JOURNAL OF DIFFERENTIAL EQUATIONS **108**, 1–35 (1994)

A Dynamical System Approach to the Self-Similar Large Time Behavior in Scalar Convection–Diffusion Equations

ENRIQUE ZUAZUA*

Departamento de Matemática Aplicada, Universidad Complutense, 28040 Madrid, Spain

Received May 28, 1991; revised February 19, 1992

We present a dynamical system method that provides both the existence of self-similar solutions and the self-similar large time behavior for convection–diffusion equations in $\mathbb{R}^N$. This method avoids a direct study of the elliptic problem related to the self-similar profiles.

We concentrate our attention on the model example

$$u_t - \Delta u = a \cdot \nabla(|u|^{1/N} u) + g(x, t) \qquad \text{in } \mathbb{R}^N \times (0, \infty)$$
$$u(0) = u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),$$

where $a \in \mathbb{R}^N$ and $g(x, t) = (t+1)^{-(N/2)-1} h(x/\sqrt{t+1})$ with $h \in L^2(\mathbb{R}^N; \exp(|x|^2/4)) \cap L^\infty(\mathbb{R}^N)$ such that $\int h(y)\, dy = 0$ and $\int |h(y)|\, |y|\, dy$ being small enough. In the natural similarity variables the self-similar profiles become stationary solutions of a new convection–diffusion equation. By using Lyapunov type arguments that rely in an essential manner on the L^1-contraction property of the system, we prove that those stationary solutions exists and that any trajectory converges to one of them. The limit of any trajectory is completely determined by its mass which is conserved along the time. In order to ensure the relative compactness of trajectories, we work in the functional framework of the weighted Sobolev spaces introduced by Escobedo and Kavian. © 1994 Academic Press, Inc.

1. INTRODUCTION

The goal of this paper is to present a dynamical system method for the study of the existence of similarity solutions and the large time behavior of convection–diffusion equations.

* Partially supported by project PB90-0245 of the "Dirección General de Investigación Científica y Técnica (MEC-España)" and by the Eurhomogenization project ERB4002PL910092 of the EEC.

0022-0396/94 $6.00

We focus on the model example

$$\begin{aligned} &u_t - \Delta u = a \cdot \nabla(|u|^{1/N} u) + (t+1)^{-(N/2)-1} h(x/\sqrt{t+1}) && \text{in } \mathbb{R}^N \times (0, \infty) \\ &u(x, 0) = u_0(x) && \text{in } \mathbb{R}^N, \end{aligned} \tag{1.1}$$

with $a \in \mathbb{R}^N$, $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $h \in L^2(\mathbb{R}^N; \exp(|x|^2/4)) \cap L^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} h(x)\, dx = 0$. In (1.1), $\cdot$ denotes the scalar product in $\mathbb{R}^N$.

Let us first recall what is known for the case where $h \equiv 0$.

In a recent joint work with Escobedo [EZ1, 2] we proved that when $h \equiv 0$, (1.1) has an unique global solution satisfying

$$u \in C([0, \infty); L^1(\mathbb{R}^N)) \cap C((0, \infty); W^{2,p}(\mathbb{R}^N)) \cap C^1((0, \infty; L^p(\mathbb{R}^N)) \tag{1.2}$$

for all $p \in (1, \infty)$ and

$$t^{(N/2)(1-1/r)} \|u(t) - w_M(t)\|_r \to 0 \qquad \text{as} \quad t \to \infty \tag{1.3}$$

for all $r \in [1, \infty]$ where $w_M(x, t) = t^{-N/2} f_M(x/\sqrt{t})$ is the unique self-similar solution of (1.1) (with $h = 0$) such that

$$\int w_M(x, t)\, dx = \int u(x, t)\, dx = \int u_0(x)\, dx = M \qquad \forall t > 0. \tag{1.4}$$

In (1.3) $\|\cdot\|_r$ denotes the norm in $L^r(\mathbb{R}^N)$. In (1.4) and in all that follows the symbol $\int$ denotes the integral in all of $\mathbb{R}^N$.

Roughly speaking, we may say that there exists a one parameter family of self-similar solutions (the distinguished parameter being the mass) that describes the large time behavior of system (1.1). This is a natural extension of well known results for Burger's equation and viscous systems of conservation laws in one space-dimension (see Kawashima [KW] and Chern and Liu [CL]).

We are interested on the self-similar behavior of the non-homogeneous problem

$$u_t - \Delta u = a \cdot \nabla(|u|^{1/N} u) + g(x, t).$$

It is easy to see that the self-similar behavior requires g to be of the form $g(x, t) = (t+1)^{-(N/2)-1} h(x/\sqrt{t+1})$ with $\int h(y)\, dy = 0$. Therefore we concentrate on system (1.1). A systematic study of the large time behavior for any g is beyond the goals of this paper.

The first difficulty encountered when proving (1.3) is the existence of self-similar solutions. Indeed, when $h = 0$, $w_M(x, t) = t^{N/2} f_M(x/\sqrt{t})$

is a self-similar solution of (1.1) if and only if the profile $f_M = f_M(y)$ satisfies

$$-\Delta f_M - y\cdot\frac{\nabla f_M}{2} - \frac{N}{2} f_M = a\cdot\nabla(|f_M|^{1/N} f_M) \qquad \text{in } \mathbb{R}^N$$
$$\int f_M\, dy = M. \tag{1.5}$$

In Aguirre, Escobedo, and Zuazua [AEZ], we used Leray–Schauder's degree theory to prove the existence of a unique and smooth solution of (1.5) that decays exponentially at infinity for every $M \in \mathbb{R}$. This solution is of constant sign and therefore $|M| = \|f\|_1$. To ensure the compactness that is needed for the application of Leray–Schauder's Theorem we worked in the functional framework of the weighted Sobolev spaces introduced by Escobedo and Kavian in [EK1, 2].

In [EZ1, 2] (always for $h = 0$) we worked in the similarity variables

$$s = \log(t+1), \qquad y = \frac{x}{(t+1)^{1/2}}. \tag{1.6}$$

Defining

$$v(y, s) = e^{sN/2} u(e^{s/2} y, e^s - 1) \tag{1.7}$$

we observed that v is solution of

$$v_s - \Delta v - y\cdot\frac{\nabla v}{2} - \frac{N}{2} v = a\cdot\nabla(|v|^{1/N} v) \qquad \text{in } \mathbb{R}^N \times (0, \infty)$$
$$v(y, 0) = u_0(y) \qquad \text{in } \mathbb{R}^N. \tag{1.8}$$

Property (1.3) is equivalent to

$$v(s) \to f_M \text{ as } s \to \infty \qquad \text{in } L^r(\mathbb{R}^N) \tag{1.9}$$

for all $r \in [1, \infty]$.

In this way, the problem is reduced to proving that every trajectory of system (1.8) converges to the stationary solution f_M. This stationary solution is defined in a unique way by the mass M of the initial datum that is conserved along the trajectory:

$$\int v(y, s)\, dy = \int u_0(y)\, dy = M, \qquad \forall s > 0.$$

In order to prove (1.9) we used La Salle's invariance principle. By working on the weighted Sobolev spaces mentioned above we proved the compactness of the trajectories in $L^r(\mathbb{R}^N)$ for all $r \in [1, \infty]$. The compactness being proved it was sufficient to construct a Lyapunov function, i.e., a functional which decreases strictly along the trajectory, the trivial case where $v(s)=f_M$ for $s \geqslant s_0$ for some $s_0 \geqslant 0$ being excepted. We proved that the function

$$\phi(s)=\int (v(y, s)-f_M(y))^+ \, dy \tag{1.10}$$

satisfies such a property. (In (1.10), by $(\cdot)^+$ we denote the positive part of the function $(\cdot)$).

The fact that ϕ decreases strictly along the trajectory is due to what is called the strict L^1-contraction property. Indeed, if v_1 and v_2 are two solutions of (1.8) with the same mass

$$\int v_1(y, s)\, dy=\int v_2(y, s)\, dy,$$

then the following alternative holds: either

(a) $\phi(s)=\int (v_1(y, s)-v_2(y, s))^+ \, dy$ is strictly decreasing with respect to s, or

(b) $v_1(y, s)=v_2(y, s)$ for $y \in \mathbb{R}^N$ and $s \geqslant s_0$ for some $s_0 \geqslant 0$ and then by backward uniqueness (see Ghidaglia [G]) $v_1 \equiv v_2$.

The main observation of this paper is that this strict L^1-contraction property and the compactness of trajectories implies both

(i) the existence of a one parameter family of self-similar solutions;

(ii) the self-similar large time behavior of solutions of (1.1), i.e., convergences (1.3) and (1.9).

The basic lines of our argument are as follows. Suppose that the trajectories of (1.8) are relatively compact in $L^1(\mathbb{R}^N)$. Given any solution v of mass M we define its ω-limit set in $L^1(\mathbb{R}^N)$: $\omega(v)$. Then, the problem reduces to proving the existence of a self-similar profile f_M with mass M such that $w(v)=\{f_M\}$. Given any $\tau>0$ we observe that $v_\tau(s)=v(s+\tau)$ is also a solution of (1.8) with the same mass M. From the strict L^1-contraction property we deduce that

$$\varphi_\tau(s)=\int (v(y, s)-v_\tau(y, s))^+ \, dy \tag{1.11}$$

is a strictly decreasing function, the case where $v \equiv v_\tau$ being excepted. By La Salle's invariance principle we deduce that any solution w of (1.8) with initial datum in $\omega(v)$ is τ-periodic. Since $\tau>0$ is arbitrary we deduce that

w is a stationary solution of (1.8), i.e., a self-similar profile. The uniqueness of the self-similar profile for a given mass is again a consequence of the strict L^1-contraction property. The ω-limit being reduced to the unique self-similar profile with the given mass, the convergence of the trajectory holds.

In fact, this is just an application of a general (and completely elementary) stability result for dynamical systems satisfying suitable compactness and contraction properties that we will prove below.

For system (1.8) the strict L^1-contraction property is a consequence of the strong maximum principle or of a unique continuation property.

We observe that by using this "dynamic argument" the direct study of the elliptic problem is avoided. This can be useful in those situations where the elliptic problem related to the self-similar solutions is hard to solve. For instance, when $h \not\equiv 0$, the methods of [AEZ] do not seem to apply to solve the corresponding elliptic problem. Indeed, when $h \not\equiv 0$ self-similar profiles are not of constant sign. Therefore the constraint $\int f_M = M$ does not provide the $L^1(\mathbb{R}^N)$-estimate which was essential for the arguments of [AEZ]. However, the dynamic argument above applies as well when $h \equiv 0$ or $h \not\equiv 0$.

When $h \not\equiv 0$ there is, however, a technical difficulty related to the boundedness of solutions of (1.1) in $L^1(\mathbb{R}^N)$. Indeed, when $h \equiv 0$, multiplying in (1.1) by $\operatorname{sgn}(u)$ and integrating in all of $\mathbb{R}^N$ we get

$$\|u(t)\|_1 \leqslant \|u_0\|_1 \qquad \forall t > 0.$$

However, when $h \not\equiv 0$ we get

$$\frac{d}{dt}\|u(t)\|_1 \leqslant (t+1)^{-(N/2)-1} \int |h(x/\sqrt{t+1})| \, dx \leqslant \|h\|_1 (t+1)^{-1} \qquad \forall t > 0,$$

and this does not provide the desired boundedness property for $\|u(t)\|_1$.

