is

$$\frac{1}{2}(sf)' + f'' - \kappa(f^2)' = 0. \tag{8.1}$$

Integrating once we get $sf + 2f' - 2\kappa f^2 = C$, and the constant must be $C = 0$ because f is integrable and goes to zero at infinity. Hence,

$$f'(s) = \frac{1}{2}\left(2\kappa f(s)^2 - sf(s)\right). \tag{8.2}$$

The solutions f are positive and tend to zero as $s \to \pm\infty$. Different curves correspond to different masses M (attention: some integral curves do not correspond to our problem setting, they may blow up or change sign). These properties guarantee that the solution curve of our problem has at least one intersection with the line $f(s) = s/2$, which locates the particle trajectories for large times. The intersection takes place at a point $c > 0$ where

$$f(c) = \frac{c}{2}, \quad f'(c) = \frac{c^2}{4}(\kappa - 1).$$

Now comes the interesting conclusion: the slope is negative, $f'(c) < 0$, if $\kappa < 1$, and in that case the intersection is unique. For $\kappa = 1$ the intersection takes place at the point of maximum of the profile. For $\kappa > 1$ the intersection must take place on the upward part of the profile, $f'(c) > 0$.

Let us eliminate the possibility of several crossings $\kappa > 1$. If there were more than points on the profile curve with $f(s) = s/2$, we would have at least one with slope $f'(c_1) \geq 1/2$ and a later one $c_2 > c_1 > 0$ with slope $f'(c_2) \leq 1/2$. But this contradicts the dependence of $f'(c)$ derived before (which is proportional to c^2). We conclude that c is unique, and also that $f'(c) < 1/2$ for all $\kappa \in \mathbb{R}$. We still may consider the modifications introduced by the presence of a viscosity

[3]This restriction was brought to our attention by Mathieu Hillariet.

coefficient μ in front of u_{xx}. Since they turn out to be minor, we review than at the end of the section.

We address now the question of the relative separation of the trajectories for large times.

Theorem 8.1 *Let u be any strong solution with initial data $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and let $h_i(t)$ and $h_j(t)$ two particle trajectories. As $t \to \infty$ we have*

$$t^{-1/2}(h_j(t) - h_i(t)) \to 0. \tag{8.3}$$

Moreover, there is a precise power rate of convergence of the form

$$\lim_{t\to\infty} \frac{\log(|h_j(t) - h_i(t)|)}{\log t} = -a, \tag{8.4}$$

where $-a = f'(c)$, f being the asymptotic similarity profile. In particular, the trajectories approach each other in real distance whenever $\kappa < 1$.

Remark. Recall that $c = \lim h(t)t^{-1/2}$. By our previous analysis $a = c^2(1-\kappa)/4 > -1/2$. Hence, the trajectories always converge in relative distance,

$$\lim_{t\to\infty} \frac{|h_j(t) - h_i(t)|}{h_j(t)} = 0.$$

Proof: We may assume that $i < j$ and even that they are consecutive. The idea is to use the differential equation satisfied by the trajectories, (2.8), to obtain an equation for the difference $h(t) = h_j(t) - h_i(t) > 0$ of the form

$$h'(t) = u(h_j(t), t) - u(h_i(t), t) = \int_{I(t)} u_x(s,t)\, ds \tag{8.5}$$

where $I(t)$ is the interval $(h_i(t), h_j(t))$. Let us give first a formal argument. We know that u converges uniformly to the selfsimilar solution U after rescaling and that

$$u(x,t) = U(x,t) + o(t^{-1/2}), \qquad h(t) = c\,t^{1/2}(1 + o(1)).$$

If the same type of uniform convergence up to scaling were true for u_x, then for all large times we would have

$$u_x(x,t) = U_x(x,t) + o(t^{-1}).$$

Now, we know that at any point in the space between the trajectories $x = ct^{1/2}(1+o(1))$, hence the asymptotic value of U_x is

$$U_x(x,t) = -\frac{a}{t}(1 + o(1)).$$

Thefore, the calculation of (8.5) can be performed as

$$h'(t) \sim -\frac{a}{t}\, h(t),$$

and it gives as $t \to \infty$ the expression

$$\frac{\log h(t)}{\log t} = -a.$$

This is a very precise form of showing how the trajectories approach.

