J. Math. Pures Appl. 79, 8 (2000) 741–808

NULL CONTROLLABILITY OF THE HEAT EQUATION AS SINGULAR LIMIT OF THE EXACT CONTROLLABILITY OF DISSIPATIVE WAVE EQUATIONS

Antonio LÓPEZ [a,1], **Xu ZHANG** [b,2], **Enrique ZUAZUA** [a,3]

[a] *Departamento de Matemática Aplicada, Universidad Complutense, 28040 Madrid, Spain*
[b] *College of Mathematics, Sichuan University, Chengdu, 610064, Sichuan, China*

Received in 10 July 1999

ABSTRACT. – In this paper we prove that the null controllability property of the heat equation may be obtained as limit of the exact controllability properties of singularly perturbed damped wave equations. We impose Dirichlet, homogeneous boundary conditions. The control is supported in a neighborhood of a subset of the boundary that satisfies the classical requirements to apply multiplier techniques. The proof needs an iterative argument that allows to treat separately the low and high frequencies and to make use of the dissipativity of the systems under consideration. This is combined with sharp observability estimates on the eigenfunctions of the Laplacian due to G. Lebeau and L. Robbiano, and global Carleman estimates.

This proof applies in any space dimension.

As a consequence of the uniform controllability we derive uniform observability estimates which can not be proved by classical methods due to the singular character of the perturbations we deal with. © 2000 Éditions scientifiques et médicales Elsevier SAS

Keywords: Null controllability, Heat equation, Dissipative wave equation, Global Carleman estimates

RÉSUMÉ. – Dans cet article on montre que la propriété de la contrôlabilité à zéro de l'équation de la chaleur peut être obtenue comme limite singulière des propriétés de contrôle exact des équations d'ondes dissipatives. On considère des conditions aux limites de Dirichlet. Le côntrole est distribué dans un voisinage d'une partie du bord qui satisfait les conditions classiques requises pour appliquer les techniques de multiplicateurs. La preuve utilise un argument itératif qui permet de traiter séparément les basses et les hautes fréquences et d'utiliser la dissipativité des systèmes considérés. Cet argument est combiné avec des estimations d'observabilité fines sur les fonctions propres du laplacien dues à G. Lebeau et L. Robbiano et aux inégalités globales de Carleman.

Cette démonstration est valable en toute dimension d'espace.

Comme conséquence de la contrôlabilité uniforme nous obtenons des estimations d'observabilité uniformes qui ne peuvent pas être obtenues par des méthodes classiques à cause du caractère singulier des perturbations que nous considérons. © 2000 Éditions scientifiques et médicales Elsevier SAS

[1] Supported by grant PB96-0663 of the DGES (Spain). E-mail: bantonio@sunma4.mat.ucm.es

[2] Supported partially by Natural Science Foundation of China and Chinese Education Ministry Science Foundation. E-mail: xuzhang@pridns.scu.edu.cn

[3] Supported by grant PB96-0663 of the DGES (Spain). E-mail: zuazua@eucmax.sim.ucm.es

1. Introduction and main results

This paper is devoted to analyze the controllability of the following damped, singularly perturbed wave equation:

$$\begin{cases} \varepsilon u_{tt} - \Delta u + u_t = f 1_\omega & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

In (1.1), Ω is a bounded domain of class C^∞ of $\mathbf{R}^n$, $u = u(x,t)$ is the state to be controlled, $f = f(x,t)$ is the control and 1_ω denotes the characteristic function of the open subset ω of Ω where the control is supported. The parameter $\varepsilon > 0$ is devoted to tend to zero. The formal limit as $\varepsilon \to 0$ of (1.1) is the controlled heat equation:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u^0(x) & \text{in } \Omega. \end{cases} \tag{1.2}$$

Given any $T > 0$ fixed, system (1.1) is known to be exactly controllable for $\varepsilon > 0$ sufficiently small when ω satisfies suitable geometric conditions. The most typical example is when ω is a neighborhood in Ω of a subset of the boundary of the form $\Gamma(x^0)$ (see J.L. Lions [8], Tome 1). More precisely, given any $x^0 \in \mathbf{R}^n$ we introduce the following subset of the boundary of Ω:

$$\Gamma\left(x^0\right) = \left\{x \in \partial\Omega\colon \left(x - x^0\right) \cdot \nu(x) > 0\right\}, \tag{1.3}$$

where $\nu(x)$ denotes the outward unit normal to Ω at $x \in \partial\Omega$ and $\cdot$ denotes the scalar product in $\mathbf{R}^n$. We then set $\omega = \Omega \cap \Theta$, where Θ is a neighborhood of $\Gamma(x^0)$ in $\mathbf{R}^n$ (see Fig. 1). All along this paper we shall work in this geometric setting.

As we said above, given $T > 0$ fixed system (1.1) is exactly controllable for $\varepsilon > 0$ sufficiently small. More precisely, for any $T > 0$ there exists $\varepsilon(T) > 0$ such that system (1.1) is exactly

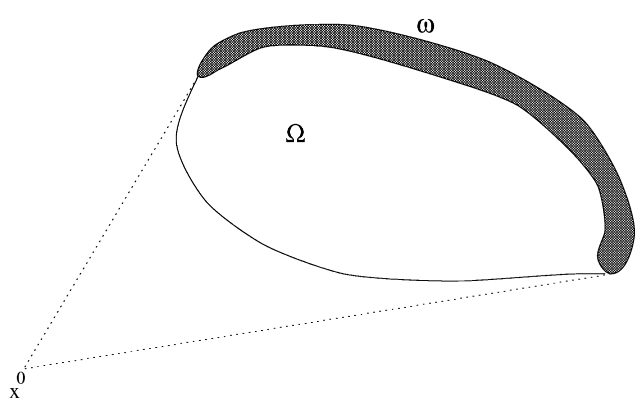

Fig. 1.

controllable for any $0 < \varepsilon < \varepsilon(T)$ in time T, i.e., for any $\{u^0, u^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ there exists $f_\varepsilon \in L^2(\omega \times (0,T))$ such that the solution of (1.1) satisfies

$$u(x,T) \equiv u_t(x,T) \equiv 0 \quad \text{in } \Omega. \tag{1.4}$$

Moreover, there exists a positive constant $C(\varepsilon, T) > 0$ such that

$$\|f_\varepsilon\|_{L^2(\omega\times(0,T))} \leqslant C(\varepsilon,T)\left\|\left\{u^0, \sqrt{\varepsilon}u^1\right\}\right\|_{H_0^1(\Omega)\times L^2(\Omega)}, \quad \forall\left\{u^0,u^1\right\} \in H_0^1(\Omega)\times L^2(\Omega). \tag{1.5}$$

Note that $C(\varepsilon, T)$ may be viewed as a measure of the cost of controllability. We refer to [8], Tome 1 for the proof of this result in the absence of damping and to R. Triggiani [12] for the case where the equation is damped but the control acts on the boundary. For the sake of completeness we give a sketch of the proof of this result in an Appendix at the end of the paper.

As far as we know, there are no estimates in the literature on how the constant $C(\varepsilon, T)$ in (1.5) depends on $\varepsilon \to 0$. More precisely, there is no result guaranteeing that the constant $C(\varepsilon, T)$ remains bounded above as $\varepsilon \to 0$. However, it is natural to expect this to be true.

As D.L. Russell in [10] pointed out, whenever the undamped wave equation is exactly controllable, the heat equation is null controllable in an arbitrarily short time. This argument applies in this geometric setting. Thus, we deduce that for any $T > 0$ and $u^0 \in L^2(\Omega)$ there exists a control $f \in L^2(\omega \times (0,T))$ such that the solution u of (1.5) satisfies

$$u(x,T) \equiv 0 \quad \text{in } \Omega. \tag{1.6}$$

Moreover, there exists $C(T) > 0$ such that

$$\|f\|_{L^2(\omega\times(0,T))} \leqslant C(T)\left\|u^0\right\|_{L^2(\Omega)}, \quad \forall u^0 \in L^2(\Omega). \tag{1.7}$$

In these circumstances it is natural to expect the constant in (1.5) to be uniformly bounded and, even more, to be able to build the null-control f of the heat Eq. (1.2) as limit when $\varepsilon \to 0$ of the controls of the wave equations (1.1).

This is precisely the main result of this paper:

THEOREM 1.1. – *Assume that Ω and ω are in the above geometric setting. Let $T > 0$. Then, there exists $\varepsilon(T) > 0$ such that for any $0 < \varepsilon < \varepsilon(T)$ system* (1.1) *is exactly controllable in time T and moreover the constant $C(\varepsilon, T)$ in* (1.5) *remains bounded as $\varepsilon \to 0$.*

Furthermore, for any $\{u^0, u^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ fixed the controls f_ε of (1.1) *may be chosen such that*:

$$f_\varepsilon \to f \quad \text{in } L^2(\omega\times(0,T)) \text{ as } \varepsilon \to 0, \tag{1.8}$$

f being a null control for the limit heat equation (1.2) *with initial datum u^0.*

Remark 1.1. – (a) Note that we assume Ω to be of class C^∞. This is only due to the fact that we are using the sharp observability estimates on the eigenfunctions of the Laplacian proved in [6] and [7]. As it was communicated to us by L. Escauriaza [2], this observability estimate (see Section 3 below) also holds when the domain is simply of class C^2 and can be proved using multipliers and the classical doubling properties of harmonic functions but, as far as we know, this proof was never published.

(b) As we mentioned above, the method developed in [10] allows to prove the null-controllability of the heat equation as a consequence of the exact controllability of the wave equation. This method applies to the moment problem formulation of the control problems and is based on the fact that, given a biorthogonal family for the wave equation, one can transform it into a biorthogonal family for the heat equation. It would be interesting to see if the methods in [10] allow to get a result similar to the one in Theorem 1.1. In other words, it would be interesting to see if the methods in [10] allow to continuously transform the biorthogonal family for the wave equation into the biorthogonal family for the heat equation passing through the biorthogonal families of equations of the form (1.1).

(c) The method we develop here allows to analyze the limit behavior of the controls when the initial data in (1.1) depend on the parameter ε as well.

(d) The one-dimensional case is studied in [9] where the same result is obtained for any open subset ω of Ω.

(e) Note that Theorem 1.1 provides the null-control of the limit heat equation only when $u_0 \in H_0^1(\Omega)$. However, as we mentioned above, the null-controllabilty of the heat equation holds for a larger class of initial data. It holds for instance for any $u_0 \in L^2(\Omega)$. Note however that, due to the regularizing effect of the heat equation, as soon as we let the heat equation to evolve freely (without control) during an arbitrarily short time interval, even if the initial datum lies in $L^2(\Omega)$ the solution enters $H_0^1(\Omega)$ and then the result above applies.

The proof of Theorem 1.1 follows the main steps developed in [9] in the one-dimensional context. Roughly, the proof is as follows. We divide the time interval $[0, T]$ in three subintervals: $I_1 = [0, T/3]$, $I_2 = [T/3, 2T/3]$ and $I_3 = [2T/3, T]$. In the time interval I_1 we control to zero the parabolic projection of the solution. This can be done uniformly with respect to $\varepsilon \to 0$ following the methods in [6]. In the time interval I_2 we let the Eq. (1.1) to evolve freely without control. In this way the parabolic components remains at rest and, due to the strong dissipativity of system (1.1) in its hyperbolic components, the size of the solution at time $t = 2T/3$ becomes exponentially small, i.e., of the order of $\mathrm{e}^{-C/\varepsilon}$ as $\varepsilon \to 0$ for a suitable constant $C > 0$. Finally, in the interval I_3 we apply a control driving the whole solution to zero. When doing this we need the following observability estimate for the solutions of the adjoint system:

$$\text{(1.9)} \qquad \begin{cases} \varepsilon\varphi_{tt} - \Delta\varphi - \varphi_t = 0 & \text{in } \Omega \times (0, T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T), \\ \varphi(x, T) = \varphi^0(x),\ \varphi_t(x, T) = \varphi^1(x) & \text{in } \Omega. \end{cases}$$

THEOREM 1.2. – *In the geometric setting above, for any* $T > 0$ *there exists* $\varepsilon(T) > 0$ *and positive constants* $C_1(T)$, $C_2(T) > 0$ *such that*:

$$\text{(1.10)} \qquad \|\varphi^0\|^2_{L^2(\Omega)} + \varepsilon\|\varphi^1\|^2_{H^{-1}(\Omega)} \leqslant C_1(T)\mathrm{e}^{C_2(T)/\sqrt{\varepsilon}} \int_0^T \int_\omega \varphi^2 \,\mathrm{d}x\,\mathrm{d}t$$

for all $0 < \varepsilon < \varepsilon(T)$ *and every solution of* (1.9).

According to this estimate, the control needed in the interval I_3 is exponentially large (of the order of $\mathrm{e}^{C_2(T)/\sqrt{\varepsilon}}$) with respect to the data of the solution at time $t = 2T/3$. However, since, in view of the analysis made in the interval I_2, these data are exponentially small, these two phenomena compensate and the control turns out to be uniformly bounded, and even exponentially small.

The proof of Theorem 1.2 is performed using global Carleman inequalities and it constitutes the main novelty of this paper with respect to the proof presented in [9] in the one-dimensional case. Indeed, in one space dimension the estimate (1.10) can be easily proved by sidewise energy estimates.

As an immediate consequence of the uniform bound on the constants $C(\varepsilon, T)$, the following uniform observability inequality holds for the solutions of the adjoint system (1.9).

THEOREM 1.3. – *Let Ω and ω be as above. Let $T > 0$. Then, there exists $\varepsilon(T) > 0$ and a constant $C(T) > 0$ such that*:

$$\varepsilon \left\| \varphi(0) \right\|^2_{L^2(\Omega)} + \left\| \varepsilon\varphi_t(0) - \varphi(0) \right\|^2_{H^{-1}(\Omega)} \leqslant C(T) \int_0^T \int_\omega \varphi^2 \,\mathrm{d}x\,\mathrm{d}t \tag{1.11}$$

for every solution of (1.9) *and every* $0 < \varepsilon < \varepsilon(T)$.

Remark 1.2. – Observe that the constant arising in (1.11) is independent of ε. It provides a uniform bound of the size of the solution at time $t = 0$ which, due to the strong dissipative character of Eq. (1.9) in the backwards sense of time, is much smaller than the norm of the solution φ at time T. Thus, one does not expect (1.11) to hold, with a constant independent of ε, if we consider $\varphi(T)$ in the place of $\varphi(0)$.

It is easy to see that (1.11) implies (1.10). Indeed, as a consequence of (1.11) one gets

$$\varepsilon \left\| \varphi(0) \right\|^2_{L^2(\Omega)} + \varepsilon^3 \left\| \varphi_t(0) \right\|^2_{H^{-1}(\Omega)} \leqslant C(T) \int_0^T \int_\omega \varphi^2 \,\mathrm{d}x\,\mathrm{d}t$$

and then (1.10) holds a consequence of the energy dissipation law (see (2.19) below).

It is also interesting to observe that the uniform inequality (1.11) is not obtained directly but rather as a consequence of the controllability result of Theorem 1.1. We do not know of any method giving directly (1.11).

It is also interesting to observe that, as a consequence of (1.11), letting $\varepsilon \to 0$ one deduces that the following observability inequality

$$\left\| \varphi(0) \right\|^2_{H^{-1}(\Omega)} \leqslant C \int_0^T \int_\omega \varphi^2 \,\mathrm{d}x\,\mathrm{d}t \tag{1.12}$$

holds for every solution of the adjoint heat equation:

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } \Omega \times (0, T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T), \\ \varphi(x, T) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{1.13}$$

The observability inequality (1.12) provides the null controllability property of the heat equation (1.2).

The rest of this paper is organized as follows. In Section 2 we describe in detail the iterative method we use to prove the uniform boundedness of the constants $C(\varepsilon, T)$ of (1.5). In Section 3 we prove the uniform boundedness of the controls of the parabolic component of solutions.

In Section 4 we analyze the dissipativity of the system over the solutions whose parabolic component vanishes. In Section 5 we prove Theorem 1.2. Section 6 is devoted to complete the proof of the uniform boundedness of the constants $C(\varepsilon, T)$ in (1.5) and of Theorem 1.3. In Section 7 we develop the limit process and complete the proof of Theorem 1.1. Finally, in Section 8 we comment on some possible extensions of the results of this paper.

It is by now well known (see for instance [3] and [6]) that the heat equation (1.2) is null controllable for any $T > 0$ and any open non-empty subset ω of Ω. However, according to the results in [1], some geometric conditions are needed on ω, the so called *geometric control conditions*, to guarantee the controllability of (1.1). Therefore we may not expect the results of this paper to hold for any ω. However, it is very likely that all the results described above remain valid provided the open subset ω of Ω is such that it satisfies the geometric control condition for the wave operator $\partial_t^2 - \Delta$ in some finite time. This is by now an open problem.

For any open subset ω of Ω, the methods of this paper allow to prove the uniform controllability of the parabolic component of the solutions of (1.1) making simultaneously its hyperbolic component to be exponentially small (with respect to ε). In this sense, one can view again the null controllability property of the solutions of (1.2) as a limit when $\varepsilon \to 0$ of partial controllability properties of the solutions of (1.1). In fact this result is an immediate consequence of the analysis made in the intervals I_1 and I_2 in the three steps method we have briefly described above. We shall discuss this question in Section 8.

We close this paper with some Appendixes devoted to prove some technical results.

2. Description of the "three-steps" controllability method

In this section we describe in the detail the "three-steps" method outlined in the introduction allowing to prove the null controllability of (1.1) in such a way that, with a further careful analysis, the controls may be shown to be uniformly bounded.

To develop this method we need to decompose the spectrum of the dissipative wave equation into its parabolic and hyperbolic components.

2.1. Spectral analysis

We consider the wave equation (1.1) without control (i.e., with $f \equiv 0$):

$$\begin{cases} \varepsilon u_{tt} - \Delta u + u_t = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \tag{2.1}$$

We also introduce the spectrum of the Laplacian:

$$\begin{cases} -\Delta e_k = \mu_k e_k & \text{in } \Omega, \\ e_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

We know that (2.2) admits an increasing sequence of positive eigenvalues of finite multiplicity

$$0 < \mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_k \leqslant \cdots$$

tending to infinity. The eigenfunctions $\{e_k\}_{k \geqslant 1}$ may be chosen to constitute an orthonormal basis of $L^2(\Omega)$.

We now analyze the spectrum of (2.1). Setting $u = e^{\lambda t} e_k(x)$ we get the following algebraic equation for λ:

$$\varepsilon\lambda^2 + \mu_k + \lambda = 0 \tag{2.3}$$

whose solutions are given by:

$$\lambda^{\varepsilon}_{k\pm} = \frac{-1 \pm \sqrt{1 - 4\varepsilon\mu_k}}{2\varepsilon}. \tag{2.4}$$

We observe that, for

$$4\varepsilon\mu_k \leqslant 1 \Leftrightarrow \mu_k \leqslant 1/4\varepsilon \tag{2.5}$$

the corresponding eigenvalues are real. We shall call them *parabolic eigenvalues*. Obviously the number of parabolic eigenvalues increases as $\varepsilon \to 0$.

When

$$\mu_k > 1/4\varepsilon \tag{2.6}$$

the corresponding eigenvalues have a non trivial imaginary part. These will be the *hyperbolic eigenvalues*.

The following properties are easy to check:

(a) for any $k > 0$ fixed:

$$\lim_{\varepsilon\to 0} \lambda^{\varepsilon}_{k+} = -\mu_k; \qquad \lim_{\varepsilon\to 0} \lambda^{\varepsilon}_{k-} = -\infty. \tag{2.7}$$

(b) For any $\varepsilon \in (0, 1/4\mu_k)$ the parabolic eigenvalues $\lambda^{\varepsilon}_{k\pm}$ satisfy

$$\lambda^{\varepsilon}_{k-} < \lambda^{\varepsilon}_{k+} < -\mu_k. \tag{2.8}$$

This property is a consequence of the fact that $\lambda^{\varepsilon}_{k+}$ decreases as ε increases.

(c) For any hyperbolic eigenvalue, we have:

$$\mathrm{Re}\big(\lambda^{\varepsilon}_{k+}\big) = \mathrm{Re}\big(\lambda^{\varepsilon}_{k-}\big) = -\frac{1}{2\varepsilon}. \tag{2.9}$$

In Fig. 2 below we describe the behavior of the eigenvalues as $\varepsilon \to 0^+$.

Note that, in view of (2.7), the limit of $\lambda^{\varepsilon}_{k+}$ as $\varepsilon \to 0$ is the eigenvalue $-\mu_k$ corresponding to the heat equation:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \tag{2.10}$$

On the other hand, $\lambda^{\varepsilon}_{k-} \to -\infty$ as $\varepsilon \to 0$. This indicates that, as $\varepsilon \to 0$, the only contribution of the parabolic eigenvalues $\lambda^{\varepsilon}_{k\pm}$ is the one that $\lambda^{\varepsilon}_{k+}$ produces which, in the limit, coincides with $-\mu_k$. This is in agreement with the fact that the corresponding solution of the limit heat equation is simply $e^{-\mu_k t} e_k(x)$.

According to (2.8) we also know that the parabolic component of the solution of the damped wave equation (2.1) decays even faster as time increases than the corresponding solution of the heat equation.

In view of (2.9) all the hyperbolic components are uniformly damped. More precisely, they decay in time exponentially uniformly with an exponential rate $-1/2\varepsilon$.

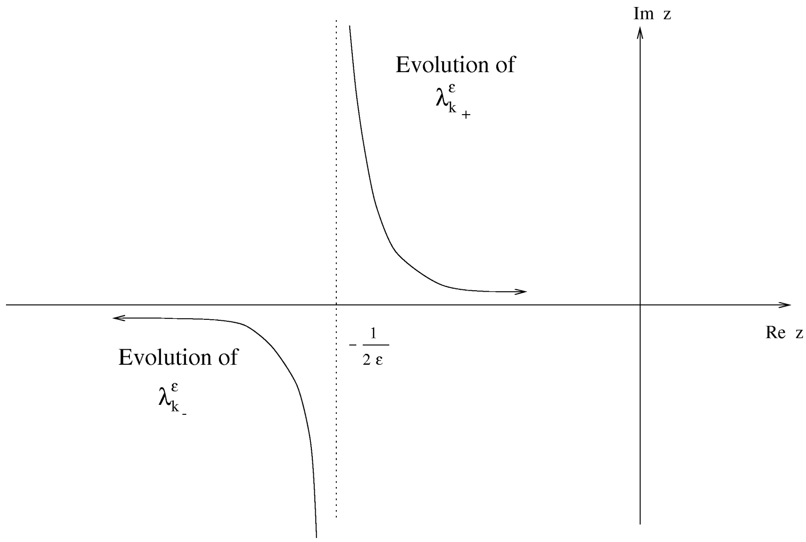

Fig. 2.

Solutions of (2.1) may now be easily developed in Fourier series.

Given initial data $\{u^0, u^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ such that:

$$u^0(x) = \sum_{k\geqslant 1} a_k e_k(x), \qquad u^1(x) = \sum_{k\geqslant 1} b_k e_k(x), \tag{2.11}$$

the solution u of (1.1) with

$$u(x,0) = u^0(x), \qquad u_t(x,0) = u^1(x) \quad \text{in } \Omega \tag{2.12}$$

can be developed in Fourier series as follows:

$$u_\varepsilon(x,t) = \sum_{k\geqslant 1} \left[\frac{a_k \lambda^\varepsilon_{k_-} - b_k}{\lambda^\varepsilon_{k_-} - \lambda^\varepsilon_{k_+}} \mathrm{e}^{\lambda^\varepsilon_{k_+} t} + \frac{b_k - a_k \lambda^\varepsilon_{k_+}}{\lambda^\varepsilon_{k_-} - \lambda^\varepsilon_{k_+}} \mathrm{e}^{\lambda^\varepsilon_{k_-} t} \right] e_k(x). \tag{2.13}$$

This formula is valid for $\varepsilon > 0$ when

$$\lambda^\varepsilon_{k_-} \neq \lambda^\varepsilon_{k_+}, \quad \forall k \in \mathbb{N}. \tag{2.14}$$

This is the generic situation since it holds for any $\varepsilon > 0$ not belonging to the sequence $\{1/4\mu_k\}_{k\geqslant 1}$. When ε is one of these exceptional values (say $\varepsilon = 1/4\mu_{k_0}$) we have

$$\lambda^\varepsilon_{k_{0,-}} = \lambda^\varepsilon_{k_{0,+}} = -\frac{1}{2\varepsilon} \tag{2.15}$$

and the corresponding expression in (2.13) has to be slightly changed.

However, the analysis we do in this paper remains essentially unchanged. For brevity we shall not discuss these exceptional cases in detail. Therefore all along the paper we will implicitly assume that $\varepsilon \neq 1/4\mu_k$ for all $k \geqslant 1$.

In the sequel, when dealing with the damped wave equation (2.1) the energy space $H = H_0^1(\Omega) \times L^2(\Omega)$ will be endowed with the norm:

$$\big\|(u,v)\big\|^2_{H_0^1(\Omega)\times L^2(\Omega)} = \|u\|^2_{H_0^1(\Omega)} + \varepsilon\|v\|^2_{L^2(\Omega)}, \tag{2.16}$$

where, obviously,

$$\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2\,\mathrm{d}x, \qquad \|v\|^2_{L^2(\Omega)} = \int_\Omega |v|^2\,\mathrm{d}x. \tag{2.17}$$

We shall denote the norm (2.16) as $\|\cdot\|_{H^\varepsilon}$ to make explicit its dependence on ε. The space H endowed with this norm will be denoted by H^ε.

We also consider the energy E_ε of solutions of (2.1):

$$E_\varepsilon(t) = \frac{1}{2}\int_\Omega \big[|\nabla u|^2 + \varepsilon|u_t|^2\big]\,\mathrm{d}x. \tag{2.18}$$

Obviously, $E_\varepsilon = \|(u,u_t)\|^2_{H^\varepsilon}/2$.

It is easy to check that, along the solutions of (2.1) we have

$$\frac{\mathrm{d}E_\varepsilon(t)}{\mathrm{d}t} = -\int_\Omega |u_t|^2\,\mathrm{d}x. \tag{2.19}$$

In our analysis we shall use the following subspaces of the space H that, according to the definition of the norm (2.16), will be denoted by H^ε to make explicit the fact that its norm is related to the wave equation (2.1) we are analyzing:

- $$H_n^\varepsilon = \Big\{U = (u,v) \in H^\varepsilon\colon\ u, v \in \operatorname*{span}_{1\leqslant k\leqslant n}(e_k)\Big\}.$$

  It is simply the subspace of H^ε generated by the first n eigenfunctions. Obviously its dimension is $2n$.

- $$H_p^\varepsilon = \Big\{U = (u,v) \in H^\varepsilon\colon\ u, v \in \operatorname*{span}_{1\leqslant k\leqslant k(\varepsilon)}(e_k)\Big\}$$

  where $k(\varepsilon)$ is such that

  $$\frac{1}{4\mu_{k(\varepsilon)+1}} < \varepsilon \leqslant \frac{1}{4\mu_{k(\varepsilon)}}.$$

  H_p^ε is the subspace of H^ε generated by the parabolic eigenfrequencies. Obviously, H_p^ε is also finite-dimensional. Its dimension increases as ε decreases. Note also that, as $\varepsilon \to 0$, H_p^ε eventually covers the whole space H^ε.

- $$H_h^\varepsilon = \Big\{U = (u,v) \in H^\varepsilon\colon\ u, v \in \operatorname*{span}_{k\geqslant k(\varepsilon)+1}(e_k)\Big\}$$

  where $k(\varepsilon)$ is as above. It is the space generated by the hyperbolic eigenfunctions.

The real part of the hyperbolic eigenvalues is $-1/2\varepsilon$ and therefore, as we shall show below, the energy of solutions in H_h^ε decays exponentially as $t \to \infty$ and more and more rapidly as $\varepsilon \to 0$.

We shall also introduce the dual space $[H^\varepsilon]' = L^2(\Omega) \times H^{-1}(\Omega)$ endowed with the dual norm associated to (2.16):

$$\|(u,v)\|_{[H^\varepsilon]'} = \left[\varepsilon\|u\|^2_{L^2(\Omega)} + \|v\|^2_{H^{-1}(\Omega)}\right]^{1/2}.$$

As above we define the corresponding finite-dimensional, parabolic and hyperbolic subspaces $[H_n^\varepsilon]'$, $[H_p^\varepsilon]'$ and $[H_h^\varepsilon]'$.

Given $(u,v) \in H^\varepsilon$ we also denote its orthogonal projections over $H_n^\varepsilon, H_p^\varepsilon$ and H_h^ε respectively by $\pi_n^\varepsilon(u,v)$, $\pi_p^\varepsilon(u,v)$ and $\pi_h^\varepsilon(u,v)$.

2.2. The control strategy

We are now in conditions to describe our control strategy.

Given $T > 0$, for $\varepsilon_0 > 0$ small enough we divide the time interval $[0,T]$ in three subintervals $[0,T] = I_1 \cup I_2 \cup I_3$ with $I_1 = [0, T/3]$, $I_2 = [T/3, 2T/3]$ and $I_3 = [2T/3, T]$.

