ARMA manuscript No.
(will be inserted by the editor)

Long Time Behavior of a Coupled Heat-Wave System Arising in Fluid-Structure Interaction

Xu Zhang, Enrique Zuazua

Contents

Abstract

This paper is devoted to analyze the long time behavior of a coupled wave-heat system in which a wave and a heat equation evolve in two bounded domains, with natural transmission conditions at a common interface. These conditions couple, in particular, the heat unknown with the velocity of the wave solution. This model may be viewed as a simplified version of linearized models arising in fluid-structure interaction. First, we show the strong asymptotic stability of solutions to this system. Then, based on the construction of ray-like solutions by means of Geometric Optics expansions and a careful analysis of the transfer of energy at the interface, we show the lack of uniform decay in general domains. Further, we obtain a polynomial decay result for smooth solutions of the system under a suitable geometric assumption guaranteeing that the heat domain envelopes the wave one. Finally, in the absence of geometric conditions we show a logarithmic decay result for the same system but with simplified transmission conditions at the interface. We also analyze the difficulty to extend this result to the more natural transmission conditions.

1. Introduction

1.1. Motivation, description of the model and formulation of the problem

Fluid-structure interaction models describe the dynamics of fluids contained inside or in contact with elastic structures, possibly along a free surface. The fluid-structure coupling is realized by matching the normal surface velocities of the elastic body and the fluid. In recent decades, this sort of problems have been extensively studied and continue to be the focus of much attention today, especially from the computational point of view ([4], [7], [14], [18], [21], [24], [26], [27], [33], [34], [35], [46], [51]). The applications include the determination of the acoustic pressure induced in an infinite medium by a totally submerged vibrating structure, the production and absorption of sound occurring when unsteady flow interacts with solid bodies, piston problems, biomedical flows in flexible pipes, ocean flows around very long risers, biomechanical systems and novel mechanical structures such as micromechanical devices, the parachute soft-landing dynamics, and so on. The book [34] and the special issue [35] give accounts of the state of the art from the engineering point of view.

The purpose of this paper is to study the long time behavior of a coupled system consisting in a wave and heat equations coupled through transmission condition along a steady interface. This model is a linearized and simplified version of more sophisticated models arising in fluid-structure interaction. Despite the effort that has been done in recent years and that has led to some significant results, mainly in what concerns existence of solutions ([3], [5], [8], [9], [10], [12], [13], [17], [19], [20], [22], [43], [44], [48], [49]), very little is known about the asymptotic behavior of solutions as time $t \to \infty$. One of the main difficulties in the analysis of these models is that they are in fact free-boundary problems, with the free boundary being the interface between the fluid and the elastic body, in which the interface evolves as time increases. In fact very few results exist on the uniqueness and global existence of solutions because of the possible collision of the interface with exterior boundaries. We refer to [52] for some of the few results in this direction concerning a $1-d$ model coupling the Burgers equation with the motion of a finite number of solid point-masses.

In this paper we deal with a linearized model for which the main goal is to perform a fine analysis of the asymptotic behavior of solutions as $t \to \infty$. After linearization around the trivial solution the interface does not move and is independent of time. The model under consideration then consists in the coupling of two different equations in two adjacent domains separated by an interface. The model we consider here couples the wave and heat equations as a prototype of more realistic models arising in this field in which the system of elasticity is coupled with the Navier- Stokes or Stokes equations. A similar model has been considered in [40] but with simpler transmission conditions. The main novelty of the model we consider in this article consists in the transmission conditions at the interface which are much more realistic and for which, the time derivative of the wave-solution is coupled with the heat-one. This type of interface condition is indeed more physical in fluid-structure interaction since both the fluid solution and the time-derivative of the elastic one constitute velocity fields (of motion of the fluid and deformations of the elastic body, respectively).

This paper is mainly devoted to analyze the rate of decay of energy of solution as $t \to \infty$. The total energy of solution is the superposition of the elastic energy and the heat one. But the energy is only dissipated through the heat domain, a fact that represents the mixed nature of the model under consideration: the wave equation is conservative while the heat one is strongly dissipative. The rate of decay of the energy constitutes a reliable test of the degree of coupling of the two components entering in the system in the sense that, if the coupling were strong enough, one could expect that the dissipative mechanism introduced by the heat equation would suffice to guarantee the uniform exponential decay of solutions. As we shall see this is not the case. The main results we prove in this paper are roughly as follows:

- Regardless whatever the geometric configuration is, the rate of decay is never uniform;
- Under suitable geometric conditions smooth solutions decay polynomially.

We also show that, for the simpler system considered in [40], smooth solutions decay logarithmically without any geometric restrictions.

Let us now describe the model under consideration and the main results in more detail.

Let $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a bounded domain with C^2 boundary $\Gamma = \partial\Omega$. Let Ω_1 be a sub-domain of Ω and set $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. We denote by γ the interface, $\Gamma_j = \partial\Omega_j \setminus \overline{\gamma}$ ($j = 1, 2$), and ν_j the unit outward normal vector to Ω_j. It is easy to see that $\Gamma = \overline{\Gamma_1 \cup \Gamma_2}$. Throughout this paper, we assume that γ is a nonempty open subset of $\partial\Omega_j$ with positive capacity (*see* [56] for the definition of capacity) and of class C^1 (unless otherwise stated).

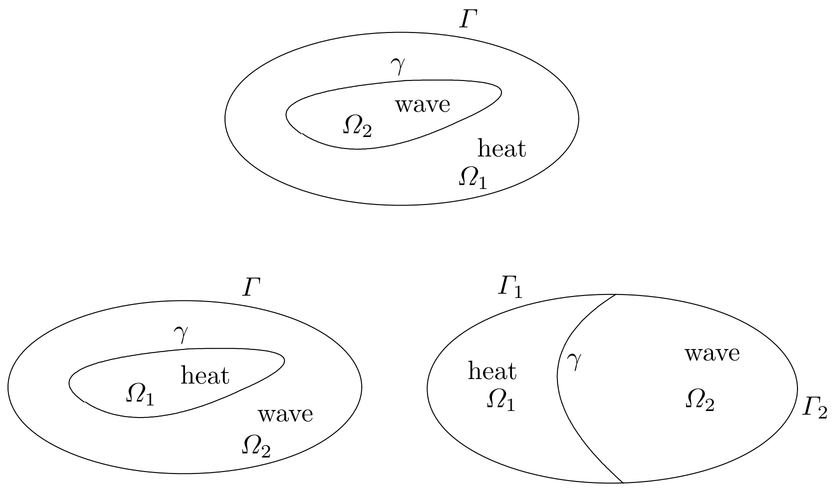

FIG. 1. GEOMETRIC DESCRIPTION OF THE FLUID-STRUCTURE INTERACTION MODEL

Denote by $\Box$ the d'Alembert operator $\partial_{tt} - \Delta$. The main purpose of this paper is to study the long time behavior of the following coupled heat-wave system:

$$
\begin{cases}
y_t - \Delta y = 0 & \text{in } (0,\infty)\times\Omega_1,\\
\Box z = 0 & \text{in } (0,\infty)\times\Omega_2,\\
y = 0 & \text{on } (0,\infty)\times\Gamma_1,\\
z = 0 & \text{on } (0,\infty)\times\Gamma_2,\\
y = z_t, \quad \dfrac{\partial y}{\partial \nu_1} = -\dfrac{\partial z}{\partial \nu_2} & \text{on } (0,\infty)\times\gamma,\\
y(0) = y_0 & \text{in } \Omega_1,\\
z(0) = z_0, \quad z_t(0) = z_1 & \text{in } \Omega_2.
\end{cases}
\tag{1.1}
$$

This is a simplified and linearized model for fluid-structure interaction, which consists of a wave and a heat equation coupled through an interface with transmission conditions. In system (1.1), y may be viewed as the velocity of the fluid; while z and z_t represent respectively the displacement and velocity of the structure. As mentioned above, more realistic models should involve the Stokes (*resp.* the elasticity) equations instead of the heat (*resp.* the wave) ones. A model similar to (1.1), but including the pressure of the fluid, is mentioned in the classical book of Dautray and Lions [11, XVIII, §7.5]. In [40], the same system was considered but for the transmission condition $y = z$ on the interface instead of $y = z_t$, i.e.,

$$
\begin{cases}
y_t - \Delta y = 0 & \text{in } (0,\infty)\times\Omega_1,\\
\Box z = 0 & \text{in } (0,\infty)\times\Omega_2,\\
y = 0 & \text{on } (0,\infty)\times\Gamma_1,\\
z = 0 & \text{on } (0,\infty)\times\Gamma_2,\\
y = z, \quad \dfrac{\partial y}{\partial \nu_1} = -\dfrac{\partial z}{\partial \nu_2} & \text{on } (0,\infty)\times\gamma,\\
y(0) = y_0 & \text{in } \Omega_1,\\
z(0) = z_0, \quad z_t(0) = z_1 & \text{in } \Omega_2.
\end{cases}
\tag{1.2}
$$

From the point of view of fluid-structure interaction, the transmission condition $y = z_t$ in (1.1) is more natural since y and z_t represent velocities of the fluid and the elastic body, respectively.

Let us now describe the functional setting in which system (1.1) is well-posed. Put

$$
H^1_{\Gamma_1}(\Omega_1) \stackrel{\triangle}{=} \Big\{ h|_{\Omega_1} \;\Big|\; h \in H^1_0(\Omega) \Big\}, \quad H^1_{\Gamma_2}(\Omega_2) \stackrel{\triangle}{=} \Big\{ h|_{\Omega_2} \;\Big|\; h \in H^1_0(\Omega) \Big\}. \tag{1.3}
$$

We refer to Theorem 1 for the existence and uniqueness of solutions of (1.1) in the Hilbert space

$$
H \stackrel{\triangle}{=} L^2(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2), \tag{1.4}
$$

which is the *finite energy space*, whose norm is given by

$$
\begin{aligned}
||f||_H = \sqrt{|f_1|^2_{L^2(\Omega_1)} + |f_2|^2_{L^2(\Omega_2)} + |\nabla f_2|^2_{(L^2(\Omega_2))^n} + |f_3|^2_{L^2(\Omega_2)}},\\
\forall\, f = (f_1, f_2, f_3) \in H.
\end{aligned}
\tag{1.5}
$$

The space H is asymmetric with respect to the wave and heat components since the regularity differs in one derivative from one side to the other. Recall that the finite energy space for system (1.2) is

$$\widetilde{H} \stackrel{\triangle}{=} \left\{(f_1, f_2) \in H^1_{\Gamma_1}(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \;\Big|\; f_1|_\gamma = f_2|_\gamma\right\} \times L^2(\Omega_2), \tag{1.6}$$

endowed with the canonical inner product. In this case, the regularity of the heat and wave components is the same.

When Γ_2 is a non-empty open subset of the boundary, the following semi-norm is in fact a norm on H, equivalent to the canonical one in (1.5):

$$|f|_H = \sqrt{|f_1|^2_{L^2(\Omega_1)} + |\nabla f_2|^2_{(L^2(\Omega_2))^n} + |f_3|^2_{L^2(\Omega_2)}}, \quad \forall\, f = (f_1, f_2, f_3) \in H. \tag{1.7}$$

In this case the only stationary solution is the trivial one and this simplifies the dynamical properties of the system. The analysis is simpler as well. The same can be said when Γ_2 has positive capacity since, then, the Poincaré inequality holds (We refer to [56] for this result). Note that when $\Gamma_2 = \emptyset$ or, more generally, when $\operatorname{Cap}\Gamma_2$, the capacity of Γ_2, vanishes, $|\cdot|_H$ defined by (1.7) is no longer a norm in H (hence one has to use the original norm $||\cdot||_H$ given in (1.5)). In this case, there are non-trivial stationary solutions of the system. Thus, the asymptotic behavior is more complex and one should rather expect the convergence of each individual trajectory with, possibly, different limit states.

In general, the energy of system (1.1) is defined as one half of the square of the semi-norm $|\cdot|_H$. More precisely, for any solution (y, z, z_t) of system (1.1), put

$$E(t) \equiv E(y, z, z_t)(t) \stackrel{\triangle}{=} \frac{1}{2}|(y(t), z(t), z_t(t))|^2_H. \tag{1.8}$$

If $\operatorname{Cap}\Gamma_2 > 0$, $E(\cdot)$ is coercive in H. When $\operatorname{Cap}\Gamma_2 = 0$, $E(\cdot)$ fails to be coercive, and it is convenient to consider the following augmented energy, which is coercive in H:

$$F(t) \equiv F(y, z, z_t)(t) \stackrel{\triangle}{=} \frac{1}{2}||(y(t), z(t), z_t(t))||^2_H \equiv E(t) + \frac{1}{2}|z(t)|^2_{L^2(\Omega_2)}. \tag{1.9}$$

By means of the classical energy method, it is easy to check that

$$\frac{d}{dt}E(t) = -\int_{\Omega_1} |\nabla y|^2 dx. \tag{1.10}$$

Therefore, the energy of (1.1) is decreasing with respect to t. The main problem we address in this paper is to analyze more precisely the asymptotic behavior of $E(t)$ as $t \to \infty$.

1.2. Description of main results

We now describe the main results in this paper.