By using more involved arguments we prove that solutions of (1.1) are bounded in $L^1(\mathbb{R}^N)$ provided h is sufficiently small for a suitable norm that is made precise below. Therefore, our results apply only for the case where h is small enough. It is important to point out that the smallness assumption on h is only necessary for proving that $u(t)$ is bounded in $L^1(\mathbb{R}^N)$. All the results of this paper would immediately extend for every h if we knew that solutions of (1.1) are bounded in $L^1(\mathbb{R}^N)$.

In order to state the main results of this paper we need to introduce the following weighted spaces (cf. Escobedo and Kavian [EK1]).

Let be

$$K(y) = \exp(|y|^2/4)$$

and let us define the following weighted L^p and Sobolev spaces:

$$L^p(K)=\left\{f\in L^p(\mathbb{R}^N):\|f\|_{L^p(K)}:=\left[\int |f(y)|^p\, K(y)\, dy\right]^{1/p}<\infty\right\}$$

$$H^m(K)=\left\{f\in H^m(\mathbb{R}^N):\|f\|_{H^m(K)}:=\left[\sum_{|\alpha|\leqslant m}\|D^\alpha f\|^2_{L^2(K)}\right]^{1/2}<\infty\right\}$$

for $m=1, 2, \ldots$ and $p\in[1, \infty)$.

The two main results of this paper are as follows.

THEOREM 1 (Existence and Uniqueness of Self-Similar Solutions). *Let be $h\in L^1(\mathbb{R}^N; 1+|x|)\cap L^\infty(\mathbb{R}^N)$ such that*

$$\int h(y)\, dy=0. \tag{1.12}$$

There exists some $\varepsilon_0>0$ such that if

$$\|h\|_{L^1(\mathbb{R}^N;\,|x|)}<\varepsilon_0, \tag{1.13}$$

then for every $M\in\mathbb{R}$ there exists a unique solution $f_M\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\cap W^{2,p}(\mathbb{R}^N)$ for every $p\in(1, \infty)$ of

$$\begin{aligned} -\Delta f_M - y\cdot\frac{\nabla f_M}{2}-\frac{N}{2}f_M&=a\cdot\nabla(|f_M|^{1/N}f_M)+h \qquad \text{in } \mathbb{R}^N\\ \int f_M(y)\, dy&=M. \end{aligned} \tag{1.14}$$

Moreover, if h satisfies the further condition $h\in L^2(K)$, then $f\in H^2(K)$.

THEOREM 2 (Self-Similar Large Time Behavior). *Let be $h\in L^2(K)\cap L^\infty(\mathbb{R}^N)$ satisfying (1.12)–(1.13). Then for every $u_0\in L^1(\mathbb{R}^N)$ there exists a unique solution u of (1.1) satisfying (1.2) and such that*

$$t^{(N/2)(1-1/r)}\|u(t)-w_M(t)\|_r\to 0 \qquad \text{as} \quad t\to\infty \tag{1.15}$$

for every $r\in[1, \infty)$, where

$$w_M(x, t)=t^{-N/2}f_M\left(\frac{x}{\sqrt{t}}\right) \tag{1.16}$$

with $f_M=f_M(y)$ solution of (1.14) and $M=\int u_0(x)\, dx$.

Remarks. (1) We observe that w_M are source type solutions of (1.1), i.e., take $M\delta$ as initial data where $M = \int f_M(y)\,dy$ and δ is the Dirac mass at the origin. More precisely,

$$w_M(x, t) \to M\delta \qquad \text{as} \quad t \to 0^+$$

in the sense of measures

$$\lim_{t \to 0^+} \int w_M(x, t)\,\psi(x)\,dx = M\psi(0)$$

for all $\psi \in C(\mathbb{R}^N)$ with compact support.

(2) Note that $\|w_M(t)\|_r = C_r t^{-(N/2)(1-1/r)}$. Therefore (1.15) asserts that the general solution u of (1.1) behaves as the self-similar one as $t \to \infty$.

(3) As we have pointed out above, we need h to be small to ensure the boundedness of solutions of (1.1) in $L^1(\mathbb{R}^N)$. Let us observe that given u and $\bar{u}$ two solutions of (1.1) with the same h but different initial data $u_0, \bar{u}_0$, by multiplying by $\operatorname{sgn}(u - \bar{u})$ the equation satisfied by $u - \bar{u}$, one gets

$$\|u(t) - \bar{u}(t)\|_1 \leqslant \|u_0 - \bar{u}_0\|_1 \qquad \forall t > 0. \tag{1.17}$$

Therefore, in order to extend Theorems 1 and 2 to the case where h is large, it would be sufficient to know that the solution of (1.1) with zero initial datum $u_0 = 0$ is uniformly bounded in $L^1(\mathbb{R}^N)$.

(4) If we assume that $u_0 \in L^2(K) \cap L^\infty(\mathbb{R}^N)$, then (1.15) holds for $r = \infty$, too. Moreover, we have

$$t^{(N/2)(1-1/r)+1/2}\,\|\nabla u(t) - \nabla w_M(t)\|_r \to 0 \qquad \text{as} \quad t \to \infty \text{ for every } r \in [1, \infty). \tag{1.18}$$

(5) Property (1.15) implies the decay estimate

$$\|u(t)\|_r \leqslant C_r t^{-(N/2)(1-1/r)}.$$

This type of decay estimate was established by Schonbek in [S1, 2] for scalar parabolic conservation laws in several space dimensions by Fourier transform methods and for initial data $u_0 \in L^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ (and $h \equiv 0$).

(6) We do not know whether (1.15) holds for every $h \in L^1(\mathbb{R}^N; 1 + |x|) \cap L^\infty(\mathbb{R}^N)$ satisfying (1.12)–(1.13), i.e., without the assumption $h \in L^2(K)$.

This paper is organized as follows. In Section 2, for the sake of completeness, we develop the "dynamic arguments" above in the abstract framework of semigroups to prove the existence of stationary states and their stability properties. In Section 3 we prove the existence and uniqueness of solutions for (1.1). We also prove some boundedness and decay properties. Section 4 is devoted to the study of system (1.1) in the similarity variables. We work in the functional framework of the weighted Sobolev spaces introduced by Escobedo and Kavian in [EK1, 2]. Some compactness properties of the trajectories are established. In Section 5 we apply the abstract results of Section 2 to Eq. (1.1) in the similarity variables to prove Theorems 1 and 2.

2. Stationary Solutions and Stability Properties for Dynamical Systems

Let (Z, d) be a complete metric space and $\{S(t)\}_{t \geqslant 0}$ a dynamic system on Z satisfying:

(i) $S(t) \in C(Z, Z)$, $\forall t \geqslant 0$

(ii) $S(0) = \text{Identity}$

(iii) $S(t+s) = S(t) \circ S(s)$, $\forall t \geqslant 0$

(iv) $S(t)z \in C([0, \infty); Z)$, $\forall z \in Z$.

We have the following result on the existence of stationary states and convergence of trajectories as $t \to +\infty$.

Proposition 1. *Assume that* $\bigcup_{t \geqslant 0} \{S(t)z\}$ *is relatively compact in* Z *for every* $z \in Z$. *Assume that there exists a continuous function* $\phi \in C(Z \times Z; \mathbb{R})$ *such that*

$$\phi(S(t)z_i, S(t)z_2) \text{ is a strictly decreasing function of } t \text{ for every } z_1, z_2 \in Z \text{ with } z_1 \neq z_2. \tag{2.1}$$

Then, there exists a unique stationary state $y \in Z$ *such that*

$$S(t)\, y = y \qquad \forall t \geqslant 0 \tag{2.2}$$

and

$$\lim_{t \to \infty} d(S(t)z, y) = 0, \forall z \in Z.$$

Proof. Given any $z \in Z$ let us introduce the ω-limit set of z:

$$\omega(z) = \{ y \in Z : \exists t_n \to +\infty \text{ such that } S(t_n)z \to y \text{ as } n \to +\infty \}.$$

Clearly $\omega(z) \neq \emptyset$. Let be any $y \in \omega(z)$. Then there exists a sequence $t_n \to +\infty$ such that

$$S(t_n)z \to y.$$

From (i) and (iii) we deduce that

$$S(t+t_n)z \to S(t)\,y, \qquad \forall t \geqslant 0.$$

From the continuity of ϕ we decude that

$$\phi(S(t+\tau)\,y, S(t)\,y) = \lim \phi(S(t+\tau+t_n)z, S(t+t_n)z), \qquad \forall t, \tau \geqslant 0. \tag{2.3}$$

In view of (2.1) we deduce that $\phi(S(t+\tau)z, S(t)z)$ is a nonincreasing function of t. Since $\bigcup_{t \geqslant 0} \{S(t)z\}$ is relatively compact, it follows that

$$\exists \phi_\infty^\tau = \lim_{t \to \infty} \phi(S(t+\tau)z, S(t)z). \tag{2.4}$$

Combining (2.3)–(2.4), we deduce that

$$\phi(S(t+\tau)\,y, S(t)\,y) = \phi_\infty^\tau \qquad \forall t \geqslant 0$$

and from (2.1) we conclude that

$$S(t+\tau)\,y = S(t)\,y \qquad \forall t, \tau \geqslant 0.$$

This implies that y is a stationary state, i.e., (2.2) holds.

Therefore, we have proved that the ω-limit set of z is contained in the (non empty) set of stationary states. On the other hand, from (2.1) we deduce that the stationary state is unique and therefore

$$\omega(z) = \{y\},$$

where y is the unique element of Z satisfying (2.2). This implies that

$$S(t)z \to y \text{ in } Z \qquad \text{as} \quad t \to +\infty$$

or equivalently,

$$\lim_{t \to +\infty} d(S(t)z, y) = 0.$$

The proof of Proposition 1 is now completed. ∎

Remark 2. As Haraux pointed out (personal communication), when $\phi(x, y) = d(x, y)$, Proposition 1 is a consequence of Theorem 4.5.1 of Haraux [H].

The following result provides the existence of a one parameter family of stationary states for those dynamical systems such that some energy is conserved along trajectories.