Given an initial datum $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ to be controlled we proceed as follows:

First step. In the first time interval I_1 we control the parabolic component of the solutions. In other words, for any $\varepsilon \in (0, \varepsilon_0)$ we build a control $f_{1,\varepsilon} \in L^2(\omega \times (0, T/3))$ such that the solution of

$$\begin{cases} \varepsilon u_{tt} - \Delta u + u_t = f_{1,\varepsilon} 1_\omega & \text{in } \Omega \times (0, T/3), \\ u = 0 & \text{on } \partial\Omega \times (0, T/3), \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in } \Omega \end{cases} \tag{2.20}$$

satisfies

$$\pi_p^\varepsilon\big(u_\varepsilon(T/3),\ u_{\varepsilon,t}(T/3)\big) = 0. \tag{2.21}$$

As we shall see, this can be done uniformly on $0 < \varepsilon < \varepsilon_0$. More precisely, we will prove the existence of a positive constant $C > 0$, independent of $0 < \varepsilon < \varepsilon_0$, such that the control $f_{1,\varepsilon}$ satisfies

$$\|f_{1,\varepsilon}\|_{L^2(\omega\times(0,T/3))} \leqslant C\big\|\big(u^0, u^1\big)\big\|_{H^\varepsilon}, \quad \forall\big(u^0, u^1\big) \in H^\varepsilon,\ \forall 0 < \varepsilon < \varepsilon_0. \tag{2.22}$$

As we mentioned in the introduction, this will be done using the iterative method introduced in [6] to prove the null controllability of the heat equation.

In view of the uniform bound (2.22) of the control we shall also show that

$$\big\|\big(u_\varepsilon(T/3), u_{\varepsilon,t}(T/3)\big)\big\|_{H^\varepsilon} \leqslant C\big\|\big(u^0, u^1\big)\big\|_{H^\varepsilon}, \quad \forall\big(u^0, u^1\big) \in H^\varepsilon,\ \forall 0 < \varepsilon < \varepsilon_0. \tag{2.23}$$

In other words, after the time interval I_1, at time $t = T/3$, the parabolic projection of the solutions vanishes, the hyperbolic one being uniformly bounded.

We denote by $(v_\varepsilon^0, v_\varepsilon^1)$ the solution obtained at the final time $T/3$, i.e.,

$$v_\varepsilon^0 = u_\varepsilon(T/3), \qquad v_\varepsilon^1 = u_{\varepsilon,t}(T/3).$$

Second step. In the time interval I_2 we let the equation to evolve freely. In other words we solve

$$\begin{cases} \varepsilon u_{tt} - \Delta u + u_t = 0 & \text{in } \Omega \times (T/3, 2T/3), \\ u = 0 & \text{on } \partial\Omega \times (T/3, 2T/3), \\ u(x, T/3) = v_\varepsilon^0,\ u_t(x, T/3) = v_\varepsilon^1 & \text{in } \Omega. \end{cases} \tag{2.24}$$

Taking into account that $\pi_p^\varepsilon(v_\varepsilon^0, v_\varepsilon^1) = 0$ and using the development of solutions of (2.24) in Fourier series we deduce that:

$$\pi_p^\varepsilon\big(u_\varepsilon(t),\ u_{\varepsilon,t}(t)\big) = 0, \quad \forall T/3 \leqslant t \leqslant 2T/3. \tag{2.25}$$

On the other hand, taking into account that the real part of the hyperbolic eigenvalues is $-1/2\varepsilon$ we deduce that there exists $C > 0$ such that:

$$\big\|\big(u_\varepsilon(t),\ u_{\varepsilon,t}(t)\big)\big\|_{H^\varepsilon} \leqslant C\mathrm{e}^{-\frac{(t-T/3)}{4\varepsilon}}\big\|\big(v_\varepsilon^0, v_\varepsilon^1\big)\big\|_{H^\varepsilon}, \quad \forall T/3 \leqslant t \leqslant 2T/3,\ \forall 0 < \varepsilon < \varepsilon_0, \tag{2.26}$$

for every solution of (2.24).

In particular, at the final time $t = 2T/3$ we deduce that:

$$\big\|\big(u_\varepsilon(2T/3), u_{\varepsilon,t}(2T/3)\big)\big\|_{H^\varepsilon} \leqslant \mathrm{e}^{-T/12\varepsilon}\big\|\big(v_\varepsilon^0,\ v_\varepsilon^1\big)\big\|_{H^\varepsilon}$$

which, according to (2.23), implies that:

$$\big\|\big(u_\varepsilon(2T/3),\ u_{\varepsilon,t}(2T/3)\big)\big\|_{H^\varepsilon} \leqslant C\mathrm{e}^{-T/12\varepsilon}\big\|\big(u^0,\ u^1\big)\big\|_{H^\varepsilon}, \quad \forall 0 < \varepsilon < \varepsilon_0. \tag{2.27}$$

Therefore, at the end of this second step, i.e., at time $t = 2T/3$, we obtain a purely hyperbolic solution with exponentially small (as $\varepsilon \to 0$) total energy.

We denote by $(w_\varepsilon^0,\ w_\varepsilon^1)$ the trace of the solution at time $t = 2T/3$, i.e.,

$$w_\varepsilon^0 = u_\varepsilon(2T/3), \qquad w_\varepsilon^1 = u_{\varepsilon,t}(2T/3).$$

Third step. In this last step we control the whole solution to zero. In other words, we prove the existence of $f_{2,\varepsilon} \in L^2(\omega \times (2T/3, T))$ such that the solution of:

$$\begin{cases} \varepsilon u_{tt} - \Delta u + u_t = f_{2,\varepsilon} 1_\omega & \text{in } \Omega \times (2T/3, T), \\ u = 0 & \text{on } \partial\Omega \times (2T/3, T), \\ u(x, 2T/3) = w_\varepsilon^0,\ u_t(x, 2T/3) = w_\varepsilon^1, & \text{in } \Omega \end{cases} \tag{2.28}$$

satisfies

$$u_\varepsilon(T) \equiv u_{\varepsilon,t}(T) \equiv 0. \tag{2.29}$$

As we shall prove in Section 5 using Carleman estimates, there exist positive constants $C_1, C_2 > 0$ such that:

$$\|f_{2,\varepsilon}\|_{L^2(\omega\times(2T/3,T))} \leqslant C_1\mathrm{e}^{C_2/\sqrt{\varepsilon}}\big\|\big(w_\varepsilon^0, w_\varepsilon^1\big)\big\|_{H^\varepsilon}, \quad \forall 0 < \varepsilon < \varepsilon_0. \tag{2.30}$$

The constants C_1, C_2 in (2.30) depend on T but they are independent of ε.

Combining (2.27) and (2.30), we get

$$\|f_{2,\varepsilon}\|_{L^2(\omega\times(2T/3,T))} \leqslant C e^{C_2/\sqrt{\varepsilon}} e^{-T/12\varepsilon} \left\|\left(u^0,u^1\right)\right\|_{H^\varepsilon}, \quad \forall 0<\varepsilon<\varepsilon_0, \tag{2.31}$$

for all initial data $\left(u^0,u^1\right)\in H^\varepsilon$.

Note that the multiplicative constant in (2.31) tends to zero exponentially as $\varepsilon\to 0$.

Conclusion. Putting all these results together we conclude that the control

$$f_\varepsilon = \begin{cases} f_{1,\varepsilon} & \text{in } [0,T/\varepsilon], \\ 0 & \text{in } [T/3,2T/3], \\ f_{2,\varepsilon} & \text{in } [2T/3,T] \end{cases} \tag{2.32}$$

is such that:

$$\|f_\varepsilon\|_{L^2(\omega\times(0,T))} \leqslant C\left\|\left(u^0,u^1\right)\right\|_{H^\varepsilon}, \quad \forall\left(u^0,u^1\right)\in H^\varepsilon, \tag{2.33}$$

with a constant C independent of $0<\varepsilon<\varepsilon_0$ and moreover the solution of (1.1) satisfies (1.4), as we wanted to prove.

Note also that the control f_ε is concentrated on the intervals $[0,T/3]\cup[2T/3,T]$ and that its restriction to $[2T/3,T]$, i.e., $f_{2,\varepsilon}$, is exponentially small with respect to $\varepsilon\to 0$.

The following three sections are devoted to rigorously prove the results stated in each of these three steps.

3. Uniform controllability of the parabolic projection

The main result of this section is as follows:

THEOREM 3.1. – *Let Ω be a bounded domain of $\mathbf{R}^n$ of class C^∞. Let ω be any open non-empty subset of Ω. Then, there exist two positive constants $\varepsilon(\Omega,\omega)$ and $C(\Omega,\omega)$ such that for all $0<\varepsilon<\varepsilon(\Omega,\omega)$, $T>C(\Omega,\omega)\sqrt{\varepsilon}$ and $(u^0,u^1)\in H^\varepsilon$ there exists a control $f_\varepsilon\in L^2(\omega\times(0,T))$ such that the solution u_ε of*:

$$\begin{cases} \varepsilon u_{tt}-\Delta u+u_t=f_\varepsilon 1_\omega & \textit{in } \Omega\times(0,T), \\ u=0 & \textit{on } \partial\Omega\times(0,T), \\ u(x,0)=u^0(x),\ u_t(x,0)=u^1(x) & \textit{in } \Omega, \end{cases} \tag{3.1}$$

satisfies

$$\pi_p^\varepsilon\left(u_\varepsilon(T),u_{\varepsilon,t}(T)\right)=0. \tag{3.2}$$

Moreover, there exists a constant $C>0$, depending on T but independent of $0<\varepsilon<\min(\varepsilon(\Omega,\omega),T^2/C^2(\Omega,\omega))$ such that:

$$\|f_\varepsilon\|_{L^2(\omega\times(0,T))} \leqslant C\left\|\left(u^0,u^1\right)\right\|_{H^\varepsilon}, \quad \forall\left(u^0,u^1\right)\in H^\varepsilon. \tag{3.3}$$

Remark 3.1. – The result above holds for every non-empty open subset ω of Ω, without any geometric restriction. However, it only provides the null-controllability of the parabolic component of the system.

Proof. – We decompose the time interval $[0,T]$ as follows. We fix two constants $\delta\in(0,T/2)$ and $\rho\in(0,1/n)$, n being the space dimension. For any $\ell\geqslant 1$ we set:

$$\sigma_\ell = 2^\ell, \qquad T_\ell = A2^{-\rho\ell} = A\sigma_\ell^{-\rho}, \tag{3.4}$$

where $A>0$ is chosen such that

$$2\sum_{\ell\geqslant 1} T_\ell = T-2\delta. \tag{3.5}$$

Taking into account that for any $\varepsilon>0$, $k(\varepsilon)$ is such that

$$\left[4\mu_{k(\varepsilon)+1}\right]^{-1} < \varepsilon \leqslant \left[4\mu_{k(\varepsilon)}\right]^{-1}, \tag{3.6}$$

we choose $\ell_0\in\mathbb{N}$ such that

$$\sigma_{\ell_0} \leqslant k(\varepsilon) < \sigma_{\ell_{0+1}}. \tag{3.7}$$

We also set

$$a_0=\delta,\ a_1=\delta+2T_1,\ \ldots,\ a_\ell = a_{\ell-1}+2T_\ell. \tag{3.8}$$

Given $(u^0,u^1)\in H^\varepsilon$ we proceed as follows. In the time interval $[0,\delta]$ we let the system to evolve freely without control, i.e., we set $f_\varepsilon=0$ in $[0,\delta]$. To simplify the notation we denote the initial data (u^0,u^1) by the vector valued function U^0 and by $\{S_\varepsilon(t)\}_{t\geqslant 0}$ the semigroup of contractions generated by system (3.1) without control (i.e., with $f_\varepsilon\equiv 0$). We denote by U^1 the state at time $t=\delta$, i.e., $U^1=S_\varepsilon(\delta)U^0$.

We now proceed as follows. The second time interval $[a_0,a_1]=[\delta,\delta+2T_1]$ is split into two parts. In the first one $[a_0,a_0+T_1]=[\delta,\delta+T_1]$ we control to zero the projection $\pi^\varepsilon_{\sigma_1}$ of the solution, using Lemma 3.2 below, which turns out to be a direct consequence of the observability estimate of Lemma C.1 in Appendix C. In the second half $[\delta+T_1,\ \delta+2T_1]$ we let the system to evolve freely. We iterate this argument for any $\ell\geqslant 2$. In this way we obtain the following sequence of states:

$$\begin{cases} U^1 = S_\varepsilon(\delta)\big(U^0\big), \\ U^{\ell+1} = S_\varepsilon(T_\ell)\big(W^\ell\big), \quad \ell=1,\ldots,\ell_0, \\ W^\ell = \mathcal{L}_\varepsilon\big(T_\ell;\ U^\ell, K^\varepsilon_{T_\ell,\sigma_\ell}\big(U^\ell\big)\big), \quad \ell=1,\ldots,\ell_0, \end{cases} \tag{3.9}$$

where $K^\varepsilon_{T_\ell,\sigma_\ell}(U^\ell)$ denotes a control acting in the time interval $[a_{\ell-1},\ a_{\ell-1}+T_\ell]$ that drives the solution U^ℓ, obtained in the previous steps at time $t=a_{\ell-1}$, to the state W^ℓ at time $t=a_{\ell-1}+T_\ell$ with $\pi^\varepsilon_{\sigma_\ell}(W^\ell)=0$. The map $\mathcal{L}_\varepsilon$ transforms the datum U^ℓ with control $K^\varepsilon_{T_\ell,\sigma_\ell}(U^\ell)$ in the datum W^ℓ. In other words, by the variation of constants formula,

$$W^\ell = S_\varepsilon(T_\ell)U^\ell + \int_0^{T_\ell} S_\varepsilon(T_\ell-s)\widetilde{K^\varepsilon_{T_\ell,\sigma_\ell}}\big(U^\ell\big)(a_{\ell-1}+s)\,\mathrm{d}s,$$

where $\widetilde{K^{\varepsilon}_{T_\ell,\sigma_\ell}}(U^\ell)$ denotes the vector valued function with components $(0, K^{\varepsilon}_{T_\ell,\sigma_\ell}(U^\ell))$.

The control f_ε of Theorem 3.1 is then as follows:

- If $\sigma_{\ell_0} = k(\varepsilon)$ we set:

$$f_\varepsilon = \begin{cases} 0, & 0 \leqslant t < a_0, \\ K^{\varepsilon}_{T_\ell,\sigma_\ell}(U^\ell), & a_{\ell-1} < t \leqslant a_{\ell-1} + T_\ell,\ \forall \ell \leqslant \ell_0, \\ 0, & a_{\ell-1} + T_\ell < t \leqslant a_{\ell-1} + 2T_\ell = a_\ell, \\ 0, & a_{\ell_0} < t \leqslant T. \end{cases} \tag{3.10}$$

By construction

$$\mathcal{L}^\varepsilon(a_\ell; U^0, f_\varepsilon) = U^{\ell+1}, \quad \forall \ell \leqslant \ell_0$$

and, in particular,

$$\mathcal{L}^\varepsilon(a_{\ell_0}; U^0, f_\varepsilon) = U^{\ell_0+1}.$$

Then

$$\pi^\varepsilon_p[\mathcal{L}^\varepsilon(a_{\ell_0}; U^0, f_\varepsilon)] = \pi^\varepsilon_{k(\varepsilon)}[\mathcal{L}^\varepsilon(a_{\ell_0}; U^0, f_\varepsilon)] = \pi^\varepsilon_{\sigma_{\ell_0}}[\mathcal{L}^\varepsilon(a_{\ell_0}; U^0, f_\varepsilon)] = \pi^\varepsilon_{\sigma_{\ell_0}} U^{\ell_0+1} = 0.$$

Taking into account that in the last time interval $[a_{\ell_0}, T]$ the control f_ε vanishes we deduce that

$$\pi^\varepsilon_p[\mathcal{L}^\varepsilon(T; U^0, f_\varepsilon)] = 0$$

as we wanted to prove.

- When $\sigma_{\ell_0} < k(\varepsilon)$ we also set

$$W^{\ell_0+1} = \mathcal{L}^\varepsilon(T_{\ell_0+1}; U^{\ell_0+1}, K^\varepsilon_{T_{l_0+1,k(\varepsilon)}}(U^{\ell_0+1})).$$

In this case we introduce the control:

$$f_\varepsilon = \begin{cases} 0, & 0 < t \leqslant a_0, \\ K^{\varepsilon}_{T_\ell,\sigma_\ell}(U^\ell), & a_{\ell-1} \leqslant t \leqslant a_{\ell-1} + T_\ell,\ \forall \ell \leqslant \ell_0, \\ 0, & a_{\ell-1} + T_\ell \leqslant t \leqslant a_{\ell-1} + 2T_\ell = a_\ell,\ \forall \ell \leqslant \ell_0, \\ K^\varepsilon_{T_{\ell_0+1,k(\varepsilon)}}(U^{\ell_0+1}), & a_{\ell_0} \leqslant t \leqslant a_{\ell_0} + T_{\ell_0+1}, \\ 0, & a_{\ell_0} + T_{\ell_0+1} \leqslant t \leqslant T. \end{cases} \tag{3.11}$$

It is easy to check, as above, that

$$\pi^\varepsilon_p[\mathcal{L}^\varepsilon(T; U^0, f_\varepsilon)] = 0.$$

We also observe that the length T_ℓ of the time intervals in which the control acts satisfies:

$$T_1 > T_2 > \cdots > T_{\ell_{0+1}} = \frac{A}{(\sigma_{\ell_{0+1}})^\rho} = \frac{A}{2^\rho(\sigma_{\ell_0})^\rho} \geqslant \frac{A}{2^\rho(k(\varepsilon))^\rho}. \tag{3.12}$$

Taking into account that $0 < \rho < 1/n$, (3.6)–(3.7) and the fact that, by Weyl's formula, $\mu_k \sim C(\Omega)k^{2/n}$ as $k \to \infty$, we deduce that

$$T_j \geqslant C\varepsilon^{\rho n/2} >> C\sqrt{\varepsilon}, \quad \forall j = 1, \dots, \ell_0 + 1 \text{ (as } \varepsilon \to 0).$$

This will be essential when applying Lemma 3.2 to get bounds on the size of the controls.

Therefore, the controls f_ε have been found so that the controllability condition (3.2) holds. Thus, it is sufficient to prove the uniform bound (3.3) on the controls.

To do this we need some technical lemmas:

LEMMA 3.1. – *There exist positive constants $C_1, C_2 > 0$ such that*

$$\big\| S_\varepsilon(t)\big(u^0, u^1\big)\big\|_{H^\varepsilon} \leqslant C_1 \mathrm{e}^{-C_2\mu_{k+1}t} \big\|\big(u^0, u^1\big)\big\|_{H^\varepsilon}, \quad \forall t > 0, \tag{3.13}$$

for all $0 < \varepsilon < 1$ and every $(u^0, u^1) \in H^\varepsilon$ such that $\pi_k^\varepsilon(u^0, u^1) = 0$, and for every $1 \leqslant k \leqslant k(\varepsilon)$.

We refer to Appendix B at the end of the paper for the proof of this lemma.

The decay rate (3.13) stated in Lemma 3.1 is natural to expect since the solutions of the uncontrolled system such that $\pi_k^\varepsilon(u^0, u^1) = 0$ satisfy $\pi_k^\varepsilon(u(t),\ u_t(t)) = 0$ for all $t \geqslant 0$. Then, in the Fourier expansion of the solution only the eigenfrequencies associated to the indexes $j \geqslant k + 1$ enter and, among them, the one of maximum real part is precisely $\lambda^\varepsilon_{k+1_+}$. Taking into account that $\lambda^\varepsilon_{j_+} < -\mu_j$, the decay rate in (3.13) is natural to expect.

Applying (3.13) we deduce that:

$$\big\|U^{\ell+1}\big\|_{H^\varepsilon} \leqslant C_1 \mathrm{e}^{-C_2 T_\ell \mu_{\sigma_\ell}} \big\|W^\ell\big\|_{H^\varepsilon}. \tag{3.14}$$

Moreover, using classical energy estimates for (3.1) we deduce that:

$$\big\|W^\ell\big\|_{H^\varepsilon} \leqslant \big\|U^\ell\big\|_{H^\varepsilon} + \frac{1}{\sqrt{2}} \big\|K^\varepsilon_{T_\ell,\sigma_\ell}\big(U^\ell\big)\big\|_{L^2(\omega\times(a_{\ell-1},a_{\ell-1}+T_\ell))}. \tag{3.15}$$

We now need an estimate on the L^2-norm of $K^\varepsilon_{T_{\ell,\sigma_\ell}}(U^\ell)$. This is provided by the following lemma:

LEMMA 3.2. – *Under the conditions above, there exist positive constants $\varepsilon(\Omega)$ and $C_1, C_2 > 0$ such that for all $0 < \varepsilon < \varepsilon(\Omega)$ and $1 \leqslant \ell \leqslant \ell_0$ we have*

$$\big\|K^\varepsilon_{T_\ell,\sigma_\ell}\big(U^\ell\big)\big\|_{L^2(\omega\times(a_{\ell-1},a_{\ell-1}+T_\ell))} \leqslant C_1 \max\left\{\frac{2}{\sqrt{T_\ell}}, \frac{1}{\sqrt{\mu_1}T_\ell}\right\} \mathrm{e}^{C_2\sqrt{\mu_{\sigma_\ell}}} \big\|U^\ell\big\|_{H^\varepsilon}. \tag{3.16}$$

We refer to Appendix C at the end of this paper for the proof of this result.

Assuming for the moment that Lemma 3.2 holds, let us go back to the estimates (3.14)–(3.15). According to (3.15)–(3.16) we have

$$\begin{aligned}\big\|W^\ell\big\|_{H^\varepsilon} &\leqslant \left[1 + \frac{C}{T_\ell}\mathrm{e}^{C_2\sqrt{\mu_{\sigma_\ell}}}\right]\big\|U^\ell\big\|_{H^\varepsilon}\\ &= \big[1 + CA^{-1}\sigma_\ell^\rho \mathrm{e}^{-C_2\sqrt{\mu_{\sigma_\ell}}}\mathrm{e}^{2C_2\sqrt{\mu_{\sigma_\ell}}}\big]\big\|U^\ell\big\|_{H^\varepsilon}\end{aligned}$$

for a suitable $C > 0$.

The function:

$$\ell \mapsto \sigma_\ell^\rho \mathrm{e}^{-C\sqrt{\mu_{\sigma_\ell}}}$$

is bounded above for any $\rho \in (0,1)$ and $C > 0$. To see this it is sufficient to observe that, according to Weyl's formula, $\mu_{\sigma_\ell} \sim C(\Omega)(\sigma_\ell)^{2/n}$.

Consequently

$$\|W^\ell\|_{H^\varepsilon} \leqslant \bigl(1 + C_1 e^{C_2\sqrt{\mu_{\sigma_\ell}}}\bigr)\|U^\ell\|_{H^\varepsilon} \tag{3.17}$$

for suitable positive constants $C_1, C_2 > 0$.

Combining (3.14) and (3.17) we deduce that:

$$\|U^{\ell+1}\|_{H^\varepsilon} \leqslant C_1 e^{-C_2 T_\ell \mu_{\sigma_\ell}}\bigl(1 + C_3 e^{C_4\sqrt{\mu_{\sigma_\ell}}}\bigr)\|U^\ell\|_{H^\varepsilon} \tag{3.18}$$

for suitable positive constants C_j, $j = 1, \ldots, 4$.

Taking into account that $T_\ell = A2^{-\rho\ell}$, $\sigma_\ell = 2^\ell$ and using Weyl's formula again we deduce that:

$$\|U^{\ell+1}\|_{H^\varepsilon} \leqslant C_1 e^{-C_2 A 2^{\ell(2/n-\rho)}}\bigl(1 + C_3 e^{C_4 2^{\ell/n}}\bigr)\|U^\ell\|_{H^\varepsilon}.$$

The function

$$g(\ell) = e^{-\frac{C_2 A}{2}2^{\ell(2/n-\rho)}}\bigl(1 + C_3 e^{C_4 2^{\ell/n}}\bigr), \quad \ell \geqslant 1,$$

is positive and bounded above for any $\rho \in (0, 1/n)$. Therefore,

$$\|U^{\ell+1}\|_{H^\varepsilon} \leqslant C_1 e^{-C_2 2^{\alpha\ell}}\|U^\ell\|_{H^\varepsilon}, \quad \forall \ell = 1, \ldots, \ell_0, \tag{3.19}$$

for suitable positive constants C_1, C_2, and $\alpha = 2/n - \rho$.

Applying (3.19) in an iterative way we deduce that:

$$\begin{aligned}
\|U^{\ell+1}\|_{H^\varepsilon} &\leqslant C_1 e^{-C_2 2^{\alpha\ell}}\|U^\ell\|_{H^\varepsilon} \\
&\leqslant C_1^2 e^{-C_2 2^{\alpha\ell}} e^{-C_2 2^{\alpha(\ell-1)}}\|U^{\ell-1}\|_{H^\varepsilon} \\
&\quad \cdots\cdots\cdots\cdots \\
&\leqslant C_1^\ell e^{-C_2 2^{\alpha\ell}(1 + 2^{-\alpha} + \cdots + 2^{-\alpha(\ell-1)})}\|U^1\|_{H^\varepsilon} \\
&\leqslant C_1^\ell e^{-C_2 2^{\alpha\ell}}\|U^1\|_{H^\varepsilon}
\end{aligned} \tag{3.20}$$

for suitable positive constants C_1, C_2.

On the other hand, the function:

$$h(\ell) = C_1^\ell e^{-\frac{C_2}{2}2^{\alpha\ell}}, \quad \ell \geqslant 1,$$

is positive and bounded above. Therefore

$$\|U^{\ell+1}\|_{H^\varepsilon} \leqslant C_1 e^{-C_2 2^{\alpha\ell}}\|U^1\|_{H^\varepsilon} \tag{3.21}$$

for suitable positive constants $C_1, C_2 > 0$. Moreover,

$$\|U^1\|_{H^\varepsilon} \leqslant \|U^0\|_{H^\varepsilon}.$$

Consequently

$$\|U^{\ell+1}\|_{H^\varepsilon} \leqslant C_1 e^{-C_2 2^{\alpha\ell}}\|U^0\|_{H^\varepsilon}. \tag{3.22}$$

We can now estimate the controls f_ε built in (3.10) and (3.11). To fix ideas we consider the case (3.10). We have

$$\|f_\varepsilon\|_{L^2(\omega\times(0,T))} \leqslant \sum_{\ell=1}^{\ell_0} \big\|K_{T_\ell,\sigma_\ell}\big(U^\ell\big)\big\|_{L^2(\omega\times(a_{\ell-1},a_{\ell-1}+T_\ell))}. \tag{3.23}$$

We now apply Lemma 3.2. We assume that $\sqrt{\mu_1}T_\ell \leqslant \sqrt{T_\ell}/2$ for all $\ell = 1,\dots,\ell_0$. Note that when $\sqrt{\mu_1}T_\ell > \sqrt{T_\ell}/2$ for a finite number of values of ℓ the same estimates apply with a larger multiplicative constant on the upper bound.

Applying Lemma 3.2 and (3.22) we get

$$\|f_\varepsilon\|_{L^2(\omega\times(0,T))} \leqslant C_1\left[\sum_{\ell=1}^{\ell_0} T_\ell^{-1}\mathrm{e}^{C_2\sqrt{\mu_{\sigma_\ell}}}\mathrm{e}^{-C_3 2^{\alpha\ell}}\right]\big\|U^0\big\|_{H^\varepsilon}. \tag{3.24}$$

Let us now analyze the sum on the right hand side of this inequality; we have:

$$\begin{aligned}&\sum_{\ell=1}^{\ell_0} T_\ell^{-1}\mathrm{e}^{C_2\sqrt{\mu_{\sigma_\ell}}}\mathrm{e}^{-C_3 2^{\alpha\ell}}\\ &\quad= A^{-1}\sum_{\ell=1}^{\ell_0} 2^{\ell\rho}\mathrm{e}^{C_2 2^{\ell/n}}\mathrm{e}^{-C_3 2^{\alpha\ell}} = A^{-1}\sum_{\ell=1}^{\ell_0}\big[2^{\ell\rho}\mathrm{e}^{C_2 2^{\ell/n}}\mathrm{e}^{-\frac{C_3}{2}2^{\alpha\ell}}\big]\mathrm{e}^{-\frac{C_3}{2}2^{\alpha\ell}}.\end{aligned} \tag{3.25}$$

The series on the right hand side of (3.25) converges as $\ell_0 \to \infty$ since the function

$$p(\ell) = 2^{\ell\rho}\mathrm{e}^{C_2 2^{\ell/n}}\mathrm{e}^{-\frac{C_3}{2}2^{\alpha\ell}}$$

is bounded above because of the fact that $\alpha = 2/n - \rho > 1/n$ in view of the choice of $\rho (\rho \in (0, 1/n))$.

This completes the proof of Theorem 3.1. □

4. Decay rates for purely hyperbolic solutions

This section is devoted to prove the estimate (2.26) on the decay of the solutions with data in H_h^ε.