To begin with, we shall show that if $\mathrm{Cap}\,\Gamma_2 > 0$, then $E(t) \to 0$ as $t \to \infty$, without any geometric conditions on the domains Ω_1 and Ω_2 (other than $\mathrm{Cap}\,(\gamma) > 0$). Note however that, unlike for system (1.2) in [40], due to the lack of compactness of the domain of the generator of the underlying semigroup of system (1.1) for $n \geq 2$, to prove this result one can not use directly the LaSalle's invariance principle. Instead, by means of the "relaxed invariance principle" ([45]), we conclude first that the first and third components of every solution (y, z, z_t) of (1.1), y and z_t, tend to zero strongly in $L^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively; while its second component z tends to zero weakly in $H^1_{\Gamma_2}(\Omega_1)$ as $t \to \infty$. Then, we use the special structure of (1.1) and the key energy dissipation law (1.10) to "recover" the desired strong convergence of z in $H^1_{\Gamma_2}(\Omega_1)$. In the case $\mathrm{Cap}\,\Gamma_2 = 0$ the dynamics of the system is more complex. Indeed, the boundedness of trajectories is not straightforward (*see* Theorem 1 ii)), and, for any constant $c \in \mathbb{R}$, $(0, c, 0)$ is a static solution of system (1.1). In this case, we show that each individual trajectory converges as $t \to \infty$ to one of these stationary solutions that can be fully identified in terms of the initial data. For this, we shall develop a systematic decomposition of the solution into the static component and the one converging to zero.

The main goal of this paper is to analyze further the longtime behavior of (1.1). Especially, we will study whether or not the energy of solutions of system (1.1) tends to zero uniformly as $t \to \infty$, i.e. whether there exist two positive constants C and α such that

$$E(t) \leq CE(0)e^{-\alpha t}, \qquad \forall\, t \geq 0 \tag{1.11}$$

for every solution of (1.1) when $\mathrm{Cap}\,\Gamma_2 > 0$. When $\mathrm{Cap}\,\Gamma_2 = 0$, one should replace (1.11) by

$$||(y(t), z(t) - c_1, z_t(t))||^2_H \leq CF(0)e^{-\alpha t}, \qquad \forall\, t \geq 0, \tag{1.12}$$

where $(0, c_1, 0)$ is a suitable static solution of (1.1) (which depends on the initial data (y_0, z_0, z_1)). The uniform exponential decay property (1.11) or (1.12) is the best one can expect for a dissipative semigroup. Whether (1.11) (or (1.12)) holds or not can be considered as a test of the robustness of the coupling between the two equations entering in the system, as explained above.

According to the energy dissipation law (1.10), when $\mathrm{Cap}\,\Gamma_2 > 0$, the uniform decay problem (1.11) is equivalent to showing that: there exists $T > 0$ and $C > 0$ such that every solution of (1.1) satisfies

$$|(y_0, z_0, z_1)|^2_H \leq C \int_0^T \int_{\Omega_1} |\nabla y|^2 dxdt, \qquad \forall\, (y_0, z_0, z_1) \in H. \tag{1.13}$$

Inequality (1.13) can be viewed as an observability estimate for system (1.1) with observation on the heat subdomain (We refer to [57] and the references cited therein for the state of art for the observability estimates for partial differential equations). When Cap $\Gamma_2 = 0$, the uniform decay problem (1.12) is also equivalent to showing a similar observability inequality (*see* (6.1) and (6.2)).

Note however that, as indicated in [55], there is no uniform decay for solutions of (1.1) even in one space dimension. This is due to the existence of a branch of eigenvectors which are mainly concentrated on the wave domain and consequently very weakly dissipated. The approach in [55], based on spectral analysis, does not apply to multidimensional situations. But the $1-d$ result in [55] is a warning in the sense that one may not expect (1.13) to hold.

The first topic we address in this paper is to show the lack of exponential decay in several space dimensions, i.e., that the exponential decay rate (1.11) (*resp.* (1.12)) never holds, as the $1-d$ spectral analysis suggests. As mentioned above, this is equivalent to showing that (1.13) (*resp.* (6.1)) fails to hold in the multidimensional case. For this purpose, following [40], we analyze carefully the interaction of the wave and heat-like solutions on the interface. The key ingredient in the proof is based on a Geometric Optics construction using plane waves or Gaussian beams ([38], [32], [40]) (depending on whether the boundary and interface are flat or not) which allows to built a family of approximate solutions of the coupled system so that their energies are mainly concentrated in the wave domain Ω_2 and are almost completely reflected on the interface γ. This is a natural extension to the multidimensional setting of the more explicit $1-d$ spectral construction. Due to the asymmetry of the energy space H, the same construction in [40] does not give the desired result. One has to introduce higher order corrector terms on the phases and amplitudes of the wave-like solutions to recover an accurate description (*see* Lemma 8).

According to the above negative result, the uniform exponential decay of solutions is excluded in any geometric configuration. This is, to some extent, an evidence of the weakness of the coupling between the heat and wave components. Indeed, the strong dissipation introduced by the heat component is insufficient to damp out uniformly the energy of the wave one. This phenomenon is similar to the classical "overdamping effect" ([39], [42]). According to it the decay rate of the energy of waves does not necessarily increase when the intensity of the damping increases. In fact, the dissipation introduced by the heat equation constitutes an unbounded perturbation of the wave-dynamics and consequently it is too strong and ends up destroying the uniform decay.

It is clear that the geometry has an important effect on the dynamic behavior of solutions. This can be easily understood when analyzing the observability inequality (1.13). Indeed, the right hand side of (1.13) provides a full measurement of the heat component. Thus, the main issue in (1.13) is whether one can recover the estimates on the wave component. The existing literature on this issue indicates that whether the heat domain satisfies the so-called Geometric Control Condition (GCC for short, *see* [2]) or not will play a key role on that. This GCC asserts that all rays of Geometric Optics enter the region where the damping is active in an uniform time and suffices to guarantee the exponential decay for the standard damped wave equation ([2]). In the present context it is natural to expect that the best decay property will hold when the heat subdomain Ω_1 satisfies the GCC in the whole domain Ω. But, even in this case, in view of the weak interaction on the interface, as we mentioned above, the uniform exponential decay fails, and one only expects the polynomial decay of smooth solutions. Proving this polynomial decay is precisely the next main objective of this paper. We show that, when the GCC is satisfied, a weakened version of (1.13) holds with a loss of a finite number of derivatives of y or, in other words, a higher order norm of the heat component y. This estimate is obtained in Theorem 12 by viewing the whole system as a perturbation of the wave equation in the whole domain Ω, a technique that was previously used in [40] for the simplified model (1.2). However, some additional important technical developments are needed to address the more complex and realistic model (1.1) and, more especially, to deal with the transmission conditions on the interface, and the necessity of distinguishing two different cases $\mathrm{Cap}\,\Gamma_2 > 0$ and $\mathrm{Cap}\,\Gamma_2 = 0$. Whenever this weakened observability inequality is established, it allows to prove the polynomial decay for smooth solutions. This result is in agreement with our previous results on the $1-d$ case obtained by spectral methods.

According to this result, despite the fact that the coupling is not strong enough to yield the uniform exponential decay, we show that its weakness is limited to a loss of a finite number of derivatives for (1.13) to hold provided that the heat domain satisfies the GCC. This is the main reason for polynomial decay.

The problem of what the asymptotic behavior is in the absence of the GCC then arises naturally. Using a Gaussian beam construction one can show that, when GCC does not hold for the heat domain, one can build a family of solutions which are exponentially concentrated in the wave domain. This means that one can not expect a weakened version of (1.13) with a loss of *any finite number* of derivatives of y to hold. However, this fact does not exclude the logarithmic decay for smooth solutions. This is the last issue we address in this paper. The most natural tool to be used in this context is the logarithmic observability estimate developed by Lebeau and Robbiano ([28], [41]) which allows quantifying the degree of observability for the wave equation in the absence of GCC. Unfortunately, we do not succeed in proving the logarithmic decay for system (1.1) because of the lack of compactness of the domain of the generator of the associated semigroup in H in several space dimensions ($n \geq 2$). Whether this is merely a technical matter or not is an open problem. Note that this lack of compactness is due to the natural transmission conditions in system (1.1). Consequently, we rather address model (1.2) with simpler transmission conditions on the interface and show the logarithmic decay property. For the proof of this result, once again we view (1.2) as a perturbation of the wave equation in the whole domain and apply the weak observability result in [41], which does not require the GCC to hold.

Our analysis shows that, for a good understanding of the coupling between the wave and heat equations in (1.1) and (1.2) the following two main ingredients have to be taken into account:

- Geometry: When the GCC does not hold for the heat domain, some wave solutions escape the heat domain and, consequently, the damping is very weak. In that case, the logarithmic decay is the rate of decay to be expected;
- Transmission condition. When the GCC holds all waves reach the interface and are damped by the heat mechanism. But overdamping occurs and the exponential decay fails. In that case the polynomial decay for smooth solutions is the best one can expect and it is satisfied, indeed.

Finally, we point out that the subject on the stability analysis for the fluid-structure interaction is full of open problems. Some of them seem to be particularly relevant and could need important new ideas and further developments. As mentioned before, in the context of fluid-structure interaction, it is more physical to replace the wave equation in system (1.1) by the system of elasticity and the heat equation by the Stokes system, and the fluid-structure interface γ by a free boundary. The techniques we employ in this article could be applied to some more realistic and coupled models coupling, for instance, the Lamé and Stokes systems. But the details are to be developed. Despite of the greater complexity of these models one expects that there will exist solutions which are mainly concentrated on the wave domain, making the uniform exponential decay impossible. On the other hand our technique for proving polynomial decay could also be extended to these models. Roughly speaking, it would suffice to replace the observability inequality we have used for the wave equation by those existing in the literature on the Lamé one (*see* [25] and the references therein). A more physical model for fluid-structure interaction would be to replace the heat and wave equations in system (1.1) by the Navier-Stokes and elasticity systems coupled through a moving boundary. To the best of our knowledge, nothing is known about the long time behavior for the solutions to the corresponding equations. The influence of the results in this paper in what concerns the dynamical properties of the free boundary problem is still to be investigated. This is a widely open subject of research.

1.3. Organization of the paper

The rest of this paper is organized as follows. In Section 2, we give some preliminary results. In Section 3 we show the existence, uniqueness and regularity of solution to system (1.1), and the non-compactness of the domain of the generator of the underlying semigroup. The strong asymptotic stability of (1.1), without the GCC, is proved in Section 4. Section 5 is devoted to the construction of approximate solutions for the transmission problem in the whole space with flat interface, which implies the lack of exponential decay for system (1.1) in polyhedral domains. This case is easier to handle and it allows a simpler presentation of the key ideas, since one needs only linear real-valued phases for constructing the approximate solutions. In Section 6 we construct highly localized approximate solutions for (1.1) in general domains, allowing to show the lack of uniform decay. The polynomial decay for smooth solutions to (1.1) under the GCC is proved in Section 7. Section 8 is devoted to showing a logarithmic decay property of smooth solutions of system (1.2) without the GCC, and analyzing the difficulty arising to apply the same method for system (1.1). Appendices A, B, C and D are devoted to prove some technical results that will be used along the paper.

2. Some preliminaries

To begin with, we fix some notation that will be used throughout this paper. First, for a subset $\omega \subset \mathbb{R}^n$, we denote its characteristic function by χ_ω. For any $\eta > 0$, the η-neighborhood of ω is denoted by $\mathcal{O}_\eta(\omega)$. Also, when writing $(w_1, w_2) \in H^s(\Omega)$ (*resp.* $H_0^s(\Omega)$) for $s \in \mathbb{R}$, we mean that the function $w \overset{\triangle}{=} w_1\chi_{\Omega_1} + w_2\chi_{\Omega_2}$ belongs to $H^s(\Omega)$ (*resp.* $H_0^s(\Omega)$).

Further, if $M \subset \mathbb{R}^m$ ($m \in \mathbb{N}$) is an open set and f^ε is a family of functions in $C^\infty(M)$ depending on $\varepsilon \in (0,1)$, we say that $f^\varepsilon \sim 0$ if and only if for all compact $K \subset M$, $\alpha \in \mathbb{N}^m$, and $N \in \mathbb{N}$ there is a constant $C_N > 0$ so that

$$\sup_{y\in K} \left|\partial_y^\alpha f^\varepsilon(y)\right| \le C_N \varepsilon^N$$

holds for all small ε. In this case we also write $f^\varepsilon = O(\varepsilon^\infty)$. For two families $f^\varepsilon, g^\varepsilon$ of smooth functions, $f^\varepsilon \sim g^\varepsilon$ means $f^\varepsilon - g^\varepsilon \sim 0$. We shall also write

$a^\varepsilon \sim \sum_{j=0}^{\infty} a_j \varepsilon^j$. This does not mean that the series converges but rather that for all $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^m$, there is a constant $C_N > 0$ so that

$$\sup_{y \in M} \Big| \partial_y^\alpha \Big(a^\varepsilon(y) - \sum_{j=0}^{N} a_j(y) \varepsilon^j \Big) \Big| \le C_N \varepsilon^{N+1}$$

holds for all small ε. In a similar way, we write $f(t, x_1, x') \sim \sum_{j=0}^{\infty} a_j(t, x') x_1^j$ and $f(t, x_1, x') = O(|x_1|^\infty)$. Here and henceforth $x = (x_1, ..., x_n) = (x_1, x')$.

Further, we will use C and c to denote generic positive constants, independent of ε, which may change from line to line.

Finally, for any nonnegative real numbers f^ε and g^ε, parameterized by $\varepsilon \in (0,1)$, we write $f^\varepsilon \approx g^\varepsilon$ if, for sufficiently small ε, one has $cf^\varepsilon \le g^\varepsilon \le Cf^\varepsilon$.

Next, we present some preliminary results.

First, we need the following simple result. Its proof is standard, see for example [31].