PROPOSITION 2. *Assume that* $\bigcup_{t\geq 0}\{S(t)z\}$ *is relatively compact in* Z *for every* $z\in Z$ *and that there exists some continuous function* $\psi\in C(Z;\mathbb{R})$ *such that*

$$\psi(S(t)z)=\psi(z), \qquad \forall t>0, \forall z\in Z. \tag{2.5}$$

Assume furthermore that there exists a continuous function $\phi\in C(Z\times Z;\mathbb{R})$ *such that*

$$\begin{aligned}&\phi(S(t)z_1, S(t)z_2) \text{ is a strictly decreasing function of } t\\ &\text{for every } z_1, z_2\in Z \text{ with } \psi(z_1)=\psi(z_2) \text{ and } z_1\neq z_2.\end{aligned} \tag{2.6}$$

Then, for every $M\in R(\psi)$ $(=$*range of* $\psi)$ *there exists a unique* $y_M\in Z$ *such that*

$$\begin{aligned}S(t)\,y_M&=y_M \qquad \forall t\geq 0\\ \psi(y_M)&=M.\end{aligned} \tag{2.7}$$

Furthermore, given any $z\in Z$ *with* $\psi(z)=M$ *we have*

$$\lim_{t\to\infty} d(S(t)z, y_M)=0 \tag{2.8}$$

Proof. Let be any $M\in R(\psi)$ and $z\in Z$ with $\psi(z)=M$. The ω-limit set $\omega(z)$ of z is non empty and by (2.5) and the continuity of ψ we have

$$\psi(y)=M \qquad \forall y\in\omega(z).$$

Proceeding as in the proof of Proposition 1, we deduce that every $y\in\omega(z)$ satisfies (2.7). The uniqueness of the solution of (2.7) follows from (2.6). Therefore, $\omega(z)=\{y_M\}$ which implies (2.8). ▌

3. Cauchy Problem: Existence, Uniqueness, and $L^1\cap L^{(N+1)/N}$ Estimates

We start this section with an existence, uniqueness, and regularity result for the following more general convection–diffusion equation

$$\begin{aligned}&u_t-\Delta u=a\cdot\nabla(\varphi(u))+g(x,t) \qquad \text{in} \quad \mathbb{R}^N\times(0,\infty)\\ &u(0)=u_0.\end{aligned} \tag{3.1}$$

PROPOSITION 3. *Assume that* $a \in \mathbb{R}^N$, $\varphi \in C^1(\mathbb{R})$, $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* $g \in C([0, \infty); L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$.

Then, there exists a unique solution $u \in C((0, \infty); L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$ *of* (3.1) *such that*

$$u \in C((0, \infty); W^{2,p}(\mathbb{R}^N)) \cap C^1((0, \infty); L^p(\mathbb{R}^N)) \quad \text{for every} \quad p \in (1, \infty). \tag{3.2}$$

Proof. Let us consider the integral equation

$$u(t) = G(t) * u_0 + \int_0^t a \cdot \nabla G(t-s) * \varphi(u(s))\, ds + \int_0^t G(t-s) * g(s)\, ds, \tag{3.3}$$

where $G = G(x, t) = (4\pi t)^{-N/2} \exp(-|x|^2/4t)$ is the heat kernel and $*$ denotes the convolution in the space variables.

Let us define the nonlinear operator

$$[\phi(u)](t) = G(t) * u_0 + \int_0^t a \cdot \nabla G(t-s) * \varphi(u(s))\, ds + \int_0^t G(t-s) * g(s)\, ds.$$

If $R > 0$ is large enough and $T > 0$ is small enough it is easy to see that ϕ is a contraction in the following closed subset of $C([0, T]; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$:

$$B = \{u \in C([0, T]; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)): \sup_{0 \leqslant t \leqslant T} (\|u(t)\|_1 + \|u(t)\|_\infty) \leqslant R\}. \tag{3.4}$$

Then, applying the Banach fixed point Theorem we deduce that the integral equation (3.3) has a unique solution in B. This solution may be extended to a maximal time interval $[0, T_{\max})$ and it is global in time (i.e., $T_{\max} = \infty$) if

$$\sup_{t \in [0, T]} (\|u(t)\|_1 + \|u(t)\|_\infty) < \infty \tag{3.5}$$

for all $T < \infty$.

In order to prove (3.5) we observe that the solution u of (3.3) is a weak solution of (3.1) in the time interval $[0, T_{\max})$. On the other hand, by classical regularity results we deduce that

$$u \in C((0, T_{\max}); W^{2,p}(\mathbb{R}^N)) \cap C^1((0, \infty); L^p(\mathbb{R}^N))$$

for every $p \in (1, \infty)$.

Multiplying in (3.1) by $\theta_\varepsilon(u)$ with $\theta_\varepsilon(s)$ a Lipschitz approximation of $\text{sgn}(s)$, integrating in $\mathbb{R}^N \times (0, T)$ and using Kato's inequality we get

$$0 \leqslant -\int \Delta u(x,t)\, \theta_\varepsilon(x,t)\, dx \qquad \forall t \in [0,T], \forall \varepsilon > 0$$

and the fact that

$$\int a \cdot \nabla(\varphi(u(x,t))\, \theta_\varepsilon(u(x,t))\, dx = 0 \qquad \forall t \in [0,T], \forall \varepsilon > 0$$

and finally, by letting $\varepsilon \to 0$, we obtain that

$$\int |u(x,T)|\, dx \leqslant \int |u_0(x)|\, dx + \int_0^t \int |g(x,t)|\, dx\, dt, \tag{3.6}$$

which implies

$$\sup_{0<t<T} \|u(t)\|_1 < \infty \tag{3.7}$$

for all $T < \infty$.

Now, observe that $\bar{u} = e^{-t}u$ satisfies

$$\begin{aligned} &\bar{u}_t - \Delta\bar{u} + \bar{u} = e^{-t}[a \cdot \nabla(\varphi(e^t\bar{u})) + g] \qquad \text{in} \quad \mathbb{R}^N \times (0, T_{\max}) \\ &\bar{u}(0) = u_0. \end{aligned} \tag{3.8}$$

Let be $T \in (0, T_{\max})$ and $m = \max(\|u_0\|_\infty, \|e^{-t}g\|_{L^\infty(\mathbb{R}^N \times (0, T_{\max}))})$.

Multiplying in (3.8) $\text{sgn}(\bar{u} - m)^+$ with

$$\text{sgn}\,(s^+) = \begin{cases} 1 & \text{if} \quad s > 0 \\ 0 & \text{if} \quad s \leqslant 0 \end{cases}$$

(as above, we multiply the equation by a Lipchitz approximation of $\text{sgn}(\bar{u} - m)^+$ and then we let $\varepsilon \to 0$) and integrating over $\mathbb{R}^N \times (0, T)$ we get

$$\begin{aligned} &\int (\bar{u}(x,T) - m)^+\, dx \\ &\qquad \leqslant \int (u_0(x) - m)^+\, dx \\ &\qquad\quad + \int_0^t \int (\bar{e}^t g(x,t) - m)\, \text{sgn}(\bar{u}(x,t) - m)^+\, dx\, dt = 0 \end{aligned}$$

and therefore $\bar{u} \leqslant m$ in $\mathbb{R}^N \times (0, T)$. An analogous argument shows that $\bar{u} \geqslant -m$. We conclude that

$$\sup_{0<t<T} (\|u(t)\|_\infty) \leqslant e^T \sup_{0<t<T} (\|\bar{u}(t)\|_\infty) \leqslant me^T. \tag{3.9}$$

Combining (3.7) and (3.9) we deduce (3.5) and therefore $T_{\max} = \infty$. This concludes the proof of the existence.

In order to show the uniqueness, let us suppose that u and $\bar{u}$ are solutions of (3.1). Then $w = u - \bar{u}$ satisfies

$$\begin{aligned} &w_t - \Delta w = a \cdot \nabla(\varphi(u) - \varphi(\bar{u})) \quad \text{in} \quad \mathbb{R}^N \times (0, \infty) \\ &w(0) = 0. \end{aligned} \tag{3.10}$$

Multiplying in (3.10) by $\operatorname{sgn}(w)$ and integrating over $\mathbb{R}^N \times (0, t)$ we easily deduce that

$$\frac{d}{dt} \|w(t)\|_1 \leqslant 0 \qquad \forall t > 0$$

and therefore $w = 0$.

We proof of the proposition is completed. ∎

Let us now consider the case where $\varphi(u) = |u|^{1/N} u$. We have the following result.

PROPOSITION 4. *Assume that* $\varphi(s) = |s|^{1/N} s$, $u_0 \in L^1(\mathbb{R}^N)$ *and* $g \in C([0, \infty); L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))$ *for some* $q > (N+1)/N$. *Then, there exists a unique solution* $u \in C([0, \infty); L^1(\mathbb{R}^N))$ *of* (3.1) *satisfying* (3.2) *for all* $p \in (1, q]$.

Proof. Let us approximate u_0 in $L^1(\mathbb{R}^N)$ and g in $C([0, \infty); L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))$ by sequences $u_{0,n} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $g_n \in C([0, \infty); L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$:

$$u_{0,n} \to u_0 \text{ in } L^1(\mathbb{R}^N) \qquad \text{as} \quad n \to \infty \tag{3.11}$$

$$g_n \to g \text{ in } C([0, \infty); L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)). \tag{3.12}$$

From Proposition 3 we know that for each $n \in \mathbb{N}$ there exists a solution $u_n \in C([0, \infty); L^1(\mathbb{R}^N))$ of (3.1) with data $u_{0,n}$, g_n that satisfies (3.2) for all $p \in (1, \infty)$.

Multiplying by $\operatorname{sgn}(u_m - u_n)$ the equation satisfied by $u_m - u_n$, we obtain

$$\|u_n(t) - u_m(t)\|_1 \leqslant \|u_{0,n} - u_{0,m}\|_1 + \int_0^t \|g_n(s) - g_m(s)\|_1 \, ds, \qquad \forall t \geqslant 0 \; \forall n, m \in \mathbb{N}. \tag{3.13}$$

In view of (3.11)–(3.13) we deduce that $\{u_n\}$ is a Cauchy sequence in $C([0, \infty); L^1(\mathbb{R}^N))$. The limit u of $\{u_n\}$ in $C([0, \infty); L^1(\mathbb{R}^N))$ satisfies $u(0) = u_0$. Let us assume for the moment that

$$\{u_n\} \text{ is uniformly bounded in } L^\infty(t_1, t_2; L^q(\mathbb{R}^N)) \qquad \text{for all} \quad t_2 > t_1 > 0. \tag{3.14}$$

Then, since $q > (N+1)/N$, by interpolation we deduce that

$$u_n \to u \text{ in } L^\infty(t_1, t_2; L^{(N+1)/N}(\mathbb{R}^N)) \qquad \text{as} \quad n \to \infty \text{ for all } t_2 > t_1 > 0. \tag{3.15}$$

This allows us to pass to the limit in Eq. (3.1) satisfied by u_n. In this way, we obtain that $u \in C([0, \infty); L^1(\mathbb{R}^N))$ solves (3.1) with the data u_0 and g. On the other hand, by classical regularity results we deduce that u satisfies (3.2) for all $p \in (1, q]$.

The uniqueness of the solution of (3.1) is proved as in the proof of Proposition 3.

Therefore it is sufficient to prove (3.14). We use the following interpolation inequality that was proved in [EZ2]:

LEMMA 1 [EZ2]. *For every* $p \in (1, \infty)$ *there exists some constant* $C = C(p, N) > 0$ *such that*

$$\|v\|_p^{(N(p-1)+2)p/N(p-1)} \leqslant C \, \|v\|_1^{2p/N(p-1)} \, \|\nabla(|v|^{p/2})\|_2^2 \tag{3.16}$$

for every $v \in W^{2,p}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.