More precisely, we prove the following result:

THEOREM 4.1. – *Let u be a solution of* (1.1) *with $f \equiv 0$ with initial data $(u^0, u^1) \in H_h^\varepsilon$. Then*

$$\big\|\big(u(t), u_t(t)\big)\big\|_{H^\varepsilon} \leqslant 2^{3/2}\mathrm{e}^{-t/4\varepsilon}\big\|\big(u^0, u^1\big)\big\|_{H^\varepsilon}. \tag{4.1}$$

Proof. – We decompose the initial data $U^0 = (u^0, u^1)$ in Fourier series:

$$U^0 = \sum_{j\geqslant k(\varepsilon)+1} (a_j, b_j)e_j(x). \tag{4.2}$$

Then the solution u of (1.1) with $f \equiv 0$ and these initial data may be written as:

$$u(x,t) = \sum_{j\geqslant k(\varepsilon)+1}\left(\frac{a_j\lambda_{j-}^\varepsilon - b_j}{\lambda_{j-}^\varepsilon - \lambda_{j+}^\varepsilon}\mathrm{e}^{\lambda_{j+}^\varepsilon t} + \frac{b_j - a_j\lambda_{j+}^\varepsilon}{\lambda_{j-}^\varepsilon - \lambda_{j+}^\varepsilon}\mathrm{e}^{\lambda_{j-}^\varepsilon t}\right)e_j(x). \tag{4.3}$$

To simplify the notation, from now on, we shall write $\lambda_{j\pm}$ instead of $\lambda^{\varepsilon}_{j\pm}$. We also introduce the notation:

$$\beta_j^{\varepsilon} = \frac{\sqrt{4\varepsilon\mu_j - 1}}{2\varepsilon} \tag{4.4}$$

and, again, we shall omit in the notation the dependence on ε. Then

$$\lambda_{j\pm} = -\frac{1}{2\varepsilon} \pm \mathrm{i}\beta_j, \quad \forall j \geqslant k(\varepsilon)+1.$$

Moreover,

$$\begin{aligned}
\|u(t)\|^2_{H_0^1(\Omega)} &= \sum_{j\geqslant k(\varepsilon)+1} \mu_j \left| \frac{a_j\lambda_{j-} - b_j}{\lambda_{j-} - \lambda_{j+}} \mathrm{e}^{\lambda_{j+}t} + \frac{b_j - a_j\lambda_{j+}}{\lambda_{j-} - \lambda_{j+}} \mathrm{e}^{\lambda_{j-}t} \right|^2 \\
&= \sum_{j\geqslant k(\varepsilon)+1} \mu_j \left| a_j \frac{\lambda_{j-}\mathrm{e}^{\lambda_{j+}t} - \lambda_{j+}\mathrm{e}^{\lambda_{j-}t}}{\lambda_{j-} - \lambda_{j+}} + b_j \frac{\mathrm{e}^{\lambda_{j-}t} - \mathrm{e}^{\lambda_{j+}t}}{\lambda_{j-} - \lambda_{j+}} \right|^2 \\
&\leqslant \frac{\mathrm{e}^{-t/\varepsilon}}{2} \sum_{j\geqslant k(\varepsilon)+1} \frac{\mu_j}{\beta_j^2} \left[a_j^2 \left|\lambda_{j-}\mathrm{e}^{\mathrm{i}\beta_j t} - \lambda_{j+}\mathrm{e}^{-\mathrm{i}\beta_j t}\right|^2 + b_j^2 \left|\mathrm{e}^{-i\beta_j t} - \mathrm{e}^{\mathrm{i}\beta_j t}\right|^2 \right] \\
&= \frac{\mathrm{e}^{-t/\varepsilon}}{2} \sum_{j\geqslant k(\varepsilon)+1} \frac{\mu_j}{\beta_j^2} \left[a_j^2 \left| \frac{\sin(\beta_j t)}{\varepsilon} + 2\beta_j \cos(\beta_j t) \right|^2 + 4b_j^2 \sin^2(\beta_j t) \right] \\
&\leqslant 2\mathrm{e}^{-t/\varepsilon} \sum_{j\geqslant k(\varepsilon)+1} \mu_j \left[2a_j^2 \left(\frac{t^2}{4\varepsilon^2} + 1 \right) + b_j^2 \left(\frac{\sin(\beta_j t)}{\beta_j} \right)^2 \right].
\end{aligned} \tag{4.5}$$

We claim that

$$\mu_j \frac{\sin^2(\beta_j t)}{\varepsilon\beta_j^2} \leqslant \frac{t^2}{2\varepsilon^2} + 2. \tag{4.6}$$

Note that combining (4.5) and (4.6) we get

$$\begin{aligned}
\|u(t)\|^2_{H_0^1(\Omega)} &\leqslant 2\left(\frac{t^2}{2\varepsilon^2} + 2 \right) \mathrm{e}^{-t/\varepsilon} \sum_{j\geqslant k(\varepsilon)+1} \left[\mu_j a_j^2 + \varepsilon b_j^2 \right] \\
&\leqslant 4\mathrm{e}^{-t/2\varepsilon} \sum_{j\geqslant k(\varepsilon)+1} \left[\mu_j a_j^2 + \varepsilon b_j^2 \right].
\end{aligned} \tag{4.7}$$

Let us now prove (4.6). When $2\mu_j\varepsilon \geqslant 1$ we have

$$\mu_j \frac{\sin^2(\beta_j t)}{\varepsilon\beta_j^2} \leqslant \frac{\mu_j}{\varepsilon\beta_j^2} = \frac{4\varepsilon\mu_j}{4\varepsilon\mu_j - 1} \leqslant 2$$

since the function $2x/(2x-1)$ is decreasing for $x \geqslant 1$. On the other hand, when $2\varepsilon\mu_j < 1$ we have

$$\mu_j \frac{\sin^2(\beta_j t)}{\varepsilon\beta_j^2} \leqslant \frac{\mu_j t^2}{\varepsilon} < \frac{t^2}{2\varepsilon^2}.$$

This completes the proof of (4.6).

Now we shall get an upper bound on $\|u_t(t)\|_{L^2(\Omega)}$.

We have

$$u_t(x,t) = \sum_{j\geqslant k(\varepsilon)+1} \left(\lambda_{j+}\left(\frac{a_j\lambda_{j-}-b_j}{\lambda_{j-}-\lambda_{j+}}\right)\mathrm{e}^{\lambda_{j+}t} + \lambda_{j-}\left(\frac{b_j-a_j\lambda_{j+}}{\lambda_{j-}-\lambda_{j+}}\right)\mathrm{e}^{\lambda_{j-}t}\right)e_j(x).$$

Therefore, taking into account that $\lambda_{j-}\lambda_{j+} = \mu_j/\varepsilon$,

$$\begin{aligned}\|u_t(t)\|^2_{L^2(\Omega)} &= \sum_{j\geqslant k(\varepsilon)+1}\left[a_j\lambda_{j+}\lambda_{j-}\frac{\mathrm{e}^{\lambda_{j+}t}-\mathrm{e}^{\lambda_{j-}t}}{\lambda_{j-}-\lambda_{j+}} + b_j\frac{\lambda_{j-}\mathrm{e}^{\lambda_{j-}t}-\lambda_{j+}\mathrm{e}^{\lambda_{j+}t}}{\lambda_{j-}-\lambda_{j+}}\right]^2\\ &\leqslant 2\sum_{j\geqslant k(\varepsilon)+1}\left[a_j^2\frac{\mu_j^2}{\varepsilon^2}\left(\frac{\mathrm{e}^{\lambda_{j+}t}-\mathrm{e}^{\lambda_{j-}t}}{\lambda_{j-}-\lambda_{j+}}\right)^2 + b_j^2\left(\frac{\lambda_{j-}\mathrm{e}^{\lambda_{j-}t}-\lambda_{j+}\mathrm{e}^{\lambda_{j+}t}}{\lambda_{j-}-\lambda_{j+}}\right)^2\right]\\ &\leqslant 2\mathrm{e}^{-\frac{t}{\varepsilon}}\sum_{j\geqslant k(\varepsilon)+1}\left[\frac{a_j^2\mu_j^2}{\varepsilon^2}\left(\frac{\sin(\beta_j t)}{\beta_j}\right)^2 + 2b_j^2\left(\frac{\sin^2(\beta_j t)}{4\varepsilon^2\beta_j^2}+\cos^2(\beta_j t)\right)\right].\end{aligned}$$

In view of (4.6) we deduce that

$$\|u_t(t)\|^2_{L^2(\Omega)} \leqslant 2\mathrm{e}^{-t/\varepsilon}\left(\frac{t^2}{2\varepsilon^2}+2\right)\sum_{j\geqslant k(\varepsilon)+1}\left[a_j^2\frac{\mu_j}{\varepsilon}+b_j^2\right]. \tag{4.8}$$

Then

$$\varepsilon\|u_t(t)\|^2_{L^2(\Omega)} \leqslant 4\mathrm{e}^{-t/2\varepsilon}\sum_{j\geqslant k(\varepsilon)+1}\left[a_j^2\mu_j+b_j^2\varepsilon\right]. \tag{4.9}$$

Combining (4.7) and (4.9) we deduce that

$$\begin{aligned}\left\|\left(u(t),u_t(t)\right)\right\|^2_{H^\varepsilon} &= \|u(t)\|^2_{H_0^1(\Omega)} + \varepsilon\|u_t(t)\|^2_{L^2(\Omega)}\\ &\leqslant 8\mathrm{e}^{-t/2\varepsilon}\sum_{j\geqslant k(\varepsilon)+1}\left[a_j^2\mu_j+\varepsilon b_j^2\right]\\ &= 8\mathrm{e}^{-t/2\varepsilon}\left\|\left(u^0,u^1\right)\right\|^2_{H^\varepsilon}.\end{aligned} \tag{4.10}$$

This completes the proof of Theorem 4.1. □

5. Proof of Theorem 1.2

This section is devoted to prove Theorem 1.2 which guarantees an observability estimate for *all solutions* of the adjoint system (1.9) with a constant of the order of $\mathrm{e}^{c/\sqrt{\varepsilon}}$.

The proof is based on the use of Carleman's inequalities.

Before getting into the proof of Theorem 1.2 let us state and prove an immediate consequence of this result on the controllability of system (1.1).

PROPOSITION 5.1. – *In the geometric setting above system* (1.1) *is exactly controllable.*

Moreover, the constant $C(\varepsilon,T)$ *in* (1.5) *measuring the cost of controlling exactly* (1.1) *may be bounded above as follows*:

$$C(\varepsilon,T) \leqslant C_1(T)\mathrm{e}^{C_2(T)/\sqrt{\varepsilon}} \tag{5.1}$$

for suitable positive constants $C_1(T)$ *and* $C_2(T)$.

Proof. – Given $(u^0, u^1) \in H^\varepsilon$, we claim that the control $f \in L^2(\omega \times (0,T))$ such that the solution of (1.1) satisfies (1.2) may be built as follows. We take $f = \varphi$ where φ is the solution of (1.9) associated with the initial data $(\varphi^0, \varphi^1) \in (H^\varepsilon)'$ that minimize the functional:

$$J(\varphi^0, \varphi^1) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 \,dx\,dt + \int_\Omega (u^0 + \varepsilon u^1)\varphi(0)\,dx - \varepsilon\langle u^0, \varphi_t(0)\rangle, \tag{5.2}$$

over $L^2(\Omega) \times H^{-1}(\Omega)$.

Indeed, the functional J is continuous, convex and coercive in $(H^\varepsilon)'$, the coercivity being a consequence of (1.10) which implies

$$\|\varphi(0)\|^2_{L^2(\Omega)} + \varepsilon\|\varphi_t(0)\|^2_{H^{-1}(\Omega)} \leqslant C_1(T)e^{C_2(T)/\sqrt{\varepsilon}} \int_0^T \int_\omega \varphi^2 \,dx\,dt \tag{5.3}$$

for every solution of (1.9) and $0 < \varepsilon < 1$, since the energy

$$F(t) = \frac{1}{2}\left[\|\varphi(t)\|^2_{L^2(\Omega)} + \varepsilon\|\varphi_t(t)\|^2_{H^{-1}(\Omega)}\right]$$

increases along the solutions of (1.9). Thus, J achieves its minimum at some $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. The fact that the differential of J vanishes at the minimizer (φ^0, φ^1) is equivalent to the fact that solution u of (1.1) with control $f = \varphi, \varphi$ being the solution of (1.9) associated to the minimizer, satisfies (1.4).

In order to estimate the norm of the control we observe that $J(\varphi^0, \varphi^1) \leqslant J(0,0) = 0$. Therefore, according to (5.3):

$$\begin{aligned}\frac{1}{2}\int_0^T \int_\omega \varphi^2 \,dx\,dt &\leqslant \left|\int_\Omega (u^0 + \varepsilon u^1)\varphi(0)\,dx - \varepsilon\langle u^0, \varphi_t(0)\rangle\right| \\ &\leqslant \left[\|u^0 + \varepsilon u^1\|_{L^2(\Omega)} + \sqrt{\varepsilon}\|u^0\|_{H_0^1(\Omega)}\right]\left[\|\varphi(0)\|_{L^2(\Omega)} + \sqrt{\varepsilon}\|\varphi_t(0)\|_{H^{-1}(\Omega)}\right] \\ &\leqslant \left[\|u^0 + \varepsilon u^1\|_{L^2(\Omega)} + \sqrt{\varepsilon}\|u^0\|_{H_0^1(\Omega)}\right]\left[C_1(T)e^{C_2(T)/\sqrt{\varepsilon}}\int_0^T \int_\omega \varphi^2 \,dx\,dt\right]^{1/2}\end{aligned}$$

which implies the upper bound (1.5) on the control with a constant $C(\varepsilon, T)$ satisfying (5.1).

This completes the proof of Proposition 5.1. □

We now prove Theorem 1.2.

We first observe that by the change of the time variable $t \to \tau = (T - t)/\sqrt{\varepsilon}$, system (1.9) becomes:

$$\begin{cases} \varphi_{\tau\tau} - \Delta\varphi + \dfrac{1}{\sqrt{\varepsilon}}\varphi_\tau = 0 & \text{in } \Omega \times (0, \widehat{T}), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, \widehat{T}), \\ \varphi(x,0) = \varphi^0(x),\ \varphi_\tau(x,0) = -\sqrt{\varepsilon}\varphi^1(x) & \text{in } \Omega, \end{cases} \tag{5.4}$$

where $\widehat{T} = T/\sqrt{\varepsilon}$. To simplify the notation we rewrite it as:

$$\begin{cases} \psi_{tt} - \Delta\psi + k\psi_t = 0 & \text{in } \Omega \times (0,T), \\ \psi = 0 & \text{on } \partial\Omega \times (0,T), \\ \psi(x,0) = \psi^0(x),\ \psi_t(x,0) = \psi^1(x) & \text{in } \Omega. \end{cases} \tag{5.5}$$

The following holds:

THEOREM 5.1. – *Let Ω be a bounded domain of $\mathbf{R}^n$ with boundary of class C^2 and ω a neighborhood of a subset of the boundary of the form $\Gamma(x^0)$ in Ω. Then, there exists $T_0 = T_0(\Omega;\omega) > 0$ such that for all $T > T_0$ there exist positive constants $C_1, C_2 > 0$ such that*

$$\|\psi^0\|^2_{L^2(\Omega)} + \|\psi^1\|^2_{H^{-1}(\Omega)} \leqslant C_1 \mathrm{e}^{C_2|k|} \int_0^T \int_\omega \psi^2 \,\mathrm{d}x\,\mathrm{d}t \tag{5.6}$$

for every solution of (5.5) *and all $k \in \mathbf{R}$.*

Remark 5.1. – The proof of Theorem 5.1 provides an explicit estimate on the minimal time $T_0 = T_0(\Omega,\omega)$. However, it is not the minimal time that is needed to apply the multiplier technique to the wave equation with $k = 0$ which is $T_0 = 2\|x - x_0\|_{L^\infty(\Omega-\omega)}$.

Proof of Theorem 5.1. – We shall use the following notations. First, for any $\delta > 0$ and any set $S \in \mathbf{R}^n$, we denote

$$\mathcal{O}_\delta(S) \triangleq \{x \in \mathbf{R}^n;\ |x-y| < \delta \text{ for some } y \in S\}.$$

Next, for $x^0 \in \mathbf{R}^n$ and for small $\delta_0 > 0$, we set:

$$\Gamma^0 \triangleq \{x \in \partial\Omega;\ (x - x^0)\cdot\nu > 0\}, \qquad \omega^0 \triangleq \Omega \cap \mathcal{O}_{\delta_0}(\Gamma^0). \tag{5.7}$$

Now, in the case $x^0 \in \bar{\Omega}$, it is easy to see that one can find another point $x^1 \in \mathbf{R}^n \setminus \{x^0\}$ (which is very close to x^0), such that

$$\omega^1 \triangleq \Omega \cap \mathcal{O}_{\delta_0/2}(\Gamma^1) \subset \omega^0, \quad \text{where } \Gamma^1 \triangleq \{x \in \partial\Omega;\ (x - x^1)\cdot\nu > 0\}. \tag{5.8}$$

Finally, we put:

$$R_0 \triangleq \min_{x\in\Omega} |x - x^0|, \quad \text{if } x^0 \in \mathbf{R}^n \setminus \bar{\Omega}; \qquad R_0 = |x^0 - x^1|/5, \quad \text{if } x^0 \in \bar{\Omega},$$

$$\begin{aligned} &R_1 \triangleq \max_{x\in\Omega} |x - x^0|, \quad \text{if } x^0 \in \mathbf{R}^n \setminus \bar{\Omega}; \\ &R_1 \triangleq \max\Bigl(\max_{x\in\Omega}|x - x^0|,\ \max_{x\in\Omega}|x - x^1|\Bigr), \quad \text{if } x^0 \in \bar{\Omega}, \end{aligned} \tag{5.9}$$

where x^1 was given in (5.8). It is easy to see that $R_0 > 0$. Thus one can choose a sufficiently small $\alpha \in (0,1)$ such that:

$$T^0 \triangleq \frac{\sqrt{(1+3\alpha)^2\alpha^2 + (1+3\alpha)(3+\alpha-\alpha^3)} - (1+3\alpha)\alpha}{\alpha(1+3\alpha)} R_0 > t^0 \triangleq 2\sqrt{\frac{2}{\alpha}} R_1 \tag{5.10}$$

and

$$\alpha t^0 R_0 < \frac{R_1^2}{8}. \tag{5.11}$$

We now distinguish two different cases.

Case 1. $x^0 \in \mathbf{R}^n \setminus \bar{\Omega}$.

We divide the proof into several steps.

Step 1. Let us make some reductions. According to the result of Appendix A it is sufficient to prove that the result holds for k sufficiently large. We define the energy:

$$E(t) = \frac{1}{2}\left(\|\psi(t)\|^2_{L^2(\Omega)} + \|\psi_t(t)\|^2_{H^{-1}(\Omega)}\right)$$

that satisfies

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -k\|\psi_t(t)\|^2_{H^{-1}(\Omega)}.$$

Observe that the left hand side of (5.6) coincides with $2E(0)$.

If $k > 0$, $E(0)$ may be replaced by $E(T)$ in (5.6) because the difference between these two energies is a multiplicative factor e^{2kT} that may be incorporated in the observability constant. Finally, according to the previous comment and reversing the sense of time in Eq. (5.5) one can reduce the case $k < 0$ to the case $\tilde{k} \stackrel{\triangle}{=} -k > 0$. So we may assume $k > 0$ in what follows. On the other hand, the energy $E(t)$ is decreasing whenever $k > 0$. Thus it suffices to prove inequality (5.6) under the assumption

$$T = t^0. \tag{5.12}$$

Step 2. We set:

$$\begin{gathered}
Q = \Omega \times (0,T); \qquad \mathcal{Q} \stackrel{\triangle}{=} Q \times (0,T), \qquad \Sigma = \partial\Omega \times (0,T), \qquad \mathcal{S} \stackrel{\triangle}{=} \Sigma \times (0,T), \\
T_i \stackrel{\triangle}{=} T/2 - \varepsilon_i T, \qquad T_i' \stackrel{\triangle}{=} T/2 + \varepsilon_i T, \\
\mathcal{Q}_i \stackrel{\triangle}{=} \Omega \times (T_i, T_i') \times (T_i, T_i'), \\
\mathcal{S}_i \stackrel{\triangle}{=} \partial\Omega \times (T_i, T_i') \times (T_i, T_i'), \\
\mathcal{S}_{i0} \stackrel{\triangle}{=} \Gamma^0 \times (T_i, T_i') \times (T_i, T_i'),
\end{gathered} \tag{5.13}$$

where $i = 0, 1, 2, 3, 4$, and $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 < 1/2$ will be given later.

We now proceed as in [13, pp. 35–36]. We introduce the following simple transformation, which will play a crucial role in the sequel. We set:

$$u(x,t,s) \stackrel{\triangle}{=} \int_s^t \psi(x,z)\,\mathrm{d}z, \quad \forall (x,t,s) \in \mathcal{Q}, \tag{5.14}$$

where $\psi(\cdot)$ is the weak solution of system (5.5). Then $u(\cdot)$ satisfies

$$\begin{cases} u_{tt} + u_{ss} - \Delta u + k(u_t + u_s) = 0, & \text{in } \mathcal{Q}, \\ u = 0, & \text{on } \mathcal{S}. \end{cases} \tag{5.15}$$

Step 3. We need the following technical result:

LEMMA 5.1. – *Let $u, \ell \in C^2(\bar{\mathcal{Q}})$. Let Ψ be a real constant. Let $\theta = \mathrm{e}^\ell$ and $v = \theta u$. Let*

$$L_k u \stackrel{\triangle}{=} u_{tt} + u_{ss} - \Delta u + k(u_t + u_s). \tag{5.16}$$

Then it holds

$$\begin{aligned}
\theta^2 |L_k u|^2 \geqslant {} & \Big[(k - 2\ell_t)\Big(|v_t|^2 - |v_s|^2 + \sum_j |v_j|^2\Big) + 2(k - 2\ell_s) v_t v_s \\
& + 4 \sum_j (\ell_j v_t v_j) + 2\Psi v_t v + (k - 2\ell_t)(A + \Psi)|v|^2\Big]_t \\
& + \Big[(k - 2\ell_s)\Big(|v_s|^2 - |v_t|^2 + \sum_j |v_j|^2\Big) + 2(k - 2\ell_t) v_t v_s \\
& + 4 \sum_j (\ell_j v_s v_j) + 2\Psi v_s v + (k - 2\ell_s)(A + \Psi)|v|^2\Big]_s \\
& - 2 \sum_j \Big[2 \sum_i (\ell_i v_i v_j) - \ell_j \sum_i |v_i|^2 + (k - 2\ell_t) v_t v_j + (k - 2\ell_s) v_s v_j \\
& + \Psi v_j v + \ell_j \big(|v_t|^2 + |v_s|^2\big) - (A + \Psi)\ell_j |v|^2\Big]_j \\
& + 2\Big(-\Psi + \sum_i \ell_{ii} + \ell_{tt} - \ell_{ss}\Big)|v_t|^2 + 2\Big(-\Psi + \sum_i \ell_{ii} + \ell_{ss} - \ell_{tt}\Big)|v_s|^2 \\
& + 8\ell_{ts} v_t v_s - 8 \sum_j (\ell_{tj} v_t v_j + \ell_{sj} v_s v_j) + 4 \sum_{i,j} (\ell_{ij} v_i v_j) \\
& + 2\Big(\Psi - \sum_i \ell_{ii} + \ell_{tt} + \ell_{ss}\Big) \sum_j |v_j|^2 + B|v|^2,
\end{aligned} \tag{5.17}$$

where

$$A \stackrel{\triangle}{=} \big(\ell_t^2 - \ell_{tt}\big) + \big(\ell_s^2 - \ell_{ss}\big) - \sum_j \big(\ell_j^2 - \ell_{jj}\big) - \Psi - k(\ell_t + \ell_s) \tag{5.18}$$

and

$$\begin{aligned}
B \stackrel{\triangle}{=} {} & 2A\Big(\Psi - \sum_i \ell_{ii}\Big) + 2(\ell_{tt} + \ell_{ss})(A + \Psi) - (k - 2\ell_t)A_t - (k - 2\ell_s)A_s \\
& - 2 \sum_j (A_j \ell_j) + \Psi^2 - 2\Psi \sum_i \ell_{ii}.
\end{aligned} \tag{5.19}$$

Proof of Lemma 5.1. – We proceed as in [5, p. 124]. Recall that:

$$v(x, t, s) = \theta(x, t, s) u(x, t, s), \quad (x, t, s) \in \mathcal{Q}. \tag{5.20}$$

Then

$$u_t = \theta^{-1}(v_t - \ell_t v), \qquad u_s = \theta^{-1}(v_s - \ell_s v), \qquad u_{tt} = \theta^{-1}\left[v_{tt} - 2\ell_t v_t + \left(\ell_t^2 - \ell_{tt}\right)v\right],$$
$$u_{ss} = \theta^{-1}\left[v_{ss} - 2\ell_s v_s + \left(\ell_s^2 - \ell_{ss}\right)v\right], \tag{5.21}$$
$$u_{jj} = \theta^{-1}\left[v_{jj} - 2\ell_j v_j + \left(\ell_j^2 - \ell_{jj}\right)v\right], \quad j = 1, \ldots, n.$$

Therefore

$$\theta^2|L_k u|^2 = \Big|\left[v_{tt} - 2\ell_t v_t + \left(\ell_t^2 - \ell_{tt}\right)v\right] + \left[v_{ss} - 2\ell_s v_s + \left(\ell_s^2 - \ell_{ss}\right)v\right]$$
$$- \sum_j \left[v_{jj} - 2\ell_j v_j + \left(\ell_j^2 - \ell_{jj}\right)v\right] + k(v_t - \ell_t v) + k\left(v_s - \ell_s v\right)\Big|^2$$
$$= |I_1 + I_2 + I_3|^2, \tag{5.22}$$

where

$$I_1 \stackrel{\triangle}{=} \left(v_{tt} + v_{ss} - \sum_j v_{jj}\right) + Av,$$
$$I_2 \stackrel{\triangle}{=} (k - 2\ell_t)v_t + (k - 2\ell_s)v_s + 2\sum_j (\ell_j v_j), \tag{5.23}$$
$$I_3 \stackrel{\triangle}{=} \Psi v.$$

Then

$$\theta^2|L_k u|^2 = |I_1|^2 + |I_2|^2 + |I_3|^2 + 2(I_1 I_2 + I_2 I_3 + I_1 I_3)$$
$$\geqslant |I_3|^2 + 2(I_1 I_2 + I_2 I_3 + I_1 I_3). \tag{5.24}$$

However

$$2I_1 I_2 = 2\left(v_{tt} + v_{ss} - \sum_j v_{jj} + Av\right)\left[(k - 2\ell_t)v_t + (k - 2\ell_s)v_s + 2\sum_j (\ell_j v_j)\right]$$
$$= (k - 2\ell_t)\left(|v_t|^2\right)_t - 2\sum_j \left[(k - 2\ell_t)v_t v_j\right]_j$$
$$+ (k - 2\ell_t)\sum_j \left(|v_j|^2\right)_t - 4\sum_j (\ell_{tj} v_t v_j) + (k - 2\ell_t)A\left(|v|^2\right)_t$$
$$+ 4\sum_j (\ell_j v_t v_j)_t - 2\sum_j \left[\ell_j\left(|v_t|^2\right)_j\right] - 4\sum_j (\ell_{tj} v_t v_j)$$
$$+ (k - 2\ell_s)\left(|v_s|^2\right)_s - 2\sum_j \left[(k - 2\ell_s)v_s v_j\right]_j$$
$$+ (k - 2\ell_s)\sum_j \left(|v_j|^2\right)_s - 4\sum_j (\ell_{sj} v_s v_j) + (k - 2\ell_s)A\left(|v|^2\right)_s$$
$$+ 4\sum_j (\ell_j v_s v_j)_s - 2\sum_j \left[\ell_j\left(|v_s|^2\right)_j\right] - 4\sum_j (\ell_{sj} v_s v_j)$$

$$
\begin{aligned}
&+2\big[v_t(k-2\ell_s)v_s\big]_t-(k-2\ell_s)\big(|v_t|^2\big)_s+2\big[v_s(k-2\ell_t)v_t\big]_s\\
&-(k-2\ell_t)\big(|v_s|^2\big)_t+8\ell_{ts}v_tv_s\\
&-4\sum_{i,j}\big[(\ell_iv_iv_j)_j-\ell_{ij}v_iv_j\big]+2\sum_{i,j}\big[\ell_i\big(|v_j|^2\big)_i\big]+2A\sum_j\big[\ell_j\big(|v|^2\big)_j\big]\\
=&\left[(k-2\ell_t)\left(|v_t|^2-|v_s|^2+\sum_j|v_j|^2+A|v|^2\right)+2(k-2\ell_s)v_tv_s+4\sum_j(\ell_jv_tv_j)\right]_t\\
&+\left[(k-2\ell_s)\left(|v_s|^2-|v_t|^2+\sum_j|v_j|^2+A|v|^2\right)+2(k-2\ell_t)v_tv_s+4\sum_j(\ell_jv_sv_j)\right]_s\\
&-2\sum_j\left[2\sum_i(\ell_iv_iv_j)-\ell_j\sum_i|v_i|^2+(k-2\ell_t)v_tv_j+(k-2\ell_s)v_sv_j\right.\\
&\left.+\ell_j\big(|v_t|^2+|v_s|^2\big)-A\ell_j|v|^2\right]_j+8\ell_{ts}v_tv_s-8\sum_j(\ell_{tj}v_tv_j+\ell_{sj}v_sv_j)\\
&+2\left(\sum_i\ell_{ii}+\ell_{tt}-\ell_{ss}\right)|v_t|^2+2\left(\sum_i\ell_{ii}+\ell_{ss}-\ell_{tt}\right)|v_s|^2\\
&+4\sum_{i,j}(\ell_{ij}v_iv_j)-2\left(\sum_i\ell_{ii}-\ell_{tt}-\ell_{ss}\right)\sum_j|v_j|^2\\
&-\left\{2\sum_i(A\ell_i)_i+\big[(k-2\ell_t)A\big]_t+\big[(k-2\ell_s)A\big]_s\right\}|v|^2. \qquad (5.25)
\end{aligned}
$$

Further (recall Ψ is a constant),

$$
\begin{aligned}
2I_1I_3&=2\left(v_{tt}+v_{ss}-\sum_jv_{jj}+Av\right)\Psi v\\
&=2(\Psi v_tv)_t-2\Psi|v_t|^2+2(\Psi v_sv)_s\\
&\quad-2\Psi|v_s|^2-2\sum_j(\Psi v_jv)_j+2\Psi\sum_j|v_j|^2+2A\Psi|v|^2. \qquad (5.26)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
2I_2I_3&=2\left[(k-2\ell_t)v_t+(k-2\ell_s)v_s+2\sum_j(\ell_jv_j)\right]\Psi v\\
&=(k-2\ell_t)\Psi\big(|v|^2\big)_t+(k-2\ell_s)\Psi\big(|v|^2\big)_s+2\Psi\sum_j\big[\ell_j\big(|v|^2\big)_j\big]\\
&=\big[(k-2\ell_t)\Psi|v|^2\big]_t+\big[(k-2\ell_s)\Psi|v|^2\big]_s+2\sum_j\big(\Psi\ell_j|v|^2\big)_j\\
&\quad+2\left(\ell_{tt}+\ell_{ss}-\sum_j\ell_{jj}\right)\Psi|v|^2. \qquad (5.27)
\end{aligned}
$$