(ii) Unfortunately, the available estimates do not allow for the convergence of $u_{\lambda,x}$ towards U_x in the space $C([0,T] : L^\infty(\mathbb{R}))$. This implies that we have to make some extra work to estimate the influence of the difference between $u_{\lambda,x}$ and U_x.

It will be more convenient to work with rescaled variables, $u_\lambda(x,t)$ and $h_\lambda(t)$, defined as in (7.13) of the previous section. Then it is enough to do it for values of time $1 \le t \le T$ for some $T > 1$, say $T = 2$, with λ very large. We recall that the asymptotic behavior for the trajectories that we have proved means that $h_{i,\lambda}(t) = (c + \delta_i(\lambda^2 t))\, t^{1/2}$ with $\delta(s) \to 0$ as $s \to \infty$, so that for $t \ge 1$

$$h_\lambda(t) = h_{j,\lambda}(t) - h_{i,\lambda}(t) = (\delta_j(\lambda^2 t) - \delta_i(\lambda^2 t))\, t^{1/2} = o(\lambda) t^{1/2}.$$

Let us call $v_\lambda(x,t) = u_\lambda(x,t) - U(x,t)$, the asymptotical error. Our main concern is estimating $v_{\lambda,x}$. We are really interested in knowing what happens when the space norm is $L^\infty(\mathbb{R})$. We will work by interpolation. On the one hand, our estimates say that as $\lambda \to \infty$, $|v_\lambda(x,t)| \to 0$ uniformly in $x \in \mathbb{R}$ and $t \ge 1$. Next, we take a look at the second derivative. It consists of the difference of derivatives in the fluid space between particles,

$$v_{\lambda,xx} = u_{\lambda,xx} - U_{\lambda,xx}, \tag{8.6}$$

which is a bounded L^2 function in space-time, plus the Dirac deltas at the particle locations $x = h_{\lambda,i}(t)$ corresponding to the $u_{\lambda,x}$ jump, with value

$$[u_{\lambda,x}](h_{i,\lambda}(t), t) = \frac{m_i}{\lambda} h''_{i,\lambda}(t) = \lambda^2 h''_i(\lambda^2 t). \tag{8.7}$$

We know from Theorem 5.1 that $t^{1/2} h''_i(t) \in L^2(1,\infty)$, hence

$$\int_1^T \lambda^2 h''_i(\lambda^2 t) dt = \int_{\lambda^2}^{\lambda^2 T} h''_i(t)\, dt \le \left(\int_{\lambda^2}^{\lambda^2 T} t (h''_i(t))^2\, dt \right)^{1/2} (\log T)^{1/2}. \tag{8.8}$$

Note that since the integral $\int t (h''_i(t))^2\, dt$ is convergent, the actual integral appearing in the right-hand side of the last formula goes to zero as $\lambda \to \infty$. According to this estimate, after scaling the effects of the jumps go to zero with a certain rate. We will be able to use this rate to eliminate their influence.

With these ingredients, we use the following interpolation inequality, valid for functions $f \in H^1(\mathbb{R})$ which have a second derivative in the sense of distributions with a continuous part $f_{xx,reg} \in L^2(\mathbb{R})$ and a Dirac part of total mass D:

$$\|f_x\|_\infty \leq C\|f\|_\infty^{1/3}\|f_{xx,reg}\|_2^{2/3} + D. \tag{8.9}$$

We postpone the proof of this technical result to the end of the section. When we apply this result to $f = v_\lambda(t)$, we find that for every t

$$\|v_{\lambda,x}(t)\|_\infty \leq C\|v_\lambda(t)\|_\infty^{1/3}\|v_{\lambda,xx,reg}(t)\|_2^{2/3} + \sum_i |h''_{\lambda,i}(t)| := \phi_\lambda(t).$$