Multiplying by $|u_n|^{q-2} u_n$ with $q > 1$ the equation satisfied by u_n and integrating in $\mathbb{R}^N$, we get

$$\frac{d}{dt} \int |u_n(x,t)|^q \, dx + \frac{4(q-1)}{q} \int |\nabla(|u_n(x,t)|^{q/2})|^2 \, dx$$
$$\leqslant q \int |g_n(x,t)| \, |u_n(x,t)|^{q-1} \, dx.$$

In view of (3.16) and using Hölder's inequality, we get

$$\frac{d}{dt} \|u_n(t)\|_q^q + \frac{C}{\|u_n(t)\|_1^{2q/N(q-1)}} \|u_n(t)\|_q^{q(N(q-1)+2)/N(q-1)}$$
$$\leqslant q \, \|g_n(t)\|_q \, \|u_n(t)\|_q^{q-1} \tag{3.17}$$

On the other hand, multiplying by $\operatorname{sgn}(u_n)$ the equation satisfied by u_n, we get

$$\|u_n(t)\| \leqslant \|u_{0,n}\|_1 + \int_0^t \|g_n(s)\|_1 \, ds \qquad \forall t > 0, \forall n \in \mathbb{N}$$

and therefore

$$\|u_n(t)\|_1 \leqslant C \qquad \forall t \in [0, t_2], \forall n \in \mathbb{N}. \tag{3.18}$$

Combining (3.17)–(3.18) and using Young's inequality on the right hand side of (3.17), we get

$$\frac{d}{dt}\|u_n(t)\|_q^q + C_1\|u_n(t)\|_q^{q(N(q-1)+2)/N(q-1)}$$
$$\leqslant C_2\|g(t)\|_q^{q(N(q-1)+2)/N(q-1)+2q} \leqslant R \qquad \forall t \in [0, t_2] \tag{3.19}$$

for $R > 0$ large enough.

The estimate (3.16) is now a consequence of (3.21) since every solution of the differential inequality

$$r'(t) + cr^{\alpha}(t) \leqslant R \qquad \forall t \in [0, t_2]$$

with $\alpha > 1$, $c > 0$, and $R > 0$ satisfies

$$r(t) \leqslant \max\left(\left(\frac{2R}{c}\right)^{1/\alpha}, \left(\frac{(\alpha-1)ct}{2}\right)^{-1/(\alpha-1)}\right) \qquad \forall t \in [0, t_2]$$

This concludes the proof of Proposition 4. ∎

We now return to system (1.1) and prove the $L^1(\mathbb{R}^N)$-estimate.

Proposition 5. *Assume that* $h \in L^1(\mathbb{R}^N; 1+|x|) \cap L^q(\mathbb{R}^N)$ *for some* $q > (1/N) + 1$ *and that* $\int h(y)\, dy = 0$. *Then, if* $\|h\|_{L^1(\mathbb{R}^N; |x|)}$ *is small enough, for every* $u_0 \in L^1(\mathbb{R}^N)$ *the solution* u *of* (1.1) *constructed in Proposition* 4 *satisfies*

$$u \in L^{\infty}(0, +\infty; L^1(\mathbb{R}^N)), \; t^{N/2(N+1)}u(t) \in L^{\infty}(0, \infty; L^{(N+1)/N}(\mathbb{R}^N)). \tag{3.20}$$

Remark 3. This proposition establishes sharp boundedness and decay estimates for u in $L^1(\mathbb{R}^N)$ and $L^{(N+1)/N}(\mathbb{R}^N)$, respectively, for h small enough but for every initial data u_0. These estimates are natural extensions of those satisfied by the heat kernel and that we extended to system (1.1) when $h = 0$ in [EZ1, 2].

As we mentioned in the Introduction, Theorems 1 and 2 would immediately extend to h large if the L^1-boundedness of solutions were known.

Proof of Proposition 5. Given two solutions u and $\bar{u}$ of (1.1) with the same h but different initial data u_0 and $\bar{u}_0$, multiplying by $\operatorname{sgn}(u - \bar{u})$ the equation satisfied by $u - \bar{u}$, we deduce that

$$\|u(t) - \bar{u}(t)\|_1 \leqslant \|u_0 - \bar{u}_0\|_1 \qquad \forall t > 0.$$

Therefore it is sufficient to prove the L^1-boundedness for $u_0 = 0$. In what follows we assume that $u_0 = 0$.

Let be u the solution of (1.1) with $u_0 = 0$ and let us define the following family of rescaled functions:

$$u_\lambda(x, t) = \lambda^N u(\lambda x, \lambda^2 t), \qquad \lambda > 0.$$

Since u solves (1.1), u_λ solves

$$\begin{aligned} &u_{\lambda,t} - \Delta u_\lambda = a \cdot \nabla(|u_\lambda|^{1/N} u_\lambda) + \left(t + \frac{1}{\lambda^2}\right)^{-(N/2)-1} h\left(\frac{x}{\sqrt{t + \lambda^{-2}}}\right) \\ &u_\lambda(0) = 0. \end{aligned} \tag{3.21}$$

On the other hand, (3.20) is equivalent to the following fact:

$$\{u_\lambda(1)\}_{\lambda \geq 1} \text{ is uniformly bounded in } L^1(\mathbb{R}^N) \cap L^{(N+1)/N}(\mathbb{R}^N). \tag{3.22}$$

We are going to prove that for $\|h\|_{L^1(\mathbb{R}^N; |x|)}$ small enough (3.21) admits a family $\{u_\lambda\}$ of solutions that is bounded in $L^\infty(0, 1; L^1(\mathbb{R}^N))$ and such that $\{t^{N/2(N+1)} u_\lambda\}$ is bounded in $L^\infty(0, 1; L^{(N+1)/N}(\mathbb{R}^N))$. By uniqueness of solutions of (3.21), $\{u_\lambda\}$ will be necessarily the rescaled family associated to u solution of (3.1) and the proof of the proposition will be completed.

Let us consider the integral equation associated to (3.21):

$$\begin{aligned} u_\lambda(t) = a \cdot \int_0^t \nabla G(t-s) * [|u_\lambda|^{1/N} u_\lambda(s)]\, ds \\ + \int_0^t G(t-s) * \left[(s + \lambda^{-2})^{-(N/2)-1} h\left(\frac{x}{\sqrt{s + \lambda^{-2}}}\right)\right] ds. \end{aligned} \tag{3.23}$$

Let us introduce the Banach space

$$X = L^\infty(0, 1; L^1(\mathbb{R}^N)) \cap \{u : t^{N/2(N+1)} u(t) \in L^\infty(0, 1; L^{(N+1)/N}(\mathbb{R}^N))\}$$

endowed with the natural norm

$$\|u\|_X = \|u\|_{L^\infty(0,1; L^1(\mathbb{R}^N))} + \|t^{N/2(N+1)} u(t)\|_{L^\infty(0,1; L^{(N+1)/N}(\mathbb{R}^N))}.$$

We are going to prove the existence of solutions of (3.23) as fixed points in X of the nonlinear operator:

$$\begin{aligned} [\phi_\lambda(u)](t) = a \cdot \int_0^t \nabla G(t-s) * [|u|^{1/N} u(s)]\, ds \\ + \int_0^t G(t-s) * \left[(s + \lambda^{-2})^{-N/2-1} h\left(\frac{x}{\sqrt{s + \lambda^{-2}}}\right)\right] ds. \end{aligned}$$

When h is sufficiently small in $L^1(\mathbb{R}^N; |x|)$, we will prove the existence of some $R>0$ small enough such that ϕ_λ is a contraction in B_R the ball of radious R and centered in the origin in X. The radius R will not depend on $\lambda \geqslant 0$ and therefore (3.22) will hold.

Let us first recall the following estimate for the heat kernel that was proved in [EZ2]:

LEMMA 2 [EZ2]. *For every $r \in [1, \infty]$ there exists some constant $C_r > 0$ such that*

$$\|G(t) * \theta\|_r \leqslant C_r \|\theta\|_{L^1(\mathbb{R}^N; |x|)} t^{-(N/2)(1-1/r)-(1/2)} \qquad \forall t>0 \tag{3.24}$$

for every $\theta \in L^1(\mathbb{R}^N; 1+|x|)$ with $\int \theta(x)\,dx = 0$.

Combining this estimate with well known L^p-estimates for the heat kernel we obtain the following estimates for ϕ_λ for $r=1$ or $(N+1)/N$:

$$\begin{aligned}
&\|[\phi_\lambda(u)](t)\|_r \\
&\quad \leqslant |a| \int_0^t \|\nabla G(t-s)\|_r \|u(s)\|_{(N+1)/N}^{(N+1)/N}\,ds \\
&\qquad + C_r \|h\|_{L^1(\mathbb{R}^N;|x|)} \int_0^t (t-s)^{-(N/2)(1-1/r)-(1/2)} (s+\lambda^{-2})^{-1/2}\,ds \\
&\quad \leqslant C'_r(|a|\, R^{(N+1)/N} + \|h\|_{L^1(\mathbb{R}^N;|x|)}) \int_0^t (t-s)^{-(N/2)(1-1/r)} s^{-1/2}\,ds \\
&\quad \leqslant C'_r(|a|\, R^{(N+1)/N} + \|h\|_{L^1(\mathbb{R}^N;|x|)})\, t^{-(N/2)(1-1/r)} \\
&\qquad \times \int_0^1 (1-\sigma)^{-(N/2)(1-1/r)-(1/2)} \sigma^{1/2}\,d\sigma \qquad \forall t \in [0,1], \forall u \in B_R, \forall \lambda \geqslant 0.
\end{aligned} \tag{3.25}$$

From (3.25) we deduce that if h is such that

$$\|h\|_{L^1(\mathbb{R}^N;|x|)} \leqslant \frac{1}{4C}\left(\frac{1}{4C\,|a|}\right)^N \tag{3.26}$$

with

$$C = \max_{r=1,(N+1)/N} \left(C'_r \int_0^1 (1-\sigma)^{-(N/2)(1-1/r)-(1/2)} \sigma^{-1/2}\,d\sigma \right), \tag{3.27}$$

then by choosing

$$R \leqslant \left(\frac{1}{4C\,|a|}\right)^N \tag{3.28}$$

ϕ_λ sends B_R into itself for all $\lambda \geqslant 0$.

On the other hand,

$$\|[\phi_\lambda(u)](t)-[\phi_\lambda(v)](t)\|_r$$
$$\leqslant |a| \int_0^t \|\nabla G(t-s)\|_r \, \| |u|^{1/N}u(s)-|v|^{1/N}v(s)\|_1 \, ds$$
$$\leqslant C_r\left(\frac{N+1}{N}\right)|a|\, R^{1/N}\, \|u-v\|_X\, t^{-(N/2)(1-1/r)}$$
$$\times \int_0^1 (1-\sigma)^{-(N/2)(1-1/r)}\, \sigma^{-1/2}\, d\sigma$$
$$\forall t\in[0,1],\ \forall u, v\in B_R,\ \forall \lambda \geqslant 0. \tag{3.29}$$

Therefore, if $R>0$, in addition to (3.28), satisfies

$$\left(C_1 \int_0^1 (1-\sigma)^{-1/2}\sigma^{-1/2}\, d\sigma \right.$$
$$\left. + C_{(N+1)/N}\int_0^1 (1-\sigma)^{-N/2(N+1)}\sigma^{-1/2}\, d\sigma\right)|a|\, R^{1/N}<1,$$

we deduce that ϕ_λ is a contraction from B_R into itself. ∎

4. The Cauchy Problem in the Similarity Variables: Decay Estimates

In this section we study system (1.1) with u_0 and $h\in L^2(K)\cap L^\infty(\mathbb{R}^N)$.