Combining (5.24)–(5.27), we obtain (5.17). So the proof of Lemma 5.1 is completed. □

As a consequence of Lemma 5.1 the following holds:

COROLLARY 5.1. – *Let* $\lambda > 0$, $\alpha \in (0,1)$, *and*

$$\phi(x,t,s) = \frac{1}{2}\left[\left|x - x^0\right|^2 - \alpha(t - T/2)^2 - \alpha(s - T/2)^2\right],$$
$$\ell = \lambda\phi, \qquad \Psi = (n - 1 + \alpha)\lambda. \tag{5.28}$$

Let $u \in C^2(\bar{Q})$, $v = \theta u$ *with* $\theta = \mathrm{e}^{\ell}$. *Then*

$$\begin{aligned}
&\theta^2\left|u_{tt} + u_{ss} - \Delta u + k(u_t + u_s)\right|^2 \\
&\quad \geqslant \bigg[(k - 2\ell_t)\bigg(|v_t|^2 - |v_s|^2 + \sum_j |v_j|^2\bigg) + 2(k - 2\ell_s)v_t v_s \\
&\qquad + 4\sum_j(\ell_j v_t v_j) + 2\Psi v_t v + (k - 2\ell_t)(A + \Psi)|v|^2\bigg]_t \\
&\qquad + \bigg[(k - 2\ell_s)\bigg(|v_s|^2 - |v_t|^2 + \sum_j |v_j|^2\bigg) + 2(k - 2\ell_t)v_t v_s \\
&\qquad + 4\sum_j(\ell_j v_s v_j) + 2\Psi v_s v + (k - 2\ell_s)(A + \Psi)|v|^2\bigg]_s \\
&\qquad - 2\sum_j\bigg[2\sum_i(\ell_i v_i v_j) - \ell_j\sum_i |v_i|^2 + (k - 2\ell_t)v_t v_j + (k - 2\ell_s)v_s v_j \\
&\qquad + \Psi v_j v + \ell_j\big(|v_t|^2 + |v_s|^2\big) - (A + \Psi)\ell_j|v|^2\bigg]_j \\
&\qquad + 2(1 - \alpha)\lambda\bigg(|v_t|^2 + |v_s|^2 + \sum_j |v_j|^2\bigg) + B|v|^2,
\end{aligned} \tag{5.29}$$

where

$$\begin{aligned}
A = {} & \lambda^2\left[\alpha^2(t - T/2)^2 + \alpha^2(s - T/2)^2 - \left|x - x^0\right|^2\right] \\
& + \alpha\lambda k(t + s - T) + (1 + \alpha)\lambda
\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}
B = {} & 2(3 + \alpha)\lambda^3\left|x - x^0\right|^2 - 2\alpha^2\lambda^3(1 + 3\alpha)\left[(t - T/2)^2 + (s - T/2)^2\right] \\
& - 2\alpha\lambda k^2 - 2\alpha\lambda^2 k(1 + 3\alpha)(t + s - T) - \left[n^2 + 4\alpha n + 1 + 2\alpha + 5\alpha^2\right]\lambda^2.
\end{aligned} \tag{5.31}$$

Proof. – Using Lemma 5.1 with ℓ and Ψ given by (5.28), Corollary 5.1 follows immediately from a direct calculation. □

Step 4. Let us use Corollary 5.1. Here we use an argument from [4]. First of all, take $\alpha \in (0,1)$ as in (5.10)–(5.11). Then, by (5.28), (5.9), (5.10) and (5.12), we get:

$$\begin{aligned}
\phi(x,0,s) = \phi(x,T,s) &\leqslant \frac{1}{2}\left(R_1^2 - \frac{\alpha T^2}{4}\right) \\
&= \frac{1}{2}\left(R_1^2 - \frac{\alpha(t^0)^2}{4}\right) = -\frac{R_1^2}{2} < 0, \quad \forall (s,x) \in Q.
\end{aligned} \tag{5.32}$$

Thus, one can find ε_1 in $(0, 1/2)$ sufficiently close to $1/2$, i.e., such that

$$1/2 - \varepsilon_1 \quad \text{is sufficiently small} \tag{5.33}$$

and a constant

$$r_0 = \frac{R_1^2}{4}\ (>0) \tag{5.34}$$

such that (recall (5.13) for T_1 and T_1')

$$\phi(x,t,s) < -r_0, \quad \forall (x,t,s) \in \Omega \times \big((0,T_1) \cup (T_1',T)\big) \times (0,T). \tag{5.35}$$

Similarly,

$$\phi(x,s,t) < -r_0, \quad \forall (x,s,t) \in \Omega \times \big((0,T_1) \cup (T_1',T)\big) \times (0,T). \tag{5.36}$$

Next, since $x^0 \in \mathbf{R}^n \setminus \bar{\Omega}$, we see that $R_0 > 0$. Thus, one can find a sufficiently small

$$\varepsilon_0 \in (0, \varepsilon_1) \tag{5.37}$$

such that (recall (5.13))

$$\phi(x,t,s) \geqslant 0, \quad \forall (x,t,s) \in \mathcal{Q}_0. \tag{5.38}$$

Further, we take

$$\lambda = \frac{k}{\alpha R_0}, \tag{5.39}$$

where $\alpha \in (0,1)$ was given in (5.10)–(5.11), R_0 was defined in (5.9). Then, by (5.31) and (5.39), we get:

$$\begin{aligned} B &= 2(3+\alpha)\lambda^3 \big|x - x^0\big|^2 - 2\alpha^2\lambda^3(1+3\alpha)\big[(t-T/2)^2 + (s-T/2)^2\big] \\ &\quad - 2\alpha\lambda k^2 - 2\alpha\lambda^2 k(1+3\alpha)(t+s-T) - \big[n^2 + 4\alpha n + 1 + 2\alpha + 5\alpha^2\big]\lambda^2 \\ &\geqslant \big[2\big(3+\alpha-\alpha^3\big)R_0^2 - 2\alpha^2(1+3\alpha)R_0 T - \alpha^2(1+3\alpha)T^2\big]\lambda^3 \\ &\quad - \big[n^2 + 4\alpha n + 1 + 2\alpha + 5\alpha^2\big]\lambda^2. \end{aligned} \tag{5.40}$$

Note that

$$x_1 \triangleq -\frac{\sqrt{(1+3\alpha)^2\alpha^2 + 2(1+3\alpha)(3+\alpha-\alpha^3)} + (1+3\alpha)\alpha}{\alpha(1+3\alpha)} R_0$$

and

$$x_2 \triangleq \frac{\sqrt{(1+3\alpha)^2\alpha^2 + 2(1+3\alpha)(3+\alpha-\alpha^3)} - (1+3\alpha)\alpha}{\alpha(1+3\alpha)} R_0$$

are the two solutions of the following equation:

$$2\big(3+\alpha-\alpha^3\big)R_0^2 - 2\alpha^2(1+3\alpha)R_0 x - \alpha^2(1+3\alpha)x^2 = 0.$$

On the other hand, by (5.10), it is easy to see that

$$x_1 < t^0 < T^0 < x_2.$$

Thus by (5.12), it is elementary to see that:

$$2(3+\alpha-\alpha\beta^2)R_0^2 - 2\alpha(1+3\alpha)\beta R_0 T - \alpha^2(1+3\alpha)T^2 > 0, \tag{5.41}$$

since the quadratic function above is positive in the interval (x_1, x_2). Thus, by (5.40)–(5.41), there exist two constants $\lambda_1 > 0$ and $c_0 > 0$, such that for any

$$\lambda > \lambda_1, \tag{5.42}$$

it holds

$$B\big(= B(x,t,s)\big) \geqslant c_0\lambda^3, \quad \forall (x,t,s) \in \mathcal{Q}. \tag{5.43}$$

Further, we take (recall (5.33))

$$\varepsilon_2 \in (\varepsilon_1, 1/2). \tag{5.44}$$

Now, let us use Corollary 5.1 with u, α and λ given by (5.14), (5.10)–(5.11) and (5.42) respectively. For any given $\tau \in (T_2, T_1)$ and $\tau' \in (T_1', T_2')$ (recall (5.13)), denote

$$\mathcal{Q}(\tau,\tau') \stackrel{\triangle}{=} \Omega \times (\tau,\tau') \times (\tau,\tau'). \tag{5.45}$$

Integrating (5.29) on $\mathcal{Q}(\tau,\tau')$, using integration by parts, by (5.30)–(5.31) and taking (5.15) into account, we arrive at (recall (5.13))

$$\begin{aligned}
&\lambda \int\limits_{\mathcal{Q}(\tau,\tau')} \Big(|v_t|^2 + |v_s|^2 + \sum_i |v_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s + \int\limits_{\mathcal{Q}(\tau,\tau')} B|v|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \leqslant C\lambda \int\limits_{\mathcal{S}_{20}} \left|\frac{\partial v}{\partial \nu}\right|^2 \mathrm{d}\mathcal{S}_{20}\\
&\quad + \lambda^3 \Bigg[\int_{T_2}^{T_2'}\int_{\Omega} \Big(\big|v(x,\tau,s)\big|^2 + \big|v_t(x,\tau,s)\big|^2 + \big|v_s(x,\tau,s)\big|^2 + \sum_i \big|v_i(x,\tau,s)\big|^2\\
&\quad + \big|v(x,\tau',s)\big|^2 + \big|v_t(x,\tau',s)\big|^2 + \big|v_s(x,\tau',s)\big|^2 + \sum_i \big|v_i(x,\tau',s)\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + \int_{T_2}^{T_2'}\int_{\Omega} \Big(\big|v(x,t,\tau)\big|^2 + \big|v_t(x,t,\tau)\big|^2 + \big|v_s(x,t,\times au)\big|^2 + \sum_i \big|v_i(x,t,\tau)\big|^2\\
&\quad + \big|v(x,t,\tau')\big|^2 + \big|v_t(x,t,\tau')\big|^2 + \big|v_s(x,t,\tau')\big|^2 + \sum_i \big|v_i(x,t,\tau')\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\Bigg].
\end{aligned} \tag{5.46}$$

Now taking into account that $v = \theta u$ with $\theta = \mathrm{e}^{\ell}$, by (5.28) and (5.15), we get

$$\int\limits_{\mathcal{S}_{20}} \left|\frac{\partial v}{\partial \nu}\right|^2 \mathrm{d}\mathcal{S}_{20} \leqslant C\mathrm{e}^{C\lambda} \int\limits_{\mathcal{S}_{20}} \left|\frac{\partial u}{\partial \nu}\right|^2 \mathrm{d}\mathcal{S}_{20}. \tag{5.47}$$

Further, by (5.28) and (5.35)–(5.36), we get

$$\begin{aligned}
&\int_{T_2}^{T_2'}\int_{\Omega} \Big(\big|v(x,\tau,s)\big|^2 + \big|v_t(x,\tau,s)\big|^2 + \big|v_s(x,\tau,s)\big|^2 + \sum_i \big|v_i(x,\tau,s)\big|^2\\
&\qquad + \big|v(x,\tau',s)\big|^2 + \big|v_t(x,\tau',s)\big|^2 + \big|v_s(x,\tau',s)\big|^2 + \sum_i \big|v_i(x,\tau',s)\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

$$+\int_{T_2}^{T_2'}\int_{\Omega}\Big(\big|v(x,t,\tau)\big|^2+\big|v_t(x,t,\tau)\big|^2+\big|v_s(x,t,\times au)\big|^2+\sum_i\big|v_i(x,t,\tau)\big|^2$$

$$+\big|v(x,t,\tau')\big|^2+\big|v_t(x,t,\tau')\big|^2+\big|v_s(x,t,\tau')\big|^2+\sum_i\big|v_i(x,t,\tau')\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant C\lambda^2\mathrm{e}^{-2r_0\lambda}\Bigg[\int_{T_2}^{T_2'}\int_{\Omega}\Big(\big|u(x,\tau,s)\big|^2+\big|u_t(x,\tau,s)\big|^2+\big|u_s(x,\tau,s)\big|^2+\sum_i\big|u_i(x,\tau,s)\big|^2$$

$$+\big|u(x,\tau',s)\big|^2+\big|u_t(x,\tau',s)\big|^2+\big|u_s(x,\tau',s)\big|^2+\sum_i\big|u_i(x,\tau',s)\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s$$

$$+\int_{T_2}^{T_2'}\int_{\Omega}\Big(\big|u(x,t,\tau)\big|^2+\big|u_t(x,t,\tau)\big|^2+\big|u_s(x,t,\times au)\big|^2+\sum_i\big|u_i(x,t,\tau)\big|^2$$

$$+\big|u(x,t,\tau')\big|^2+\big|u_t(x,t,\tau')\big|^2+\big|u_s(x,t,\tau')\big|^2+\sum_i\big|u_i(x,t,\tau')\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\Bigg]. \tag{5.48}$$

On the other hand, by (5.43), for any $\lambda > \lambda_1$, we have:

$$\lambda\int_{\mathcal{Q}(\tau,\tau')}\Big(|v_t|^2+|v_s|^2+\sum_i|v_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\int_{\mathcal{Q}(\tau,\tau')}B|v|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$

$$\geqslant\lambda\int_{\mathcal{Q}_0}\Big(|v_t|^2+|v_s|^2+\sum_i|v_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+c_0\lambda^3\int_{\mathcal{Q}_0}|v|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s. \tag{5.49}$$

Thus, combining (5.46)–(5.49), we conclude that:

$$\int_{\mathcal{Q}_0}\Big(|v_t|^2+|v_s|^2+\sum_i|v_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\lambda^2\int_{\mathcal{Q}_0}\theta^2|v|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$

$$\leqslant C\mathrm{e}^{C\lambda}\int_{\mathcal{S}_{20}}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}\mathcal{S}_{20}+C\lambda^4\mathrm{e}^{-2r_0\lambda}\Bigg[\int_{T_2}^{T_2'}\int_{\Omega}\Big(\big|u(x,\tau,s)\big|^2+\big|u_t(x,\tau,s)\big|^2$$

$$+\big|u_s(x,\tau,s)\big|^2+\sum_i\big|u_i(x,\tau,s)\big|^2$$

$$+\big|u(x,\tau',s)\big|^2+\big|u_t(x,\tau',s)\big|^2+\big|u_s(x,\tau',s)\big|^2+\sum_i\big|u_i(x,\tau',s)\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s$$

$$+\int_{T_2}^{T_2'}\int_{\Omega}\Big(\big|u(x,t,\tau)\big|^2+\big|u_t(x,t,\tau)\big|^2+\big|u_s(x,t,\times au)\big|^2+\sum_i\big|u_i(x,t,\tau)\big|^2$$

$$+\big|u(x,t,\tau')\big|^2+\big|u_t(x,t,\tau')\big|^2+\big|u_s(x,t,\tau')\big|^2+\sum_i\big|u_i(x,t,\tau')\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\Bigg]. \tag{5.50}$$

Integrating (5.50) with respect to τ and τ' from T_2 to T_1 and from T_1' to T_2' respectively, we get

$$\int_{\mathcal{Q}_0}\Big(|v_t|^2+|v_s|^2+\sum_i|v_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\lambda^2\int_{\mathcal{Q}_0}|v|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$

$$\leqslant C\mathrm{e}^{C\lambda}\int_{\mathcal{S}_{20}}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}\mathcal{S}_{20}+C\lambda^4\mathrm{e}^{-2r_0\lambda}\int_{\mathcal{Q}_2}\Big(|u|^2+|u_t|^2+|u_s|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s. \tag{5.51}$$

So, by (5.28) and (5.38), recalling that $u=\theta^{-1}v$ with $\theta=\mathrm{e}^{\ell}$, and using (5.51), we see that for any $\lambda>\lambda_1$, it holds

$$\begin{aligned}
&\int_{\mathcal{Q}_0}\Big(|u_t|^2+|u_s|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
&\quad\leqslant\int_{\mathcal{Q}_0}\theta^2\Big(|u_t|^2+|u_s|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
&\quad\leqslant C\left[\int_{\mathcal{Q}_0}\Big(|v_t|^2+|v_s|^2+\sum_i|v_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\lambda^2\int_{\mathcal{Q}_0}|v|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\right]\\
&\quad\leqslant C\mathrm{e}^{C\lambda}\int_{\mathcal{S}_{20}}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}\mathcal{S}_{20}+C\lambda^4\mathrm{e}^{-2r_0\lambda}\int_{\mathcal{Q}_2}\Big(|u|^2+|u_t|^2+|u_s|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned} \tag{5.52}$$

Step 5. We now estimate

$$\int_{\mathcal{S}_{20}}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}\mathcal{S}_{20}\quad\text{and}\quad\int_{\mathcal{Q}_2}\sum_i|u_i|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s.$$

For this purpose, we need the following identity from [13]:

LEMMA 5.2. – *Let $h\overset{\triangle}{=}(h^1,\dots,h^n):\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}^n$ be a vector field of class C^1. Then for any $z\in C^2(\mathbf{R}^n\times\mathbf{R}\times\mathbf{R})$, we have*:

$$\begin{aligned}
&\nabla\cdot\left\{2(h\cdot\nabla z)(\nabla z)+h\Big[z_t^2+z_s^2-\sum_i z_i^2\Big]\right\}\\
&\quad=-2(z_{tt}+z_{ss}-\Delta z)h\cdot\nabla z+(2z_t h\cdot\nabla z)_t+(2z_s h\cdot\nabla z)_s\\
&\qquad-2z_t h_t\cdot\nabla z-2z_s h_s\cdot\nabla z+(\nabla\cdot h)\Big[z_t^2+z_s^2-\sum_i z_i^2\Big]+2\sum_{i,j}\left(\frac{\partial h^j}{\partial x_i}z_i z_j\right).
\end{aligned} \tag{5.53}$$

Now let us return to the proof of Theorem 5.1. Choose $h_1=h_1(x)\in C^1(\bar{\Omega};\mathbf{R}^n)$ such that $h_1=\nu$ on $\partial\Omega$ (cf. [8]), and choose $h_i=h_i(x)\in C^{\mathrm{i}}(\bar{\Omega};[0,1])$ $(i=2,3)$ such that:

$$\begin{aligned}
&0\leqslant h_i(x)\leqslant 1,\quad x\in\bar{\Omega},\\
&h_i(x)\equiv 1,\quad x\in\omega_{i-1},\\
&h_i(x)\equiv 0,\quad x\in\Omega\setminus\omega_i,
\end{aligned} \tag{5.54}$$

where $\omega_1 \subset \omega_2 \subset \omega_3 \equiv \omega^0 (\subset \Omega)$ are all intersections of some neighborhood of Γ^0 and Ω.

Let us fix ε_3 and ε_4 such that $\varepsilon_2 < \varepsilon_3 < \varepsilon_4 < 1/2$. Now, we apply identity (5.53) with

$$h = (t - T_3)(T_3' - t)(s - T_3)(T_3' - s)h_1 h_2. \tag{5.55}$$

Integrating (5.53) (with $z(\cdot)$ replaced by $u(\cdot)$) on $\mathcal{Q}_3$, using (5.15) and (5.54)–(5.55), and integrating by parts, we obtain

$$\begin{aligned} &\int_{\mathcal{S}_{30}} (t - T_3)(T_3' - t)(s - T_3)(T_3' - s)\left|\frac{\partial u}{\partial \nu}\right|^2 \mathrm{d}\mathcal{S}_{30} \\ &\quad \leqslant C(1+k)\int_{T_3}^{T_3'}\int_{T_3}^{T_3'}\int_{\omega_2}\left(|u_t|^2 + |u_s|^2 + \sum_i |u_i|^2\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s. \end{aligned} \tag{5.56}$$

On the other hand, denote:

$$\eta = \eta(x,t,s) \overset{\triangle}{=} (t - T_4)(T_4' - t)(s - T_4)(T_4' - s)h_3(x), \tag{5.57}$$

where h_3 is given in (5.54). By (5.15) we get

$$\begin{aligned} &-\int_{\mathcal{Q}_4} \eta u k(u_t + u_s)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \\ &\quad = \int_{\mathcal{Q}_4} \eta u(u_{tt} + u_{ss} - \Delta u)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \\ &\quad = -\int_{\mathcal{Q}_4} \big[u_t(\eta_t u + \eta u_t) + u_s(\eta_s u + \eta u_s)\big]\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \\ &\qquad + \int_{\mathcal{Q}_4} \eta|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s + \int_{\mathcal{Q}_4} (\nabla u)\cdot(\nabla\eta)u\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \\ &\quad \geqslant -\int_{\mathcal{Q}_4} \big[u_t(\eta_t u + \eta u_t) + u_s(\eta_s u + \eta u_s)\big]\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \\ &\qquad + \frac{1}{2}\int_{\mathcal{Q}_4} \eta|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s - C\int_{T_4}^{T_4'}\int_{T_4}^{T_4'}\int_{\omega^0} |u|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \\ &\quad \geqslant -\int_{\mathcal{Q}_4} \big[u_t(\eta_t u + \eta u_t) + u_s(\eta_s u + \eta u_s)\big]\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \\ &\qquad + \frac{C}{2}\int_{T_3}^{T_3'}\int_{T_3}^{T_3'}\int_{\omega_2}\sum_i |u_i|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s - C\int_{T_4}^{T_4'}\int_{T_4}^{T_4'}\int_{\omega^0} |u|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s, \end{aligned} \tag{5.58}$$

provided $\nabla\eta/\sqrt{\eta}$ is bounded. Obviously η can be chosen so that this holds. Thus

$$\int_{T_3}^{T_3'}\int_{T_3}^{T_3'}\int_{\omega_2}\sum_i |u_i|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \leqslant C(1+k)\int_{T_4}^{T_4'}\int_{T_4}^{T_4'}\int_{\omega^0}\big(|u_t|^2+|u_s|^2+|u|^2\big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s. \tag{5.59}$$

Similarly,

$$\int_{\mathcal{Q}_2}\sum_i |u_i|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \leqslant C(1+k)\int_{\mathcal{Q}_4}\big(|u_t|^2+|u_s|^2+|u|^2\big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s. \tag{5.60}$$

So by (5.52), (5.56) and (5.59)–(5.60), we conclude that for any $\lambda>\lambda_0$, it holds

$$\begin{aligned}\int_{\mathcal{Q}_0}&\big(|u_t|^2+|u_s|^2\big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\ &\leqslant C\big(1+k^2\big)\mathrm{e}^{C\lambda}\int_0^T\int_0^T\int_{\omega^0}\big(|u_t|^2+|u_s|^2+|u|^2\big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\ &\quad+C(1+k)\lambda^4\mathrm{e}^{-2r_0\lambda}\int_{\mathcal{Q}}\big(|u_t|^2+|u_s|^2+|u|^2\big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s.\end{aligned} \tag{5.61}$$

Step 6. Let us return to the original function ψ. By (5.14) and (5.61), we get:

$$\begin{aligned}\int_{T_0}^{T_0'}\int_{\Omega}\psi^2\,\mathrm{d}x\,\mathrm{d}t &\leqslant C\big(1+k^2\big)\mathrm{e}^{C\lambda}\int_0^T\int_0^T\int_{\omega^0}\left(|\psi|^2+\left|\int_s^t\psi(z)\,\mathrm{d}z\right|^2\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\ &\quad+C(1+k)\lambda^4\mathrm{e}^{-2r_0\lambda}\int_{\mathcal{Q}}\left(|\psi|^2+\left|\int_s^t\psi(z)\,\mathrm{d}z\right|^2\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\ &\leqslant C(1+k^2)\left[\mathrm{e}^{C\lambda}\int_0^T\int_{\omega^0}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t+\lambda^4\mathrm{e}^{-2r_0\lambda}\int_Q|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right].\end{aligned} \tag{5.62}$$

On the other hand, it is easy to see that for any $0\leqslant S_1<S_2<T_2<T_1\leqslant T$, there is a constant $C>0$, which is independent of k, such that

$$\int_{S_2}^{T_2}\big|\psi_t(\cdot,t)\big|^2_{H^{-1}(\Omega)}\,\mathrm{d}t\leqslant C\big(1+k^2\big)\int_{S_1}^{T_1}\big|\psi(\cdot,t)\big|^2_{L^2(\Omega)}\,\mathrm{d}t, \tag{5.63}$$

for every $\psi(\cdot)$ solution of system (5.5).

Further, using the usual energy estimate and noting the time reversibility of Eq. (5.5), one also has

$$E(t)\leqslant E(s)\mathrm{e}^{2kT},\quad \forall t,s\in[0,T]. \tag{5.64}$$

Now, choose $S_0\in(T_0,T/2)$ and $S_0'\in(T/2,T_0')$. By (5.62)–(5.63) we obtain that:

$$\int_{S_0}^{S_0'} \big[\big|\psi_t(\cdot,t)\big|^2_{H^{-1}(\Omega)} + \big|\psi(\cdot,t)\big|^2_{L^2(\Omega)}\big]\,\mathrm{d}t$$

$$\leqslant C\big(1+k^4\big)\left[\mathrm{e}^{C\lambda}\int_0^T\int_{\omega^0} |\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \lambda^4\mathrm{e}^{-2r_0\lambda}\int_Q |\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right]. \tag{5.65}$$

Thus

$$\int_{S_0}^{S_0'} E(t)\,\mathrm{d}t \leqslant C\big(1+k^4\big)\left[\mathrm{e}^{C\lambda}\int_0^T\int_{\omega^0} |\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \lambda^4\mathrm{e}^{-2r_0\lambda}\int_0^T E(t)\,\mathrm{d}t\right]. \tag{5.66}$$

Finally, by (5.66) and (5.64) and noting that

$$E(t) \geqslant E(T), \quad \forall t\in[0,T], \tag{5.67}$$

we obtain

$$E(T) \leqslant C\big(1+k^4\big)\mathrm{e}^{C\lambda}\int_0^T\int_{\omega^0} |\psi|^2\,\mathrm{d}x\,\mathrm{d}t + C\lambda^4\mathrm{e}^{-2r_0\lambda}\mathrm{e}^{2Tk}\big(1+k^4\big)E(T). \tag{5.68}$$

However, by (5.11), (5.12), (5.34) and (5.39), we have

$$\mathrm{e}^{-2r_0\lambda}\mathrm{e}^{2Tk} = \exp\left\{2\lambda\left(\alpha t_0 R_0 - \frac{R_1^2}{4}\right)\right\} < \mathrm{e}^{-\lambda R_1^2/8}. \tag{5.69}$$

Thus, by (5.68)–(5.69), (5.64) and (5.39), we can find a constant $C_1 > 0$ and a sufficiently large constant $k_0 > 0$, such that for any $k > k_0$, it holds

$$E(T) \leqslant C_1\mathrm{e}^{C_1 k}\int_0^T\int_{\omega^0} |\psi|^2\,\mathrm{d}x\,\mathrm{d}t, \quad \forall(\psi^0,\psi^1)\in L^2(\Omega)\times H^{-1}(\Omega). \tag{5.70}$$

Step 7. Finally, as a consequence of Appendix A we know that (5.6) for any $0 \leqslant k \leqslant k_0$ as well. This concludes the proof of Theorem 5.1 in Case 1.

Case 2. $x^0 \in \bar{\Omega}$.

As in the first case, we divide the proof into several steps.

Step 1′. As in Step 1 of the previous case we reduce the problem to analyze the case where $k > 0$ and

$$T = t^0. \tag{5.71}$$

Step 2′. We first introduce some notation. First, for any fixed $x \in \mathbf{R}^n$ and $\delta > 0$, we set

$$B_\delta(x) \stackrel{\triangle}{=} \big\{y\in\mathbf{R}^n;\ |y-x|<\delta\big\}. \tag{5.72}$$

Next, we set:

$$\Omega^{2j} \triangleq \Omega \setminus \overline{B_{|x^0-x^1|/(5-j)}(x^0)}, \qquad \Omega^{2j+1} \triangleq \Omega \setminus \overline{B_{|x^0-x^1|/(5-j)}(x^1)}, \quad j = 0, 1, 2, \tag{5.73}$$

where x^1 is as in (5.8). So, we have

$$\Omega^4 \subset \Omega^2 \subset \Omega^0, \qquad \Omega^5 \subset \Omega^3 \subset \Omega^1. \tag{5.74}$$

Finally, denote:

$$\begin{gathered} \mathcal{Q}^j \triangleq \Omega^j \times (0,T) \times (0,T), \quad j = 0,1,2,3,4,5, \\ T_i \triangleq T/2 - \varepsilon_i T, \qquad T_i' \triangleq T/2 + \varepsilon_i T, \\ \mathcal{Q}_i^j \triangleq \Omega^j \times (T_i, T_i') \times (T_i, T_i'), \quad j = 0,1,2,3,4,5, \\ \mathcal{S}_i \triangleq \partial\Omega \times (T_i, T_i') \times (T_i, T_i'), \\ \mathcal{S}^j \triangleq \Gamma^j \times (0,T) \times (0,T), \qquad \mathcal{S}_{i0}^j \triangleq \Gamma^j \times (T_i, T_i') \times (T_i, T_i'), \quad j = 0,1, \end{gathered} \tag{5.75}$$

where Γ^0 and Γ^1 were given in (5.7) and (5.8) respectively, $i = 0,1,2,3,4$, and $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 < 1/2$ will be given later.

We introduce u as in (5.14).