Lemma 1. *Let Ω be a bounded domain with piecewise C^1 boundary Γ. Then there is a constant $C > 0$ such that for all $\zeta \in \mathcal{D}'(\Omega)$ with $\nabla \zeta \in (L^2(\Omega))^n$ and $\Delta \zeta \in L^2(\Omega)$, it holds*

$$\Big| \frac{\partial \zeta}{\partial \nu} \Big|_{H^{-1/2}(\Gamma)} \le C \Big[|\nabla \zeta|_{(L^2(\Omega))^n} + |\Delta \zeta|_{L^2(\Omega)} \Big]. \tag{2.1}$$

For any open subset $\operatorname{Cap} \Gamma_0 > 0$ of Γ, we denote by $H_{\Gamma_0}^{-1/2}(\Gamma)$ the completion of $C(\overline{\Gamma_0}\,)$ with respect to the norm:

$$\begin{aligned} &|u|_{H_{\Gamma_0}^{-1/2}(\Gamma)} \\ &= \sup \left\{ \frac{|\int_\Gamma u f d\Gamma|}{|f|_{H^{1/2}(\Gamma)}} \;\middle|\; f \in H^{1/2}(\Gamma) \setminus \{0\} \text{ and } f = 0 \text{ on } \Gamma \setminus \overline{\Gamma}_0 \right\}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, u \in C(\overline{\Gamma_0}\,). \end{aligned} \tag{2.2}$$

One can check that $H_{\Gamma_0}^{-1/2}(\Gamma)$ is the dual space of $\Big\{ f \in H^{1/2}(\Gamma) \;\Big|\; f = 0 \text{ on } \Gamma \setminus \overline{\Gamma}_0 \Big\}$ with respect to the pivot $L^2(\Gamma)$. Since $\Big\{ f \in H^{1/2}(\Gamma) \;\Big|\; f = 0 \text{ on } \Gamma \setminus \overline{\Gamma}_0 \Big\}$ is closed in $H^{1/2}(\Gamma)$, we deduce that $H^{-1/2}(\Gamma) \subset H_{\Gamma_0}^{-1/2}(\Gamma)$ and the embedding is continuous. Further, since $H_\gamma^{-1/2}(\partial \Omega_1)$ can be identified with $H_\gamma^{-1/2}(\partial \Omega_2)$ (algebraically and topologically), in the sequel we shall denote them simply by $H_\gamma^{-1/2}$, i.e.,

$$H_\gamma^{-1/2} \stackrel{\triangle}{=} H_\gamma^{-1/2}(\partial \Omega_1) \equiv H_\gamma^{-1/2}(\partial \Omega_2). \tag{2.3}$$

One can check that for $i = 1, 2$, it holds

$$|u|_{H_\gamma^{-1/2}} \leq |u|_{H^{-1/2}(\partial\Omega_i)}, \qquad \forall\, u \in H^{-1/2}(\partial\Omega_i). \tag{2.4}$$

Also, when γ is a manifold without boundary, it is easy to show that $H_\gamma^{-1/2} = H^{-1/2}(\gamma)$ (algebraically and topologically).

Next, the following lemma is an easy generalization of the known results in [29] and [54] (which can be proved by using Lax-Milgram Theorem):

Lemma 2. *If* $\operatorname{Cap} \Gamma_2 = 0$ *(see the first geometric configuration in Figure* 1*), then the following elliptic boundary-value problem with equivalued surface*

$$\begin{cases} \Delta g = 0 & \text{in } \Omega_1, \\ g = 0 & \text{on } \Gamma_1, \\ g = C_0 \quad \text{(an unknown constant)} & \text{on } \gamma, \\ \displaystyle\int_\gamma \frac{\partial g}{\partial \nu_1} d\gamma = A_0 \quad \text{(a known constant)} & \end{cases} \tag{2.5}$$

admits a unique weak solution $g \in H^1(\Omega_1)$.

Remark 1. It is easy to see that, in Lemma 2, $C_0 = 0$ if and only if $A_0 = 0$. Especially for the case $A_0 = 1$, we shall denote by $c_0(\neq 0)$ the corresponding constant C_0, the trace of g on γ.

Further, we recall the following elementary result (*see*, for example, [53, Theorem 6, p. 296]):

Lemma 3. *Let* V *be an infinite dimensional separable Hilbert space. Then there is a sequence* $\{v_k\}_{k=1}^\infty \subset V$ *such that* $|v_k|_V = 1$ *for each* k *and* $v_k \rightharpoonup 0$ *in* V *as* $k \to \infty$.

Finally, we show a result, which is a generalization of [50, pp. 42, Proposition 3.3].

Lemma 4. *Let* A *generate a bounded* C_0*-semigroup* $\{e^{At}\}_{t\geq 0}$ *on a Banach space* V. *Then there is a constant* $C > 0$ *such that, for any* $v \in D(A^2)$, *one has*

$$|Av|_V^2 \leq C|v|_V|A^2v|_V. \tag{2.6}$$

Proof. The proof is very close to that of [50, pp. 42, Proposition 3.3]. But, for completeness, we give the details here. Since $\{e^{At}\}_{t\geq 0}$ is uniformly bounded (with respect to $t \geq 0$) on $\mathcal{L}(V)$, the Banach space of all linear continuous operators from V into itself, by Hille-Yosida's Theorem ([36]), there is a constant $C > 0$ such that

$$||(\lambda I - A)^{-1}||_{\mathcal{L}(V)} \leq C\lambda^{-1}, \qquad \forall\, \lambda > 0. \tag{2.7}$$

Here I denotes the identity operator in V.

Now, for any $v \in D(A^2)$,

$$
\begin{aligned}
&(\lambda I - A)^{-1} A^2 v \\
&= (\lambda I - A)^{-1}(A - \lambda I)Av + \lambda(\lambda I - A)^{-1}Av = -Av + \lambda(\lambda I - A)^{-1}Av \\
&= -Av + \lambda(\lambda I - A)^{-1}(A - \lambda I)v + \lambda^2(\lambda I - A)^{-1}v \\
&= -Av - \lambda v + \lambda^2(\lambda I - A)^{-1}v.
\end{aligned}
$$

Hence,

$$
Av = -(\lambda I - A)^{-1}A^2 v - \lambda v + \lambda^2(\lambda I - A)^{-1}v. \tag{2.8}
$$

From (2.8) and (2.7), we have

$$
\begin{aligned}
|Av|_V &\le |(\lambda I - A)^{-1}A^2 v|_V + \lambda |v|_V + \lambda^2 |(\lambda I - A)^{-1} v|_V \\
&\le C\Big[\lambda^{-1}|A^2 v|_V + \lambda |v|_V\Big].
\end{aligned} \tag{2.9}
$$

Choosing λ in (2.9) so that $\lambda^{-1}|A^2 v|_V = \lambda |v|_V$, we conclude the desired inequality (2.6). □

As a consequence of Lemma 4, we have the following result.

Corollary 1. *Let A generate a bounded C_0-semigroup $\{e^{At}\}_{t\ge 0}$ on a Banach space V. The following two conclusions hold:*

i) If $0 \in \rho(A)$, the resolvent of A, then, for any $u \in D(A)$, one has

$$
|u|_V \le C|A^{-3}u|_V^{1/4}|Au|_V^{3/4}; \tag{2.10}
$$

ii) For $u \in D(A^4)$, one has

$$
|A^3 u|_V \le C|u|_V^{1/4}|A^4 u|_V^{3/4}. \tag{2.11}
$$

Proof. By Lemma 4, one has

$$
\begin{aligned}
&|u|_V^2 \le C|A^{-1}u|_V|Au|_V, \qquad |A^{-1}u|_V^2 \le C|A^{-2}u|_V|u|_V, \\
&|A^{-2}u|_V^2 \le C|A^{-3}u|_V|A^{-1}u|_V,
\end{aligned}
$$

which gives (2.10). Similarly, one gets (2.11). □

3. Existence, uniqueness, regularity and non-compactness

This section is devoted to show the existence, uniqueness and regularity of solutions to system (1.1), and the non-compactness of the domain of the generator of the underlying semigroup.

We define an unbounded operator $\mathcal{A}:\ D(\mathcal{A}) \subset H \to H$ by

$$
\mathcal{A}Y = (\mathit{\Delta} Y_1, Y_3, \mathit{\Delta} Y_2), \tag{3.1}
$$

where $Y = (Y_1, Y_2, Y_3) \in D(\mathcal{A})$, and

$$D(\mathcal{A}) = \Big\{(Y_1, Y_2, Y_3) \in H \;\Big|\; \Delta Y_1 \in L^2(\Omega_1),\ \Delta Y_2 \in L^2(\Omega_2),\ Y_3 \in H^1(\Omega_2),\ Y_1|_{\Gamma_1} = Y_3|_{\Gamma_2} = 0,\ Y_1|_\gamma = Y_3|_\gamma,\ \frac{\partial Y_1}{\partial \nu_1}\Big|_\gamma = -\frac{\partial Y_2}{\partial \nu_2}\Big|_\gamma \Big\}. \tag{3.2}$$

Note that, by Lemma 1, both $\frac{\partial Y_1}{\partial \nu_1}$ and $\frac{\partial Y_2}{\partial \nu_2}$ in (3.2) are well-defined on γ.

Remark 2. Obviously, in one space dimension, i.e. $n = 1$, we have

$$D(\mathcal{A}) = \Big\{(Y_1, Y_2, Y_3) \in H \;\Big|\; Y_1 \in H^2(\Omega_1),\ Y_2 \in H^2(\Omega_2),\ Y_3 \in H^1(\Omega_2),\ Y_1|_{\Gamma_1} = Y_3|_{\Gamma_2} = 0,\ Y_1|_\gamma = Y_3|_\gamma,\ \frac{\partial Y_1}{\partial \nu_1}\Big|_\gamma = -\frac{\partial Y_2}{\partial \nu_2}\Big|_\gamma \Big\} \subset H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2).$$

But this is not longer true in several space dimensions.

It is easy to see that system (1.1) can be re-written as an abstract Cauchy problem in H as

$$\begin{cases} X_t = \mathcal{A}X, & t > 0, \\ X(0) = X_0, \end{cases}$$

where $X = (y, z, z_t)$ and $X_0 = (y_0, z_0, z_1)$.

In the case $\operatorname{Cap} \Gamma_2 = 0$, we shall need to consider the following closed subspace of H:

$$\overset{\circ}{H} = \Big\{(f_1, f_2, f_3) \in H \;\Big|\; \int_{\Omega_2} f_2 dx = 0\Big\}.$$

On $\overset{\circ}{H}$, $|\cdot|_H$ is a norm, equivalent to $||\cdot||_H$. As we shall see later, $\mathcal{A} : \overset{\circ}{H} \bigcap D(\mathcal{A}) \to H$ is one to one.

We have the following result, which, among others, gives the existence and uniqueness of solutions to system (1.1).

Theorem 1. *i) If* $\operatorname{Cap} \Gamma_2 > 0$, *then the operator* $\mathcal{A}$ *generates a contractive* C_0-*semigroup in* H, *and* $0 \in \rho(\mathcal{A})$.

ii) If $\operatorname{Cap} \Gamma_2 = 0$, *then* $\mathcal{A}$ *generates a bounded* C_0-*semigroup in* H, *but* $0 \notin \rho(\mathcal{A})$. *The range of* $\mathcal{A}$ *is given by*

$$\mathcal{R}(\mathcal{A}) = \Big\{(f_1, f_2, f_3) \in H \;\Big|\; \int_{\Omega_2} f_3 dx + \int_\gamma \frac{\partial G(f_1, f_2)}{\partial \nu_1} d\gamma = 0\Big\}, \tag{3.3}$$

where $G = G(f_1, f_2)$ solves

$$\begin{cases} \Delta G = f_1 & \text{in } \Omega_1, \\ G = 0 & \text{on } \Gamma_1, \\ G = f_2 & \text{on } \gamma. \end{cases} \tag{3.4}$$

Moreover, $\mathcal{R}(\mathcal{A})$ is a closed subspace of H, and $\mathcal{A} : \overset{\circ}{H} \bigcap D(\mathcal{A}) \to \mathcal{R}(\mathcal{A})$ is a one to one and onto linear continuous map. Here, $D(\mathcal{A})$ and $\mathcal{R}(\mathcal{A})$ are equipped with the graph and inherited norms, respectively.

Furthermore,

$$H = \mathcal{R}(\mathcal{A}) \bigoplus \left\{ (0, c, 0) \;\middle|\; c \in \mathbb{C} \right\} \quad \textit{algebraically}, \tag{3.5}$$

and for any $(y_0, z_0, z_1) \in H$, putting (recall Remark 1 for the definition of c_0)

$$c_1 = c_0 \Big[\int_{\Omega_2} z_1 dx + \int_\gamma \frac{\partial G(y_0, z_0)}{\partial \nu_1} d\gamma \Big], \tag{3.6}$$

one has $(y_0, z_0 - c_1, z_1) \in \mathcal{R}(\mathcal{A})$ and the decomposition

$$(y_0, z_0, z_1) = (y_0, z_0 - c_1, z_1) + (0, c_1, 0).$$

Remark 3. By Lemma 1, we see that $\frac{\partial G(f_1,f_2)}{\partial \nu_1} \in H^{-1/2}(\gamma)$, and noting that $1 \in H^{1/2}(\gamma)$, one sees that the duality product $\langle \frac{\partial G(f_1,f_2)}{\partial \nu_1}, 1 \rangle_{H^{-1/2}(\gamma), H^{1/2}(\gamma)}$ makes sense. However, to simplify the notation, in (3.3) and in the sequel we denote it by $\int_\gamma \frac{\partial G(f_1,f_2)}{\partial \nu_1} d\gamma$.

We refer to Appendix A for the proof of Theorem 1.

Remark 4. By Theorem 1, $\mathcal{A}^{-1}$ exists whenever $\operatorname{Cap} \Gamma_2 > 0$. When $n = 1$, in view of the embedding in Remark 2, it is easy to check that $\mathcal{A}^{-1}$ is compact (if it exists). However, in view of the structure of $D(A)$, $\mathcal{A}^{-1}$ is not guaranteed to be compact in several space dimensions. Indeed, for any $F = (F_1, F_2, F_3) \in H$, from the proof of Theorem 1, we see that the second component Y_2 of $\mathcal{A}^{-1}F (\equiv (Y_1, Y_2, Y_3) \in D(\mathcal{A}))$ belongs to $H^1_{\Gamma_2}(\Omega_2)$, which has the same regularity as the second component F_2 of F. According to the regularity theory of elliptic equations, this regularity property for Y_2 is sharp. As we shall see below, in fact $D(A)$ fails to be compact.