We now recall that $u_{\lambda,x} = v_{\lambda,x} + U_x$, and go back to estimate (8.5) which now becomes

$$h'_\lambda(t) = \int_{I_\lambda(t)} u_{\lambda,x}(s,t)\,ds = \int_{I_\lambda(t)} v_{\lambda,x}(s,t)\,ds + U(h_{j,\lambda}(t),t) - U(h_{i,\lambda}(t),t). \tag{8.10}$$

where $I_\lambda(t)$ is the interval $(h_{i,\lambda}(t), h_{j,\lambda}(t))$. Therefore,

$$h'_\lambda(t) + \frac{a}{t}h_\lambda(t) = \int_{I_\lambda(t)} v_{\lambda,x}(s,t)\,ds + (U(h_{j,\lambda}(t),t) - U(h_{i,\lambda}(t),t) - U_x(c,t)h_\lambda(t))\,.$$

In the last term we are working in a neighborhood of c. Since $U_x = (1/t)f'(x/t^{1/2})$ and $U_{xx} = O(t^{-3/2})$, we obtain the estimate

$$|h'_\lambda(t) + \frac{a}{t}h_\lambda(t)| \leq \frac{C}{t^{3/2}}\,h_\lambda(t)^2 + \phi_\lambda(t)\,h_\lambda(t) \leq (\frac{o(\lambda)}{t} + \phi_\lambda(t))\,h_\lambda(t).$$

Integration between $t = 1$ and $t = T$ after dividing by $h_\lambda(t)$ gives

$$|\log(h_\lambda(T)/h_\lambda(1)) + a\log(T)| \leq o(\lambda)\,\log(T) + \int_1^T \phi_\lambda(t)\,dt,$$

In view of these estimates and the uniform estimates on v_λ which are in fact a consequence of the uniform estimates on u_λ, which turn out to be consequence of the global estimates on u) we get

$$\int_1^T \phi_\lambda(t)\,dt \leq C_1\epsilon_1(\lambda)^{1/3}T^{2/3} + \epsilon_2(\lambda)(\log(T))^{1/2},$$

where $\epsilon_1(\lambda)$ is a bound for the L^∞ norm of v_λ, C_1 comes from the L^2 norm of $v_{xx,reg}$ in space-time, and $\epsilon_2(\lambda)$ comes from the L^2 norm of h''_λ as carefully studied above.

Summing up, we can write

$$|\log(h_\lambda(T)/h_\lambda(1)) + a\log(T)| \leq \epsilon((\log(T))^{1/2} + T^{2/3}).$$

Moreover, we can be more precise as $\lambda \to \infty$. Since we have $\epsilon \to 0$, the right-hand side becomes negligible compared with $-a \log(T)$ uniformly on sets of the form $[t_1, t_2]$ for all $1 < t_1 < t_2$. Indeed, for every $b = a - \mu \le a$ and for fixed T there exists a λ_0 large enough so that

$$\log(h_\lambda(T)/h_\lambda(1)) + b\log(T) \le 0$$

if $\lambda \ge \lambda_0$ or, in other words,

$$h(\lambda^2 T)T^b \le h(\lambda^2).$$

Putting now $\lambda^2 = t_1$ and $T\lambda^2 = t_2$ we get the inequality

$$h(t_2)t_2^b \le h(t_1)t_1^b,$$

whenever $t_1 \le t_* = \lambda_0^2$ and $t_2 = Tt_1$. This implies that the function $H(t) = h(t)t^b$ is bounded for all values $t \le t_*$ by its values in the interval $t_* \le t \le Tt_*$. We conclude that there exists $C = C(b)$ such that

$$h(t) \le C(b)\, t^{-b} = C_1(\mu)t^{\mu - a}.$$

In a similar way we may operate for $b = a + \mu$ with $\mu > 0$ small and get an estimate of the form

$$h(t) \ge C_2(\mu)t^{-\mu - a}.$$

This ends the proof. ■

Proof of Estimate (8.9). We assume that f is a bounded function with derivative f' that has a maximum, say, at $x = 0$, and we assume without loss of generality that $f'(0) = a > 0$. When $a \le D$ there is nothing to prove, hence we assume that $a > D$. Then we have for $x > 0$