Let us consider the similarity variables (y, s) introduced in (1.6) and the function $v=v(y, s)$ defined in (1.7). The function v solves

$$\begin{aligned} &v_s-\Delta v-y\cdot\frac{\nabla v}{2}-\frac{N}{2}v=a\cdot\nabla(|v|^{1/N}v)+h(y) && \text{in } \mathbb{R}^N\times(0,\infty)\\ &v(y,0)=u_0(y) && \text{in } \mathbb{R}^N. \end{aligned} \tag{4.1}$$

From Proposition 4 we know that u satisfies (1.2) for all $p\in(1,\infty)$. Therefore

$$v\in C([0,\infty); L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))$$
$$\cap\, C((0,\infty); W^{2,p}(\mathbb{R}^N))\cap C^1((0,\infty); L^p(\mathbb{R}^N)) \quad \text{for all } p\in(1,\infty). \tag{4.2}$$

Let us now recall some well-known facts about the weighted Sobolev spaces mentioned in the Introduction and about the heat equation in the similarity variables.

From Corollary 1.11 and Proposition 1.12 of [EK1] it is easy to see that

$$\forall \varepsilon > 0, \forall p > 2, \exists C(\varepsilon, p) > 0\colon \|f\|_{L^2(K)} \leqslant \varepsilon \|f\|_{H^1(K)} + C \|f\|_p, \qquad \forall f \in H^1(K) \cap L^p(\mathbb{R}^N). \tag{4.3}$$

The operator

$$Lf := -\Delta f - \frac{1}{2} y \cdot \nabla f = -\frac{1}{K} \operatorname{div}(K \nabla f)$$

is self-adjoint in $L^2(K)$. If we consider L as an unbounded operator in $L^2(K)$, then $D(L) = H^2(K)$ (see Kavian [K]). The inverse of L, L^{-1}, is bounded and compact from $L^2(K)$ into itself. The sequence of eigenvalues of L is

$$\lambda_k = \frac{N+k-1}{2}, \qquad k = 1, 2, 3, \ldots. \tag{4.4}$$

The first eigenvalue $\lambda_1 = N/2$ is simple and the corresponding eigenspace E_1 is spanned by $K^{-1}(y)$. We set

$$\varphi_1(y) = c \exp(-|y|^2/4)$$

with $c > 0$ such that $\int \varphi_1(y)\, dy = 1$.

The operator L is an isomorphism from $H^1(K)$ into its dual $(H^1(K))^*$ and $\|\nabla v\|_{L^2(K)}$ defines a norm in $H^1(K)$ which is equivalent to that given in the Introduction.

Denote by S the analytic semigroup generated by $L - (N/2)I$ in $L^2(K)$. This semigroup is given by the following formula (see [K]:

$$(S(s)f)(y) = e^{SN/2}(G(e^s - 1) * g)(e^{s/2} y) \qquad \forall s > 0, \forall y \in \mathbb{R}^N, \forall f \in L^2(K). \tag{4.5}$$

From (4.1) we deduce that v solves the following integral equation:

$$v(s) = S(s)\, u_0 + \int_0^s S(s-\sigma)\, a \cdot \nabla(|v|^{1/N} v(\sigma))\, d\sigma + \int_0^s S(s-\sigma)\, h\, d\sigma. \tag{4.6}$$

In addition to (4.2) we have the following regularity and boundedness properties for v.

PROPOSITION 6. *Assume that* $u_0, h \in L^2(K) \cap L^\infty(\mathbb{R}^N)$ *and* $\int h(y)\,dy = 0$, $\|h\|_{L^1(\mathbb{R}^N;\,|x|)} < \varepsilon_0$ *so that Proposition* 5 *holds.*

Then if u *is the solution of* (1.1) *and* v *is given by* (1.7), v *is the unique solution of* (4.1) *such that*

$$v \in C([0, \infty); L^2(K) \cap L^\infty(\mathbb{R}^N)) \\ \cap C((0, \infty); H^2(K)) \cap C^1((0, \infty); L^2(K)). \tag{4.7}$$

Moreover, we have the following uniform estimates for v:

$$v(s) \in L^\infty((s_0, \infty); W^{1,p}(\mathbb{R}^N)), \qquad \forall s_0 > 0, \forall p \in [1, \infty) \tag{4.8}$$

$$v \in L^\infty((0, \infty); L^2(K)) \tag{4.9}$$

$$v \in L^\infty((s_0, \infty); H^1(K)) \qquad \forall s_0 > 0. \tag{4.10}$$

Proof. We proceed in several steps.

Step 1. First we observe that, since $S(\cdot)$ is a semigroup generated by a self-adjoint operator we have

$$\|S(s)v\|_{L^2(K)} \leqslant \|v\|_{L^2(K)}, \ \|S(s)v\|_{H^1(K)} \leqslant C\left(1 + \frac{1}{\sqrt{s}}\right)\|v\|_{L^2(K)}, \\ \|S(s)v\|_{H^2(K)} \leqslant C\left(1 + \frac{1}{s}\right)\|v\|_{L^2(K)} \qquad \forall s > 0, \forall v \in L^2(K). \tag{4.11}$$

Taking $H^1(K)$-norms in the integral equation (4.6) we get

$$\begin{aligned} \|v(s)\|_{H^1(K)} &\leqslant C\left(1 + \frac{1}{\sqrt{s}}\right)\|u_0\|_{L^2(K)} + \left(\frac{N+1}{N}\right)|a| \\ &\quad \times \int_0^s (1 + (s-\sigma)^{-1/2})\,\|v(\sigma)\|_\infty^{1/N}\,\|v(\sigma)\|_{H^1(K)}\,d\sigma \\ &\quad + \int_0^s (s-\sigma)^{-1/2}\,\|h\|_{L^2(K)}\,d\sigma \\ &\leqslant C\left(1 + \frac{1}{\sqrt{s}}\right)\|u_0\|_{L^2(K)} + 2\sqrt{s}\,\|h\|_{L^2(K)} \\ &\quad + \left(\frac{N+1}{N}\right)|a|\,\|v\|_{L^\infty(\mathbb{R}^N \times (0,s))} \\ &\quad \times \int_0^s (1 + (s-\sigma)^{-1/2})\,\|v(\sigma)\|_{H^1(K)}\,d\sigma. \end{aligned}$$

Applying Gronwall's lemma, we get

$$\|v(s)\|_{H^1(K)} \leqslant C\left(\left(1+\frac{1}{\sqrt{s}}\right)\|u_0\|_{L^2(K)} + 2\sqrt{s}\,\|h\|_{L^2(K)}\right)$$

$$\forall s \in (0, s_0] \tag{4.12}$$

for every $s_0 > 0$ with some $C > 0$ that depends on $\|v\|_{L^\infty(\mathbb{R}^N \times (0,s_0))}$. The fact that $v \in C((0, \infty); H^1(K))$ follows in a standard way.

Since $S(s)u_0$, $S(s)h \in C([0, \infty); L^2(K))$, $v \in C([0, \infty); L^\infty(\mathbb{R}^N))$, and the estimate (4.12) holds, from the integral equation (4.6) we easily deduce that $v \in C([0, \infty); L^2(K))$.

We now estimate the $L^2(K)$-norm of $w = v_s$ that satisfies for all $s_0 > 0$,

$$w_s + Lw - \frac{N}{2}w = \left(\frac{N+1}{N}\right) a \cdot \nabla(|v|^{1/N} w) \qquad \text{in } \mathbb{R}^N \times (s_0, \infty)$$

$$w(s_0) = w_0 = -Lw(s_0) + \frac{N}{2} w(s_0)$$
$$+ a \cdot \nabla(|v(s_0)|^{1/N} v(s_0)) + h \in (H^1(K))^*,$$

and therefore

$$w(s+s_0) = S(s) w_0 + \frac{N+1}{N}\int_0^s S(s-\sigma)\, a \cdot \nabla(|v|^{1/N} w(\sigma + s_0))\, d\sigma.$$

Taking $L^2(K)$-norms in this integral equation and using the fact that

$$\|S(s)v\|_{L^2(K)} \leqslant C\left(1+\frac{1}{\sqrt{s}}\right)\|v\|_{(H^1(K))^*} \qquad \forall v \in (H^1(K))^*, \forall s > 0,$$

we get the following estimate:

$$\|w(s+s_0)\|_{L^2(K)} \leqslant C\left(1+\frac{1}{\sqrt{s}}\right)\|w_0\|_{(H^1(K))^*}$$
$$+ \frac{N+1}{N}|a|\,\|v\|^{1/N}_{L^\infty(\mathbb{R}^N \times (0, s_0+s_1))}$$
$$\times \int_0^s (1 + (s-\sigma)^{-1/2})\,\|w(\sigma+s_0)\|_{L^2(K)}\, d\sigma \qquad \forall s \in (0, s_1).$$

Applying Gronwall's inequality, we get

$$\|w(s+s_0)\|_{L^2(K)} \leqslant C\left(1+\frac{1}{\sqrt{s}}\right)\|w_0\|_{(H^1(K))^*}, \qquad \forall s \in (0, s_1]$$

with $C>0$ depending on $\|v\|_{L^\infty(\mathbb{R}^N\times(0,s_0+s_1))}$. Therefore $v_s(s)\in L^2(K)$ for all $s>0$.

The continuity of the map $s\in(0,\infty)\to v_s(s)\in L^2(K)$ follows in a standard way.

We finally observe that

$$Lv=\frac{N}{2}v-v_s+a\cdot\nabla(|v|^{1/N}v)+h\in C((0,\infty;L^2(K))$$

and therefore $v\in C((0,\infty);H^2(K))$.

We have proved that the solution v of (4.1) given by (1.7) satisfies (4.7).

Step 2. In order to prove the uniqueness, let us assume that v and $\bar{v}$ are solutions of (4.1) satisfying (4.7). Multiplying by $\operatorname{sgn}(v-\bar{v})$ the equation satisfied by $v-\bar{v}$ and integrating over $\mathbb{R}^N$, we get

$$\frac{d}{ds}\|v(s)-\bar{v}(s)\|_1\leqslant 0 \qquad \forall s\geqslant 0,$$

and therefore $v\equiv\bar{v}$.

Step 3. Let us now prove the L^p-estimates for $p\in[1,\infty)$.