Theorem 2. *For dimensions $n \geq 2$, $D(\mathcal{A})$ is noncompact in H.*

Proof. It suffices to show that there exists a sequence $\{(Y_1^k, Y_2^k, Y_3^k)\}_{k=1}^\infty \subset D(\mathcal{A})$ such that $(Y_1^k, Y_2^k, Y_3^k) \rightharpoonup 0$ in $D(\mathcal{A})$ as $k \to \infty$ and $\inf\limits_{k \in \mathbb{N}} |(Y_1^k, Y_2^k, Y_3^k)|_H \geq c$ for some constant $c > 0$. We distinguish two different cases.

i) **The case that both** $\mathrm{Cap}\,\Gamma_1$ **and** $\mathrm{Cap}\,\Gamma_2$ **are positive.** Recall the definition of $H_\gamma^{-1/2}$ in (2.3) and (2.2). It is easy to see that $H_\gamma^{-1/2}$ is an infinite dimensional separable Hilbert space whenever $n \ge 2$. Applying Lemma 3 to $H_\gamma^{-1/2}$, we deduce that there is a sequence $\{\beta^k\}_{k=1}^\infty \subset H_\gamma^{-1/2}$ such that $|\beta^k|_{H_\gamma^{-1/2}} = 1$ for each k and $\beta^k \rightharpoonup 0$ in $H_\gamma^{-1/2}$ as $k \to \infty$.

By Lax-Milgram Theorem, we may solve the following two systems

$$\begin{cases} \Delta Y_1^k = 0 & \text{in } \Omega_1, \\ Y_1^k = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial Y_1^k}{\partial \nu_1} = -\beta^k & \text{on } \gamma, \end{cases} \qquad \begin{cases} \Delta Y_2^k = 0 & \text{in } \Omega_2, \\ Y_2^k = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial Y_2^k}{\partial \nu_2} = \beta^k & \text{on } \gamma \end{cases} \tag{3.7}$$

and get $Y_i^k \in H_{\Gamma_i}^1(\Omega_i)$ $(i = 1, 2)$. Since $\beta^k \rightharpoonup 0$ in $H_\gamma^{-1/2}$, we deduce that

$$Y_i^k \rightharpoonup 0 \quad \text{in } H_{\Gamma_i}^1(\Omega_i) \text{ as } k \to \infty. \tag{3.8}$$

On the other hand, noting (2.4) and using Lemma 1, we conclude that

$$1 = |\beta^k|_{H_\gamma^{-1/2}} \le \Big|\frac{\partial Y_2^k}{\partial \nu_2}\Big|_{H^{-1/2}(\partial\Omega_2)} \le C|Y_2^k|_{H_{\Gamma_2}^1(\Omega_2)}. \tag{3.9}$$

Further, we solve

$$\begin{cases} \Delta Y_3^k = 0 & \text{in } \Omega_2, \\ Y_3^k = 0 & \text{on } \Gamma_2, \\ Y_3^k = Y_1^k & \text{on } \gamma \end{cases}$$

and get $Y_3^k \in H_{\Gamma_2}^1(\Omega_2)$. Noting (3.8), we conclude that

$$Y_3^k \rightharpoonup 0 \quad \text{in } H_{\Gamma_2}^1(\Omega_2) \text{ as } k \to \infty. \tag{3.10}$$

It is clear that $(Y_1^k, Y_2^k, Y_3^k) \subset D(\mathcal{A})$. By (3.8) and (3.10), $(Y_1^k, Y_2^k, Y_3^k) \rightharpoonup 0$ in $D(\mathcal{A})$ as $k \to \infty$; while by (3.9), $\inf\limits_{k\in\mathbb{N}} |(Y_1^k, Y_2^k, Y_3^k)|_H \ge C^{-1}$.

ii) **The case that either** $\mathrm{Cap}\,\Gamma_1$ **or** $\mathrm{Cap}\,\Gamma_2$ **is zero.** In this case, γ is a manifold without boundary. Replacing $H_\gamma^{-1/2}$ in the first case by the infinite dimensional separable Hilbert space $\Big\{f \in H^{-1/2}(\gamma) \;\Big|\; \int_\gamma f d\gamma = 0\Big\}$, and proceeding as above, one can find the desired sequence $\{(Y_1^k, Y_2^k, Y_3^k)\}_{k=1}^\infty$. Here, another small needed change is that, in the case $\mathrm{Cap}\,\Gamma_1 = 0$ (*resp.* $\mathrm{Cap}\,\Gamma_2 = 0$), one has to choose $Y_1^k \in \Big\{f \in H^1(\Omega_1) \;\Big|\; \int_{\Omega_1} f dx = 0\Big\}$ for the first system of (3.7) (*resp.* $Y_2^k \in \Big\{f \in H^1(\Omega_2) \;\Big|\; \int_{\Omega_2} f dx = 0\Big\}$ for the second one). □

Remark 5. When Ω is a square domain in $\mathbb{R}^2$ and γ is a straight line dividing Ω into two equal rectangle, the construction in the proof of Theorem 2 can be completely explicit. One can take for instance $\{\beta^k\}_{k=1}^{\infty}$ to be a family of rapidly oscillating functions. The proof we adopt here is due to Thomas Duyckaerts ([16]).

Remark 6. i) The lack of compactness of the domain of the generator of the semigroup is specific to system (1.1). The situation is different for system (1.2), whose generator $\widetilde{\mathcal{A}}: D(\widetilde{\mathcal{A}}) \subset \widetilde{H} \to \widetilde{H}$ is defined as follows:

$$\widetilde{\mathcal{A}}Y = (\Delta Y_1, Y_3, \Delta Y_2), \tag{3.11}$$

where $Y = (Y_1, Y_2, Y_3) \in D(\widetilde{\mathcal{A}})$, and

$$D(\widetilde{\mathcal{A}}) = \Big\{(Y_1, Y_2, Y_3) \in \widetilde{H} \ \Big|\ \begin{array}{l}(Y_1, Y_2) \in H^2(\Omega),\ Y_3 \in H^1(\Omega_2),\\ \Delta Y_1 \in H^1(\Omega_1),\ \Delta Y_1|_{\Gamma_1} = Y_3|_{\Gamma_2} = 0\\ \text{and}\ \ \Delta Y_1|_{\gamma} = Y_3|_{\gamma}\Big\}.\end{array} \tag{3.12}$$

As shown in [40], $\widetilde{\mathcal{A}}$ generates a contractive C_0-semigroup and $\widetilde{\mathcal{A}}^{-1}$ is always compact, which differs from the property of (1.1). As we shall see in Section 8, the compactness of $D(\widetilde{\mathcal{A}})$ plays a crucial role in analyzing the long time behavior of solutions of system (1.2) without the GCC and proving the logarithmic decay result.

ii) By Lemma 1 and noting the structure of $D(\mathcal{A})$, it is easy to see that

$$D(\mathcal{A}) \subset H^1_{\Gamma_1}(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times H^1_{\Gamma_2}(\Omega_2). \tag{3.13}$$

This guarantees the H^1-regularity for the heat and wave components of system (1.1) whenever its initial data belong to $D(\mathcal{A})$. But this does not suffice to guarantee the H^2-regularity for the heat and wave components unless for one space dimension (as we have seen in the proof of Proposition 2). This is another difference between systems (1.1) and (1.2).

In order to prove the existence of smooth solutions of (1.1), we introduce the following Hilbert space:

$$V = \Big\{(y_0, z_0, z_1) \in D(\mathcal{A}) \ \Big|\ y_0 \in H^2(\Omega_1), z_0 \in H^2(\Omega_2)\Big\}, \tag{3.14}$$

with the canonical norm. Obviously, V is a closed subspace of $D(\mathcal{A})$. Note however that, according to Proposition 2, $D(\mathcal{A}^k)$ is not necessarily a subspace of V even if $k \in \mathbb{N}$ is large.

By using derivation along the tangential direction of the interface, we have the following regularity result (The proof is standard and we omit the details):

Theorem 3. *Let $\Gamma \cap \gamma = \emptyset$ and $\gamma \in C^2$. Then for any $(y_0, z_0, z_1) \in V$, the solution of (1.1) satisfies $(y, z, z_t) \in C([0, \infty); V)$, and, for any $T \in (0, \infty)$, there is a constant $C_T > 0$ such that*

$$|(y, z, z_t)|_{C([0,T];V)} \leq C_T |(y_0, z_0, z_1)|_V. \tag{3.15}$$

4. Strong asymptotic stability

In this section, we shall show the strong asymptotic stability of (1.1) without the GCC. Our result is as follows.

Theorem 4. *i) If $\operatorname{Cap} \Gamma_2 > 0$, then for any given $(y_0, z_0, z_1) \in H$, the solution (y, z, z_t) of (1.1) tends to 0 strongly in H as $t \to \infty$.*

ii) If $\operatorname{Cap} \Gamma_2 = 0$, then for any given $(y_0, z_0, z_1) \in H$, the solution (y, z, z_t) of (1.1) tends to $(0, c_1, 0)$ strongly in H as $t \to \infty$, where $c_1 = c_1(y_0, z_0, z_1)$ is the constant given in (3.6).

Proof. We distinguish two different cases.

i) **The case** $\operatorname{Cap} \Gamma_2 > 0$**.** The proof is divided into several steps.

Step 1. For any given $(y_0, z_0, z_1) \in H$, we show first that the solution (y, z, z_t) of (1.1) satisfies

$$\begin{aligned} &(y(t), z_t(t)) \to 0 \text{ strongly in } L^2(\Omega_1) \times L^2(\Omega_2), \\ &z(t) \rightharpoonup 0 \text{ weakly in } H^1_{\Gamma_2}(\Omega_2), \end{aligned} \tag{4.1}$$

as $t \to \infty$.

For this, we use an argument inspired in [45]. By density, it suffices to consider $(y_0, z_0, z_1) \in D(\mathcal{A})$ and to prove that the "relaxed" ω-limit set of (y_0, z_0, z_1) defined by

$$\begin{aligned} \omega(y_0, z_0, z_1) \triangleq \Big\{ (\hat{y}_0, \hat{z}_0, \hat{z}_1) \Big| \; &\text{for some } t_n \to \infty, \text{ the solution} \\ &(y, z, z_t) \text{ of (1.1) satisfies } (y(t_n), z_t(t_n)) \\ &\to (\hat{y}_0, \hat{z}_1) \text{ strongly in } L^2(\Omega_1) \times L^2(\Omega_2), \\ &z(t_n) \rightharpoonup \hat{z}_0 \text{ weakly in } H^1_{\Gamma_2}(\Omega_2) \Big\} \end{aligned} \tag{4.2}$$

is equal to $\{(0, 0, 0)\}$. In view of (3.13) and noting that the solution (y, z, z_t) remains bounded in $D(\mathcal{A})$ for all $t \geq 0$, one concludes that $\omega(y_0, z_0, z_1) \neq \emptyset$. Let us choose a $(\hat{y}_0, \hat{z}_0, \hat{z}_1) \in \omega(y_0, z_0, z_1)$. Then, for some $t_n \to \infty$, one knows that, $e^{t_n \mathcal{A}}(y_0, z_0, z_1) \rightharpoonup (\hat{y}_0, \hat{z}_0, \hat{z}_1)$ weakly in H. Moreover, using the semigroup property of $e^{t\mathcal{A}}$ and noting that (1.8)–(1.10) yields

$$|e^{t\mathcal{A}}(y_0, z_0, z_1)|_H \leq |e^{s\mathcal{A}}(y_0, z_0, z_1)|_H$$

for any $t \geq s \geq 0$, we see that, for each $T \geq 0$,

$$\lim_{t_n\to\infty} |e^{t_n\mathcal{A}}(y_0, z_0, z_1)|_H = \lim_{t_n\to\infty} |e^{(T+t_n)\mathcal{A}}(y_0, z_0, z_1)|_H. \tag{4.3}$$

Put $(y^n(t), z^n(t), z_t^n(t)) \triangleq e^{(t+t_n)\mathcal{A}}(y_0, z_0, z_1)$ and $(\hat{y}(t), \hat{z}(t), \hat{z}_t(t)) \triangleq e^{t\mathcal{A}}(\hat{y}_0, \hat{z}_0, \hat{z}_1)$. Then, as $n \to \infty$, $(y^n(t), z^n(t), z_t^n(t)) \rightharpoonup (\hat{y}(t), \hat{z}(t), \hat{z}_t(t))$ weakly in H. On the other hand, according to the dissipation law (1.10) and noting (1.8), we get

$$|e^{t_n\mathcal{A}}(y_0, z_0, z_1)|_H^2 - |e^{(T+t_n)\mathcal{A}}(y_0, z_0, z_1)|_H^2 = 2\int_0^T \int_{\Omega_1} |\nabla y^n(t)|^2 dx. \tag{4.4}$$

Combining (4.3) and (4.4), we conclude that

$$y^n(\cdot) \to 0 \quad \text{strongly in } L^2(0, T; H^1_{\Gamma_1}(\Omega_1)) \quad \text{as } n \to \infty.$$

By definition, $(\hat{y}, \hat{z}, \hat{z}_t)$ solves (1.1) with initial data $(\hat{y}_0, \hat{z}_0, \hat{z}_1)$, and $\hat{y} \equiv 0$ in $(0,T) \times \Omega_1$. From this, it is easy to conclude that $(\hat{y}_0, \hat{z}_0, \hat{z}_1) = 0$ provided that T is large enough, which gives (4.1). (Here, due to the fact that solutions to the wave equation propagate with a finite velocity, T has to be taken large enough).