Obviously one can find two functions $g^j \in C^1(\bar{\Omega};\ [0,1])$ $(j = 0,1)$ such that

$$g^j(x) \equiv 1, \quad x \in \Omega^{j+4}; \qquad g^j(x) \equiv 0, \quad x \in \Omega \setminus \Omega^{j+2}. \tag{5.76}$$

We set

$$u^j = g^j u, \quad j = 0,1. \tag{5.77}$$

Then u^j satisfies $(j = 0,1)$

$$\begin{cases} (u^j)_{tt} + (u^j)_{ss} - \Delta u^j + k\big((u^j)_t + (u^j)_s\big) & \\ \quad = -u\Delta g^j - 2\nabla g^j \cdot \nabla u & \text{in } \mathcal{Q}^j, \\ u^j(x) \equiv 0, & x \in \Omega \setminus \Omega^{j+2}, \\ u^j = 0, & \text{on } \mathcal{S}^j. \end{cases} \tag{5.78}$$

Step 3′. We shall use Lemma 5.1 and Corollary 5.1 without change.

Step 4′. As in Step 4, we use Corollary 5.1. First of all, we take $\alpha \in (0,1)$ as in (5.10), and we introduce:

$$\begin{gathered} \phi^j(x,t,s) = \frac{1}{2}\big[|x - x^j|^2 - \alpha(t - T/2)^2 - \alpha(s - T/2)^2\big], \\ \Psi = (n - 1 + \alpha)\lambda, \\ v^j = \mathrm{e}^{\lambda\phi^j} u^j, \end{gathered} \tag{5.79}$$

where $j=0,1$. By (5.79), (5.9), (5.10) and (5.71), as in (5.32), we get

$$\phi^j(x,0,s)=\phi^j(x,T,s)=-\frac{R_1^2}{2}<0,\quad \forall (x,s)\in Q. \tag{5.80}$$

Thus, one can find ε_1 in $(0,1/2)$ close enough to $1/2$, or, in other words,

$$1/2-\varepsilon_1 \quad \text{small enough} \tag{5.81}$$

and a constant

$$r_0=\frac{R_1^2}{4}\ (>0) \tag{5.82}$$

such that:

$$\phi^j(x,t,s)<-r_0,\quad \forall (x,t,s)\in \Omega\times\big((0,T_1)\cup(T_1',T)\big)\times(0,T). \tag{5.83}$$

Similarly,

$$\phi^j(x,t,s)<-r_0,\quad \forall (x,s,t)\in \Omega\times\big((0,T_1)\cup(T_1',T)\big)\times(0,T). \tag{5.84}$$

Next, recalling (5.73) and (5.75), it is easy to see that one can find a sufficiently small

$$\varepsilon_0\in(0,\varepsilon_1) \tag{5.85}$$

such that

$$\phi^j(x,t,s)\geqslant 0,\quad \forall (x,t,s)\in \mathcal{Q}_0^j\ (j=0,1). \tag{5.86}$$

Further, we take

$$\lambda=\frac{k}{\alpha R_0}, \tag{5.87}$$

where $\alpha\in(0,1)$ was given in (5.10)–(5.11). Then, as in (5.40)–(5.43), we conclude that there exist two constants $\lambda_1>0$ and $c_0>0$, such that for any $\lambda>\lambda_1$, it holds

$$B^j\big(=B^j(x,t,s)\big)\geqslant c_0\lambda^3,\quad \forall (x,t,s)\in \mathcal{Q}^j\ (j=0,1), \tag{5.88}$$

where B^j was given by (5.31) (with B and x^0 replaced by B^j and x^j respectively).

Further, we choose $\varepsilon_2\in(\varepsilon_1,1/2)$.

Now, we use Corollary 5.1 with u^j, α, ϕ^j and v^j given by (5.77), (5.10)–(5.11) and (5.79) respectively. For any given $\tau\in(T_2,T_1)$ and $\tau'\in(T_1',T_2')$ denote

$$\mathcal{Q}^j(\tau,\tau')\stackrel{\triangle}{=}\Omega^j\times(\tau,\tau')\times(\tau,\tau')\quad (j=0,1). \tag{5.89}$$

Integrating (5.29) (with u and v replaced by u^j and v^j $(j=0,1)$ respectively) on $\mathcal{Q}^j(\tau,\tau')$, using integration by parts, and by (5.78) and (5.88), we arrive at:

$$\int_{\mathcal{Q}^j(\tau,\tau')}\Big(\big|(v^j)_t\big|^2+\big|(v^j)_s\big|^2+\sum_i\big|(v^j)_i\big|^2\Big)\,dx\,dt\,ds+\lambda^2\int_{\mathcal{Q}^j(\tau,\tau')}\big|v^j\big|^2\,dx\,dt\,ds$$
$$\leqslant C\bigg\{\lambda^{-1}\int_{\mathcal{Q}^j(\tau,\tau')}e^{2\lambda\phi^j}\Big(|u|^2+\sum_i|u_i|^2\Big)\,dx\,dt\,ds+\int_{\mathcal{S}_{20}^j}\left|\frac{\partial v^j}{\partial\nu}\right|^2 d\mathcal{S}_{20}^j$$

$$+\lambda^2\Bigg[\int_{T_2}^{T_2'}\int_{\Omega^j}\Big(\big|v^j(x,\tau,s)\big|^2+\big|(v^j)_t(x,\tau,s)\big|^2+\big|(v^j)_s(x,\tau,s)\big|^2$$

$$+\sum_i\big|(v^j)_i(x,\tau,s)\big|^2+\big|v^j(x,\tau',s)\big|^2+\big|(v^j)_t(x,\tau',s)\big|^2+\big|(v^j)_s(x,\tau',s)\big|^2$$

$$+\sum_i\big|(v^j)_i(x,\tau',s)\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s+\int_{T_2}^{T_2'}\int_{\Omega^j}\Big(\big|v^j(x,t,\tau)\big|^2$$

$$+\big|(v^j)_t(x,t,\tau)\big|^2+\big|(v^j)_s(x,t,\tau)\big|^2+\sum_i\big|(v^j)_i(x,t,\tau)\big|^2$$

$$+\big|v^j(x,t,\tau')\big|^2+\big|(v^j)_t(x,t,\tau')\big|^2+\big|(v^j)_s(x,t,\tau')\big|^2$$

$$+\sum_i\big|(v^j)_i(x,t,\tau')\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\Bigg]\Bigg\}, \tag{5.90}$$

where ϕ^j was given in (5.79), u was given in (5.14). However, by (5.79) and (5.76)–(5.78), it is easy to see that

$$\int_{\mathcal{S}_{20}^j}\left|\frac{\partial v^j}{\partial \nu}\right|^2\mathrm{d}\mathcal{S}_{20}^j\leqslant C\mathrm{e}^{C\lambda}\int_{\mathcal{S}_{20}^j}\left|\frac{\partial u}{\partial \nu}\right|^2\mathrm{d}\mathcal{S}_{20}^j. \tag{5.91}$$

Further, by (5.83)–(5.84), (5.79) and (5.77), we get

$$\int_{T_2}^{T_2'}\int_{\Omega^j}\Big(\big|v^j(x,\tau,s)\big|^2+\big|(v^j)_t(x,\tau,s)\big|^2+\big|(v^j)_s(x,\tau,s)\big|^2+\sum_i\big|(v^j)_i(x,\tau,s)\big|^2$$

$$+\big|v^j(x,\tau',s)\big|^2+\big|(v^j)_t(x,\tau',s)\big|^2+\big|(v^j)_s(x,\tau',s)\big|^2+\sum_i\big|(v^j)_i(x,\tau',s)\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s$$

$$+\int_{T_2}^{T_2'}\int_{\Omega^j}\Big(\big|v^j(x,t,\tau)\big|^2+\big|(v^j)_t(x,t,\tau)\big|^2+\big|(v^j)_s(x,t,\tau)\big|^2+\sum_i\big|(v^j)_i(x,t,\tau)\big|^2$$

$$+\big|v^j(x,t,\tau')\big|^2+\big|(v^j)_t(x,t,\tau')\big|^2+\big|(v^j)_s(x,t,\tau')\big|^2+\sum_i\big|(v^j)_i(x,t,\tau')\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant C\lambda^2\mathrm{e}^{-2r_0\lambda}\Bigg[\int_{T_2}^{T_2'}\int_{\Omega^j}\Big(\big|u(x,\tau,s)\big|^2+\big|u_t(x,\tau,s)\big|^2+\big|u_s(x,\tau,s)\big|^2+\sum_i\big|u_i(x,\tau,s)\big|^2$$

$$+\big|u(x,\tau',s)\big|^2+\big|u_t(x,\tau',s)\big|^2+\big|u_s(x,\tau',s)\big|^2+\sum_i\big|u_i(x,\tau',s)\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s$$

$$+\int_{T_2}^{T_2'}\int_{\Omega^j}\Big(\big|u(x,t,\tau)\big|^2+\big|u_t(x,t,\tau)\big|^2+\big|u_s(x,t,\tau)\big|^2+\sum_i\big|u_i(x,t,\tau)\big|^2$$

$$+\left|u(x,t,\tau')\right|^2+\left|u_t(x,t,\tau')\right|^2+\left|u_s(x,t,\tau')\right|^2+\sum_i\left|u_i(x,t,\tau')\right|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\Big]. \tag{5.92}$$

Thus, combining (5.90)–(5.92), we conclude that:

$$\int\limits_{Q^j(\tau,\tau')}\Big(\left|(v^j)_t\right|^2+\left|(v^j)_s\right|^2+\sum_i\left|(v^j)_i\right|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\lambda^2\int\limits_{Q^j(\tau,\tau')}|v^j|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$
$$\leqslant C\Big\{\lambda^{-1}\int\limits_{Q^j(\tau,\tau')}\mathrm{e}^{2\lambda\phi^j}\Big(|u|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\mathrm{e}^{C\lambda}\int\limits_{\mathcal{S}^j_{20}}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}\mathcal{S}^j_{20}$$
$$+\lambda^4\mathrm{e}^{-2r_0\lambda}\Big[\int_{T_2}^{T_2'}\int\limits_{\Omega^j}\Big(\left|u(x,\tau,s)\right|^2+\left|u_t(x,\tau,s)\right|^2+\left|u_s(x,\tau,s)\right|^2+\sum_i\left|u_i(x,\tau,s)\right|^2$$
$$+\left|u(x,\tau',s)\right|^2+\left|u_t(x,\tau',s)\right|^2+\left|u_s(x,\tau',s)\right|^2+\sum_i\left|u_i(x,\tau',s)\right|^2\Big)\,\mathrm{d}x\,\mathrm{d}s$$
$$+\int_{T_2}^{T_2'}\int\limits_{\Omega^j}\Big(\left|u(x,t,\tau)\right|^2+\left|u_t(x,t,\tau)\right|^2+\left|u_s(x,t,\tau)\right|^2+\sum_i\left|u_i(x,t,\tau)\right|^2$$
$$+\left|u(x,t,\tau')\right|^2+\left|u_t(x,t,\tau')\right|^2+\left|u_s(x,t,\tau')\right|^2+\sum_i\left|u_i(x,t,\tau')\right|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\Big]\Big\}. \tag{5.93}$$

Integrating (5.93) with respect to τ and τ' from T_2 to T_1, and from T_1' to T_2' respectively, we get:

$$\int\limits_{Q^j_1}\Big(\left|(v^j)_t\right|^2+\left|(v^j)_s\right|^2+\sum_i\left|(v^j)_i\right|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\lambda^2\int\limits_{Q^j_1}\left|v^j\right|^2\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$
$$\leqslant C\Big[\lambda^{-1}\int\limits_{Q^j_2}\mathrm{e}^{2\lambda\phi^j}\big(|u|^2+\sum_i|u_i|^2\big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\mathrm{e}^{C\lambda}\int\limits_{\mathcal{S}^j_{20}}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}\mathcal{S}^j_{20}$$
$$+\lambda^4\mathrm{e}^{-2r_0\lambda}\int\limits_{Q^j_2}\Big(|u|^2+|u_t|^2+|u_s|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\Big]. \tag{5.94}$$

So, by (5.80) and (5.78), and in view of (5.94), we see that for any $\lambda>\lambda_1$, it holds

$$\int\limits_{Q^j_1}\mathrm{e}^{2\lambda\phi^j}\Big(\left|(u^j)_t\right|^2+\left|(u^j)_s\right|^2+\sum_i\left|(u^j)_i\right|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\lambda^2\int\limits_{Q^j_1}\mathrm{e}^{2\lambda\phi^j}\left|u^j\right|^2\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$
$$\leqslant C\Big[\int\limits_{Q^j_1}\Big(\left|(v^j)_t\right|^2+\left|(v^j)_s\right|^2+\sum_i\left|(v^j)_i\right|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\lambda^2\int\limits_{Q^j_1}\left|v^j\right|^2\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\Big]$$
$$\leqslant C\Big[\lambda^{-1}\int\limits_{Q^j_2}\mathrm{e}^{2\lambda\phi^j}\Big(|u|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\mathrm{e}^{C\lambda}\int\limits_{\mathcal{S}^j_{20}}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}\mathcal{S}^j_{20}$$

$$+\lambda^4 e^{-2r_0\lambda} \int_{Q_2^j} \left(|u|^2 + |u_t|^2 + |u_s|^2 + \sum_i |u_i|^2\right) dt\, ds\, dx \Bigg], \tag{5.95}$$

where $j = 0, 1$.

However, by (5.75) and (5.76)–(5.77), and taking into account that

$$Q_1^4 \cup Q_1^5 \supset Q_1, \tag{5.96}$$

where Q_1 was defined in (5.13), we get:

$$\begin{aligned}
&\sum_{j=0}^{1} \left[\int_{Q_1^j} e^{2\lambda\phi^j} \left(\left|(u^j)_t\right|^2 + \left|(u^j)_s\right|^2 + \sum_i \left|(u^j)_i\right|^2 \right) dx\, dt\, ds + \lambda^2 \int_{Q_1^j} e^{2\lambda\phi^j} \left|u^j\right|^2 dx\, dt\, ds \right] \\
&\quad \geqslant \sum_{j=0}^{1} \left[\int_{Q_1^{j+4}} e^{2\lambda\phi^j} \left(\left|(u^j)_t\right|^2 + \left|(u^j)_s\right|^2 + \sum_i \left|(u^j)_i\right|^2 \right) dx\, dt\, ds \right. \\
&\qquad \left. + \lambda^2 \int_{Q_1^{j+4}} e^{2\lambda\phi^j} \left|u^j\right|^2 dx\, dt\, ds \right] \\
&\quad = \sum_{j=0}^{1} \left[\int_{Q_1^{j+4}} e^{2\lambda\phi^j} \left(\left|u_t\right|^2 + \left|u_s\right|^2 + \sum_i \left|u_i\right|^2 \right) dx\, dt\, ds \right. \\
&\qquad \left. + \lambda^2 \int_{Q_1^{j+4}} e^{2\lambda\phi^j} |u|^2 dx\, dt\, ds \right] \\
&\quad \geqslant \int_{Q_1} e^{2\lambda \max(\phi^0,\phi^1)} \left(|u_t|^2 + |u_s|^2 + \sum_i |u_i|^2 \right) dx\, dt\, ds \\
&\qquad + \lambda^2 \int_{Q_1} e^{2\lambda \max(\phi^0,\phi^1)} |u|^2 dx\, dt\, ds \\
&\quad \geqslant \frac{1}{2} \sum_{j=0}^{1} \left[\int_{Q_1^j} e^{2\lambda\phi^j} \left(|u_t|^2 + |u_s|^2 + \sum_i |u_i|^2 \right) dx\, dt\, ds \right. \\
&\qquad \left. + \lambda^2 \int_{Q_1^j} e^{2\lambda\phi^j} |u|^2 dx\, dt\, ds \right].
\end{aligned} \tag{5.97}$$

Now, combining (5.95) and (5.97), we conclude that for any $\lambda > \lambda_1$, the following holds:

$$\sum_{j=0}^{1} \left[\int_{Q_1^j} e^{2\lambda\phi^j} \left(|u_t|^2 + |u_s|^2 + \sum_i |u_i|^2 \right) dt\, ds\, dx + \lambda^2 \int_{Q_1^j} e^{2\lambda\phi^j} |u|^2 dx\, dt\, ds \right]$$

$$\leqslant C\sum_{j=0}^{1}\Bigg[\lambda^{-1}\int\limits_{\mathcal{Q}_2^j}\mathrm{e}^{2\lambda\phi^j}\Big(|u|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\mathrm{e}^{C\lambda}\int\limits_{\mathcal{S}_{20}^j}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}\mathcal{S}_{20}^j$$

$$+\lambda^4\mathrm{e}^{-2r_0\lambda}\int\limits_{\mathcal{Q}_2^j}\Big(|u|^2+|u_t|^2+|u_s|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x\Bigg]. \tag{5.98}$$

However, by (5.75) and (5.83)–(5.84), we have:

$$\sum_{j=0}^{1}\int\limits_{\mathcal{Q}_2^j}\mathrm{e}^{2\lambda\phi^j}\Big(|u|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$

$$=\sum_{j=0}^{1}\Bigg[\Bigg(\int\limits_{\mathcal{Q}_1^j}+\int\limits_{\mathcal{Q}_2^j\setminus\mathcal{Q}_1^j}\Bigg)\Big(\mathrm{e}^{2\lambda\phi^j}\Big(|u|^2+\sum_i|u_i|^2\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\Bigg]$$

$$\leqslant\sum_{j=0}^{1}\Bigg[\int\limits_{\mathcal{Q}_1^j}\mathrm{e}^{2\lambda\phi^j}\Big(|u|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$

$$+\mathrm{e}^{-2r_0\lambda}\int\limits_{\mathcal{Q}_2^j}\Big(|u|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\Bigg]. \tag{5.99}$$

So, combining (5.98)–(5.99), we see that there exists a constant λ_2 $(\geqslant\lambda_1)$ such that whenever $\lambda>\lambda_2$, it holds

$$\sum_{j=0}^{1}\Bigg[\int\limits_{\mathcal{Q}_1^j}\mathrm{e}^{2\lambda\phi^j}\Big(|u_t|^2+|u_s|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x+\lambda^2\int\limits_{\mathcal{Q}_1^j}\mathrm{e}^{2\lambda\phi^j}|u|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\Bigg]$$

$$\leqslant C\sum_{j=0}^{1}\Bigg[\mathrm{e}^{C\lambda}\int\limits_{\mathcal{S}_{20}^j}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}\mathcal{S}_{20}^j$$

$$+\lambda^4\mathrm{e}^{-2r_0\lambda}\int\limits_{\mathcal{Q}_2^j}\Big(|u|^2+|u_t|^2+|u_s|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\Bigg]. \tag{5.100}$$

However, by (5.75) and (5.86), and taking into account that

$$\mathcal{Q}_0^0\cup\mathcal{Q}_0^1\supset\mathcal{Q}_0, \tag{5.101}$$

we get

$$\sum_{j=0}^{1}\Bigg[\int\limits_{\mathcal{Q}_1^j}\mathrm{e}^{2\lambda\phi^j}\Big(|u_t|^2+|u_s|^2+\sum_i|u_i|^2\Big)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x+\lambda^2\int\limits_{\mathcal{Q}_1^j}\mathrm{e}^{2\lambda\phi^j}|u|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\Bigg]$$

$$\geqslant \sum_{j=0}^{1}\left[\int_{\mathcal{Q}_0^j}\left(|u_t|^2+|u_s|^2+\sum_i |u_i|^2\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\lambda^2\int_{\mathcal{Q}_0^j}|u|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\right]$$

$$\geqslant \frac{1}{2}\left[\int_{\mathcal{Q}_0}\left(|u_t|^2+|u_s|^2+\sum_i |u_i|^2\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s+\lambda^2\int_{\mathcal{Q}_0}|u|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\right]. \tag{5.102}$$

Thus, by (5.100) and (5.102), we get

$$\int_{\mathcal{Q}_0}\left(|u_t|^2+|u_s|^2+\sum_i |u_i|^2\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$

$$\leqslant C\left[\mathrm{e}^{C\lambda}\sum_{j=0}^{1}\int_{\mathcal{S}_{20}^j}\left|\frac{\partial u}{\partial \nu}\right|^2\mathrm{d}\mathcal{S}_{20}^j\right.$$

$$\left.+\lambda^4\mathrm{e}^{-2r_0\lambda}\int_{\mathcal{Q}_2}\left(|u|^2+|u_t|^2+|u_s|^2+\sum_i |u_i|^2\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\right], \tag{5.103}$$

where u was defined in (5.14), and $\mathcal{Q}_2$ was defined in (5.13).

Step 5′. Proceeding as in Steps 5–7 of the previous case the proof may be completed easily. □

6. Proof of the main controllability and observability results

This section is devoted to complete the proofs of the uniform controllability result of Theorem 1.1 and its consequence in the context of the observability inequalities. Note that the proof of the convergence of the controls stated in Theorem 1.1 is left to Section 7. We divide it in two subsections.

6.1. Proof of the uniform controllability results of Theorem 1.1

We follow the control stategy presented in Section 2.2.

We fix a control time $T>0$. We then choose $\varepsilon(T)$ such that Theorem 3.1 on the uniform controllability of the parabolic projections holds in the time interval $[0,T/3]$ for any $0<\varepsilon<\varepsilon(T)$.

We now divide the control time in three subintervals $I_1=[0,T/3]$, $I_2=[T/3,2T/3]$ and $I_3=[2T/3,T]$.

In the first time interval I_1 we apply Theorem 3.1. We deduce that, for any $(u^0,u^1)\in H^\varepsilon$ there exists a control $f_{1,\varepsilon}\in L^2(\omega\times(0,T/3))$ such that the solution of (2.20) satisfies (2.21). Moreover, according to Theorem 3.1, we deduce the existence of a positive constant $C>0$ which depends, in particular, on T but is independent of $0<\varepsilon<\varepsilon(T)$, such that:

$$\|f_{1,\varepsilon}\|_{L^2(\omega\times(0,T/3))}\leqslant C\left\|\left(u^0,u^1\right)\right\|_{H^\varepsilon}, \tag{6.1}$$

for all $0<\varepsilon<\varepsilon(T)$ and every $(u^0,u^1)\in H^\varepsilon$.

In the second time interval $I_2=[T/3,2T/3]$ we let the system evolve freely. In other words, we solve the uncontrolled system (2.24) with data $(v_\varepsilon^0,v_\varepsilon^1)=(u(T/3),\,u_t(T/3)),u$ being the solution of (2.20) obtained in the first time interval I_1. According to (2.21), i.e., to the fact that

$$\pi_p^\varepsilon\big(v_\varepsilon^0, v_\varepsilon^1\big) = \pi_p^\varepsilon\big(u(T/3),\ u_t(T/3)\big), \tag{6.2}$$

we have

$$\big(v_\varepsilon^0, v_\varepsilon^1\big) = \big(u(T/3),\ u_t(T/3)\big) \in H_h^\varepsilon. \tag{6.3}$$

Therefore, in the time interval I_2 we may apply Theorem 4.1. We obtain

$$\big\|(u(t),\ u_t(t))\big\|_{H^\varepsilon} \leqslant 2^{3/2}\mathrm{e}^{-t/(4\varepsilon)}\big\|\big(v_\varepsilon^0, v_\varepsilon^1\big)\big\|_{H^\varepsilon}, \quad \forall T/3 < t < 2T/3,\ \forall 0 < \varepsilon < \varepsilon(T). \tag{6.4}$$

Moreover,

$$\big(u(t),\ u_t(t)\big) \in H_h^\varepsilon, \quad \forall T/3 < t < 2T/3, \tag{6.5}$$

since the condition

$$\pi_p^\varepsilon\big(u(t),\ u_t(t)\big) = 0, \quad \forall T/3 < t < 2T/3, \tag{6.6}$$

is kept along the trajectory.

We set

$$\big(w_\varepsilon^0, w_\varepsilon^1\big) = \big(u(2T/3),\ u_t(2T/3)\big). \tag{6.7}$$

We have

$$\big\|\big(w_\varepsilon^0, w_\varepsilon^1\big)\big\|_{H^\varepsilon} \leqslant 2^{3/2}\mathrm{e}^{-T/(12\varepsilon)}\big\|\big(v_\varepsilon^0, v_\varepsilon^1\big)\big\|_{H^\varepsilon}. \tag{6.8}$$

On the other hand, taking into account that $f_{1,\varepsilon}$ is bounded in $L^2(\omega \times (0, T/3))$ (see (6.1)) we have, by classical energy estimates:

$$\big\|\big(v_\varepsilon^0, v_\varepsilon^1\big)\big\|_{H^\varepsilon} \leqslant C\big[\big\|\big(u^0, u^1\big)\big\|_{H^\varepsilon} + \|f_{1,\varepsilon}\|_{L^2(\omega\times(0,T/3))}\big] \leqslant C\big\|\big(u^0, u^1\big)\big\|_{H^\varepsilon}. \tag{6.9}$$

Combining (6.8)–(6.9) we deduce that:

$$\big\|\big(w_\varepsilon^0, w_\varepsilon^1\big)\big\|_{H^\varepsilon} \leqslant C\mathrm{e}^{-T/(12\varepsilon)}\big\|\big(u^0, u^1\big)\big\|_{H^\varepsilon}. \tag{6.10}$$

In the last time interval $I_3 = [2T/3, T]$ we control to zero the data $(w_\varepsilon^0, w_\varepsilon^1)$ obtained in the previous two steps. In order words, we solve (2.28) with the control $f_{2,\varepsilon} \in L^2(\omega \times (2T/3, T))$ obtained as a consequence of Proposition 5.1. We then have

$$u(T) \equiv u_t(T) \equiv 0 \quad \text{in } \Omega \tag{6.11}$$

as we wanted to prove and, on the other hand,

$$\|f_{2,\varepsilon}\|_{L^2(\omega\times(2T/3,T))} \leqslant C_1(T)\mathrm{e}^{C_2(T)/\sqrt{\varepsilon}}\big\|\big(w_\varepsilon^0, w_\varepsilon^1\big)\big\|_{H^\varepsilon} \leqslant C_1(T)\mathrm{e}^{C_2(T)/\sqrt{\varepsilon}}\mathrm{e}^{-T/12\varepsilon}\big\|\big(u^0, u^1\big)\big\|_{H^\varepsilon}. \tag{6.12}$$

This shows that the control:

$$f_\varepsilon = \begin{cases} f_{1,\varepsilon}, & 0 \leqslant t \leqslant T/3, \\ 0, & T/3 \leqslant t \leqslant 2T/3, \\ f_{2,\varepsilon}, & 2T/3 \leqslant t \leqslant T \end{cases} \tag{6.13}$$

is such that the null controllability condition (6.11) holds and, according to (6.1) and (6.12),

$$\|f_\varepsilon\|_{L^2(\omega\times(0,T))} \leqslant C\left\|\left(u^0,u^1\right)\right\|_{H^\varepsilon}. \tag{6.14}$$

This completes the proof of the uniform controllability result of Theorem 1.1. □

Remark 6.1. – In view of the developments of the proof we see that the control $f_{2,\varepsilon}$ acting on the last time interval $[2T/3, T]$, is exponentially small with respect to ε.

6.2. Proof of Theorem 1.3

In this section we prove Theorem 1.3 as a direct consequence of Theorem 1.1.

First of all we observe that, multiplying in (1.1) by φ solution of (1.9) and integrating by parts the following holds:

$$\int_0^T\int_\omega f\varphi\,\mathrm{d}x\,\mathrm{d}t = \varepsilon\int_\Omega u_t\varphi\,\mathrm{d}x\Big|_0^T - \varepsilon\int_\Omega u\varphi_t\,\mathrm{d}x\Big|_0^T + \int_\Omega u\varphi\,\mathrm{d}x\Big|_0^T.$$

Now, if $f\in L^2(\omega\times(0,T))$ is the control of (1.1) such that the solution u satisfies (1.4) we deduce that

$$\int_0^T\int_\omega f\varphi\,\mathrm{d}x\,\mathrm{d}t = \varepsilon\int_\Omega\left[u^0\varphi_t(0) - u^1\varphi(0)\right]\mathrm{d}x - \int_\Omega u^0\varphi(0)\,\mathrm{d}x. \tag{6.15}$$

We now observe that, in view of the uniform bound (6.14) we have obtained on the control, it follows that

$$\left|\varepsilon\int_\Omega\left(u^0\varphi_t(0) - u^1\varphi(0)\right)\mathrm{d}x - \int_\Omega u^0\varphi(0)\,\mathrm{d}x\right|$$
$$\leqslant C\left\|\left(u^0,u^1\right)\right\|_{H^\varepsilon}\left[\int_0^T\int_\omega \varphi^2\,\mathrm{d}x\,\mathrm{d}t\right]^{1/2},\quad \forall\left(u^0,u^1\right)\in H^\varepsilon,\ \forall\left(\varphi^0,\varphi^1\right)\in\left(H^\varepsilon\right)'. \tag{6.16}$$

Taking into account that

$$\left\|\left(u^0,u^1\right)\right\|_{H^\varepsilon} = \left[\left\|u^0\right\|^2_{H^1_0(\Omega)} + \varepsilon\left\|u^1\right\|^2_{L^2(\Omega)}\right]^{1/2}$$

and rewriting the left hand side of (6.16) as

$$\left|\int_\Omega u^0\left(\varepsilon\varphi_t(0) - \varphi(0)\right)\mathrm{d}x - \sqrt{\varepsilon}\int_\Omega u^1\sqrt{\varepsilon}\varphi(0)\,\mathrm{d}x\right|$$

we deduce that:

$$\left\|\varepsilon\varphi_t(0) - \varphi(0)\right\|_{H^{-1}(\Omega)} + \sqrt{\varepsilon}\left\|\varphi(0)\right\|_{L^2(\Omega)} \leqslant C\left[\int_0^T\int_\omega\varphi^2\,\mathrm{d}x\,\mathrm{d}t\right]^{1/2},$$

or, in other words,

$$\varepsilon\|\varphi(0)\|^2_{L^2(\Omega)} + \|\varepsilon\varphi_t(0) - \varphi(0)\|^2_{H^{-1}(\Omega)} \leqslant C\int_0^T\int_\omega \varphi^2\,\mathrm{d}x\,\mathrm{d}t, \quad \forall(\varphi^0,\varphi^1) \in (H^\varepsilon)',$$

with a constant $C > 0$ which is independent of $0 < \varepsilon < \varepsilon(T)$. This completes the proof of Theorem 1.3.