We first observe that, as a consequence of (3.20) we have

$$v\in L^\infty(0,\infty;L^1(\mathbb{R}^N)\cap L^{(N+1)/N}(\mathbb{R}^N)). \tag{4.13}$$

Multiplying by $|v|^{q-1}v$ the equation satisfied by v and integrating in $\mathbb{R}^N$, we get

$$\begin{aligned}\frac{d}{ds}\|v(s)\|_q^q+4\frac{(q-1)}{q}\|\nabla(|v|^{q/2}(s))\|_2^2 &\leqslant\frac{N}{2}(q-1)\|v(s)\|_q^q+\int|h|\,|v(s)|^{q-1}\,dy\\ &\leqslant C_q(\|h\|_q^q+\|v(s)\|_q^q).\end{aligned}$$

In view of (3.16) and using the fact that $v\in L^\infty(0,\infty;L^1(\mathbb{R}^N))$, we get, as in the proof of Proposition 4,

$$\frac{d}{ds}\|v(s)\|_q^q+C'_q\|v(s)\|_q^{q(N(q-1)+2)/N(q-1)}\leqslant C'_q(\|h\|_q^q+\|v(s)\|_q^q),$$

and therefore

$$\frac{d}{ds}\|v(s)\|_q^q+\frac{C'_q}{2}\|v(s)\|_q^{q(N(q-1)+2)/N(q-1)}\leqslant C''_q(1+\|h\|_q^q), \qquad \forall s>0.$$

As we pointed out in the proof of Proposition 4, this differential inequality implies that

$$v(s) \in L^\infty(0, \infty; L^q(\mathbb{R}^N)), \qquad \forall q \in [1, \infty), \tag{4.14}$$

which is equivalent to

$$t^{(N/2)(1-1/q)} u(t) \in L^\infty(0, \infty; L^q(\mathbb{R}^N)), \qquad \forall q \in [1, \infty) \tag{4.15}$$

with u a solution of (1.1).

Step 4. Let us now prove that

$$v \in L^\infty(\mathbb{R}^N \times (0, \infty)), \tag{4.16}$$

which is equivalent to

$$t^{N/2} u(t) \in L^\infty(\mathbb{R}^N \times (0, \infty)). \tag{4.17}$$

Let us write u as $u = z + w$ where w solves

$$\begin{aligned} w_t - \Delta w &= (t+1)^{-N/2-1} h\left(\frac{x}{\sqrt{t}}\right) \\ w(0) &= u_0 \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} z_t - \Delta z &= a \cdot \nabla(|u|^{1/N} u) \\ z(0) &= 0. \end{aligned} \tag{4.19}$$

Let us assume for the moment that

$$t^{(N/2)(1-1/p)} w(t) \in L^\infty(0, \infty; L^p(\mathbb{R}^N)), \qquad \forall p \in [1, \infty]. \tag{4.20}$$

Then, it is sufficient to prove that

$$t^{N/2} z(t) \in L^\infty(\mathbb{R}^N \times (0, \infty)). \tag{4.21}$$

For any $t > 0$, z satisfies the following integral equation:

$$z(2t) = \int_0^t a \cdot \nabla G(t-s) * (|u|^{1/N} u(s+t))\, ds.$$

Taking L^∞-norms in this identity and using (4.15) we get

$$\|z(2t)\|_\infty \leqslant |a| \int_0^t \|\nabla G(t-s)\|_r \|u(s+t)\|_{(N+1)r/N(r-1)}^{(N+1)/N} ds$$

$$\leqslant C_r t^{-(1/2+N/2r)} \int_0^t (t-s)^{-(N/2)(1-1/r)-(1/2)} ds$$

for any $r>1$ with $(N/2)(1-1/r)+1/2<1$, and therefore (4.21) follows.

In order to prove (4.20) we observe that, passing to the similarity variables, (4.20) is equivalent to the fact that the solution ω of

$$\omega_s + L\omega - \frac{N}{2}\omega = h$$

$$\omega(0) = u_0$$

belongs to $L^\infty(0, \infty; L^p(\mathbb{R}^N))$ for all $p \in [1, \infty]$.

Proceeding as in the proof of Proposition 5 and Step 3 above, it is easy to prove that

$$\omega \in L^\infty(0, \infty; L^2(K)), \qquad \omega \in L^\infty(s_0, \infty; H^1(K)), \qquad \forall s_0 > 0.$$

In particular, $\nabla\omega \in L^\infty(s_0, \infty; L^1(\mathbb{R}^N))$. Then, writing the equation satisfied by $\partial_i \omega$ and proceeding as in Step 3 above for all $i \in \{1, ..., N\}$, one deduces that

$$\nabla\omega \in L^\infty(s_0, \infty; L^p(\mathbb{R}^N)), \qquad \forall p \in [1, \infty), \forall s_0 > 0, \tag{4.22}$$

and this implies (4.20).

Step 5. $L^2(K)$-estimate.

Let us denote by $(\cdot, \cdot)$ the scalar product in $L^2(K)$. Integrating (4.1) over $\mathbb{R}^N$, we get

$$(v(s), \varphi_1) = c \int v(y, s)\, dy = c \int u_0(y)\, dy = cM, \qquad \forall s \geqslant 0,$$

and therefore

$$v(y,s) = M\varphi_1(y) + \tilde{v}(y, s), \qquad \forall y \in \mathbb{R}^N, s \geqslant 0 \text{ with } \tilde{v}(s) \in E_1^\perp, \forall s > 0. \tag{4.23}$$

Observe that $E_1^\perp = \{\psi \in L^2(K): \int \psi = 0\}$.

In view of (4.23), (4.9) is equivalent to

$$\tilde{v} \in L^{\infty}(0, \infty; L^2(K)). \tag{4.24}$$

Observe that

$$\left(\left(L-\frac{N}{2}I\right)\psi, \psi\right) \geqslant \frac{1}{N+1} \|\psi\|^2_{H^1(K)} \qquad \forall \psi \in E_1^{\perp} \cap H^1(K).$$

Multiplying in (4.1) by vK and integrating in $\mathbb{R}^N$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\|\tilde{v}(s)\|^2_{L^2(K)} + \frac{1}{N+1}\|\tilde{v}(s)\|^2_{H^1(K)} \\ &\quad \leqslant \int (a\cdot\nabla(|v|^{1/N}v) + h)\, vK\,dy \\ &\quad \leqslant \left(\frac{N+1}{N}\right) |a|\ \|v\|^{1/N}_{L^{\infty}(\mathbb{R}^N\times(0,\infty))} \|\nabla v\|_{L^2(K)} \|v\|_{L^2(K)} + \|h\|_{L^2(K)} \|v\|_{L^2(K)}. \end{aligned} \tag{4.25}$$

Combining (4.3), (4.14), and (4.25), we get

$$\frac{d}{ds}\|\tilde{v}(s)\|^2_{L^2(K)} + \frac{1}{N+1}\|\tilde{v}(s)\|^2_{H^1(K)} \leqslant C, \tag{4.26}$$

which implies (3.24).

Step 6. $H^1(K)$-estimate.
Given any $s_1 > 0$, v satisfies the following integral identity:

$$\begin{aligned} v(s+s_1)) = S(s)\,v(s_1) + \int_0^s S(s-\sigma)\, a\cdot\nabla(|v|^{1/N}v(s_1+\sigma))\,d\sigma \\ + \int_0^s S(s-\sigma)\,h\,d\sigma. \end{aligned}$$

Taking $H^1(K)$-norms in this identity and using (4.11) and (4.14), we get

$$\begin{aligned} \|v(s+s_1)\|_{H^1(K)} \leqslant C\left(1+\frac{1}{\sqrt{s}}\right) \|v(s_1)\|_{L^2(K)} \\ + C\int (1+(s-\sigma)^{-1/2})\, \|v(s_1+\sigma)\|_{H^1(K)}\,d\sigma \\ + C\,\|h\|_{L^2(K)} \int_0^s (1+(s+\sigma)^{-1/2})\,d\sigma. \end{aligned}$$

Applying Gronwall's inequality, we deduce that

$$\|v(s+s_1)\|_{H^1(K)} \leqslant C\left\{\left(1+\frac{1}{\sqrt{s}}\right)\|v(s_1)\|_{L^2(K)}+\sqrt{s}\,\|h\|_{L^2(K)}\right\}$$
$$\forall s_1>0,\ s\in(0,s_1). \tag{4.27}$$

Combining (4.9) and (4.27), (4.10) follows.

Step 7. $W^{1,p}(\mathbb{R}^N)$-estimates.
In view of (4.14), (4.8) is equivalent to

$$\nabla v\in L^{\infty}(s_0,\infty;L^p(\mathbb{R}^N)),\qquad \forall p\in[1,\infty),\ \forall s_0>0, \tag{4.28}$$

and this is a consequence of

$$t^{(N/2)(1-1/p)+1/2}\,\nabla u(t)\in L^{\infty}(t_0,\infty,L^p(\mathbb{R}^N)),\qquad \forall p\in[1,\infty),\ \forall t_0>0. \tag{4.29}$$

From (4.10) we know that (4.29) holds for all $p\in[1,2]$. In order to prove this estimate for $p\in(2,\infty)$ we return to the decomposition $u=w+z$ where w and z are respectively solutions of (4.18) and (4.19). In view of (4.22) we know that w satisfies

$$t^{(N/2)(1-1/p)+1/2}\,\nabla w(t)\in L^{\infty}(t_0,\infty,L^p(\mathbb{R}^N)),\qquad \forall p\in[1,\infty),\ \forall t_0>0. \tag{4.30}$$

Therefore it is sufficient to prove it for z. For any $t_0>0$ we have

$$\nabla z(2t+t_0)=\int_0^t \nabla G(t-s)*\left[\frac{N+1}{N}\,|u|^{1/N}(s+t+t_0)\,a\cdot\nabla u(s+t+t_0)\right]ds.$$

Taking $L^r(\mathbb{R}^N)$-norms in this identity, we get

$$\|\nabla z(2t+t_0)\|_r\leqslant C\int_0^t \|\nabla G(t-s)\|_q\,\|u(s+t+t_0)\|_{\infty}^{1/N}\,\|\nabla u(s+t+t_0)\|_2\,ds$$

with $(1/q)+(1/2)=1+(1/r)$. Combining (4.17) and (4.29) with $p=2$ we get

$$\|\nabla z(2t+t_0)\|_r\leqslant Ct^{-(N/4)-1}\int_0^t (t-s)^{-(N/2)(1-1/q)-(1/2)}ds$$
$$\leqslant Ct^{-(N/2)(1-1/r)-(1/2)}$$

if $(N/2)(1-1/q)+1/2<1$, i.e., $q>N/(N-1)$. In this way we get (4.29) for any $p<2N/(N-2)$. A standard boot-strap argument allows us to conclude (4.29) for all $p\in[1,\infty)$. ∎

Remark 4.1. As we pointed out in the proof of Proposition 6, (4.8) provides sharp decay estimates for u, namely,

$$\|u(t)\|_p \leqslant C_p t^{-(N/2)(1-1/p)}, \qquad \forall t>0, \forall p\in[1,\infty]$$
$$\|\nabla u(t)\|_p \leqslant C_{p,t_0} t^{-(N/2)(1-1/p)-(1/2)}, \qquad \forall t\geqslant t_0, \forall p\in[1,\infty].$$

In Step 3 of the proof of Proposition 6 we have only used that

$$u_0\in L^1(\mathbb{R}^N), \qquad h\in L^1(\mathbb{R}^N;1+|x|)\cap L^\infty(\mathbb{R}^N),$$
$$\int h(y)\,dy=0,\ \|h\|_{L^1(\mathbb{R}^N;|x|)}<\varepsilon_0. \tag{4.31}$$

Therefore, under assumption (4.31) we have

$$t^{(N/2)(1-1/p)}u(t)\in L^\infty(0,\infty;L^p(\mathbb{R}^N)), \qquad \forall p\in[1,\infty).\ ∎ \tag{4.32}$$

5. Proofs of Theorems 1 and 2

This section is devoted to the proofs of Theorems 1 and 2. We proceed in several steps.