Step 2. Now, for any $(y_0, z_0, z_1) \in H$, put

$$(Y, Z, Z_t)(t) = \int_0^t (y, z, z_t)(s)ds + \mathcal{A}^{-1}(y_0, z_0, z_1). \tag{4.5}$$

It is obvious that

$$Y_t = y, \qquad Z_t = z. \tag{4.6}$$

Also, from (1.1), we see that (Y, Z, Z_t) solves (1.1) with initial data $\mathcal{A}^{-1}(y_0, z_0, z_1)$.

By the dissipation law (1.10), one has

$$\begin{aligned} &|(Y(0), Z(0), Z_t(0))|_H^2 - |(Y(T), Z(T), Z_t(T))|_H^2 \\ &= 2\int_0^T \int_{\Omega_1} |\nabla Y|^2 dxdt, \qquad \forall\, T \geq 0. \end{aligned} \tag{4.7}$$

Combining (1.10) and (4.7), we conclude that

$$\int_0^T \int_{\Omega_1} (|\nabla y|^2 + |\nabla Y|^2) dxdt \leq \frac{1}{2}\Big[|(y_0, z_0, z_1)|_H^2 + |(Y(0), Z(0), Z_t(0))|_H^2\Big].$$

Hence,

$$(\nabla y, \nabla Y) \in L^2(0, \infty; (L^2(\Omega_1))^n \times (L^2(\Omega_1))^n). \tag{4.8}$$

Also, by the standard semigroup theory and (3.13) in Remark 6 ii), we see that $(\nabla y, \nabla Y) \in C([0,\infty); (L^2(\Omega_1))^n \times (L^2(\Omega_1))^n)$. Therefore, (4.8) implies that there is a sequence $\{s_n\}_{n=1}^{\infty}$ which tends to ∞ such that

$$(\nabla y(s_n), \nabla Y(s_n)) \to 0 \ \text{ strongly in } \ (L^2(\Omega_1))^n \times (L^2(\Omega_1))^n \tag{4.9}$$

as $n \to \infty$.

Step 3. We now show that, for any given $(y_0, z_0, z_1) \in H$, the energy $E(t)$ of the solution $(y(t), z(t), z_t(t))$ of (1.1) tends to zero as $t \to \infty$. By density, it suffices to assume $(y_0, z_0, z_1) \in D(\mathcal{A})$. In this case, (y, z, z_t) is a strong solution of (1.1), and (y_t, z_t, z_{tt}) solves (1.1) with initial data in H replaced by $\mathcal{A}(y_0, z_0, z_1)$.

Applying the conclusion in Step 1 to (Y, Z, Z_t), (y, z, z_t) and (y_t, z_t, z_{tt}), we see that, as $t \to \infty$, Y, y and y_t tend to zero strongly in $L^2(\Omega_1)$, and Z_t, z_t and z_{tt} tend to zero strongly in $L^2(\Omega_2)$. In particular, we have

$$\begin{aligned} &(Y(s_n), y(s_n), y_t(s_n)) \to 0 \ \text{ strongly in } \ (L^2(\Omega_1))^3, \\ &(Z_t(s_n), z_t(s_n), z_{tt}(s_n)) \to 0 \ \text{ strongly in } \ (L^2(\Omega_2))^3, \end{aligned} \tag{4.10}$$

as $n \to \infty$.

On the other hand, by system (1.1) and the analogue that Y satisfies, using integration by parts and noting (4.6), we get

$$\begin{aligned} 0 &= \int_{\Omega_1} Y(y_t - \Delta y)dx + \int_{\Omega_2} Z_t(z_{tt} - \Delta z)dx \\ &= \int_{\Omega_1} Y y_t dx - \int_\gamma Y \frac{\partial y}{\partial \nu_1} d\gamma + \int_{\Omega_1} \nabla Y \cdot \nabla y dx + \int_{\Omega_2} Z_t z_{tt} dx \\ &\quad - \int_\gamma Z_t \frac{\partial z}{\partial \nu_2} d\gamma + \int_{\Omega_2} |\nabla z|^2 dx \\ &= \int_{\Omega_1} Y y_t dx + \int_{\Omega_1} \nabla Y \cdot \nabla y dx + \int_{\Omega_2} Z_t z_{tt} dx + \int_{\Omega_2} |\nabla z|^2 dx. \end{aligned} \tag{4.11}$$

By (4.11), we get

$$\begin{aligned} \int_{\Omega_2} |\nabla z(s_n)|^2 dx = -\Bigg[&\int_{\Omega_1} Y(s_n) y_t(s_n) dx + \int_{\Omega_1} \nabla Y(s_n) \cdot \nabla y(s_n) dx \\ &+ \int_{\Omega_2} Z_t(s_n) z_{tt}(s_n) dx \Bigg]. \end{aligned} \tag{4.12}$$

Now, from (4.12) and noting (4.9)–(4.10), we conclude that $\nabla z(s_n) \to 0$ strongly in $(L^2(\Omega_2))^n$ as $n \to \infty$, which combined with (4.10) yields $E(s_n) \to 0$ as $n \to \infty$. Note however that $E(t)$ is decreasing with respect to t. Therefore, $\lim\limits_{t\to\infty} E(t) = 0$.

ii) **The case** $\operatorname{Cap} \Gamma_2 = 0$**.** The proof is very close to that of the first case and we only give the sketch.

For any given $(y_0, z_0, z_1) \in H$, by Theorem 1 ii), there is a constant $c_1 = c_1(y_0, z_0, z_1)$ defined by (3.6) such that $(y_0, z_0 - c_1, z_1) \in \mathcal{R}(\mathcal{A})$. We decompose the solution as $(y, z, z_t)(t) = (y, z - c_1, z_t)(t) + (0, c_1, 0)$. Clearly, $(0, c_1, 0)$ is a stationary solution. Thus, it suffices to show that $(y, z - c_1, z_t)$ tends to zero as $t \to \infty$. This can be done as in the previous case. This completes the proof of Theorem 4. $\square$

5. The transmission problem in the whole space with flat interface

We consider the case of a flat interface γ, say $\gamma = \{x_1 = 0\}$, Ω being the whole space $\mathbb{R}^n$. In this case, system (1.1) may be written as follows:

$$\begin{cases} y_t - \Delta y = 0 & \text{in } (0,\infty) \times \{x_1 < 0\}, \\ \Box z = 0 & \text{in } (0,\infty) \times \{x_1 > 0\}, \\ y = z_t, \quad y_{x_1} = z_{x_1} & \text{on } (0,\infty) \times \{x_1 = 0\}. \end{cases} \tag{5.1}$$

We will use the WKB approach developed in [40] to construct approximate solutions of (5.1). Throughout this section, $\tau \in \mathbb{R}\setminus\{0\}$ and $\xi = (\xi_1, \xi') \in \mathbb{R}^n$ with $\xi_1 \neq 0$ are given and are assumed to satisfy the condition

$$\tau^2 - |\xi|^2 = 0, \quad \text{i.e.,} \quad \tau = \pm|\xi|. \tag{5.2}$$

Let

$$e^{i(\tau t + \xi\cdot x)/\varepsilon} \sum_{j=0}^{\infty} \varepsilon^j a_j \tag{5.3}$$

be the *incoming wave*. For simplicity, we only consider the case of non-normal incidence, i.e., $|\tau| > |\xi'| > 0$. We now seek an approximate solution to the wave equation $\Box z = 0$ in $\{x_1 > 0\}$ of the form

$$z^\varepsilon(t,x) \sim e^{i(\tau t + \xi\cdot x)/\varepsilon} \sum_{j=0}^{\infty} \varepsilon^j a_j + e^{i(\tau t + \tilde{\xi}\cdot x)/\varepsilon} \sum_{j=0}^{\infty} \varepsilon^j b_j. \tag{5.4}$$

The second component in the right hand side of (5.4) is referred to as the *outgoing wave*. We refer to [40] for the detailed constructions of a_j and b_j. According to [40, Remark 3.1], the profiles a_j and b_j are determined by their values at the interface $x_1 = 0$: $a_j^0 \equiv a_j^0(t, x') \stackrel{\triangle}{=} a_j(t, 0, x')$ and $b_j^0 \equiv b_j^0(t, x') \stackrel{\triangle}{=} b_j(t, 0, x')$, and can be chosen to have support in a tube of rays (i.e., characteristic curves of $\partial_t - \frac{\xi}{\tau} \cdot \partial_x$). In order to construct approximate solutions to the transmission problem (5.1), we shall glue (5.4) to an approximate solution y^ε of the heat equation $y_t - \Delta y = 0$ in $\{x_1 < 0\}$ given by

$$y^\varepsilon \sim e^{|\xi'|x_1/\varepsilon} e^{i(\tau t + \xi' \cdot x')/\varepsilon} \sum_{j=0}^{\infty} \varepsilon^j A_j(t, x). \tag{5.5}$$

By [40], the above A_j are also determined by the initial condition imposed at the interface $x_1 = 0$: $A_j^0 = A_j(t, 0, x')$ $(j = 0, 1, 2, \cdots)$. Therefore, it remains to determine all b_j^0 and A_j^0 from a_j^0. For this, we use the following two transmission conditions

$$y^\varepsilon(t, 0, x') = z_t^\varepsilon(t, 0, x'), \qquad \partial_{x_1} y^\varepsilon(t, 0, x') = \partial_{x_1} z^\varepsilon(t, 0, x'). \tag{5.6}$$

From (5.4), we see that

$$\begin{aligned}\partial_{x_1} z^\varepsilon(t, x_1, x') \sim \frac{1}{\varepsilon} \Bigg[& e^{i(\tau t + \xi \cdot x)/\varepsilon} \left(i\xi_1 a_0 + \sum_{j=1}^{\infty} \varepsilon^j (i\xi_1 a_j + \partial_{x_1} a_{j-1}) \right) \\ & + e^{i(\tau t + \tilde{\xi} \cdot x)/\varepsilon} \left(-i\xi_1 b_0 + \sum_{j=1}^{\infty} \varepsilon^j (-i\xi_1 b_j + \partial_{x_1} b_{j-1}) \right) \Bigg],\end{aligned}$$

and

$$\begin{aligned}\partial_t z^\varepsilon(t, x_1, x') \sim \frac{1}{\varepsilon} \Bigg[& e^{i(\tau t + \xi \cdot x)/\varepsilon} \left(i\tau a_0 + \sum_{j=1}^{\infty} \varepsilon^j (i\tau a_j + \partial_t a_{j-1}) \right) \\ & + e^{i(\tau t + \tilde{\xi} \cdot x)/\varepsilon} \left(i\tau b_0 + \sum_{j=1}^{\infty} \varepsilon^j (i\tau b_j + \partial_t b_{j-1}) \right) \Bigg].\end{aligned}$$

Similarly, from (5.5), we have

$$\begin{aligned}& \partial_{x_1} y^\varepsilon(t, x_1, x') \\ & \sim \frac{1}{\varepsilon} e^{|\xi'|x_1/\varepsilon} e^{i(\tau t + \xi' \cdot x')/\varepsilon} \left[|\xi'| A_0 + \sum_{j=1}^{\infty} \varepsilon^j \left(|\xi'| A_j + \partial_{x_1} A_{j-1} \right) \right].\end{aligned}$$

Clearly, the first condition in (5.6) holds, if and only if,

$$\begin{cases} 0 = i\tau(a_0^0 + b_0^0), \\ A_{j-1}^0 = i\tau(a_j^0 + b_j^0) + \partial_t a_{j-1}^0 + \partial_t b_{j-1}^0, \qquad \forall\, j = 1, 2, \cdots. \end{cases} \tag{5.7}$$

The second condition in (5.6) holds, if and only if,

$$\begin{cases} |\xi'| A_0^0 = i\xi_1(a_0^0 - b_0^0), \\ |\xi'| A_j^0 + \partial_{x_1} A_{j-1}(t, 0, x') \\ = i\xi_1(a_j^0 - b_j^0) + \partial_{x_1} a_{j-1}(t, 0, x') + \partial_{x_1} b_{j-1}(t, 0, x'), \\ \qquad \forall\, j = 1, 2, \cdots. \end{cases} \tag{5.8}$$

Note that by [40], one may express $\partial_{x_1}A_{j-1}(t,0,x')$ in terms of $A_0^0,\cdots,A_{j-1}^0$, and $\partial_{x_1}b_{j-1}(t,0,x')$ in terms of $b_0^0,\cdots,b_{j-1}^0$. Consequently, by induction, equations (5.7) and (5.8) uniquely determine all b_j^0 and A_j^0 in terms of the incoming coefficients $a_0^0,\cdots,a_j^0$. This gives an infinitely accurate solution of the transmission problem for the incoming wave of geometric optics type (5.3).

Further, the first equation in (5.7) and that in (5.8) read

$$\begin{cases} 0 = i\tau(a_0^0 + b_0^0), \\ |\xi'|A_0^0 = i\xi_1(a_0^0 - b_0^0), \end{cases} \tag{5.9}$$

from which, we find that

$$b_0^0 = -a_0^0, \qquad A_0^0 = 2i\xi_1 a_0^0/|\xi'|. \tag{5.10}$$

The reflection coefficient is $b_0^0/a_0^0 = -1$. By this we conclude that the incoming wave is almost completely reflected, and that only a negligible high frequency wave enters the heat domain. A similar result for curved interfaces will be shown in the following section (*see* (6.80)).

Note also that the above reflection law coincides with the observation in [55] in the sense that the first order approximation of the high frequency hyperbolic eigenfunctions are eigenfunctions of the wave equation in the wave domain with Dirichlet boundary conditions in the interface.