7. The limit process

This section is devoted to prove the convergence of the controls stated in Theorem 1.1.

We fix $T > 0$. We then consider $0 < \varepsilon < \varepsilon(T)$ so that the uniform bound on the controls stated in Theorem 1.1 holds, as proved in Section 6.1.

We also fix $u^0 \in H_0^1(\Omega)$ and $u^1 \in L^2(\Omega)$.

For any $0 < \varepsilon < \varepsilon(T)$ we consider the control $f_\varepsilon \in L^2(\omega \times (0,T))$ such that the solution of (1.1) satisfies (1.4). Note that

$$\|f_\varepsilon\|_{L^2(\omega\times(0,T))} \leqslant C\|(u^0,u^1)\|_{H^\varepsilon}, \quad \forall 0 < \varepsilon < \varepsilon(T), \tag{7.1}$$

with C independent of $0 < \varepsilon < \varepsilon(T)$.

Moreover, according to the construction of the control of Section 5, we know that we may choose a control of minimal norm of the form

$$f_\varepsilon = \varphi_\varepsilon \quad \text{in } \omega \times (0,T), \tag{7.2}$$

where φ_ε is the solution of (1.9) with initial data $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in (H^\varepsilon)'$ minimizing the functional

$$J_\varepsilon(\varphi^0,\varphi^1) = \frac{1}{2}\int_0^T\int_\omega \varphi^2\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega (u^0 + \varepsilon u^1)\varphi(0)\,\mathrm{d}x - \varepsilon\langle u^0, \varphi_t(0)\rangle \tag{7.3}$$

in $(H^\varepsilon)'$. Obviously, the control in (7.2) is not the one we built in the three steps control method but we keep the same notation since it has the same essential properties: for instance, the bound (7.1) is kept.

In view of the uniform observability inequality of Theorem 1.3 it is easy to see that the minimizers are such that

$$\int_0^T\int_\omega \varphi_\varepsilon^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C, \quad \forall 0 < \varepsilon < \varepsilon(T), \tag{7.4}$$

and consequently

$$\varepsilon\|\varphi_\varepsilon(0)\|^2_{L^2(\Omega)} + \|\varepsilon\varphi_{\varepsilon,t}(0) - \varphi_\varepsilon(0)\|^2_{H^{-1}(\Omega)} \leqslant C, \tag{7.5}$$

as well.

Note however that we may not deduce a uniform bound of the form (7.5) for the initial data $(\varphi_\varepsilon^0, \varphi_\varepsilon^1)$.

According to (7.2) and (7.4) the controls $f_\varepsilon = \varphi_\varepsilon$ are uniformly bounded in $L^2(\omega \times (0,T))$. By extracting subsequences (that we still denote by the index ε) we deduce that

$$\varphi_\varepsilon \rightharpoonup \varphi \quad \text{weakly in } L^2(\omega \times (0,T)). \tag{7.6}$$

Let us now observe that the Euler equation satisfied by the minimizer $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in (H^\varepsilon)'$ of the functional J_ε may also be written as:

$$\begin{aligned} &\int_0^T \int_\omega \varphi_\varepsilon \psi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \left(u^0 + \varepsilon u^1\right)\psi_\varepsilon(0)\,\mathrm{d}x - \varepsilon\left\langle u^0, \psi_{\varepsilon,t}(0)\right\rangle = 0, \\ &\quad \forall \psi_\varepsilon \text{ solution of (1.9).} \end{aligned} \tag{7.7}$$

Given $(\psi^0, \psi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ fixed, we denote by ψ_ε the solution of (1.9) with those data at time $t = T$. It is then easy to see that

$$\left(\psi_\varepsilon(0), \sqrt{\varepsilon}\psi_{\varepsilon,t}(0)\right) \to \left(\psi(0), 0\right) \quad \text{strongly in } L^2(\Omega) \times H^{-1}(\Omega). \tag{7.8}$$

where ψ is the solution of (1.13) with datum ψ^0 at time $t = T$. Moreover

$$\psi_\varepsilon \to \psi \quad \text{strongly in } L^2\left(\Omega \times (0,T)\right). \tag{7.9}$$

We refer to Appendix D for a proof of these convergence results.

Using (7.6), (7.8) and (7.9) and passing to the limit in (7.7) we deduce that:

$$\int_0^T \int_\omega \varphi\psi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u^0 \psi(0)\,\mathrm{d}x = 0. \tag{7.10}$$

But then the solution u of the controlled heat equation (1.2) with control $f = \varphi$ is such that

$$u(T) \equiv 0 \quad \text{in } \Omega. \tag{7.11}$$

Indeed, multipling in (1.2) by ψ solution of (1.13) and integrating by parts we deduce that

$$\int_0^T \int_\omega f\psi \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega u\psi \,\mathrm{d}x \Big|_0^T,$$

which combined with (7.10) implies that

$$\int_\Omega u(T)\psi^0 \,\mathrm{d}x = 0, \quad \forall \psi^0 \in L^2(\Omega),$$

and this implies (7.11).

Consequently, the weak limit φ in (7.6) is a null control for the limit heat equation (1.2).

We are now going to prove the following two facts to complete the proof of Theorem 1.1:

(a) The limit φ is uniquely determined, i.e., independently of the subsequence;
(b) The convergence (7.6) holds in the strong topology of $L^2(\omega \times (0,T))$.

To prove (a) we first observe that, according to Theorem 1.3, the uniform observability inequality (1.11) holds for any $T > 0$. In particular it also applies in any interval of the form (τ, T) with $0 < \tau < T$.

Then, we deduce that

$$\left\| \varepsilon\varphi_{\varepsilon,t}(\tau) - \varphi_\varepsilon(\tau) \right\|^2_{H^{-1}(\Omega)} + \varepsilon \left\| \varphi_\varepsilon(\tau) \right\|^2_{L^2(\Omega)} \leqslant C(T-\tau)$$

with $C(T-\tau)$ that becomes singular when $\tau = T$. Nevertheless, we get that

$$\varphi_\varepsilon - \varepsilon\varphi_{\varepsilon,t} \quad \text{is bounded in } L^\infty\big(0,\tau;\ H^{-1}(\Omega)\big) \tag{7.12}$$

$$\sqrt{\varepsilon}\varphi_\varepsilon \quad \text{is bounded in } L^\infty\big(0,\tau;\ L^2(\Omega)\big) \tag{7.13}$$

for any $0 < \tau < T$.

Combining (7.12) and (7.13) we deduce that

$$\varphi_\varepsilon \quad \text{is bounded in } H^{-1}\big(0,\tau;\ H^{-1}(\Omega)\big) \tag{7.14}$$

for any $0 < \tau < T$.

Then, in addition to the convergence in (7.6), we may guarantee that:

$$\varphi_\varepsilon \rightharpoonup \varphi \quad \text{weakly in } H^{-1}\big(0,\tau;\ H^{-1}(\Omega)\big). \tag{7.15}$$

In fact

$$\varphi_\varepsilon - \varepsilon\varphi_{\varepsilon,t} \rightharpoonup \varphi \quad \text{weakly in } H^{-1}\big(0,\tau;\ H^{-1}(\Omega)\big) \tag{7.16}$$

as well since, in view of (7.13), $\varepsilon\varphi_{\varepsilon,t} \rightharpoonup 0$ weakly in $H^{-1}(0,\tau;\ H^{-1}(\Omega))$.

Combining (7.16) and the boundedness condition (7.12) we deduce that

$$\varphi_\varepsilon - \varepsilon\varphi_{\varepsilon,t} \rightharpoonup \varphi \quad \text{weakly-}* \text{ in } L^\infty\big(0,\tau;\ H^{-1}(\Omega)\big), \tag{7.17}$$

for all $0 < \tau < T$.

Consequently $\varphi \in C([0,\tau];\ H^{-1}(\Omega))$, but since it is a weak solution of:

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } \Omega \times (0,T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{7.18}$$

by the classical regularizing effect we deduce that

$$\varphi \in C\big([0,\tau];\ H_0^1(\Omega)\big), \quad \forall 0 < \tau < T. \tag{7.19}$$

In order to identify the limit φ we define the following Hilbert space:

$$F = \big\{ f \in L^2\big(\omega \times (0,T)\big):\ f = \varphi|_{\omega \times (0,T)} \text{ for some solution } \varphi \text{ of (7.18) with } \varphi \in L^\infty_{\text{loc}}\big(0,T;\ H_0^1(\Omega)\big) \big\}, \tag{7.20}$$

endowed with the norm

$$\|f\|_F = \left(\int_0^T \int_\omega \varphi^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}. \tag{7.21}$$

We define the following quadratic functional on F:

$$J(f) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u^0 \varphi(0)\,\mathrm{d}x. \tag{7.22}$$

We now make use of the following observability inequality that solutions of (7.18) satisfy:

$$\|\varphi(0)\|^2_{H^{-1}(\Omega)} \leqslant C \int_0^T \int_\omega \varphi^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{7.23}$$

with C independent of the solution φ of (7.18).

This inequality is well-known to hold for any open non-empty subset ω of Ω (see [3] and [6]). But in the present geometric setting it can also be derived as limit when $\varepsilon \to 0$ of the uniform inequalities (1.11).

In view of (7.23) it is easy to see that J achieves its minimum at a single point $f = \varphi$ which is characterized by the Euler equation:

$$\int_0^T \int_\omega f\psi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u^0 \psi(0)\,\mathrm{d}x = 0, \quad \forall \psi \text{ solution of (7.18).} \tag{7.24}$$

Taking into account that the limit φ of the sequence φ_ε also satisfies (7.10), we deduce that the limit φ is precisely the minimizer of J in F. Consequently, the limit in (7.6) holds for the whole sequence φ_ε.

Let us finally prove that the convergence in (7.6) holds in the strong topology of $L^2(\omega \times (0,T))$.

The solutions φ_ε, since they minimize the functionals J_ε, satisfy

$$\int_0^T \int_\omega |\varphi_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega \left(u^0 + \varepsilon u^1\right)\varphi_\varepsilon(0) + \varepsilon \int_\Omega \varphi_{\varepsilon,t}(0) u^0. \tag{7.25}$$

Passing to the limit in (7.25) we deduce that:

$$\begin{aligned} \int_0^T \int_\omega \varphi^2 \,\mathrm{d}x\,\mathrm{d}t &\leqslant \liminf_{\varepsilon\to 0} \int_0^T \int_\omega |\varphi_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{\varepsilon\to 0}\left[-\int_\Omega \left(u^0 + \varepsilon u^1\right)\varphi_\varepsilon(0) + \varepsilon \int_\Omega \varphi_{\varepsilon,t}(0) u^0\right] \\ &= -\int_\Omega u^0 \varphi(0)\,\mathrm{d}x, \end{aligned} \tag{7.26}$$

since one can easily check that $\varphi_\varepsilon(0) - \varepsilon\varphi_{\varepsilon,t}(0) \rightharpoonup \varphi(0)$ weakly in $H^{-1}(\Omega)$, as $\varepsilon \to 0$.

But

$$\int_0^T\int_\omega \varphi^2\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u^0\varphi(0)\,\mathrm{d}x = 0. \tag{7.27}$$

Combining (7.26) and (7.27) we deduce that

$$\int_0^T\int_\omega |\varphi_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\omega \varphi^2\,\mathrm{d}x\,\mathrm{d}t$$

which, combined with the weak convergence (7.6), implies that $\varphi_\varepsilon \to \varphi$ strongly in $L^2(\omega \times (0,T))$.

8. Further comments and results

In this section we discuss some variants and extensions of the results of this paper and also present an open problem.

8.1. Sharp geometric conditions on ω

Let ω be an open non-empty subset of Ω. Suppose that ω does not satisfy the geometric conditions imposed in this paper. This is for instance the case when Ω is convex and $\bar\omega \subset \Omega$. In this situation one does not expect, in general, the results of this paper to hold. In fact, one can not even guarantee the exact controllability of system (1.1) with $\varepsilon > 0$ fixed.

However, Theorem 3.1 on the uniform controllability of the parabolic projections does apply. Moreover, if we apply the control strategy described in Section 2 but taking $f_{2,\varepsilon} = 0$, i.e., letting the equation evolve freely in the interval $I_2 \cup I_3 = [T/3, T]$, at time $t = T$ we get

$$\pi_p^\varepsilon\big(u_\varepsilon(T),\, u_{\varepsilon,t}(T)\big) = 0 \tag{8.1}$$

and also

$$\big\|\big(u_\varepsilon(T),\, u_{\varepsilon,t}(T)\big)\big\|_{H^\varepsilon} \leqslant C_1\mathrm{e}^{-C_2T/\varepsilon}\big\|\big(u^0, u^1\big)\big\|_{H^\varepsilon}. \tag{8.2}$$

More precisely the following holds:

THEOREM 8.1. – *Let Ω be a bounded domain of $\mathbf{R}^n$ of class C^∞. Let ω be any open non-empty subset Ω. Let $T > 0$. Then, for all $0 < \varepsilon < \min(\varepsilon(\Omega,\omega),\, T^2/C^2(\Omega,\omega))$ and $(u^0, u^1) \in H^\varepsilon$ there exists a control $f_\varepsilon \in L^2(\omega \times (0,T))$ such that the solution of* (1.1) *satisfies* (8.1).

Moreover, there exist positive constants $C_1, C_2, C_3 > 0$ such that (8.2) *holds and*

$$\|f_\varepsilon\|_{L^2(\omega\times(0,T))} \leqslant C_3\big\|\big(u^0, u^1\big)\big\|_{H^\varepsilon} \tag{8.3}$$

for all $(u^0, u^1) \in H^\varepsilon$ and ε as above.

The constants $\varepsilon(\Omega,\omega)$ and $C(\Omega,\omega)$ in this statement are those of Theorem 3.1. This theorem provides a partial and uniform controllability result in which the parabolic component of the solutions is driven to zero and the hyperbolic one is exponentially small as $\varepsilon \to 0$.

Obviously, the null controllability of the heat equation from an arbitrary open non-empty subset ω of Ω may also be viewed as the limit as $\varepsilon \to 0$ of this partial controllability property.

However, it makes sense to analyze whether (8.1)–(8.2) may be replaced by the stronger condition

$$u(T) \equiv u_t(T) \equiv 0$$

keeping the uniform boundedness condition (8.3) on the control.

For this to hold Ω and ω have to satisfy the geometric control condition introduced in [1]. Roughly, this condition asserts that every ray of geometric optics has to enter the subdomain ω in time T. Indeed, this is a sharp sufficient condition for the controllability of system (1.1) for any $\varepsilon > 0$ fixed. Note that the velocity of propagation of the rays of geometric optics depends on ε. Thus, if this geometric control condition is satisfied for some $\varepsilon > 0$ it also holds for all $0 < \varepsilon' < \varepsilon$.

This paper provides a positive answer in the particular case where ω is a neighborhood of a subset of the boundary of the form $\Gamma(x^0)$.

One may expect the results of this paper to hold under the sharp geometric control condition of [1] on ω and Ω. But this is by now an open problem.

8.2. Variable coefficients

The three steps control strategy we have used along this paper may be applied to more general variable coefficients wave equations admiting Fourier series decompositions. It would be interesting to see what are the minimal regularity conditions on the coefficients $\rho(x)$ and $a_{ij}(x)$ so that the results of this paper apply to the more general equation:

$$\varepsilon\rho(x)u_{tt} - \partial_j\big(a_{ij}(x)\partial_i u\big) + \rho(x)u_t = 0 \tag{8.4}$$

with

$$0 < \rho_0 \leqslant \rho(x) \leqslant \rho_1 < \infty, \quad \forall x \in \Omega,$$

$$\alpha|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant A|\xi|^2, \quad \forall x \in \Omega,\ \forall \xi \in \mathbf{R}^n$$

with $0 < \alpha \leqslant A < \infty$ and

$$a_{ij} = a_{ji}.$$

8.3. Initial data depending on ε

In Theorem 1.1 we have stated the strong convergence of the controls f_ε of (1.1) towards a null control for the limit heat equation when the initial data $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ are fixed. The result may be easily extended to initial data depending on ε, i.e., $(u_\varepsilon^0, u_\varepsilon^1) \in H^\varepsilon$ provided

$$\big\|\big(u_\varepsilon^0, u_\varepsilon^1\big)\big\|_{H^\varepsilon} \leqslant C \tag{8.5}$$

and

$$u_\varepsilon^0 \to u^0 \quad \text{in } H_0^1(\Omega), \tag{8.6}$$

$$\sqrt{\varepsilon}u_\varepsilon^1 \to 0 \quad \text{in } L^2(\Omega). \tag{8.7}$$

Weak convergence results may also be established when the initial data converge weakly.

8.4. Boundary control

The same problems arise in the context of boundary control. In that case the most natural functional framework is when $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and the boundary control lies in L^2 and is supported on a subset of the boundary of the form $\Gamma(x^0) \times (0,T)$. Most of the developments of this paper apply in this context too. However, the strategy to uniformly control the low frequencies has to be slightly modified. This will be done in a future paper.

Appendix A. Proof of the exact controllability of the damped wave equation

In the geometric setting we are working in, the exact controllability of (1.1) can be easily obtained as a combination of HUM (Hilbert Uniqueness Method) (see J.L. Lions [8]) and multiplier techniques.

Indeed, applying HUM one easily sees that system (1.1) is exactly controllable if and only if there exists a constant $C(\varepsilon, T)$ such that the following observability estimate holds:

$$\|\varphi^0\|^2_{L^2(\Omega)} + \varepsilon\|\varphi^1\|^2_{H^{-1}(\Omega)} \leqslant C(\varepsilon, T)\int_0^T\int_\omega \varphi^2\,\mathrm{d}x\,\mathrm{d}t \tag{A.1}$$

for any solution of the adjoint wave equation

$$\begin{cases} \varepsilon\varphi_{tt} - \Delta\varphi - \varphi_t = 0 & \text{in } \Omega\times(0,T),\\ \varphi = 0 & \text{on } \partial\Omega\times(0,T),\\ \varphi(x,T) = \varphi^0(x),\ \varphi_t(x,T) = \varphi^1(x) & \text{in } \Omega. \end{cases} \tag{A.2}$$

The following holds:

LEMMA A.1. – *In the geometric setting of Theorem* 1.1*, for any* ε *such that*:

$$0 < \varepsilon < \left[T/\left(2\|x - x^0\|_{L^\infty(\Omega\setminus\omega)}\right)\right]^2 \tag{A.3}$$

there exists a positive constant $C(\varepsilon, T)$ *such that* (A.1) *holds for any solution of* (A.2)*.*

Remark A.1. – Note that this result does not provide any information on how the constant in (A.1) depends on ε.

Proof. – We perform the change of variables

$$\tau = t/\sqrt{\varepsilon}. \tag{A.4}$$

Then Eq. (A.2) reads as

$$\begin{cases} \varphi_{\tau\tau} - \Delta\varphi - \dfrac{1}{\sqrt{\varepsilon}}\varphi_\tau = 0 & \text{in } \Omega\times(0,\widehat{T}),\\ \varphi = 0 & \text{on } \partial\Omega\times(0,\widehat{T}),\\ \varphi(x,\widehat{T}) = \varphi^0(x),\ \varphi_t(x,\widehat{T}) = \sqrt{\varepsilon}\varphi^1(x) & \text{in } \Omega, \end{cases} \tag{A.5}$$

with $\widehat{T} = T/\sqrt{\varepsilon}$.

The problem is then reduced to show that for any

$$\widehat{T} > 2\|x - x_0\|_{L^\infty(\Omega\setminus\omega)} \tag{A.6}$$

there exists a constant $C(\varepsilon, \widehat{T}) > 0$ such that (A.1) holds for any solution of (A.5).

As proved in [8], Tome 1, when

$$\widehat{T} > 2\|x - x_0\|_{L^\infty(\Omega)} \tag{A.7}$$

the estimate

$$\|\varphi^0\|^2_{L^2(\Omega)} + \varepsilon\|\varphi^1\|^2_{H^{-1}(\omega)} \leqslant C \int_0^{\widehat{T}}\int_\omega \varphi^2\,\mathrm{d}x\,\mathrm{d}t \tag{A.8}$$

holds for the solutions of

$$\begin{cases} \varphi_{\tau\tau} - \Delta\varphi = 0 & \text{in } \Omega \times (0, \widehat{T}), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, \widehat{T}). \end{cases} \tag{A.9}$$

By a slight change in the proof of [8], Tome 1, it can be seen that (A.5) actually holds for the solutions of (A.8) under the weaker restriction (A.6) on the control time $\widehat{T}$.

We prove (A.1) for the solutions of (A.5) as a consequence of the fact that (A.8) holds for the solutions of the purely conservative wave equation.

We make the following change of variables

$$\psi = \mathrm{e}^{-\tau/2\sqrt{\varepsilon}}\varphi. \tag{A.10}$$

Then φ satisfies (A.5) if and only if ψ solves:

$$\begin{cases} \psi_{\tau\tau} - \Delta\psi - \dfrac{1}{4\varepsilon}\psi = 0 & \text{in } \Omega \times (0, \widehat{T}), \\ \psi = 0 & \text{on } \partial\Omega \times (0, \widehat{T}). \end{cases} \tag{A.11}$$

In view of the nature of the change of variables (A.10) we deduce that (A.1) holds for the solutions φ of (A.5) if and only if

$$\|\psi^0\|^2_{L^2(\Omega)} + \|\psi^1\|^2_{H^{-1}(\Omega)} \leqslant C(\varepsilon, \widehat{T}) \int_0^{T}\int_\omega \psi^2\,\mathrm{d}x\,\mathrm{d}t \tag{A.12}$$

holds for every solution ψ of (A.11) (with possibly a different constant).

We finally claim that (A.12) holds for the solutions of (A.11) as a consequence of the estimate (A.8) that we know is valid for the solutions of (A.9). To see this it is sufficient to apply the "compacteness-uniqueness" method described in Appendix I of [8], Tome 1.

Note however that this method does not provide an explicit bound on the observability constant $C(\varepsilon, T)$. The proof we develop in Section 5 using global Carleman estimates shows that $C(\varepsilon, T)$

grows as $\mathrm{e}^{C/\sqrt{\varepsilon}}$ as $\varepsilon \to 0$ for a suitable $C > 0$. Note however that, as a consequence of Theorem 1.2 the observability constant ends up being uniformly bounded above provided the dissipativity of the systems under consideration is suitably taken into account.

Appendix B. Proof of Lemma 3.1

Assume that $(u^0, u^1) \in H^\varepsilon$ is such that $\pi_k(u^0, u^1) = 0$ with $k \leqslant k(\varepsilon)$. Then we decompose $U^0 = (u^0, u^1)$ into its parabolic and hyperbolic components:

$$U^0 = U_p^0 + U_h^0, \quad U_p^0 = \pi_p^\varepsilon(U^0),\ U_h^0 = \pi_h^\varepsilon(U^0). \tag{B.1}$$

Obviously both U_p^0 and U_h^0 depend also on ε but we do not make this dependence explicit in the notation to simplify it.

Then

$$\|S_\varepsilon(t)(U^0)\|_{H^\varepsilon} \leqslant \|S_\varepsilon(t)(U_p^0)\|_{H^\varepsilon} + \|S_\varepsilon(t)(U_h^0)\|_{H^\varepsilon}. \tag{B.2}$$

We claim that:

$$\|S_\varepsilon(t)(U_p^0)\|_{H^\varepsilon} \leqslant C_1 \mathrm{e}^{-C_2\mu_{k+1}t} \|U_p^0\|_{H^\varepsilon} \tag{B.3}$$

and

$$\|S_\varepsilon(t)(U_h^0)\|_{H^\varepsilon} \leqslant C_1 \mathrm{e}^{-C_2 t/\varepsilon} \|U_h^0\|_{H^\varepsilon}. \tag{B.4}$$

Obviously combining (B.2)–(B.4) we deduce that the inequality (3.10) of Lemma 3.1 holds.

Note that (B.4) states that the hyperbolic component of the solution decays at an exponential rate of the order of $1/\varepsilon$. This is in agreement with the fact that the real part of all the hyperbolic eigenvalues is $-1/2\varepsilon$.

The estimate (B.4) on the decay of the hyperbolic components was proved in Section 4. The decay rate (B.3) of the parabolic components is proved in the following lemma.

LEMMA B.1. – *There exist positive constants $C_1, C_2 > 0$ such that*:

$$\|S_\varepsilon(t)(U^0)\|_{H^\varepsilon} \leqslant C_1 \mathrm{e}^{-C_2\mu_{k+1}t} \|U^0\|_{H^\varepsilon}, \quad \forall t > 0, \tag{B.5}$$

for every $0 < \varepsilon < 1$, $k < k(\varepsilon)$ and $U^0 \in H_p^\varepsilon$ such that $\pi_k^\varepsilon(U^0) = 0$.

Proof. – The solution $(u_\varepsilon, u_{\varepsilon,t}) = S_\varepsilon(t)(U^0)$ may be decomposed in Fourier series as follows:

$$u_\varepsilon(x,t) = \sum_{j=k+1}^{k(\varepsilon)} \left[\frac{a_j\lambda_{j-}^\varepsilon - b_j}{\lambda_{j-}^\varepsilon - \lambda_{j+}^\varepsilon} \mathrm{e}^{\lambda_{j+}^\varepsilon t} + \frac{b_j - b_j\lambda_{j+}^\varepsilon}{\lambda_{j-}^\varepsilon - \lambda_{j+}^\varepsilon} \mathrm{e}^{\lambda_{j-}^\varepsilon t} \right] e_j(x),$$

where $\{a_j, b_j\}$ are the Fourier coefficients of the initial data, i.e.,

$$(u^0, u^1) = \sum_{j=k+1}^{k(\varepsilon)} (a_j, b_j) e_j(x).$$

Using the orthogonality of the eigenfunctions $\{e_j\}$ in $L^2(\Omega)$ and $H_0^1(\Omega)$ we deduce that:

$$\begin{aligned}
\big\|S_\varepsilon(t)\big(U^0\big)\big\|^2_{H^\varepsilon} &= \sum_{j=k+1}^{k(\varepsilon)} \mu_j \left[\frac{a_j\lambda^\varepsilon_{j-} - b_j}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}} \mathrm{e}^{\lambda^\varepsilon_{j+}t} + \frac{b_j - a_j\lambda^\varepsilon_{j+}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}} \mathrm{e}^{\lambda^\varepsilon_{j-}t}\right]^2 \\
&\quad + \varepsilon \sum_{j=k+1}^{k(\varepsilon)} \left[\frac{a_j\lambda^\varepsilon_{j-} - b_j}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}} \lambda^\varepsilon_{j+}\mathrm{e}^{\lambda^\varepsilon_{j+}t} + \frac{b_j - a_j\lambda^\varepsilon_{j+}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}} \lambda^\varepsilon_{j-}\mathrm{e}^{\lambda^\varepsilon_{j-}t}\right]^2 \\
&\leqslant 2 \sum_{j=k+1}^{k(\varepsilon)} \mu_j \left\{ a_j^2 \left[\frac{\lambda^\varepsilon_{j-}\mathrm{e}^{\lambda^\varepsilon_{j+}t} - \lambda^\varepsilon_{j+}\mathrm{e}^{\lambda^\varepsilon_{j-}t}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}}\right]^2 + b_j^2 \left[\frac{\mathrm{e}^{\lambda^\varepsilon_{j-}t} - \mathrm{e}^{\lambda^\varepsilon_{j+}t}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}}\right]^2 \right\} \\
&\quad + 2\varepsilon \sum_{j=k+1}^{k(\varepsilon)} \left\{ a_j^2 \left[\lambda^\varepsilon_{j-}\lambda^\varepsilon_{j+} \frac{\mathrm{e}^{\lambda^\varepsilon_{j+}t} - \mathrm{e}^{\lambda^\varepsilon_{j-}t}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}}\right]^2 + b_j^2 \left[\frac{\lambda^\varepsilon_{j+}\mathrm{e}^{\lambda^\varepsilon_{j+}t} - \lambda^\varepsilon_{j-}\mathrm{e}^{\lambda^\varepsilon_{j-}t}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}}\right]^2 \right\}. \qquad \text{(B.6)}
\end{aligned}$$

Let us assume that there exist positive constants $A_1, A_2 > 0$ such that:

$$\mu_j \left[\frac{\lambda^\varepsilon_{j-}\mathrm{e}^{\lambda^\varepsilon_{j+}t} - \lambda^\varepsilon_{j+}\mathrm{e}^{\lambda^\varepsilon_{j-}t}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}}\right]^2 + \varepsilon \left[\lambda^\varepsilon_{j-}\lambda^\varepsilon_{j+} \frac{\mathrm{e}^{\lambda^\varepsilon_{j+}t} - \mathrm{e}^{\lambda^\varepsilon_{j-}t}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}}\right]^2 \leqslant \mu_j A_1^2 \mathrm{e}^{-2A_2\mu_j t} \qquad \text{(B.7)}$$

and

$$\mu_j \left[\frac{\mathrm{e}^{\lambda^\varepsilon_{j-}t} - \mathrm{e}^{\lambda^\varepsilon_{j+}t}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}}\right]^2 + \varepsilon \left[\frac{\lambda^\varepsilon_{j+}\mathrm{e}^{\lambda^\varepsilon_{j+}t} - \lambda^\varepsilon_{j-}\mathrm{e}^{\lambda^\varepsilon_{j-}t}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}}\right]^2 \leqslant \varepsilon A_1^2 \mathrm{e}^{-2A_2\mu_j t} \qquad \text{(B.8)}$$

for all $j = k+1, \ldots, k(\varepsilon)$.