Step 1. The case u_0, $h\in L^2(K)\cap L^\infty(\mathbb{R}^N)$.

Assume that $u_0, h\in L^2(K)\cap L^\infty(\mathbb{R}^N)$ with $\int h(x)\,dx=0$ and $\|h\|_{L^1(\mathbb{R}^N;|x|)}$ being small enough so that, by Proposition 5, the solution u of (1.1) is bounded in $L^1(\mathbb{R}^N)$.

We will apply Proposition 2 to the semigroup generated by system (4.1). We set $Z=L^2(K)\cap L^\infty(\mathbb{R}^N)$ endowed with the natural norm. By Proposition 6 we know that

$$v\in L^\infty(0,\infty;L^2(K)\cap L^\infty(\mathbb{R}^N))$$
$$v\in L^\infty(s_0,\infty;H^1(K)\cap W^{1,p}(\mathbb{R}^N)), \qquad \forall s_0>0, \forall p\in[1,\infty).$$

As a consequence of the following lemma, it follows that $\{v(s)\}_{s\geqslant 0}$ is relatively compact in Z.

LEMMA 3. *Let $(v_n)_n$ be a bounded sequence in the space $H^1(K)\cap W^{1,p}(\mathbb{R}^N)$. Then, for any $\varepsilon>0$ and $1\leqslant q<p$ there exists $R=R(\varepsilon,q)>0$ such that, for all $n\geqslant 1$ one has $\|v_n\|_{W^{1,q}([|x|>R])}\leqslant\varepsilon$. As a consequence, the imbedding $H^1(K)\cap W^{1,p}(\mathbb{R}^N)\subset L^2(K)\cap L^\infty(\mathbb{R}^N)$ is compact for any $p>N$.*

The proof of this lemma is straightforward.
On the other hand,

$$\int v(y, s)\, dy = \int u_0(y)\, dy \qquad \forall s \geqslant 0.$$

Therefore, we may take

$$\psi(z) = \int z(y)\, dy \qquad \forall z \in Z.$$

Let us now construct the function ϕ satisfying (2.6). We set

$$\phi(z_1, z_2) = \int (z_1(y) - z_2(y))^+\, dy, \qquad \forall z_1, z_2 \in Z. \tag{5.1}$$

Assume for the moment that ϕ satisfies (2.6), i.e., given $v_{0,1}, v_{0,2} \in Z$ with $\int v_{0,1}(y)\, dy = \int v_{0,2}(y)\, dy$, then if $v_{0,1} \neq v_{0,2}$ and v_1, v_2 are the corresponding solutions of (4.1):

$$\phi(v_1(s), v_2(s)) \text{ is strictly decreasing with respect to } s. \tag{5.2}$$

Then, as a consequence of (2.6), we deduce that (1.14) has a unique solution $f_M \in L^2(K) \cap L^\infty(\mathbb{R}^N)$ for all $M \in \mathbb{R}$ and that

$$v(s) \to f_M \text{ in } L^2(K) \cap L^\infty(\mathbb{R}^N) \qquad \text{as} \quad s \to \infty \tag{5.3}$$

with $M = \int u_0(y)$. Then by elliptic regularity one easily sees that $f_M \in H^2(K)$. On the other hand, since $w_M(x, t) = (t+1)^{-N/2} f_M(x/\sqrt{t+1})$ solves (1.1) with $u_0 = f_M$, by Proposition 4, we deduce that $f_M \in W^{2,p}(\mathbb{R}^N)$ for all $p \in (1, \infty)$.

As we pointed out in the Introduction, (5.3) implies (1.15) for all $r \in [1, \infty]$. On the other hand, from (4.8), (4.10) and (5.3), by interpolation, we deduce that

$$v(s) \to f_M \text{ in } W^{1,r}(\mathbb{R}^N) \text{ as } s \to \infty, \qquad \text{for all} \quad r \in [1, \infty),$$

and this implies (1.18).

Therefore, once (2.6) is proved, Theorem 1 will be proved for $h \in L^2(K) \cap L^\infty(\mathbb{R}^N)$ satisfying (1.12)–(1.13) and Theorem 2 for $u_0 \in L^2(K) \cap L^\infty(\mathbb{R}^N)$.

Step 2. The case $u_0 \in L^1(\mathbb{R}^N)$, $h \in L^2(K) \cap L^\infty(\mathbb{R}^N)$.

In view of (1.17), and by the density of $L^2(K) \cap L^\infty(\mathbb{R}^N)$ in $L^1(\mathbb{R}^N)$, we deduce that

$$\|u(t) - w_M(t)\|_1 \to 0 \qquad \text{as} \quad t \to \infty \tag{5.4}$$

for every $u_0 \in L^1(\mathbb{R}^N)$ with $\int u_0(x)\, dx = M$.

Combining (4.32) and (5.4), by interpolation we obtain (1.15) for every $r \in [1, \infty)$.

Step 3. Strictly decreasing character of ϕ.

As we have seen in Steps 1 and 2 above, Theorems 1 and 2 will be proved if we show that (5.2) holds.

Let be $v_{0,1}, v_{0,2} \in L^2(K) \cap L^\infty(\mathbb{R}^N)$ with $\int v_{0,1}(y)\, dy = \int v_{0,2}(y)\, dy$ and set for simplicity

$$\varphi(s) = \phi(v_1(s), v_2(s)) = \int (v_1(y, s) - v_2(y, s))^+ \, dy,$$

where v_1 and v_2 are the solutions of (4.1) with initial data $v_{0,1}$ and $v_{0,2}$, respectively.

Multiplying by $\operatorname{sgn}(v_1 - v_2)^+$ the equation satisfied by $v_1 - v_2$, integrating over $\mathbb{R}^N$ and using Kato's inequality, we get

$$\varphi'(s) = \int \frac{d}{ds}(v_1 - v_2) \operatorname{sgn}(v_1 - v_2)^+ \, dy \leqslant 0$$

and therefore φ is nonincreasing. Moreover, we have

$$\varphi'(s) = \int_{\Omega(s)} \Delta(v_1(y, s) - v_2(y, s))\, dy \tag{5.5}$$

where $\Omega(s) = \{y \in \mathbb{R}^N : v_1(y, s) > v_2(y, s)\}$.

In view of (4.2) we know that $v_1, v_2 \in C(\mathbb{R}^N \times (0, \infty))$ and therefore $\Omega(s)$ is an open set for all $s > 0$.

On the other hand, both $\Omega(s)$ and $\Omega^*(s) = \mathbb{R}^N \backslash \overline{\Omega(s)}$ are nonempty for all $s > 0$. Indeed, assume that $\Omega(s_0) = \varnothing$ (or $\Omega^*(s_0) = \varnothing$) for some $s_0 > 0$. Then, since

$$\int v_1(y, s_0)\, dy = \int v_2(y, s_0)\, dy,$$

we have $v_1(s_0) = v_2(s_0)$. By uniqueness of solutions of (4.1) we deduce that $v_1(s) = v_2(s)$ for every $s \geqslant s_0$.

In order to show that $v_1(s) = v_2(s)$ for $s \in [0, s_0]$ we use backward uniqueness results. We set $w(s) = v_1(s) - v_2(s)$. Then, by (4.7)

$$w \in C([s_1, s_0]; H^2(K)) \qquad \text{for all} \quad s_1 \in (0, s_0)$$

and

$$w_s - \Delta w - y \cdot \frac{\nabla w}{2} - \frac{N}{2} w = a \cdot \nabla(b(y, s) w) \qquad \text{in} \quad \mathbb{R}^N \times (s_1, s_0)$$

$$w(s_0) = 0,$$

with

$$b(y, s) = \frac{|v_1|^{1/N} v_1(y, s) - |v_2|^{1/N} v_2(y, s)}{w(y, s)}$$

In view of (4.7) we have $b \in L^\infty(\mathbb{R}^N \times (s_1, s_0))$ and therefore

$$\begin{aligned} \|a \cdot \nabla(b(s)\, w(s))\|_{(H^1(K))^*} &\leqslant |a| \; \|b(s)\, w(s)\|_{L^2(K)} \\ &\leqslant |a| \; \|b(s)\|_\infty \; \|w(s)\|_{L^2(K)}. \end{aligned}$$

Applying Theorem 1.1 of Ghidaglia [G] with $A = L - (N/2) I$, $V = L^2(K)$, $D(A) = H^1(K)$, and $H = (H^1(K))^*$, we deduce that $w(s) = 0$ for all $s \in [s_1, s_0]$ and this for all $s_1 > 0$. Therefore $v_1(s) = v_2(s)$ for every $s > 0$ and since $v_1, v_2 \in C([0, \infty); L^2(K))$ we deduce that $v_{0,1} = v_{0,2}$.

Let us now see that φ is strictly decreasing. If not, by (5.5), there exist $0 \leqslant s_0 < s_1$ such that

$$\int_{s_0}^{s_1} \int_{\Omega(s)} \Delta(v_1(y, s) - v_2(y, s))\, dy\, ds = 0. \tag{5.6}$$

Now define

$$w(y, s) = (v_1(y, s) - v_2(y, s))\, \chi_\Omega, \qquad \forall s \in (s_0, s_1), \forall y \in \mathbb{R}^N,$$

with χ_Ω the characteristic function of the set

$$\Omega = \bigcup_{s \in (s_0, s_1)} \Omega(s) \times \{s\} = \{(y, s) \in \mathbb{R}^N \times (s_0, s_1) : v_1(y, s) > v_2(y, s)\}. \tag{5.7}$$

We claim that it is sufficient to prove that $w \equiv 0$. Indeed, then $\Omega = \varnothing$ and therefore $\Omega(s) = \varnothing$ for all $s \in (s_0, s_1)$ which contradicts the fact that $v_{0,1} \neq v_{0,2}$, as we have seen above.

At was shown in [AEZ, EZ2], from (5.6) we obtain that

$$w \in C([s_0, s_1]; H^2(K))$$

and that w solves

$$\begin{gathered} w_s - \Delta w - \frac{y \cdot \nabla w}{2} - \frac{N}{2} w = a \cdot \nabla(b(y, s) w) \quad \text{in} \quad \mathbb{R}^N \times (s_0, s_1) \\ w(s_0) = (v_1(s_0) - v_2(s_0))^+, \end{gathered} \tag{5.8}$$

with

$$b = \frac{|v_1|^{1/N} v_1 - |v_2|^{1/N} v}{w} \chi_\Omega \in L^\infty(\mathbb{R}^N \times (s_0, s_1)). \tag{5.9}$$

Clearly Ω is a open subset of $\mathbb{R}^N \times (s_0, s_1)$ since $v_1, v_2 \in C(\mathbb{R}^N \times (0, \infty))$. On the other hand, as we observed above, Ω must be non empty. Then, there exists a ball $B(y_0, \varepsilon)$ of $\mathbb{R}^N$ and some $s_2, s_3 \in (s_0, s_1)$ with $s_2 < s_3$ such that

$$B(y_0, \varepsilon) \times (s_2, s_3) \subset \Omega.$$

Let

$$\eta(y, s) = \int_0^{((y_0 - y) \cdot a)/|a|^2} w(y + ta, s)\, dt. \tag{5.10}$$

In this identity, $\cdot$ and $|\cdot|$ denote the scalar product and the norm in $\mathbb{R}^N$, respectively. In (5.10) we have a line integral over the segment relying y to $y + (((y - y_0) \cdot a)/|a|^2) a$, its projection to the hyperplane $y \cdot a = y_0 \cdot a$ containing y_0 and orthogonal to the vector a.