Let us analyze the energy absorbed upon reflection. The dissipated component y^ε is localized in a boundary layer of width $O(\varepsilon)$ as $\varepsilon \to 0$. From $|\nabla y^\varepsilon|^2 = O(1/\varepsilon^2)$, we know that the energy dissipated between $t = 0$ and $t = T$ is

$$\int_0^T \int_{\{x_1<0\}} |\nabla y^\varepsilon|^2 \, dtdx \le C \int_0^T \int_{\{-C\varepsilon<x_1<0\}} \frac{1}{\varepsilon^2} dtdx = O\Big(\frac{1}{\varepsilon}\Big). \tag{5.11}$$

However the total energy of $(y^\varepsilon, z^\varepsilon)$ is $O(1/\varepsilon^2)$. Hence the dissipated energy is negligible, and the negligible loss can be quantified as $\varepsilon\%$ of the total energy. A similar computation shows that

$$\int_0^T \int_{\{x_1<0\}} |\nabla y_t^\varepsilon|^2 \, dtdx = O\Big(\frac{1}{\varepsilon^3}\Big). \tag{5.12}$$

By interpolation, from (5.11) and (5.12), we get

$$|\nabla y^\varepsilon|^2_{H^{1/2}(0,T;L^2(\{x_1<0\}))} = O\Big(\frac{1}{\varepsilon^2}\Big), \tag{5.13}$$

which is the same order of the total energy of $(y^\varepsilon, z^\varepsilon)$. Hence, we deduce that, at least $1/2$ derivative is lost in (1.13). The above analysis will also be useful in Section 7 for one to guess the possible sharp polynomial decay rate and the weakened observability inequality one may expect for smooth solutions of system (1.1) under the GCC.

Similar to [40], as a consequence of the above analysis, we conclude that

Theorem 5. *Let the wave domain Ω_2 be polyhedral in $\mathbb{R}^n$. Then*

i) For any given $T > 0$, there is no constant $C > 0$ such that (1.13) holds for all solutions of (1.1);

ii) The energy $E(t)$ of solutions of system (1.1) does not decay exponentially.

The proof of Theorem 5 is roughly as follows. It suffices to show that (1.13) fails for each $T > 0$. Given any $T > 0$ we consider a ray of length T in the wave domain Ω_2 which is reflected at the exterior boundary Γ_2 and the interface γ according to the law of Geometric Optics. We then build a family of solutions which is more and more concentrated along that ray as the frequency $1/\varepsilon$ increases. When the ray hits the exterior boundary, this can be done by simply using the reflection rule at the Dirichlet boundary. When the ray intersects the interface the reflection occurs according to the construction above.

Theorem 5 shows the non-uniform decay in polyhedral wave domains. However, for more general domains, one needs a different construction of approximate solutions for the transmission problem. Especially, when treating general interfaces and boundaries, to avoid caustics, we have to introduce nonlinear and complex-valued phases for the approximate wave solutions. This will be done in the next section.

6. Non-uniform decay in general domains

In this section, by means of Geometric Optics constructions, we perform a careful analysis of the interaction of waves at a general interface to show that there is no uniform decay for solutions of system (1.1).

6.1. Statement of the main result

The main non-uniform decay result in this paper is as follows:

Theorem 6. *Let* $\mathrm{Cap}\,\Gamma_2 > 0$*, and both Γ_2 and γ be of class C^4. Then*

i) For any given $T > 0$, there is no constant $C > 0$ such that (1.13) holds for all solutions of (1.1);

ii) The energy $E(t)$ of solutions of system (1.1) does not decay exponentially as $t \to \infty$.

Theorem 7. *Let* $\operatorname{Cap}\Gamma_2 = 0$ *and* γ *be of class* C^4. *Then*

i) For any given $T > 0$, *there is no constant* $C > 0$ *such that*

$$|(y_0, z_0, z_1)|_H^2 + |\mathcal{A}(y_0, z_0, z_1)|_H^2 \le C \int_0^T \int_{\Omega_1} [|\nabla y|^2 + |\nabla y_t|^2] dxdt, \quad \forall\ (y_0, z_0, z_1) \in D(\mathcal{A}) \tag{6.1}$$

holds for all solutions of (1.1);

ii) For any given $T > 0$, *there is no constant* $C > 0$ *such that*

$$||(y_0, z_0, z_1)||_H^2 \le C \int_0^T \int_{\Omega_1} [|\nabla y|^2 + |\nabla Y|^2] dxdt, \quad \forall\ (y_0, z_0, z_1) \in \mathcal{R}(\mathcal{A}) \tag{6.2}$$

holds for all solutions of (1.1), where Y *is given by* $(Y, Z, Z_t)(t) = \int_0^t (y, z, z_t)(s) ds + \mathcal{A}^{-1}(y_0, z_0, z_1)$;

iii) Solutions (y, z, z_t) *of (1.1) do not tend to the stationary solution* $(0, c_1, 0)$ *exponentially in* H *as* $t \to \infty$, *where* $c_1 = c_1(y_0, z_0, z_1)$ *is the constant defined in (3.6).*

In view of Lemma 5 in the next subsection, Theorem 6 is a direct consequence of Theorem 9 at the end of this section, which is based on the construction of highly localized solutions of system (1.1). The proof of Theorem 9 uses Gaussian beams to construct solutions of system (1.1) which are supported near rays. The main idea is similar to [40]. However, as we mentioned before, significant corrections on the phases and amplitudes of the wave-like approximate solutions are necessary to treat the asymmetry on the regularity properties of the wave and heat components in the energy space H of (1.1). Nevertheless, the conclusion on the lack of exponential decay is similar to [40]. As we shall see, for any given $T > 0$, and any ray of Geometric Optics in the wave domain which, for $0 \le t \le T$, reflects transversally and non-normally at the exterior boundary $\partial\Omega_2$ or at the interface γ, we construct a family of solutions $(y_\varepsilon, z_\varepsilon)$ of system (1.1) whose energy is concentrated in the wave domain Ω_2, for which a negligible part of the whole energy enters the heat domain Ω_1. As a consequence of this, (1.13) fails necessarily.

On the other hand, it is easy to show that the three assertions in Theorem 7 are equivalent. Indeed, i) ⇔ ii) is clear. To see ii)⇔ iii), it suffices to note that

$$\widetilde{F}(t) \triangleq \frac{1}{2} \left[|(y(t), z(t), z_t(t))|_H^2 + |(Y(t), Z(t), Z_t(t))|_H^2 \right]$$

is equivalent to the energy $F(t)$ defined by (1.9), and the fact that, via (1.10), the dissipation law reads

$$\frac{d}{dt}\widetilde{F}(t) = -\int_{\Omega_1} [|\nabla y|^2 + |\nabla Y|^2] dx.$$

Consequently, similar to Theorem 6, it is easy to see that the first conclusion in Theorem 7 is a direct consequence of Theorem 10 at the end of this section. Hence, the main task in the rest of this section is to derive Theorems 9–10.

6.2. Multiply reflected rays

We recall that a *null bicharacteristic* for $\Box$ in $\mathbb{R}^n$ may be defined to be a solution of the ODE:

$$\begin{cases} \dot{x}(t) = 2\xi(t), \quad \dot{\xi}(t) = 0, \\ x(0) = x^0, \quad \xi(0) = \xi^0, \end{cases} \tag{6.3}$$

where the initial data ξ^0 are chosen such that $|\xi^0| = 1/2$. It is easy to check that

$$|\xi(t)| = \frac{1}{2}, \qquad \forall\, t \in \mathbb{R}. \tag{6.4}$$

Clearly, $(t, x(t))$, the projection of the null bicharacteristic to the physical time-space, traces a line in $\mathbb{R}^{1+n}$ (starting from $(0, x^0)$), which is called a *ray* for $\Box$ in the sequel. Sometimes, we also refer to $(t, x(t), \xi(t))$ as the ray. Obviously, rays for $\Box$ in $\mathbb{R}^n$ are simply straight lines.

In the presence of boundaries, rays, when reaching the boundary, are reflected following the usual rules of Geometric Optics. For a $T > 0$ and a bounded domain $M \subset \mathbb{R}^n$ with piecewise C^1 boundary ∂M, the singular set being localized on a closed (topological) sub-manifold S with $\dim S \le n-2$, we introduce the following definition of multiply reflected rays.

Definition 1. A parametric curve: $[0, T] \ni t \mapsto (t, x(t), \xi(t)) \in [0, T] \times \overline{M} \times \mathbb{R}^n$, with $x(0) \in M$ and $x(T) \in M$, is called a multiply reflected ray for the operator $\Box$ in $[0, T] \times \overline{M}$ if there exist $m \in \mathbb{N}$, $0 < t_0 < t_1 < \cdots < t_m = T$ such that each $(t, x(t), \xi(t))|_{t_i < t < t_{i+1}}$ is a ray for $\Box$ ($i = 0, 1, 2, \cdots, m-1$), which arrives at $\partial M \setminus S$ at time $t = t_{i+1}$, and is reflected by $(t, x(t), \xi(t))|_{t_{i+1} < t < t_{i+2}}$ by the usual geometric optics law whenever $i < m-1$.

In view of [47, Lemma 1.7.1] and [40, Lemma 2.2 and Remark 2.4], we have the following geometric lemma.

Lemma 5. *For each $T > 0$, there is a multiply reflected ray for the operator $\Box$ in M which meets always $\partial M \setminus S$ transversally and non-normally.*

6.3. Gaussian beams for the wave equation in the whole space

Now, given a ray $(t, x(t), \xi(t))$ of $\Box$ in $[0,T] \times \mathbb{R}^n$, one may construct a family of highly localized approximate solutions of the equation

$$\Box u = 0, \qquad \text{in } [0,T] \times \mathbb{R}^n \tag{6.5}$$

in the following form

$$u^\varepsilon(t,x) = \varepsilon^{1-n/4} a(t) e^{i\varphi(t,x)/\varepsilon}, \qquad \varepsilon > 0. \tag{6.6}$$

In (6.6), we take the phase function φ to be of the form

$$\varphi(t,x) = \xi(t)^T (x - x(t)) + \frac{1}{2}\Big(x - x(t)\Big)^T M(t) \Big(x - x(t)\Big), \tag{6.7}$$

where $M(t)$ is a $n \times n$ complex symmetric matrix with positive definite imaginary part. The construction of approximate solutions (6.6) requires an appropriate selection of $a(t)$ and $M(t)$.

Denote the perturbed energy of (6.5) by

$$e(t) \equiv e(u)(t) = \frac{1}{2}\int_{\mathbb{R}^n} \Big[|u(t,x)|^2 + |\nabla u(t,x)|^2 + |u_t(t,x)|^2\Big] dx. \tag{6.8}$$

The following result can be found in [38] and [32]:

Theorem 8. *Let $T > 0$ be given, and $(t, x(t))$ be a ray for $\Box$. Then for any $n \times n$ complex symmetric matrix M_0 with $\operatorname{Im} M_0 > 0$ and any $a_0 \in \mathbb{C} \setminus \{0\}$, there is a complex-valued symmetric matrix $M(\cdot) \in C^2([0,T]; \mathbb{C}^{n\times n})$ and a complex-valued function $a(\cdot) \in C^2([0,T]; \mathbb{C} \setminus \{0\})$ with*

$$M(0) = M_0, \qquad \operatorname{Im} M(t) > 0, \qquad a(0) = a_0, \tag{6.9}$$

such that

1) The functions u^ε as in (6.6) are approximate solutions of (6.5):

$$\sup_{t\in[0,T]} |\Box u^\varepsilon(t,\cdot)|_{L^2(\mathbb{R}^n)} = O(\varepsilon^{1/2}), \qquad \text{as } \varepsilon \to 0; \tag{6.10}$$

2) The initial energy of u^ε is bounded below as $\varepsilon \to 0$, i.e.,

$$e^\varepsilon(0) \equiv e(u^\varepsilon)(0) \approx 1; \tag{6.11}$$

3) The energy of u^ε is exponentially small off the ray $(t, x(t))$, i.e., there is a constant $\beta > 0$ such that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^n \setminus B_{\varepsilon^{1/4}}(t)} \Big[|u_t^\varepsilon(t,x)|^2 + |\nabla u^\varepsilon(t,x)|^2\Big] dx = O(e^{-\beta/\varepsilon}) \tag{6.12}$$

as $\varepsilon \to 0$, where $B_{\varepsilon^{1/4}}(t)$ is the ball centered at $x(t)$ with radius $\varepsilon^{1/4}$.

Remark 7. We recall that, by [38] and [32], the amplitude $a(t)$ in Theorem 8 is determined by the ODE:

$$\begin{cases} \dfrac{d}{dt}a(t) = a(t)\Box\varphi(t, x(t)), \\ a(t_0) = a_0. \end{cases}$$

On the other hand, $M(t)$ in Theorem 8 is determined by the Riccati equation:

$$\begin{cases} \dfrac{dM(t)}{dt} + M(t)C(t)M(t) + B(t)M(t) + M(t)B(t)^T + A(t) = 0, \\ M(0) = M_0, \end{cases}$$

where $C(t)$, $B(t)$ and $A(t)$ are $n \times n$ matrices whose coefficients are determined by the first and second derivatives of $|\xi|^2$ evaluated along the ray $(t, x(t), \xi(t))$. We refer to [38] for the global existence of solutions to this nonlinear ODE with initial datum M_0 so that $\operatorname{Im} M_0 > 0$.