Then by (B.6) we deduce that:

$$\begin{aligned}
\big\|S_\varepsilon(t)\big(U^0\big)\big\|^2_{H^\varepsilon} &\leqslant 2 \sum_{j=k+1}^{k(\varepsilon)} \big[\mu_j a_j^2 A_1^2 \mathrm{e}^{-2A_2\mu_j t} + \varepsilon b_j^2 A_1^2 \mathrm{e}^{-2A_2\mu_j t}\big] \\
&= 2A_1^2 \sum_{j=k+1}^{k(\varepsilon)} \big[\mu_j a_j^2 + \varepsilon b_j^2\big] \mathrm{e}^{-2A_2\mu_j t} \\
&\leqslant 2A_1^2 \mathrm{e}^{-2A_2\mu_{k+1}t} \sum_{j=k+1}^{k(\varepsilon)} \big[\mu_j a_j^2 + \varepsilon b_j^2\big] \\
&= 2A_1^2 \mathrm{e}^{-2A_2\mu_{j+1}t} \big\|U^0\big\|^2_{H^\varepsilon}. \qquad \text{(B.9)}
\end{aligned}$$

Thus, Lemma B.1 will be proved if we are able to prove (B.7) and (B.8). Obviously, (B.7) and (B.8) will hold provided the following four inequalities hold:

$$2 \left[\frac{\lambda^\varepsilon_{j-}\mathrm{e}^{\lambda^\varepsilon_{j+}t} - \lambda^\varepsilon_{j+}\mathrm{e}^{\lambda^\varepsilon_{j-}t}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}}\right]^2 \leqslant A_1^2 \mathrm{e}^{-2A_2\mu_j t}, \qquad \text{(B.10)}$$

$$2\varepsilon \left[\lambda^\varepsilon_{j-}\lambda^\varepsilon_{j+} \frac{\mathrm{e}^{\lambda_{j+}t} - \mathrm{e}^{\lambda^\varepsilon_{j-}t}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}}\right]^2 \leqslant A_1^2 \mu_j \mathrm{e}^{-2A_2\mu_j t}, \qquad \text{(B.11)}$$

$$2\mu_j \left[\frac{\mathrm{e}^{\lambda^\varepsilon_{j-}t} - \mathrm{e}^{\lambda^\varepsilon_{j+}t}}{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}}\right]^2 \leqslant \varepsilon A_1^2 \mathrm{e}^{-2A_2\mu_j t}, \qquad \text{(B.12)}$$

$$2\left[\frac{\lambda^\varepsilon_{j+}\mathrm{e}^{\lambda^\varepsilon_{j+}t}-\lambda^\varepsilon_{j-}\mathrm{e}^{\lambda^\varepsilon_{j-}t}}{\lambda^\varepsilon_{j-}-\lambda^\varepsilon_{j+}}\right]^2 \leqslant A_1^2\mathrm{e}^{-2A_2\mu_j t}, \tag{B.13}$$

for all $j=k+1,\dots,n(\varepsilon)$

We now prove these inequalities.

Inequality (B.10).

It is equivalent to

$$\left[\frac{\lambda^\varepsilon_{j-}\mathrm{e}^{(\lambda^\varepsilon_{j+}+A_2\mu_j)t}-\lambda^\varepsilon_{j+}\mathrm{e}^{(\lambda^\varepsilon_{j-}+A_2\mu_j)t}}{\lambda^\varepsilon_{j-}-\lambda^\varepsilon_{j+}}\right]^2 \leqslant \frac{A_1^2}{2}. \tag{B.14}$$

We set

$$f(t)=\frac{\lambda^\varepsilon_{j-}\mathrm{e}^{(\lambda^\varepsilon_{j+}+A_2\mu_j)t}-\lambda^\varepsilon_{j+}\mathrm{e}^{(\lambda^\varepsilon_{j-}+A_2\mu_j)t}}{\lambda^\varepsilon_{j-}-\lambda^\varepsilon_{j+}}. \tag{B.15}$$

This function f satisfies the following properties:

- $f(0)=1$ and $f(t)\geqslant 0$ for all $t\geqslant 0$; indeed

$$f(t)\geqslant \mathrm{e}^{(\lambda^\varepsilon_{j+}+A_2\mu_j)t}\frac{(\lambda^\varepsilon_{j-}\lambda^\varepsilon_{j+})}{\lambda^\varepsilon_{j-}-\lambda^\varepsilon_{j+}}=\mathrm{e}^{(\lambda^\varepsilon_{j+}+A_2\mu_j)t}\geqslant 0.$$

- Obviously, $f(t)\to 0$ as $t\to\infty$ whenever $\lambda^\varepsilon_{j+}+A_2\mu_j<0$. According to (2.8) this condition holds with $A_2=1$.
- f achieves its maximum at the point

$$t_0=\log\left(\frac{\lambda^\varepsilon_{j+}(\lambda^\varepsilon_{j-}+A_2\mu_j)}{\lambda^\varepsilon_{j-}(\lambda^\varepsilon_{j+}+A_2\mu_j)}\right)\Big/\left(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-}\right), \tag{B.16}$$

  so that

$$f(t)\leqslant f(t_0)\leqslant 2/(2-A_2). \tag{B.17}$$

Assuming for the moment that (B.16) and (B.17) hold we deduce that a sufficient condition for (B.14) to hold is to take $A_2=1$, as above, and $A_1\geqslant 2\sqrt{2}$.

Let us now check (B.16) and (B.17).

It is easy to see that t_0 as in (B.16) is the unique critical point of f. On the other hand:

$$\begin{aligned}
f(t_0)&=\left(\lambda^\varepsilon_{j-}-\lambda^\varepsilon_{j+}\right)^{-1}\left\{\lambda^\varepsilon_{j-}\left[\frac{\lambda^\varepsilon_{j+}(\lambda^\varepsilon_{j-}+A_2\mu_j)}{\lambda^\varepsilon_{j-}(\lambda^\varepsilon_{j+}+A_2\mu_j)}\right]^{(\lambda^\varepsilon_{j+}+A_2\mu_j)/(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-})}\right.\\
&\qquad\left.-\lambda^\varepsilon_{j+}\left[\frac{\lambda^\varepsilon_{j+}(\lambda^\varepsilon_{j-}+A_2\mu_j)}{\lambda^\varepsilon_{j-}(\lambda^\varepsilon_{j+}+A_2\mu_j)}\right]^{(\lambda^\varepsilon_{j-}+A_2\mu_j)/(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-})}\right\}\\
&=\left[\frac{\lambda^\varepsilon_{j+}(\lambda^\varepsilon_{j-}+A_2\mu_j)}{\lambda^\varepsilon_{j-}(\lambda^\varepsilon_{j+}+A_2\mu_j)}\right]^{(\lambda^\varepsilon_{j+}+A_2\mu_j)/(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-})}\left[\frac{\lambda^\varepsilon_{j-}-\lambda^\varepsilon_{j+}\left(\frac{\lambda^\varepsilon_{j+}(\lambda^\varepsilon_{j-}+A_2\mu_j)}{\lambda^\varepsilon_{j-}(\lambda^\varepsilon_{j-}+A_2\mu_j)}\right)^{-1}}{\lambda^\varepsilon_{j-}-\lambda^\varepsilon_{j+}}\right].
\end{aligned}$$

We have, in view of (2.8),

$$\big(\lambda^{\varepsilon}_{j+} + A_2\mu_j\big)/\big(\lambda^{\varepsilon}_{j+} - \lambda^{\varepsilon}_{j-}\big) \leqslant 0$$

and

$$\frac{\lambda^{\varepsilon}_{j+}(\lambda^{\varepsilon}_{j-} + A_2\mu_j)}{\lambda^{\varepsilon}_{j-}(\lambda^{\varepsilon}_{j+} + A_2\mu_j)} \geqslant \frac{\lambda^{\varepsilon}_{j-}(\lambda^{\varepsilon}_{j+} + A_2\mu_j)}{\lambda^{\varepsilon}_{j-}(\lambda^{\varepsilon}_{j+} + A_2\mu_j)} = 1.$$

Therefore

$$\begin{aligned} f(t_0) &\leqslant \frac{\lambda^{\varepsilon}_{j-} - \lambda^{\varepsilon}_{j+}\big(\frac{\lambda^{\varepsilon}_{j+}(\lambda^{\varepsilon}_{j-}+A_2\mu_j)}{\lambda^{\varepsilon}_{j-}(\lambda^{\varepsilon}_{j+}+A_2\mu_j)}\big)^{-1}}{\lambda^{\varepsilon}_{j-} - \lambda^{\varepsilon}_{j+}} \\ &= \frac{\lambda^{\varepsilon}_{j-}\lambda^{\varepsilon}_{j+}(\lambda^{\varepsilon}_{j-} + A_2\mu_j) - \lambda^{\varepsilon}_{j+}\lambda^{\varepsilon}_{j-}(\lambda^{\varepsilon}_{j+} + A_2\mu_j)}{(\lambda^{\varepsilon}_{j-} - \lambda^{\varepsilon}_{j+})\lambda^{\varepsilon}_{j+}(\lambda^{\varepsilon}_{j-} + A_2\mu_j)} \\ &= \frac{\lambda^{\varepsilon}_{j-}\lambda^{\varepsilon}_{j+}(\lambda^{\varepsilon}_{j-} - \lambda^{\varepsilon}_{j+})}{\lambda^{\varepsilon}_{j+}(\lambda^{\varepsilon}_{j-} + A_2\mu_j)(\lambda^{\varepsilon}_{j-} - \lambda^{\varepsilon}_{j+})} = \frac{\lambda^{\varepsilon}_{j-}}{\lambda^{\varepsilon}_{j-} + A_2\mu_j}. \end{aligned}$$

Taking into account that $\lambda^{\varepsilon}_{j-}\lambda^{\varepsilon}_{j+} = \mu_j/\varepsilon$ we deduce that

$$f(t) \leqslant f(t_0) \leqslant \frac{\lambda^{\varepsilon}_{j-}}{\lambda^{\varepsilon}_{j-} + A_2\varepsilon\lambda^{\varepsilon}_{j-}\lambda^{\varepsilon}_{j+}} = \frac{1}{1 + A_2\varepsilon\lambda^{\varepsilon}_{j+}} = \frac{2}{2 - A_2 + A_2\sqrt{1 - 4\varepsilon\mu_j}} \leqslant \frac{2}{2 - A_2}.$$

This completes the proof of (B.10).

Inequality (B.11).

Taking into account that $\lambda^{\varepsilon}_{j-}\lambda^{\varepsilon}_{j+} = \mu_j/\varepsilon$ we deduce that (B.11) is equivalent to

$$2\left[\sqrt{\lambda^{\varepsilon}_{j-}\lambda^{\varepsilon}_{j+}}\,\frac{e^{\lambda^{\varepsilon}_{j+}t} - e^{\lambda^{\varepsilon}_{j-}t}}{\lambda^{\varepsilon}_{j-} - \lambda^{\varepsilon}_{j+}}\right]^2 \leqslant A_1^2 e^{-2A_2\mu_j t},$$

or, equivalently,

$$\sqrt{\lambda^{\varepsilon}_{j-}\lambda^{\varepsilon}_{j+}}\,\frac{e^{(\lambda^{\varepsilon}_{j+}+A_2\mu_j)t} - e^{(\lambda^{\varepsilon}_{j-}+A_2\mu_j)t}}{\lambda^{\varepsilon}_{j+} - \lambda^{\varepsilon}_{j-}} \leqslant A_1/\sqrt{2}. \tag{B.18}$$

We now define the function

$$f(t) = \sqrt{\lambda^{\varepsilon}_{j-}\lambda^{\varepsilon}_{j+}}\,\frac{e^{(\lambda^{\varepsilon}_{j+}+A_2\mu_j)t} - e^{(\lambda^{\varepsilon}_{j-}+A_2\mu_j)t}}{\lambda^{\varepsilon}_{j+} - \lambda^{\varepsilon}_{j-}}.$$

It satisfies the following properties:

- $f(0) = 0,\ f(t) \geqslant 0$ for all $t \geqslant 0$,
- $f(t) \to 0$ as $t \to +\infty$,
- f achieves it maximum value at

$$t_0 = \log\big((\lambda^{\varepsilon}_{j-} + A_2\mu_j)\big/(\lambda^{\varepsilon}_{j+} + A_2\mu_j)\big)\big/\big(\lambda^{\varepsilon}_{j+} - \lambda^{\varepsilon}_{j-}\big) > 0 \tag{B.19}$$

and

$$f(t) \leqslant f(t_0) \leqslant 2/(2 - A_2). \tag{B.20}$$

Once (B.19)–(B.20) are proved (B.18) can be obtained immediately by a suitable choice of the constants A_1 and A_2.

Let us now check (B.19) and (B.20). The fact that the maximum is achieved in t_0 as in (B.19) is a simple computation. Moreover we have:

$$f(t_0) = \frac{\sqrt{\lambda^\varepsilon_{j-}\lambda^\varepsilon_{j+}}}{\lambda^\varepsilon_{j+} - \lambda^\varepsilon_{j-}}\left[\left(\frac{\lambda^\varepsilon_{j-} + A_2\mu_j}{\lambda^\varepsilon_{j+} + A_2\mu_j}\right)^{(\lambda^\varepsilon_{j+}+A_2\mu_j)/(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-})} - \left(\frac{\lambda^\varepsilon_{j-} + A_2\mu_j}{\lambda^\varepsilon_{j+} + A_2\mu_j}\right)^{(\lambda^\varepsilon_{j-}+A_2\mu_j)/(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-})}\right]$$
$$= \frac{\sqrt{\lambda^\varepsilon_{j-}\lambda^\varepsilon_{j+}}}{\lambda^\varepsilon_{j+} - \lambda^\varepsilon_{j-}}\left(\frac{\lambda^\varepsilon_{j-} + A_2\mu_j}{\lambda^\varepsilon_{j+} + A_2\mu_j}\right)^{(\lambda^\varepsilon_{j+}+A_2\mu_j)/(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-})}\left[1 - \frac{\lambda^\varepsilon_{j+} + A_2\mu_j}{\lambda^\varepsilon_{j-} + A_2\mu_j}\right].$$

We also have

$$\frac{\lambda^\varepsilon_{j+} + A_2\mu_j}{\lambda^\varepsilon_{j+} - \lambda^\varepsilon_{j-}} \leqslant 0, \qquad \frac{\lambda^\varepsilon_{j-} + A_2\mu_j}{\lambda^\varepsilon_{j+} + A_2\mu_j} \geqslant 1, \qquad 1 - \frac{\lambda^\varepsilon_{j+} + A_2\mu_j}{\lambda^\varepsilon_{j-} + A_2\mu_j} = \frac{\lambda^\varepsilon_{j-} - \lambda^\varepsilon_{j+}}{\lambda^\varepsilon_{j-} + A_2\mu_j}.$$

Therefore

$$f(t_0) \leqslant -\frac{\sqrt{\lambda^\varepsilon_{j-}\lambda^\varepsilon_{j+}}}{\lambda^\varepsilon_{j-} + A_2\mu_j} \leqslant \frac{|\lambda^\varepsilon_{j-}|}{|\lambda^\varepsilon_{j-}| - A_2\mu_j}.$$

Taking into account that

$$|\lambda^\varepsilon_{j-}| = \frac{1 + \sqrt{1 - 4\varepsilon\mu_j}}{2\varepsilon}$$

and since $4\varepsilon\mu_j \leqslant 1$ for the parabolic eigenvalues we are considering, we deduce that:

$$f(t_0) \leqslant \frac{1 + \sqrt{1 - 4\varepsilon\mu_j}}{1 + \sqrt{1 - 4\varepsilon\mu_j} - 2\varepsilon A_2\mu_j} \leqslant \frac{1 + \sqrt{1 - 4\varepsilon\mu_j}}{1 + \sqrt{1 - 4\varepsilon\mu_j} - A_2/2}.$$

Taking into account that the function $h(t) = t/(t-a)$ is nonincreasing for $t \geqslant 0$ provided $a > 0$, we deduce that

$$f(t_0) \leqslant \frac{1 + \sqrt{1 - 4\varepsilon\mu_j}}{1 + \sqrt{1 - 4\varepsilon\mu_j} - A_2/2} \leqslant \frac{1}{1 - A_2/2} = \frac{2}{2 - A_2}.$$

This completes the proof of (B.20) and therefore the proof of (B.11) as well.

Inequality (B.12).

Taking into account that $\lambda^\varepsilon_{j-}\lambda^\varepsilon_{j+} = \mu_j/\varepsilon$, inequality (B.12) is equivalent to

$$\sqrt{\lambda^\varepsilon_{j-}\lambda^\varepsilon_{j+}}\frac{e^{\lambda^\varepsilon_{j+}t} - e^{\lambda^\varepsilon_{j-}t}}{\lambda^\varepsilon_{j+} - \lambda^\varepsilon_{j-}} \leqslant \frac{A_1}{\sqrt{2}}e^{-A_2\mu_j t}.$$

This inequality is equivalent to (B.18).

Inequality (B.13).
This inequality is equivalent to

$$\frac{\lambda^\varepsilon_{j+}\mathrm{e}^{(\lambda^\varepsilon_{j+}+A_2\mu_j)t}-\lambda^\varepsilon_{j-}\mathrm{e}^{(\lambda^\varepsilon_{j-}+A_2\mu_j)t}}{\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-}}\leqslant A_1/\sqrt{2}. \tag{B.21}$$

We define the function

$$f(t)=\frac{\lambda^\varepsilon_{j+}\mathrm{e}^{(\lambda^\varepsilon_{j+}+A_2\mu_j)t}-\lambda^\varepsilon_{j-}\mathrm{e}^{(\lambda^\varepsilon_{j-}+A_2\mu_j)t}}{\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-}}.$$

Provided $A_2<1$, this function satisfies the following properties:

- $f(0)=1$, f vanishes at

  $$t_1=\log\bigl(\lambda^\varepsilon_{j-}/\lambda^\varepsilon_{j+}\bigr)\big/\bigl(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-}\bigr),$$

  and $f>0$ in $(0,t_1)\cup(t_1,\infty)$,
- $f(t)\to 0$ as $t\to+\infty$,
- Finally,

  $$f(t)\leqslant 2/(2-A_2),\quad \forall t\geqslant 0. \tag{B.22}$$

Obviously, once (B.22) is proved (B.21) holds by a suitable choice of the constants A_1 and A_2. Thus, let us focus on the proof of (B.22).

It is easy to check that f' vanishes only at the point:

$$t_0=\log\left[\frac{\lambda^\varepsilon_{j-}(\lambda^\varepsilon_{j-}+A_2\mu_j)}{\lambda^\varepsilon_{j+}(\lambda^\varepsilon_{j+}+A_2\mu_j)}\right]\Big/\bigl(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-}\bigr). \tag{B.23}$$

The maximum of f is achieved at this point t_0. We also have:

$$\begin{aligned} f(t_0)&=\frac{\lambda^\varepsilon_{j+}\Bigl[\frac{\lambda^\varepsilon_{j-}(\lambda^\varepsilon_{j-}+A_2\mu_j)}{\lambda^\varepsilon_{j+}(\lambda^\varepsilon_{j+}+A_2\mu_j)}\Bigr]^{(\lambda^\varepsilon_{j+}+A_2\mu_j)/(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-})}-\lambda^\varepsilon_{j-}\Bigl[\frac{\lambda^\varepsilon_{j-}(\lambda^\varepsilon_{j-}+A_2\mu_j)}{\lambda^\varepsilon_{j+}(\lambda^\varepsilon_{j+}+A_2\mu_j)}\Bigr]^{(\lambda^\varepsilon_{j-}+A_2\mu_j)/(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-})}}{\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-}}\\ &=\lambda^\varepsilon_{j+}\left[\frac{\lambda^\varepsilon_{j-}(\lambda^\varepsilon_{j-}+A_2\mu_j)}{\lambda^\varepsilon_{j+}(\lambda^\varepsilon_{j+}+A_2\mu_j)}\right]^{(\lambda^\varepsilon_{j+}+A_2\mu_j)/(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-})}\left[\frac{1-(\lambda^\varepsilon_{j+}+A_2\mu_j)/(\lambda^\varepsilon_{j-}+A_2\mu_j)}{\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-}}\right]. \end{aligned} \tag{B.24}$$

We also have

$$0\leqslant\left[\frac{\lambda^\varepsilon_{j-}(\lambda^\varepsilon_{j-}+A_2\mu_j)}{\lambda^\varepsilon_{j+}(\lambda^\varepsilon_{j+}+A_2\mu_j)}\right]^{(\lambda^\varepsilon_{j+}+A_2\mu_j)/(\lambda^\varepsilon_{j+}-\lambda^\varepsilon_{j-})}\leqslant 1.$$

Therefore

$$f(t_0)\leqslant\left|\frac{\lambda^\varepsilon_{j+}}{\lambda_{j-}+A_2\mu_j}\right|=\frac{1-\sqrt{1-4\varepsilon\mu_j}}{1+\sqrt{1-4\varepsilon\mu_j}-2\varepsilon A_2\mu_j}\leqslant\frac{1}{1-2\varepsilon A_2\mu_j},$$

and taking into account that $4\mu_j\varepsilon \leqslant 1$ we deduce that

$$f(t_0) \leqslant \frac{1}{1 - A_2/2} = \frac{2}{2 - A_2}.$$

This concludes the proof of (B.22). Consequently (B.13) and Lemma B.1 are proved. □

Appendix C. Proof of Lemma 3.2

This Appendix is devoted to prove Lemma 3.2 which provides an upper bound on the cost of controlling a finite number of parabolic components of the system.

To simplify the notation we replace $T_\ell, U^\ell, \sigma_\ell$ and the interval $(a_{\ell-1}, a_{\ell-1} + T_\ell)$ by T, U^0, k and $(0, T)$ respectively.

We claim that Lemma 3.2 is consequence of the following observability inequality for the solutions of the adjoint system (1.9).

LEMMA C.1. – *Under the assumptions of Theorem* 3.1, *there exist positive constants* $\varepsilon(\Omega)$, $C(\Omega)$ *and* C_1, $C_2 > 0$ *such that for all* $0 < \varepsilon < \varepsilon(\Omega)$ *and* $T > C(\Omega)\sqrt{\varepsilon}$ *the inequality*

$$\|\varphi(0)\|^2_{L^2(\Omega)} + \varepsilon\|\varphi_t(0)\|^2_{H^{-1}(\Omega)} \leqslant C_1 \max\left\{\frac{4}{T}, \frac{1}{\mu_1 T^2}\right\} \mathrm{e}^{C_2\sqrt{\mu_k}} \int_0^T \int_\omega \varphi^2 \,\mathrm{d}x\,\mathrm{d}t \tag{C.1}$$

holds for every solution φ *of* (1.9) *with initial data* $(\varphi^0, \varphi^1) \in (H_k^\varepsilon)'$ *and* $1 \leqslant k \leqslant k(\varepsilon)$.

Remark C.1. – (C.1) provides an observability estimate for the solutions of (1.9) with only a finite number (k) of parabolic components and null hyperbolic component. This estimate is uniform on $0 < \varepsilon < 1$. However, the observability constant grows exponentially with k.

Before getting into the proof of Lemma C.1, let us show that it implies Lemma 3.2. To do this we indicate how we may build the control $f_\varepsilon \in L^2(\omega \times (0, T))$ such that the solution of (3.1) with initial datum $U^0 = (u^0, u^1)$ satisfies

$$\pi_k^\varepsilon\big(u_\varepsilon(T), u_{\varepsilon,t}(T)\big) = 0. \tag{C.2}$$

To build the control we consider the quadratic, convex and continuous functional $J_{\varepsilon,k} : (H_k^\varepsilon)' \to$ **R** defined as:

$$J_{\varepsilon,k}\big(\varphi^0, \varphi^1\big) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \big(u^0 + \varepsilon u^1\big)\varphi(0)\,\mathrm{d}x - \varepsilon\big\langle u^0, \varphi_t(0)\big\rangle, \tag{C.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, and φ is the solution of (1.9) corresponding to the data $(\varphi^0, \varphi^1) \in (H_k^\varepsilon)'$.

According to (C.1), the functional $J_{\varepsilon,k}$ is also coercive. Therefore $J_{\varepsilon,k}$ achieves its minimum at some point $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in (H_k^\varepsilon)'$. It is easy to check that the control $f_\varepsilon = \widehat{\varphi}_\varepsilon$, where $\widehat{\varphi}_\varepsilon$ is the solution of (1.9) with the minimizer $(\widehat{\varphi}_\varepsilon^0, \widehat{\varphi}_\varepsilon^1)$ as data, is the control we are looking for.

On the other hand, taking into account that:

$$J_{\varepsilon,k}\big(\widehat{\varphi}_\varepsilon^0, \widehat{\varphi}_\varepsilon^1\big) = \min_{(\varphi^0,\varphi^1)\in(H_k^\varepsilon)'} J_{\varepsilon,k}\big(\varphi^0, \varphi^1\big) \leqslant J_{\varepsilon,k}(0,0) = 0$$

and using the bound (C.1) we obtain:

$$\int_0^T\int_\omega |\widehat{\varphi}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant 2\big[\sqrt{\varepsilon}\|u^0\|_{H_0^1(\Omega)} + \|u^0+\varepsilon u^1\|_{L^2(\Omega)}\big]\big[\|\widehat{\varphi}_\varepsilon(0)\|_{L^2(\Omega)} + \sqrt{\varepsilon}\|\widehat{\varphi}_{\varepsilon,t}(0)\|_{H^{-1}(\Omega)}\big]$$

$$\leqslant C\|U^0\|_{H^\varepsilon}\left[\max\left(\frac{4}{T},\frac{1}{\mu_1 T^2}\right)\mathrm{e}^{C_2\sqrt{\mu_k}}\int_0^T\int_\omega |\widehat{\varphi}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\right]^{1/2}. \tag{C.4}$$

From (C.4) we get

$$\|f_\varepsilon\|_{L^2(\omega\times(0,T))} \leqslant C_1\left(\max\left(\frac{4}{T},\frac{1}{\mu_1 T^2}\right)\right)^{1/2}\mathrm{e}^{C_2\sqrt{\mu_k}}\|U^0\|_{H^\varepsilon}$$

for suitable positive constants C_1 and C_2.

Therefore the proof of Lemma 3.2 is reduced to prove Lemma C.1.

Proof of Lemma C.1. – The proof of (C.1) uses in an essential way the following result due to G. Lebeau and L. Robbiano [6] (see also [7]). □

PROPOSITION C.1. – *Let Ω be a bounded domain of $\mathbf{R}^n$ of class C^∞. Let ω be an open non-empty subset of Ω. Let $\{\mu_j\}_{j\geqslant 1}$ be the eigenvalues of $-\Delta$ with Dirichlet boundary conditions and $\{e_j\}_{j\geqslant 1}$ the corresponding eigenfunctions, constituting an orthonormal basis of $L^2(\Omega)$.*

Then, there exist two positive constants $C_1, C_2 > 0$ such that

$$\sum_{\mu_j\leqslant\gamma}|a_j|^2 \leqslant C_1\mathrm{e}^{C_2\sqrt{\gamma}}\int_\omega\Big|\sum_{\mu_j\leqslant\gamma}a_je_j(x)\Big|^2\,\mathrm{d}x \tag{C.5}$$

for every finite $\gamma > 0$ and every choice of the coefficients $\{a_j\}_{\mu_j\leqslant\gamma}$.

This inequality has to be combined with the following result on the time-independent exponentials arising in the development of the solutions of (1.9) in Fourier series.

LEMMA C.2. – *There exist two positive constants $\varepsilon_0(\Omega)$, $C(\Omega)$ such that for all $0<\varepsilon<\varepsilon_0(\Omega)$ and $T > C(\Omega)\sqrt{\varepsilon}$,*

$$\big|A\mathrm{e}^{\lambda^\varepsilon_{j+}T} + B\mathrm{e}^{\lambda^\varepsilon_{j-}T}\big|^2 + \frac{1}{\lambda^\varepsilon_{j-}\lambda^\varepsilon_{j+}}\big|A\lambda^\varepsilon_{j+}\mathrm{e}^{\lambda^\varepsilon_{j+}T} + B\lambda^\varepsilon_{j-}\mathrm{e}^{\lambda^\varepsilon_{j-}T}\big|^2$$

$$\leqslant 2\max\left\{\frac{4}{T},\frac{1}{\mu_1T^2}\right\}\int_0^T\big|A\mathrm{e}^{\lambda^\varepsilon_{j+}(T-t)} + B\mathrm{e}^{\lambda^\varepsilon_{j-}(T-t)}\big|^2\,\mathrm{d}t \tag{C.6}$$

for all $j = 1,\dots,k(\varepsilon)$ and $A, B\in\mathbf{R}$.

Assuming for the moment that Lemma C.2 holds let us complete the proof of (C.1) and therefore of Lemma C.1 as well. We shall return later to the proof of Lemma C.2.

Recall that

$$\lambda^\varepsilon_{j+}\lambda^\varepsilon_{j-} = \frac{\mu_j}{\varepsilon}.$$

Therefore, for any $1\leqslant k\leqslant k(\varepsilon)$ it follows that:

$$\sum_{j=1}^{k}\Big[\big|A_j e^{\lambda^{\varepsilon}_{j+}T}+Be^{\lambda^{\varepsilon}_{j-}T}\big|^2+\frac{\varepsilon}{\mu_j}\big|A_j\lambda^{\varepsilon}_{j+}e^{\lambda^{\varepsilon}_{j+}T}+B_j\lambda^{\varepsilon}_{j-}e^{\lambda^{\varepsilon}_{j-}T}\big|^2\Big]$$

$$\leqslant 2\max\Big\{\frac{4}{T},\frac{1}{\mu_1T^2}\Big\}\int_0^T\sum_{j=1}^{k}\big|A_j e^{\lambda^{\varepsilon}_{j+}(T-t)}+B_j e^{\lambda^{\varepsilon}_{j-}(T-t)}\big|^2\,dt \tag{C.7}$$

for all $A_j, B_j\in\mathbf{R}$, $j=1,\dots,k$.