$$f'(x) \ge a + \int_0^x f''(x)\, dx - D,$$

where D represents the sum of the absolute values of the jumps and f'' stands for the regular part of the second derivative. From this we get

$$f'(x) \ge a - \|f''\|_2 x^{1/2} - D.$$

Integrating once again we get

$$2||f||_\infty \ge f(x) - f(0) \ge (a - D)x - \frac{2}{3}\|f''\|_2 x^{3/2}.$$

This inequality has to be true for all $x > 0$. Obtaining the maximum of the right-hand side, we get

$$\frac{1}{3}\frac{(a-D)^3}{\|f''\|_2^2} \le 2\|f\|_\infty.$$

The desired inequality follows in that case. ■

The effect of viscosity. One may wonder if the introduction of yet another parameter, the viscosity coefficient, in front of the Laplacian of the equation, may alter the discussion of the location of c on the Burgers profiles and the slope a. We show here that this is not so. We consider the modified equation

$$u_t - \mu u_{xx} + \kappa\,(u^2)_x = 0, \tag{8.11}$$

where μ and κ are positive constants. The rest of the equations of system (1.2) is kept untouched. A simple scaling in time $\tau = \mu\, t$ allows to eliminate the new viscosity coefficient μ at the price of altering a bit everything. We get the modified system,

$$\begin{cases} u_\tau - u_{xx} + \frac{\kappa}{\mu}(u^2)_x = 0, & x \in I_i(\tau), \quad i = 0, \cdots, N, \quad \tau > 0 \\ h_i'(\tau) = \frac{1}{\mu} u(h_i(\tau), \tau), & i = 1, \cdots, N, \quad \tau > 0 \\ m_i \mu^2\, h_i''(\tau) = [u_x](h_i(\tau), \tau), & i = 1, \cdots, N, \quad \tau > 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \\ h_i(0) = h_{i,0}, & h_i'(0) = h_{i,1}, \quad i = 1, \cdots, N. \end{cases} \tag{8.12}$$

where now primes denote differentiation with respect to τ. The analysis of the behavior of $h_i(\tau)$ shows again that it grows like $\tau^{1/2}$ and this, together with equation $h_i'(\tau) = \frac{1}{\mu} u(h_i(\tau), \tau)$ and the asymptotic selfsimilarity of u lead to the equation

$$f(c) = \frac{\mu c}{2}.$$

The profile f satisfies

$$f' = \frac{f}{2}\,(\frac{2\kappa}{\mu} f - s).$$

Combining both formulas we get the value of the critical slope

$$f'(c) = \frac{\mu c^2}{4}(\kappa - 1).$$

We see that the sign only depends on $\kappa - 1$. In particular, it is independent of the viscosity coefficient $\mu > 0$.

9 Comment on Navier-Stokes

One can expect the results on the lack of collision to hold in the context of Navier-Stokes equations in 2 space dimensions. But this work remains to be done. The key point to be developed, following the arguments in this paper, would be the obtention of H^2 estimates in space up to the possible collision time. H^2-regularity on the nonlinearity is exactly critical for the uniqueness of systems of ODE's consisting on two equations, but uniqueness still holds in that case as pointed out in [Z]. But obtaining these H^2-estimates for the Navier-Stokes equations certainly requires of important additional developments. We underline however that the $3D$

problem would still remain out of reach since H^2-regularity is insufficient there. One would rather need H^3-estimates in that case in order to apply the uniqueness arguments developed in this article to exclude collision in three space dimensions.

Added to the preprint. As natural extensions of the work done in this paper, we mention the asymptotic behavior for a fluid confined by a wall, and the asymptotics in the line, $x \in \mathbb{R}$ when the mass vanishes $M = 0$. For this last problem M. Hillariet has informed us that the particles do not approach asymptotically if the initial distribution is antisymmetric, $u_0(-x) = -u_0(x)$.

Acknowledgments. The work of the first author has been supported by grant BFM 2002-04572-C02-02 of MCYT (Spain) and Research Training Network HYKE, HPRN-CT-2002-00282. The second author has been supported by grant BFM 2002-03345 of MCYT (Spain) and the EU Networks "Homogenization and Multiple Scales" and "Smart Systems".

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