Remark 8. Let $a_{i_1\cdots i_k}, b_{i_1\cdots i_k} \in \mathbb{C}$, $i_1, \cdots, i_k = 1, 2, \cdots, n$ $(k \in \mathbb{N})$, be given so that $a_{i_1\cdots i_k} = a_{i'_1\cdots i'_k}$ and $b_{i_1\cdots i_k} = b_{i'_1\cdots i'_k}$ for any permutation $i'_1, \cdots, i'_k$ of $i_1, \cdots, i_k$. Put $(x_1(t), \cdots, x_n(t)) \equiv x(t)$. From [38] and [32], we see that the conclusions in Theorem 8 still hold true if

i) for any given $k_0 \geq 1$, one replaces the amplitude function $a(t)$ in (6.6) by

$$a(t, x) = a(t) + \sum_{k=1}^{k_0} \sum_{i_1, \cdots, i_k = 1}^{n} a_{i_1\cdots i_k}(x_{i_1} - x_{i_1}(t)) \cdots (x_{i_k} - x_{i_k}(t)); \tag{6.13}$$

and/or

ii) for any given $k_1 \geq 3$, one replaces the phase function φ given by (6.7) by

$$\begin{aligned} \phi(t, x) = \xi(t)^T(x - x(t)) + \frac{1}{2}\Big(x - x(t)\Big)^T M(t)\Big(x - x(t)\Big) \\ + \sum_{k=3}^{k_1} \sum_{i_1, \cdots, i_k = 1}^{n} b_{i_1\cdots i_k}(x_{i_1} - x_{i_1}(t)) \cdots (x_{i_k} - x_{i_k}(t)). \end{aligned} \tag{6.14}$$

Indeed, this follows from the fact that the difference between those two amplitude and phases are of order $|x - x(t)|$ and $|x - x(t)|^3$, respectively. Hence, for any fixed $s_0 \in [0, T]$, the perturbed phase ϕ satisfies

$$\operatorname{Im} \phi(s_0, x) \geq c|x - x(s_0)|^2, \qquad \text{as } x \to x(s_0). \tag{6.15}$$

This means that the factor $e^{i\phi(s_0,x)/\varepsilon}$ localizes $u^\varepsilon(s_0,x) = \varepsilon^{1-n/4}a(s_0,x)e^{i\varphi(s_0,x)/\varepsilon}$ in the region

$$|x - x(s_0)|^2 = O(\varepsilon). \tag{6.16}$$

Furthermore, the conclusions in Theorem 8 still hold if for any given C^2 functions $a^1(t,x)$ and $a^2(t,x)$, one replaces the amplitude function $a(t,x)$ in (6.13) by

$$A(t,x) = a(t,x) + \sum_{k=1}^{2} \varepsilon^k a^k(t,x), \tag{6.17}$$

by adding C^2-smooth $O(\varepsilon^k)$-terms ($k = 1,2$).

The above observations will play a key role in the sequel.

Remark 9. Let $\chi \in C_0^\infty(\mathbb{R}^{1+n})$ be any given cut-off function which is identically equal to 1 in a neighborhood of the ray $\Big\{(t,x(t)) \;\Big|\; t \in [0,T]\Big\}$. Then the functions χu^ε also satisfy (6.10)–(6.12). In view of this, we may choose u^ε such that they are supported in any given (small) neighborhood of the ray.

6.4. Gaussian beams for the wave equation with curved wavefronts

From now on to the rest of this section, we shall construct highly localized solutions to system (1.1).

Assume $(t, x^-(t), \xi^-(t))$ is an incoming ray for $\Box \equiv \partial_{tt} - \Delta$ starting from some point in Ω_2 at time $t = 0$, i.e., $x^-(0) \in \Omega_2$, and arriving at the boundary $\partial\Omega_2$ at time $t = t_0$, i.e., $x_0 \stackrel{\triangle}{=} x^-(t_0) \in \partial\Omega_2$. According to (6.3), $(x^-(t), \xi^-(t))$ satisfies

$$\begin{cases} \dot{x}^-(t) = 2\xi^-(t), \quad \dot{\xi}^-(t) = 0, \\ x^-(t_0) = x_0, \quad \xi^-(t_0) = \xi^-(t_0). \end{cases} \tag{6.18}$$

Assume $(t, x^+(t), \xi^+(t))$ to be the reflected ray. More precisely, we choose $(x^+(t), \xi^+(t))$ to be the solution of the system

$$\begin{cases} \dot{x}^+(t) = 2\xi^+(t), \quad \dot{\xi}^+(t) = 0, \\ x^+(t_0) = x_0, \quad \xi^+(t_0) = \xi^-(t_0) - 2\xi^-(t_0)\cdot\nu_2(x_0)\nu_2(x_0), \end{cases} \tag{6.19}$$

where $\nu_2(x_0)$ is the unit outward normal vector of Ω_2 at x_0. The choice of the initial datum $\xi^+(t_0)$ is such that the directions of the incoming and reflected rays satisfy the usual "*geometric optics law*", i.e.,

$$\dot{x}^+(t_0) = \dot{x}^-(t_0) - 2\dot{x}^-(t_0)\cdot\nu_2(x_0)\nu_2(x_0). \tag{6.20}$$

On the other hand, from (6.4), one has $|\xi^-(t_0)| = 1/2$. Hence, noting $|\nu_2(x_0)| = 1$, it is easy to check that $|\xi^+(t_0)| = 1/2$. Therefore,

$$|\xi^{\pm}(t)| = \frac{1}{2}, \qquad \forall\, t \in \mathbb{R}. \tag{6.21}$$

Denote by T_1 the instant when the reflected ray arrives at $\partial\Omega_2$, i.e., $x^+(T_1) \in \partial\Omega_2$ (Note that, of course, $0 < t_0 < T_1$). In the rest of this section, we fix any

$$T^* \in (t_0, T_1). \tag{6.22}$$

Also, by Lemma 5, we may assume that $(t, x^-(t), \xi^-(t))$ is transversal and non-normal to the boundary $\partial\Omega_2$ at time $t = t_0$, i.e.,

$$\xi^-(t_0) \cdot \nu_2(x_0) \neq 0 \quad \text{ and } \quad \xi^-(t_0) \not\parallel \nu_2(x_0). \tag{6.23}$$

We now construct approximate solutions to system (1.1), which are highly localized in a small neighborhood of the rays $(t, x^{\pm}(t))$.

We distinguish two cases, i.e., either x_0 belongs to the exterior boundary Γ_2 of the wave domain Ω_2, or to the interface γ. In view of Lemma 5, we can always choose the ray such that $x_0 \notin \overline{\Gamma_2} \cap \overline{\gamma}$. The case where $x_0 \in \gamma$ will be analyzed in the Subsection 6.6. In this subsection, we focus on the first case, i.e., $x_0 \in \Gamma_2$. The construction of reflected beams is very close to [38], [32] and [40]. Therefore, we only give a sketch.

Let us introduce geodesic normal coordinates near the reflection point $x_0 \in \Gamma_2$, called henceforth $\tilde{x} \equiv \tilde{x}(x) = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \equiv (\tilde{x}_1, \tilde{x}')$, centered at the reflection point

$$\tilde{x}_0 \equiv (0, \tilde{x}_0'), \text{ the new coordinate of } x_0, \tag{6.24}$$

such that Ω_2 is locally given by $\tilde{x}_1 \geq 0$ and Γ_2 is flat near $\tilde{x}_0$. Denote the inverse Jacobian matrix of $\tilde{x} = \tilde{x}(x)$ by $J(\tilde{x})$, i.e.

$$J(\tilde{x}) \equiv \Big(g_{ij}(\tilde{x})\Big)_{1 \leq i,j \leq n} \stackrel{\triangle}{=} \frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)}. \tag{6.25}$$

In this subsection, we only need to assume the boundary $\partial\Omega_2$ of Ω_2 to be C^2. This means that $J(\tilde{x}) \in C^1$.

Put

$$\begin{aligned} \sigma^{\pm} &\equiv (\sigma_1^{\pm}, \sigma_2^{\pm}, \cdots, \sigma_n^{\pm}) \equiv (\sigma_1^{\pm}, \sigma_{\pm}') \stackrel{\triangle}{=} (J(\tilde{x}_0))^T \xi^{\pm}(t_0), \\ \eta^{\pm} &\equiv (\eta_1^{\pm}, \eta_2^{\pm}, \cdots, \eta_n^{\pm}) \equiv (\eta_1^{\pm}, \eta_{\pm}') \stackrel{\triangle}{=} (J(\tilde{x}_0))^{-1} \xi^{\pm}(t_0), \end{aligned} \tag{6.26}$$

where $\sigma_{\pm}', \eta_{\pm}' \in \mathbb{R}^{n-1}$. By [40, Proposition 4.1], the following is known to hold.

Proposition 1. *Under the assumption (6.23), it holds:*

$$\begin{aligned} &\eta_1^\pm \neq 0, \qquad (\eta_2^+,\cdots,\eta_n^+) = (\eta_2^-,\cdots,\eta_n^-),\\ &(\sigma_2^+,\cdots,\sigma_n^+) = (\sigma_2^-,\cdots,\sigma_n^-) \neq 0. \end{aligned} \tag{6.27}$$

Remark 10. From the proof of [40, Proposition 4.1], we see that $\eta_1^\pm \neq 0$ and $(\sigma_2^+,\cdots,\sigma_n^+) = (\sigma_2^-,\cdots,\sigma_n^-) \neq 0$ are implied respectively by the first and the second assumptions in (6.23). The first assumption in (6.23) means that the incoming ray $(t, x^-(t))$ reaches the boundary transversally; while the second one means it arrives at the boundary non-normally. Note however that in this subsection we only need the first assumption.

Fix any $k_0 \geq 1$, $\ell_0 \geq 1$, $k_1 \geq 3$, and functions $a_\pm^1(t,x), a_\pm^2(t,x) \in C^2([0,T]\times\overline{\Omega_2};\mathbb{C})$, and any $a^\pm_{i_1\cdots i_k}, b^\pm_{i_1\cdots i_k} \in \mathbb{C}$, $i_1,\cdots,i_k = 1,2,\cdots,n$, $k\in\mathbb{N}$ so that $a^\pm_{i_1\cdots i_k} = a^\pm_{i_1'\cdots i_k'}$ and $b^\pm_{i_1\cdots i_k} = b^\pm_{i_1'\cdots i_k'}$ for any permutation $i_1',\cdots,i_k'$ of $i_1,\cdots,i_k$. According to (6.6) and Theorem 8, and noting Remark 8, we may assume the incoming and reflected waves to be of the form

$$z_\varepsilon^\pm(t,x) = \varepsilon^{1-n/4}A^\pm(t,x)e^{i\phi^\pm(t,x)/\varepsilon}, \tag{6.28}$$

where

$$A^\pm(t,x) = a^\pm(t,x) + \sum_{k=1}^{2}\varepsilon^k a_\pm^k(t,x), \tag{6.29}$$

$$a^\pm(t,x) = a^\pm(t) + \sum_{k=1}^{k_0}\sum_{i_1,\cdots,i_k=1}^{n} a^\pm_{i_1\cdots i_k}(x_{i_1}-x^\pm_{i_1}(t))\cdots(x_{i_k}-x^\pm_{i_k}(t)), \tag{6.30}$$

and

$$\begin{aligned} \phi^\pm(t,x) &= \xi^\pm(t)^T(x-x^\pm(t)) + \frac{1}{2}\Big(x-x^\pm(t)\Big)^T M^\pm(t)\Big(x-x^\pm(t)\Big)\\ &\quad + \sum_{k=3}^{k_1}\sum_{i_1,\cdots,i_k=1}^{n} b^\pm_{i_1\cdots i_k}(x_{i_1}-x^\pm_{i_1}(t))\cdots(x_{i_k}-x^\pm_{i_k}(t)). \end{aligned} \tag{6.31}$$

In (6.31), $M^\pm(t)$ are some $n\times n$ complex symmetric matrices with positive definite imaginary parts. According to Remark 7, $M^-(t)$ is known (it is determined by its initial data $M^-(0)$ and the incoming ray $(t, x^-(t), \xi^-(t))$); while $M^+(t)$ will be determined by its initial data $M^+(t_0)$ and the reflected ray $(t, x^+(t), \xi^+(t))$. Obviously, we need to determine $M^+(t_0)$ in terms of the incoming wave. This can be done following the procedure in [40] that we recall here for the reader's convenience.

Denote

$$\widetilde{M}^\pm(t_0) \stackrel{\triangle}{=} (J(\tilde{x}_0))^T M^\pm(t_0) J(\tilde{x}_0). \tag{6.32}$$

Obviously, determining $M^+(t_0)$ is equivalent to choosing $\widetilde{M}^+(t_0)$.

Denote

$$\begin{aligned}&\nabla_{\tilde{x}}\Big((J(\tilde{x}_0))^T\xi^{\pm}(t_0)\Big) \equiv \Big(h^{\pm}_{ij}\Big)_{1\le i,j\le n},\\ &\widetilde{M}^{\pm}(t_0) \equiv \Big(m^{\pm}_{ij}\Big)_{1\le i,j\le n} \equiv \begin{pmatrix} m^{\pm}_{11} & \vartheta^T_{\pm} \\ \vartheta_{\pm} & \hat{M}^{\pm}\end{pmatrix},\end{aligned} \tag{6.33}$$

where $\vartheta_{\pm} = (m^{\pm}_{21}, \cdots, m^{\pm}_{n1})^T$ and $\hat{M}^{\pm} = \Big(m^{\pm}_{ij}\Big)_{2\le i,j\le n}$. Note that all m^{-}_{ij} are known. We now assign all m^{+}_{ij}, and in this way one obtains $\widetilde{M}^{+}(t_0)$ in (6.32). First of all, we choose

$$m^{+}_{ij} = h^{-}_{ij} + m^{-}_{ij} - h^{+}_{ij}, \qquad 2 \le i, j \le n. \tag{6.34}$$

This determines $\hat{M}^{+}$. Next, we choose

$$\vartheta_{+} = \frac{\eta^{-}_1\vartheta_{-} + \hat{M}^{-}\eta'_{-} - \hat{M}^{+}\eta'_{+}}{\eta^{+}_1}. \tag{6.35}$$

This determines $m^{+}_{j1} = m^{+}_{1j}$ for $j = 2, \cdots, n$. Finally, we choose

$$m^{+}_{11} = \frac{m^{-}_{11}|\eta^{-}_1|^2 + 2\eta^{-}_1\vartheta^T_{-}\eta'_{-} + (\eta'_{-})^T\hat{M}^{-}\eta'_{-} - 2\eta^{+}_1\vartheta^T_{+}\eta'_{+} - (\eta'_{+})^T\hat{M}^{+}\eta'_{+}}{|\eta^{+}_1|^2}. \tag{6.36}$$

This completes the assignment of $\widetilde{M}^{+}(t_0)$, and hence, according to (6.32), of $M^{+}(t_0)$.