Combining (C.5) and (C.7) we deduce that:

$$\sum_{j=1}^{k}\Big[\big|A_j e^{\lambda^{\varepsilon}_{j+}T}+B_je^{\lambda^{\varepsilon}_{j-}T}\big|^2+\frac{\varepsilon}{\mu_j}\big|A_j\lambda^{\varepsilon}_{j+}e^{\lambda^{\varepsilon}_{j+}t}+B_j\lambda^{\varepsilon}_{j-}e^{\lambda^{\varepsilon}_{j-}t}\big|^2\Big]$$

$$\leqslant 2C_1\max\Big\{\frac{4}{T},\frac{1}{\mu_1T^2}\Big\}e^{C_2\sqrt{\mu_k}}\int_0^T\int_\omega\Big|\sum_{j=1}^{k}\big(A_j e^{\lambda^{\varepsilon}_{j+}(T-t)}+B_j e^{\lambda^{\varepsilon}_{j-}(T-t)}\big)e_j(x)\Big|^2\,dx\,dt \tag{C.8}$$

for all $A_j, B_j\in\mathbf{R}$, $j=1,\dots,k$.

Applying (C.8) with

$$A_j=\frac{a_j\lambda^{\varepsilon}_{j-}+b_j}{\lambda^{\varepsilon}_{j-}-\lambda^{\varepsilon}_{j+}},\qquad B_j=\frac{-b_j-a_j\lambda^{\varepsilon}_{j+}}{\lambda^{\varepsilon}_{j-}-\lambda^{\varepsilon}_{j+}}$$

and taking into account that the solution φ of (1.9) with initial data

$$\varphi^0=\sum_{j=1}^{k}a_je_j,\qquad \varphi^1=\sum_{k=1}^{k}b_je_j$$

may be developed in Fourier series as

$$\varphi=\sum_{j=1}^{k}\Big[\frac{a_j\lambda^{\varepsilon}_{j-}+b_j}{\lambda^{\varepsilon}_{j-}-\lambda^{\varepsilon}_{j+}}e^{\lambda^{\varepsilon}_{j+}(T-t)}-\frac{b_j+a_j\lambda^{\varepsilon}_{j+}}{\lambda^{\varepsilon}_{j-}-\lambda^{\varepsilon}_{j+}}e^{\lambda^{\varepsilon}_{j-}(T-t)}\Big]e_j(x)$$

we deduce that (C.1) holds.

Therefore, the proof of Lemma C.1 is reduced to prove Lemma C.2.

Proof of Lemma C.2. – We proceed in several steps.

Step 1. First of all we prove the following inequality:

$$\big|Ae^{\lambda^{\varepsilon}_{j+}T}+Be^{\lambda^{\varepsilon}_{j-}T}\big|^2\leqslant\frac{4}{T}\int_0^T\big|Ae^{\lambda^{\varepsilon}_{j+}(T-t)}+Be^{\lambda^{\varepsilon}_{j-}(T-t)}\big|^2\,dt, \tag{C.9}$$

which holds for all $T>0$.

To prove (C.9) we consider the function

$$f(x)=\frac{1-e^{-x}}{x}. \tag{C.10}$$

Then, (C.9) is equivalent to

$$\begin{aligned} &\left|A\mathrm{e}^{\lambda^{\varepsilon}_{j+}T}+B\mathrm{e}^{\lambda^{\varepsilon}_{j-}T}\right|^2\\ &\quad\leqslant 4\left[A^2f\left(-2\lambda^{\varepsilon}_{j+}T\right)+B^2f\left(-2\lambda^{\varepsilon}_{j-}T\right)+2ABf\left(-\left(\lambda^{\varepsilon}_{j+}+\lambda^{\varepsilon}_{j-}\right)T\right)\right]. \end{aligned}\tag{C.11}$$

From now on, to simplify the notation, we denote $\lambda^{\varepsilon}_{j+}$ and $\lambda^{\varepsilon}_{j-}$ by λ_+ and λ_- respectively.

Obviously, (C.11) is equivalent to

$$\begin{aligned} &A^2\left[4f(-2\lambda_+T)-\mathrm{e}^{2\lambda_+T}\right]+B^2\left[4f(-2\lambda_-T)-\mathrm{e}^{2\lambda_-T}\right]\\ &\quad+2AB\left[4f\left(-(\lambda_++\lambda_-)T\right)-\mathrm{e}^{(\lambda_++\lambda_-)T}\right]\geqslant 0. \end{aligned}\tag{C.12}$$

To analyze (C.12) we introduce the function

$$g(x)=4f(x)-\mathrm{e}^{-x}.$$

Obviously $g(x)\geqslant 0$ for all $x\geqslant 0$.

Then, inequality (C.12) may be rewritten as:

$$A^2g(-2\lambda_+T)+B^2g(-2\lambda_-T)+2ABg\left(-(\lambda_++\lambda_-)T\right)\geqslant 0.\tag{C.13}$$

For any $\eta>0$ we have:

$$\begin{aligned} &A^2\left(g(-2\lambda_+T)-\eta g\left(-(\lambda_++\lambda_-)T\right)\right)+B^2\left(g(-2\lambda_-T)-g\left(-(\lambda_-+\lambda_+)T\right)/\eta\right)\\ &\quad\leqslant A^2g(-2\lambda_+T)+B^2g(-2\lambda_-T)+2ABg(-(\lambda_++\lambda_-)T). \end{aligned}\tag{C.14}$$

Thus, in order to show (C.13) it is sufficient to prove the existence of a positive constant $\eta>0$ such that $g(-2\lambda_+T)-\eta g(-(\lambda_-+\lambda_+)T)\geqslant 0$ and $g(-2\lambda_-T)-g(-(\lambda_-+\lambda_+)T)/\eta\geqslant 0$, or, in other words,

$$\frac{g(-(\lambda_++\lambda_-)T)}{g(-2\lambda_-T)}\leqslant\eta\leqslant\frac{g(-2\lambda_+T)}{g(-(\lambda_-+\lambda_+)T)}.\tag{C.15}$$

Thus, the problem is reduced to show that

$$\left(g\left(-(\lambda_++\lambda_-)T\right)\right)^2\leqslant g(-2\lambda_+T)g(-2\lambda_-T),\tag{C.16}$$

or, equivalently,

$$2\log\left(g\left(-(\lambda_++\lambda_-)T\right)\right)\leqslant\log\left(g(-2\lambda_+T)\right)+\log\left(g(-2\lambda_-T)\right).\tag{C.17}$$

Clearly, in order to show that (C.17) holds it is sufficient to prove that the function $\log(g(x))$ is convex.

Step 2. We now prove the convexity of $\log(g(x))$.

We have

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\log\left(g(x)\right)\right)=\frac{g(x)g''(x)-(g'(x))^2}{g^2(x)}.$$

Therefore, the problem is reduced to prove that:

$$g(x)g''(x)-\left(g'(x)\right)^2\geqslant 0.\tag{C.18}$$

Moreover:

$$g'(x)=\frac{x^2+4x+4}{x^2}\mathrm{e}^{-x}-\frac{4}{x^2};$$
$$g''(x)=\frac{8}{x^3}-\left(\frac{x^3+4x^2+8x+8}{x^3}\right)\mathrm{e}^{-x}.$$

Therefore

$$g(x)g''(x)-\big(g'(x)\big)^2=\frac{16}{x^4}-4\mathrm{e}^{-x}\frac{(x^3+2x^2+2x+8)}{x^4}+4\mathrm{e}^{-2x}\frac{(2x+4)}{x^4},$$

and it is sufficient to check that

$$16-4\mathrm{e}^{-x}\big(x^3+2x^2+2x+8\big)+4\mathrm{e}^{-2x}(2x+4)\geqslant 0. \tag{C.19}$$

Inequality (C.19) holds since

$$\begin{aligned}
&4\mathrm{e}^{-x}\big[4\big[\mathrm{e}^x-x^2-2+\mathrm{e}^{-x}\big]+\big[-x^3+2x^2-2x+2x\mathrm{e}^{-x}\big]\big]\\
&\quad=4\mathrm{e}^{-x}\big[4\big(2\cosh x-\big(x^2+2\big)\big)+\big(-x^3+2x^2-2x+2x\mathrm{e}^{-x}\big)\big]\\
&\quad=4\mathrm{e}^{-x}\left[8\sum_{j\geqslant 1}\frac{x^{2j}}{(2j)!}-4x^2-8-x^3+2x^2-2x+2x\sum_{j\geqslant 0}(-1)\frac{jx^j}{j!}\right)\\
&\quad=4\mathrm{e}^{-x}\left[8\sum_{j\geqslant 3}\frac{x^{2j}}{(2j)!}+2x\sum_{j\geqslant 4}(-1)\frac{jx^j}{j!}\right]\geqslant 0,\quad \forall x\geqslant 0.
\end{aligned}$$

This completes the proof of the convexity of $\log(g(x))$ and that of (C.9).

Step 3. We now prove the inequality:

$$\big|A\lambda_+\mathrm{e}^{\lambda_+T}+B\lambda_-\mathrm{e}^{\lambda_-T}\big|^2\leqslant\lambda_+\lambda_-\max\left(\frac{1}{T},\frac{1}{\mu_1T^2}\right)\int_0^T\big|A\mathrm{e}^{\lambda_+t}+B\mathrm{e}^{\lambda_-t}\big|^2\,\mathrm{d}t, \tag{C.20}$$

provided $T>C(\Omega)\sqrt{\varepsilon}$ for all $0<\varepsilon<\varepsilon(\Omega)$.

Obviously, (C.6) is a consequence of (C.9) and (C.20).

Using the functions f and g introduced in Step 1 above, (C.20) may be rewritten as

$$\begin{aligned}
&A^2\big(\tau\lambda_+\lambda_-f(-2\lambda_+T)-(\lambda_+)^2\mathrm{e}^{2\lambda_+T}\big)+B^2\big(\tau\lambda_+\lambda_-f(-2\lambda_-T)-(\lambda_-)^2\mathrm{e}^{2\lambda_-T}\big)\\
&\quad+2AB\big(\tau\lambda_+\lambda_-f\big(-(\lambda_++\lambda_-)T\big)-\lambda_+\lambda_-\mathrm{e}^{(\lambda_++\lambda_-)T}\big)\geqslant 0,
\end{aligned}\tag{C.21}$$

with $\tau=\max(1,1/(\mu_1T))$. Taking into account that $\lambda_+\lambda_-=\mu_j/\varepsilon$, this is equivalent to

$$\begin{aligned}
&A^2\left(\tau f(-2\lambda_+T)-\frac{\lambda_+}{\lambda_-}\mathrm{e}^{2\lambda_+T}\right)+B^2\left(\tau f(-2\lambda_-T)-\frac{\lambda_-}{\lambda_+}\mathrm{e}^{2\lambda_-T}\right)\\
&\quad+2AB\big(\tau f\big(-(\lambda_++\lambda_-)T\big)-\mathrm{e}^{(\lambda_++\lambda_-)T}\big)\geqslant 0.
\end{aligned}\tag{C.22}$$

We now check that the numbers multiplying A^2, B^2 and AB in (C.22) are all positive. More precisely:

$$\tau f(-2\lambda_+T)-\frac{\lambda_+}{\lambda_-}\mathrm{e}^{2\lambda_+T}\geqslant 0; \tag{C.23}$$

$$\tau f(-2\lambda_- T) - \frac{\lambda_-}{\lambda_+} e^{2\lambda_- T} \geqslant 0; \tag{C.24}$$

$$\tau f\big(-(\lambda_+ + \lambda_-)T\big) - e^{(\lambda_+ + \lambda_-)T} \geqslant 0. \tag{C.25}$$

Proof of (C.23). – Obviously

$$\tau f(-2\lambda_+ T) - \frac{\lambda_+}{\lambda_-} e^{2\lambda_+ T} \geqslant \tau f(-2\lambda_+ T) - e^{2\lambda_+ T} = g_\tau(-2\lambda_+ T)$$

with $g_\tau(x) = \tau f(x) - e^{-x}$. It is easy to check that $g_\tau(x) \geqslant 0$ for all $x \geqslant 0$ and $\tau \geqslant 1$. This completes the proof of (C.23).

The same argument shows that (C.25) is also true.

Proof of (C.24). – Inequality (C.24) is equivalent to

$$\tau \geqslant 2 \frac{\lambda_-}{\lambda_+} \frac{|\lambda_-| T e^{2\lambda_- T}}{(1 - e^{2\lambda_- T})}. \tag{C.26}$$

According to (2.8), $-\lambda_+ > \mu_j > \mu_1$. Consequently the following is a sufficient condition for (C.26) to hold:

$$\tau T \geqslant \frac{2|\lambda_- T|^2}{\mu_1} \frac{e^{2\lambda_- T}}{(1 - e^{2\lambda_- T})} = \frac{1}{\mu_1} m(-2\lambda_- T) \tag{C.27}$$

with

$$m(x) = \frac{x^2}{2} \frac{e^{-x}}{1 - e^{-x}}. \tag{C.28}$$

The function m in (C.28) is bounded above by 1 for all $x \geqslant 0$.

Thus, it is sufficient to guarantee that

$$\tau T \geqslant \frac{1}{\mu_1} \Leftrightarrow \tau \geqslant \frac{1}{\mu_1 T}, \tag{C.29}$$

but this is obviously true by the choice of τ we have done: $\tau = \max(1, /(\mu_1 T))$.

This completes the proof of (C.24).

Let us go back now to (C.22).

We set:

$$\delta_+ = \frac{\lambda_- - \lambda_+}{\lambda_-}, \qquad \delta_- = \frac{\lambda_- - \lambda_+}{\lambda_+}.$$

Observe that $\delta_-, \delta_+ > 0$. With this notation, the quantity in (C.22) may be rewritten and bounded below as follows:

$$\begin{aligned}
&A^2\big(g_\tau(-2\lambda_+ T) + \delta_+ e^{2\lambda_+ T}\big) + B^2\big(g_\tau(-2\lambda_- T) - \delta_- e^{2\lambda_- T}\big) + 2ABg_\tau\big(-(\lambda_- + \lambda_+)T\big) \\
&\quad \geqslant A^2\big[g_\tau(-2\lambda_+ T) + \delta_+ e^{2\lambda_+ T} - \eta g_\tau\big(-(\lambda_+ + \lambda_-)T\big)\big] \\
&\qquad + B^2\left[g_\tau(-2\lambda_- T) - \delta_- e^{2\lambda_- T} - \frac{1}{\eta} g_\tau\big(-(\lambda_+ + \lambda_-)T\big)\right]
\end{aligned}$$

for any $\eta > 0$.

Thus, it is sufficient to check whether

$$\frac{g_\tau(-(\lambda_+ + \lambda_-)T)}{g_\tau(-2\lambda_- T) - \delta_- e^{2\lambda_- T}} \leqslant \frac{g_\tau(-2\lambda_+ T) + \delta_+ e^{2\lambda_+ T}}{g_\tau(-(\lambda_+ + \lambda_-)T)}. \tag{C.30}$$

In view of (C.23)–(C.24) and taking into account that $g_\tau(x) \geqslant 0$ for all $x \geqslant 0$, we deduce that each of the four quantities involved in (C.20) are nonnegative.

Therefore, (C.30) is equivalent to

$$\begin{aligned}\big[g\big(-(\lambda_+ + \lambda_-)T\big)\big]^2 \leqslant{}& g_\tau(-2\lambda_+ T) g_\tau(-2\lambda_- T) - \delta_- \delta_+ e^{2(\lambda_+ + \lambda_-)T} \\ & - \delta_- e^{2\lambda_- T} g_\tau(-2\lambda_+ T) + \delta_+ e^{2\lambda_+ T} g_\tau(-2\lambda_- T).\end{aligned} \tag{C.31}$$

The same proof as for (C.16) shows that

$$\big[g\big(-(\lambda_- + \lambda_+)T\big)\big]^2 \leqslant g(-2\lambda_+ T) g(-2\lambda_- T) \tag{C.32}$$

for any $\tau \geqslant 1$. Then, in order to prove (C.31) it is sufficient to show that the following holds:

$$\delta_+ e^{2\lambda_+ T} g_\tau(-2\lambda_- T) - \delta_- \delta_+ e^{2(\lambda_+ + \lambda_-)T} - \delta_- e^{2\lambda_- T} g_\tau(-2\lambda_+ T) \geqslant 0. \tag{C.33}$$

The left hand side in (C.33) may be rewritten as:

$$\tau \delta_+ e^{2\lambda_+ T} f(-2\lambda_- T) + e^{(\lambda_+ + \lambda_-)T}[\delta_- - \delta_+ - \delta_- \delta_+] - \tau \delta_- e^{2\lambda_- T} f(-2\lambda_+ T). \tag{C.34}$$

The second term on the right hand side of (C.34) vanishes since $\delta_- - \delta_+ - \delta_- \delta_+ = 0$. Therefore, (C.33) is equivalent to

$$\delta_+ e^{2\lambda_+ T} f(-2\lambda_- T) - \delta_- e^{2\lambda_- T} f(-2\lambda_+ T) \geqslant 0. \tag{C.35}$$

Inequality (C.35) may be rewritten as

$$h(-2\lambda_- T) - h(-2\lambda_+ T) \geqslant 0 \tag{C.36}$$

where

$$h(x) = \frac{e^x - 1}{x^2}. \tag{C.37}$$

It is easy to check that h satisfies the following properties:

- h is strictly convex for $x > 0$,
- h achieves its minimum on a single point $x_0 > 0$, it is strictly decreasing (resp. increasing) in $(0, x_0)$ (resp. (x_0, ∞)),
- $h(x) \geqslant x$ for $x \geqslant x_1$ with $x_1 \geqslant x_0$ large enough.

Let us also recall that the eigenvalues $\lambda^\varepsilon_{j\pm}$ are ordered in the following way:

$$\lambda^\varepsilon_{1_-} < \cdots < \lambda^\varepsilon_{k(\varepsilon)_-} < -\frac{1}{2\varepsilon} < \lambda^\varepsilon_{k(\varepsilon)_+} < \cdots < \lambda^\varepsilon_{1_+}. \tag{C.38}$$

Consequently

$$-2\lambda^\varepsilon_{1_-} T > \cdots > -2\lambda^\varepsilon_{k(\varepsilon)_-} T > \frac{T}{\varepsilon} > -2\lambda_{k(\varepsilon)_+} T > \cdots > -2\lambda^\varepsilon_{1_+} T. \tag{C.39}$$

According to this and the properties of h mentioned above we see that (C.36) holds immediately for all $j=1,\dots,k(\varepsilon)$ provided $-2\lambda^{\varepsilon}_{1_+}T \geqslant x_0$.

Let us now analyze the case where

$$-2\lambda^{\varepsilon}_{1_+}T < x_0. \tag{C.40}$$

We shall suppose that

$$\frac{T}{\varepsilon} \geqslant x_1, \tag{C.41}$$

or, in other words, $\varepsilon \leqslant T/x_1$.

In view of (C.39), (C.41) and taking into account that h is increasing in (x_0,∞) it is then sufficient to prove that

$$h\left(\frac{T}{\varepsilon}\right) \geqslant h\big(-2\lambda^{\varepsilon}_{1_+}T\big). \tag{C.42}$$

Indeed, once (C.42) holds, (C.36) holds as well since

$$h\big(-2\lambda^{\varepsilon}_{j_-}T\big) \geqslant h(T/\varepsilon), \quad \forall j=1,\dots,k(\varepsilon),$$

and

$$h\big(-2\lambda^{\varepsilon}_{j_+}T\big) \leqslant \max\big(h\big(-2\lambda^{\varepsilon}_{1_+}T\big),\, h(T/\varepsilon)\big), \quad \forall j=1,\dots,k(\varepsilon).$$

Let us now check (C.42). Taking into account that $h(T/\varepsilon) \geqslant T/\varepsilon$ (since, by assumption, $T/\varepsilon \geqslant x_1$) and $-\lambda^{\varepsilon}_{1_+} > \mu_1$ and therefore, $h(-2\lambda^{\varepsilon}_{1_-}T) < h(2\mu_1 T)$, it is sufficient to prove that

$$T/\varepsilon \geqslant h(2\mu_1 T), \tag{C.43}$$

which is equivalent to

$$\frac{T^2}{\varepsilon} \geqslant \frac{1}{2\mu_1}\left(\frac{\mathrm{e}^{2\mu_1 T}-1}{2\mu_1 T}\right). \tag{C.44}$$

But, recall, we are dealing with the case where

$$-2\lambda^{\varepsilon}_{1_+}T < x_0 \Leftrightarrow T < -x_0/2\lambda^{\varepsilon}_{1_+} < x_0/2\mu_1.$$

Thus, taking into account that the function $(\mathrm{e}^x-1)/x$ is increasing for $x \geqslant 0$, it is sufficient to prove that

$$T^2/\varepsilon \geqslant \frac{1}{2\mu_1}\left(\frac{\mathrm{e}^{x_0}-1}{x_0}\right). \tag{C.45}$$

Thus, inequality (C.20) is proved under the condition (C.45). Note that, for $\varepsilon > 0$ fixed, (C.45) may be viewed as a lower bound on the control time T. Obviously (C.45) implies (C.41) provided $\varepsilon > 0$ is taken sufficiently small. Note that (C.45) provides the value of the constant $C(\Omega)$ in the statement of Lemma C.1 and C.2.

This completes the proof of Lemma C.2 and therefore that of Lemma 3.2 as well, which was the goal of this Appendix. □

Appendix D. Convergence on the uncontrolled system

The goal of this section is to prove a convergence result of the solutions of:

$$\begin{cases} \varepsilon\varphi_{tt} - \Delta\varphi - \varphi_t = 0 & \text{in } \Omega \times (0,T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0,T), \\ \varphi(T) = \varphi^0,\ \varphi_t(T) = \varphi^1 & \text{in } \Omega \end{cases} \tag{D.1}$$

towards the solutions of:

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } \Omega \times (0,T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0,T), \\ \varphi(T) = \varphi^0 & \text{in } \Omega \end{cases} \tag{D.2}$$

that has been used in Section 7.

The following holds:

LEMMA D.1. – *Let $\varphi^0 \in L^2(\Omega)$ and $\varphi^1 \in H^{-1}(\Omega)$ be fixed. Then the solutions φ_ε of* (D.1) *satisfy, as $\varepsilon \to 0$:*

$$\varphi_\varepsilon \to \varphi \quad \textit{strongly in } L^2\big(\Omega \times (0,T)\big), \tag{D.3}$$

$$\varphi_\varepsilon(0) \to \varphi(0) \quad \textit{strongly in } L^2(\Omega), \tag{D.4}$$

$$\sqrt{\varepsilon}\varphi_{\varepsilon,t}(0) \to 0 \quad \textit{strongly in } H^{-1}(\Omega), \tag{D.5}$$

where φ is the solution of (D.2).

Proof. – The energy

$$F_\varepsilon(t) = \frac{\varepsilon}{2}\|\varphi_t(t)\|^2_{H^{-1}(\Omega)} + \frac{1}{2}\|\varphi(t)\|^2_{L^2(\Omega)} \tag{D.6}$$

satisfies

$$\frac{\mathrm{d}F_\varepsilon(t)}{\mathrm{d}t} = \|\varphi_t(t)\|^2_{H^{-1}(\Omega)}. \tag{D.7}$$

Therefore

$$F_\varepsilon(t) = F_\varepsilon(T) - \int_t^T \|\varphi_{\varepsilon,t}(s)\|^2_{H^{-1}(\Omega)}\,\mathrm{d}s. \tag{D.8}$$

Taking into account that

$$F_\varepsilon(T) = \frac{\varepsilon}{2}\|\varphi^1\|^2_{H^{-1}(\Omega)} + \frac{1}{2}\|\varphi^0\|^2_{L^2(\Omega)} \to \frac{1}{2}\|\varphi^0\|^2_{L^2(\Omega)} \tag{D.9}$$

we deduce that

$$F_\varepsilon \text{ is bounded in } L^\infty(0,T). \tag{D.10}$$

Consequently

$$\varphi_\varepsilon(t) \text{ is bounded in } L^\infty\big(0,T;L^2(\Omega)\big) \tag{D.11}$$

and

$$\sqrt{\varepsilon}\varphi_{\varepsilon,t}(t) \text{ is bounded in } L^\infty\big(0,T;H^{-1}(\Omega)\big). \tag{D.12}$$

In view of (D.8) we also have

$$\int_0^T \|\varphi_{\varepsilon,t}(t)\|^2_{H^{-1}(\Omega)}\,\mathrm{d}t = F_\varepsilon(T) - F_\varepsilon(0) \leqslant F_\varepsilon(T). \tag{D.13}$$

Therefore

$$\varphi_{\varepsilon,t} \text{ is bounded in } L^2\big(0,T;H^{-1}(\Omega)\big). \tag{D.14}$$

By extracting subsequences (that we still denote by the index ε), we deduce that

$$\varphi_\varepsilon \rightharpoonup \varphi \quad \text{weakly in } L^2\big(\Omega\times(0,T)\big), \tag{D.15}$$

$$\varphi_{\varepsilon,t} \rightharpoonup \varphi_t \quad \text{weakly in } L^2\big(0,T;H^{-1}(\Omega)\big). \tag{D.16}$$

It is easy to check that φ is a weak solution of (D.2). In view of the uniqueness of the solution of (D.2) we deduce that the convergences in (D.15)–(D.16) hold for the whole sequence φ_ε.

In view of the boundedness of $(\varphi_\varepsilon(0), \sqrt{\varepsilon}\varphi_{\varepsilon,t}(0))$ in $L^2(\Omega)\times H^{-1}(\Omega)$, by extracting subsequences we also have that, for a suitable subsequence,

$$(\varphi_\varepsilon(0), \sqrt{\varepsilon}\varphi_{\varepsilon,t}(0)) \rightharpoonup (\xi,\eta) \quad \text{weakly in } L^2(\Omega)\times H^{-1}(\Omega). \tag{D.17}$$

Combining (D.10) and (D.13) and Aubin–Lions' compactness lemma (see for instance Corollary 4 in J. Simon [11]) we have:

$$\varphi_\varepsilon \to \varphi \quad \text{in } C\big([0,T];\ H^{-1}(\Omega)\big) \text{ strongly}. \tag{D.18}$$

As a consequence of (D.17)–(D.18) we deduce that

$$\xi = \varphi(0). \tag{D.19}$$

Passing to the limit in the identity (D.13), using (D.9), (D.14), (D.17) and (D.19) and the lower weak semicontinuity of the norms we deduce that:

$$\frac{1}{2}\|\varphi(0)\|^2_{L^2(\Omega)} + \int_0^T \|\varphi_t(t)\|^2_{H^{-1}(\Omega)}\,\mathrm{d}t \leqslant \frac{1}{2}\|\varphi^0\|^2_{L^2(\Omega)}, \tag{D.20}$$

and, also:

$$\frac{1}{2}\|\varphi(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\eta\|^2_{H^{-1}(\Omega)} \leqslant \liminf_{\varepsilon\to 0} F_\varepsilon(0), \tag{D.21}$$

$$\int_0^T \left\|\varphi_t(t)\right\|^2_{H^{-1}(\Omega)}\,\mathrm{d}t \leqslant \liminf_{\varepsilon\to 0}\int_0^T \left\|\varphi_t(t)\right\|^2_{H^{-1}(\Omega)}\,\mathrm{d}t. \tag{D.22}$$

The energy law for the limit system (D.2) guarantees that

$$\frac{1}{2}\left\|\varphi(0)\right\|^2_{L^2(\Omega)} + \int_0^T \left\|\varphi_t(t)\right\|^2_{H^{-1}(\Omega)}\,\mathrm{d}t = \frac{1}{2}\left\|\varphi^0\right\|^2_{L^2(\Omega)}. \tag{D.23}$$

Then, in view of (D.20)–(D.23) we deduce that:

$$\int_0^T \left\|\varphi_{\varepsilon,t}(t)\right\|^2_{H^{-1}(\Omega)}\,\mathrm{d}t \to \int_0^T \left\|\varphi_t(t)\right\|^2_{H^{-1}(\Omega)}\,\mathrm{d}t, \tag{D.24}$$

$$\left\|\varphi_\varepsilon(0)\right\|_{L^2(\Omega)} \to \left\|\varphi(0)\right\|_{L^2(\Omega)}, \tag{D.25}$$

$$\varepsilon\left\|\varphi_{\varepsilon,t}(0)\right\|^2_{H^{-1}(\Omega)} \to 0. \tag{D.26}$$

Combining (D.16), (D.17), (D.19), (D.24)–(D.26), we immediately deduce the strong convergences (D.4)–(D.5).

The same argument shows that

$$\varphi_\varepsilon(t) \to \varphi(t) \quad \text{strongly in } L^2(\Omega) \text{ as } \varepsilon \to 0 \tag{D.27}$$

for all $0 \leqslant t \leqslant T$.

To conclude the proof of (D.3) it is sufficient to show that

$$\int_0^T\int_\Omega \varphi_\varepsilon^2\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \varphi^2\,\mathrm{d}x\,\mathrm{d}t$$

but this is an immediate consequence of (D.27), of the Dominated Convergence Theorem and of the fact that

$$\frac{1}{2}\left\|\varphi_\varepsilon(t)\right\|^2_{L^2(\Omega)} \leqslant F_\varepsilon(t) \leqslant F_\varepsilon(t) \leqslant C,$$

for all $0 \leqslant t \leqslant T$ and $\varepsilon > 0$.

This completes the proof of Lemma D.1. □

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