If $y \in \mathbb{R}^N$ belongs to the cylinder

$$\zeta = \left\{ y \in \mathbb{R}^N : \left| y + \frac{a}{|a|^2} ((y_0 - y) \cdot a) - y_0 \right| < \varepsilon \right\}$$

then, integrating over this segment the equation (5.8) we obtain that

$$\eta_s - \Delta \eta - y \cdot \frac{\nabla \eta}{2} - \frac{(N-1)}{2} \eta = -ba \cdot \nabla \eta \quad \text{in} \quad \xi \times (s_2, s_3).$$

On the other hand,

$$\eta = 0 \quad \text{in} \quad [\zeta \cap B(y_0, \varepsilon)] \times (s_2, s_3).$$

Therefore, since $b \in L^\infty$ and η is smooth, by parabolic unique continuation (see Hörmander [Ho, Theorem 8.9.1, p. 224]), we deduce that

$$\eta \equiv 0 \quad \text{in} \quad \zeta \times (s_2, s_3)$$

and therefore

$$w(y, s) = -a \cdot \nabla\eta(y, s) \equiv 0 \quad \text{in} \quad \zeta \times (s_2, s_3). \tag{5.11}$$

Given any $y \in \mathbb{R}^N$ we write

$$y = \left(\bar{y}, \frac{(y \cdot a)}{|a|}\right)$$

where $\bar{y}$ is an $(N-1)$-dimensional vector on the hyperplane Π orthogonal to a and containing the origin.

Let us define

$$\rho(\bar{y}, s) = \int_{-\infty}^{\infty} w(\bar{y}, \tau, s)\, d\tau = \int_{-\infty}^{\infty} w(y + ta, s)\, dt.$$

From (4.11) we deduce that

$$\rho(\bar{y}, s) = 0 \qquad \forall \bar{y} \in \Pi \cap B(0, \varepsilon), \forall s \in (s_2, s_3).$$

On the other hand, integrating (5.8) with respect to the direction a, we get

$$\rho_s - \Delta_{\bar{y}}\rho - \bar{y} \cdot \frac{\nabla\rho}{2} - \left(\frac{N-1}{2}\right)\rho = 0 \qquad \forall \bar{y} \in \Pi, \forall s \in (s_2, s_3).$$

By unique continuation we get

$$\rho \equiv 0 \quad \text{in} \quad \Pi \times (s_2, s_3),$$

and since $w \geqslant 0$ we deduce

$$w \equiv 0 \quad \text{in} \quad \mathbb{R}^N \times (s_2, s_3).$$

Therefore (5.2) holds.

Step 4. Self-similar solutions for $h \in L^1(\mathbb{R}^N; 1 + |x|) \cap L^\infty(\mathbb{R}^N)$.

Let us approximate h in $L^1(\mathbb{R}^N; 1 + |x|) \cap L^\infty(\mathbb{R}^N)$ by a sequence $h_\varepsilon \in L^2(K) \cap L^\infty(\mathbb{R}^N)$ such that

$$\begin{aligned} &\text{(i)} && \int h_\varepsilon(y)\, dy = 0 \\ &\text{(ii)} && \|h_\varepsilon\|_\infty \leqslant C, \qquad \forall \varepsilon > 0 \\ &\text{(iii)} && h_\varepsilon \to h \text{ in } L^1(\mathbb{R}^N; 1 + |x|) \qquad \text{as} \quad \varepsilon \to 0. \end{aligned} \tag{5.12}$$

Since h is assumed to be small in $L^1(\mathbb{R}^N; |x|)$, by (5.12) (iii) we may assume it also for h_ε. Then, by Step 1, for every $M \in \mathbb{R}$ and $\varepsilon > 0$ there exists a unique self-similar profile satisfying

$$\begin{gathered} Lf_{M,\varepsilon} - \frac{N}{2} f_{M,\varepsilon} = a \cdot \nabla(|f_{M,\varepsilon}|^{1/N} f_{M,\varepsilon}) + h_\varepsilon \\ \int f_{M,\varepsilon}(y)\,dy = M, \qquad f_{M,\varepsilon} \in H^2(K) \cap L^\infty(\mathbb{R}^N). \end{gathered} \tag{5.13}$$

For any $M \in \mathbb{R}$ let us fix some $u_0 \in L^2(K) \cap L^\infty(\mathbb{R}^N)$ such that $\int u_0(x)\,dx = M$. By Step 1, we know that

$$v_\varepsilon(s) \to f_{M,\varepsilon} \text{ in } L^1(\mathbb{R}^N) \qquad \text{as} \quad s \to \infty,$$

where v_ε is the solution of (4.1) with right-hand side h_ε and initial data u_0. By Remark 4.2 we know that v_ε is uniformly bounded in $L^\infty(0, \infty; L^p(\mathbb{R}^N))$ for every $p \in [1, \infty]$; we deduce that

$$\|f_{M,\varepsilon}\|_p \leqslant C(M, p), \qquad \forall \varepsilon > 0, \forall p \in [1, \infty). \tag{5.14}$$

On the other hand, $w_{M,\varepsilon}(x, t) = (t+1)^{-N/2} f_{M,\varepsilon}(x/\sqrt{t+1})$ is solution of (1.1) with right-hand side h_ε. Therefore, by parabolic regularity and using (5.12), (5.14), we deduce that $w_{M,\varepsilon}$ is uniformly bounded in $L^\infty(0, \infty; W^{2,p}(\mathbb{R}^N))$ for every $p \in (1, \infty)$. This implies that

$$\|f_{M,\varepsilon}\|_{W^{2,p}(\mathbb{R}^N)} \leqslant C(M, p), \qquad \forall \varepsilon > 0, \forall p \in (1, \infty). \tag{5.15}$$

On the other hand, multiplying in (5.13) by $\operatorname{sgn}(f)\,|x|$ and integrating over $\mathbb{R}^N$, we deduce that

$$\frac{N}{2} \int |f|\,|x|\,dx \leqslant -\int \frac{a \cdot x}{|x|} |f_{M,\varepsilon}|^{1+1/N}\,dx + \int |h_\varepsilon|\,|x|\,dx.$$

In view of (5.12)(iii) and (5.14), we deduce that $f_{M,\varepsilon}$ is bounded in $L^1(\mathbb{R}^N; 1+|x|)$. This fact, combined with (5.15), allows us to deduce that, up to a subsequence,

$$f_{M,\varepsilon} \to f_M \text{ in } L^p(\mathbb{R}^N), \qquad \text{for every} \quad p \in [1, \infty],$$

where the limit $f_M \in L^1(\mathbb{R}^N) \cap W^{2,p}(\mathbb{R}^N)$ for every $p \in (1, \infty)$ solves (1.14).

Let us now sketch the proof of the uniqueness of the solution of (1.14). If f_M and $\tilde{f}_M$ are two distinct solutions of (1.14), then $w_M(x, t) = (t+1)^{-N/2} f_M(x/\sqrt{t+1})$ and $\tilde{w}_M(x, t) = (t+1)^{-N/2} f_M(x/\sqrt{t+1})$ are solutions of (1.1) with distinct initial data $w_M(0) = f_M$, $\tilde{w}_M(0) = \tilde{f}_M$.

Clearly

$$\varphi(t)=\int_{\mathbb{R}^N}(w_M(x,t)-\tilde{w}_M(x,t))^+\,dx$$

does not depend on t. However,

$$\varphi'(t)=\int_{\Omega(t)}\Delta(w_M(x,t)-\tilde{w}_M(x,t))^+\,dx,$$

with $\Omega(t)=\{x\in\mathbb{R}^N: w_M(x,t)>\tilde{w}_M(x,t)\}$. The unique continuation arguments of Step 3 applied directly in (1.1) show that φ must be strictly decreasing, the case where $w_M=\tilde{w}_M$ being excepted. This concludes the proof of the uniqueness.

Acknowledgments

The author gratefully acknowledges T. Cazenave and S. Kamin for fruitful discussions.

References

[AEZ] J. Aguirre, M. Escobedo, and E. Zuazua, Self-similar solutions of a convection-diffusion equation and related elliptic problems, *Comm. Partial Differential Equations* **15**, No. 2, (1990), 139–157.

[CL] I. L. Chern and T. P. Liu, Convergence to diffusion waves of solutions for viscous conservation laws, *Comm. Math. Phys.* **110** (1987), 503–517.

[EK1] M. Escobedo and O. Kavian, Variational problems related to self-similar solutions of the heat equation, *Nonlinear Anal. TMA* **11**, No. 10 (1987), 1103–1133.

[EK2] M. Escobedo and O. Kavian, Asymptotic behavior of positive solutions of a nonlinear heat equation, *Houston J. Math.* **13**, No. 4 (1987), 39–50.

[EZ1] M. Escobedo and E. Zuazua, Comportement asymptotique des solutions d'une équation de convection–diffusion, *C. R. Acad. Sci. Paris* **309** (1989), 329–334.

[EZ2] M. Escobedo and E. Zuazua, Large time behavior for convection–diffusion equations in $\mathbb{R}^N$, *J. Funct. Anal.* **100** (1991), 119–161.

[G] J. M. Ghidaglia, Some backward uniqueness results, *Nonlinear Anal. TMA* **10** (1986), 777–790.

[H] A. Haraux, "Systèmes dynamiques, processus et applications aux équations aux dérivées partielles," Masson, RMA 17, 1991.

[Ho] L. Hörmander, "Linear Partial Differential Operators," Springer-Verlag, Berlin/New York, 1976.

[K] O. Kavian, Remarks on the large time behavior of a nonlinear diffusion equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **4**, No. 5 (1987), 423–452.

[KW] S. Kawashima, Large-time behaviour of solutions to hyperbolic–parabolic systems of conservation laws and applications, *Proc. Roy. Soc. Edinburgh Sect. A* **106A** (1987), 169–194.

[L] T. P. Liu, Nonlinear stability of shock waves for viscous conservation laws, *Mem. Amer. Math. Soc.* **56**, No. 328 (1985), 1–108.

[S1] M. E. Schonbek, Decay of solutions to parabolic conservation laws, *Comm. Partial Differential Equations* **7**, No. 1 (1980), 449–473.

[S2] M. E. Schonbek, Uniform decay rates for parabolic conservation laws, *Nonlinear Anal. TMA* **10**, No. 9 (1986), 943–956.