As mentioned in [40], the choice of $M^{+}(t_0)$, is such that the incoming and reflected phases $\phi^{\pm}(t,x)$ coincide at the reflection point up to second order in time and tangential derivatives. More precisely, denote the expressions of $\phi^{\pm}(t,x)$ in the $\tilde{x}$-coordinates by

$$\begin{aligned}\tilde{\phi}^{\pm}(t,\tilde{x}) = {}& \xi^{\pm}(t)^T(x(\tilde{x}) - x^{\pm}(t))\\ &+\frac{1}{2}\Big(x(\tilde{x}) - x^{\pm}(t)\Big)^T M^{\pm}(t)\Big(x(\tilde{x}) - x^{\pm}(t)\Big)\\ &+\sum_{k=3}^{k_1}\sum_{i_1,\cdots,i_k=1}^{n} b^{\pm}_{i_1\cdots i_k}(x_{i_1}(\tilde{x}) - x^{\pm}_{i_1}(t))\cdots(x_{i_k}(\tilde{x}) - x^{\pm}_{i_k}(t)).\end{aligned} \tag{6.37}$$

By [40, Propositions 4.2, 4,4 and 4.6], and noting that the corrector terms in (6.37) are of order $|x(\tilde{x}) - x^{\pm}(t)|^3$, the following result holds (recall (6.24) for the definition of $\tilde{x}'_0$):

Proposition 2. *As $(t, \tilde{x}')$ tends to $(t_0, \tilde{x}'_0)$, one has*

$$\partial_t\tilde{\phi}^{\pm}(t, 0, \tilde{x}') = -\frac{1}{2} + O(|t - t_0| + |\tilde{x}' - \tilde{x}'_0|), \tag{6.38}$$

$$\nabla_{\tilde{x}}\tilde{\phi}^{\pm}(t, 0, \tilde{x}') = \sigma^{\pm} + O(|t - t_0| + |\tilde{x}' - \tilde{x}'_0|), \tag{6.39}$$

$$\tilde{\phi}^{+}(t, 0, \tilde{x}') - \tilde{\phi}^{-}(t, 0, \tilde{x}') = O(|t - t_0|^3 + |\tilde{x}' - \tilde{x}'_0|^3), \tag{6.40}$$

$$\operatorname{Im}\tilde{\phi}^{\pm}(t, 0, \tilde{x}') \ge c(|t - t_0|^2 + |\tilde{x}' - \tilde{x}'_0|^2). \tag{6.41}$$

Remark 11. From (6.41) in Proposition 2, it is easy to see that the factor $e^{i\tilde{\phi}^{\pm}(t,0,\tilde{x}')/\varepsilon}$ localizes $\tilde{z}_\varepsilon^{\pm}$, which corresponds to $z_\varepsilon^{\pm}$ in (6.28) in the $\tilde{x}$-coordinates, in the region:

$$|t-t_0|^2+|\tilde{x}'-\tilde{x}_0'|^2=O(\varepsilon). \tag{6.42}$$

Denote the expressions of $a^{\pm}(t,x)$ in the $\tilde{x}$-coordinates by

$$\begin{aligned}&\tilde{a}^{\pm}(t,\tilde{x})\\&=a^{\pm}(t)+\sum_{k=1}^{k_0}\sum_{i_1,\cdots,i_k=1}^{n}a^{\pm}_{i_1\cdots i_k}(x_{i_1}(\tilde{x})-x^{\pm}_{i_1}(t))\cdots(x_{i_k}(\tilde{x})-x^{\pm}_{i_k}(t)).\end{aligned} \tag{6.43}$$

Similar to [40] and in view of Remarks 7 and 11, choosing the initial value of $a^+(t)$ at $t=t_0$ by

$$a^+(t_0)=-a^-(t_0), \tag{6.44}$$

we conclude that

Lemma 6. *The approximate solutions $z_\varepsilon^{\pm}(t,x)$ of $\Box u=0$, constructed by (6.28), (6.30) and (6.31), satisfy*

$$|z_\varepsilon^-+z_\varepsilon^+|_{H^1((0,T^*)\times\partial\Omega_2)}=O(\varepsilon^{1/2}). \tag{6.45}$$

The proof of Lemma 6 is the same as [40, Lemma 4.1]. Hence we omit the details. Also, recall that the considered case in this subsection is $x_0\in\Gamma_2$. Hence, similar to [40], as a consequence of Lemma 6, we arrive at the following conclusion (recall (6.22) for the definition of T^*):

Lemma 7. *Let $(t,x^-(t),\xi^-(t))$, with $x^-(0)\in\Omega_2$, be an incoming ray for $\Box$, which arrives transversally at Γ_2 at time $t=t_0$, i.e., the first assumption in (6.23) holds. Let $(t,x^+(t),\xi^+(t))$ be the reflected ray constructed above, with reflection point $x_0\in\Gamma_2$. Then there is a family of solutions $\{(y_\varepsilon,z_\varepsilon)\}_{\varepsilon>0}$ of system (1.1) in $(0,T^*]\times\Omega_2$ (the initial conditions being excepted), such that*

$$|\nabla y_\varepsilon|^2_{(L^2((0,T^*)\times\Omega_1))^n}=O(\varepsilon),\quad E_\varepsilon(0)=E(y_\varepsilon,z_\varepsilon,\partial_t z_\varepsilon)(0)\approx 1. \tag{6.46}$$

6.5. Higher order matching of amplitudes and phases

We recall that, as for Proposition 2 and Lemma 6, the choices of $a^{\pm}_{i_1\cdots i_k}$ in (6.43) and $b^{\pm}_{i_1\cdots i_k}$ in (6.37) are arbitrary. From (6.44), we see that the incoming and reflected amplitudes $a^{\pm}(t,x)$ anti-coincide at the reflection point. On the other hand, from (6.40) in Proposition 2, it is clear that the incoming and reflected phases $\phi^{\pm}(t,x)$ coincide at the reflection point, up to the second order derivatives. This was enough in Subsection 6.4 for the case where the reflection point $x_0 \in \Gamma_2$. However, in order to consider the case in which the reflection point x_0 belongs to the interface γ, one needs higher order anti-coincidence and coincidence properties of amplitudes $a^{\pm}(t,x)$ and phases $\phi^{\pm}(t,x)$ at the reflection point, respectively. As we shall see in this subsection, this is actually possible by suitably choosing the coefficients $a^{+}_{i_1\cdots i_k}$ and $b^{+}_{i_1\cdots i_k}$ in (6.30) and (6.31).

We have the result:

Proposition 3. *(1) For any given $k_0 \geq 1$, and any given $a^{-}_{i_1\cdots i_k} \in \mathbb{C}$, $i_1,\cdots,i_k = 1,2,\cdots,n$ and $k = 1,\cdots,k_0$, so that $a^{-}_{i_1\cdots i_k} = a^{-}_{i'_1\cdots i'_k}$ for any permutation $i'_1,\cdots,i'_k$ of $i_1,\cdots,i_k$, there exist $a^{+}_{i_1\cdots i_k} \in \mathbb{C}$, $i_1,\cdots,i_k = 1,2,\cdots,n$ and $k = 1,\cdots,k_0$, with the same symmetry property, such that $\tilde{a}^{\pm}$ as in (6.43) fulfill*

$$\begin{aligned}&\tilde{a}^{+}(t,0,\tilde{x}') + \tilde{a}^{-}(t,0,\tilde{x}')\\&= a^{+}(t_0) + a^{-}(t_0) + O(|t-t_0|^{k_0+1} + |\tilde{x}' - \tilde{x}'_0|^{k_0+1}),\end{aligned} \tag{6.47}$$

as $(t,\tilde{x}') \to (t_0,\tilde{x}'_0)$.

(2) For any given $k_1 \geq 3$, $b^{-}_{i_1\cdots i_k} \in \mathbb{C}$, $i_1,\cdots,i_k = 1,2,\cdots,n$, and $k = 3,\cdots,k_1$, so that $b^{-}_{i_1\cdots i_k} = b^{-}_{i'_1\cdots i'_k}$ for any permutation $i'_1,\cdots,i'_k$ of $i_1,\cdots,i_k$, there exist $b^{+}_{i_1\cdots i_k} \in \mathbb{C}$, $i_1,\cdots,i_k = 1,2,\cdots,n$ and $k = 3,\cdots,k_1$, with the same symmetry property, such that $\tilde{\phi}^{\pm}$ as in (6.37) satisfy

$$\tilde{\phi}^{+}(t,0,\tilde{x}') - \tilde{\phi}^{-}(t,0,\tilde{x}') = O(|t-t_0|^{k_1+1} + |\tilde{x}' - \tilde{x}'_0|^{k_1+1}), \tag{6.48}$$

as $(t,\tilde{x}') \to (t_0,\tilde{x}'_0)$.

Proof. We only prove the first statement (6.47) (The second one can be proved similarly). By induction, assume that we have chosen the desired coefficients $a^{+}_{i_1\cdots i_k}$ $(k = 1,2,\cdots,k_0-1)$ such that

$$\begin{aligned}&\varrho^{+}(t,0,\tilde{x}') + \varrho^{-}(t,0,\tilde{x}')\\&= a^{+}(t_0) + a^{-}(t_0) + O(|t-t_0|^{k_0} + |\tilde{x}' - \tilde{x}'_0|^{k_0})\end{aligned} \tag{6.49}$$

as $(t,\tilde{x}') \to (t_0,\tilde{x}'_0)$, where

$$\varrho^{\pm}(t,\tilde{x}) \triangleq \begin{cases} a^{\pm}(t) + \displaystyle\sum_{k=1}^{k_0-1} \sum_{i_1,\cdots,i_k=1}^{n} a^{\pm}_{i_1\cdots i_k}(x_{i_1}(\tilde{x}) - x^{\pm}_{i_1}(t))\cdots(x_{i_k}(\tilde{x}) - x^{\pm}_{i_k}(t)), & \text{if } k_0 \geq 2,\\ a^{\pm}(t), & \text{if } k_0 = 1.\end{cases} \tag{6.50}$$

Then, according to (6.43), one has

$$\begin{aligned}&\tilde a^{\pm}(t,\tilde x)\\&=\varrho^{\pm}(t,\tilde x)+\sum_{i_1,\cdots,i_{k_0}=1}^{n}a^{\pm}_{i_1\cdots i_{k_0}}(x_{i_1}(\tilde x)-x^{\pm}_{i_1}(t))\cdots(x_{i_{k_0}}(\tilde x)-x^{\pm}_{i_{k_0}}(t)).\end{aligned}\tag{6.51}$$

It remains to choose $a^{+}_{i_1\cdots i_{k_0}}$ such that (6.47) holds. For this purpose, for any $s_1,\cdots,s_{k_0}\in\{1,2,\cdots,n\}$, put

$$\tilde a^{\pm}_{s_1\cdots s_{k_0}}\stackrel{\triangle}{=}\sum_{i_1,\cdots,i_{k_0}=1}^{n}a^{\pm}_{i_1\cdots i_{k_0}}g_{i_1s_1}(\tilde x_0)\cdots g_{i_{k_0}s_{k_0}}(\tilde x_0),\tag{6.52}$$

where the matrix $\Big(g_{ij}(\tilde x_0)\Big)_{1\le i,j\le n}$ is as in (6.25). Obviously, all $\tilde a^{-}_{s_1\cdots s_{k_0}}$ are known. From (6.52) and noting that the matrix $\Big(g_{ij}(\tilde x_0)\Big)_{1\le i,j\le n}$ is invertible, it suffices to determine $\tilde a^{+}_{i_1i_2\cdots i_{k_0}}$ for all $i_1,\cdots,i_{k_0}\in\{1,2,\cdots,n\}$.

From (6.51), recalling (6.24) for the definition of $\tilde x_0$ and noting that $x(\tilde x_0)=x_0=x^{\pm}(t_0)$, and recalling the definition of $g_{ij}(\tilde x_0)$ in (6.25), we get

$$\begin{aligned}&\partial_{\tilde x_{j_1}\cdots\tilde x_{j_{k_0}}}\tilde a^{\pm}(t_0,\tilde x_0)\\&=\partial_{\tilde x_{j_1}\cdots\tilde x_{j_{k_0}}}\varrho^{\pm}(t_0,\tilde x_0)+k_0!\sum_{i_1,\cdots,i_{k_0}=1}^{n}a^{\pm}_{i_1\cdots i_{k_0}}g_{i_1j_1}(\tilde x_0)\cdots g_{i_{k_0}j_{k_0}}(\tilde x_0),\end{aligned}\tag{6.53}$$

where $j_1,j_2,\cdots,j_{k_0}\in\{2,3,\cdots,n\}$. From (6.53) and (6.52), for $j_1,j_2,\cdots,j_{k_0}$ $\in\{2,3,\cdots,n\}$, one chooses

$$\tilde a^{+}_{j_1\cdots j_{k_0}}=-\frac{\partial_{\tilde x_{j_1}\cdots\tilde x_{j_{k_0}}}\varrho^{+}(t_0,\tilde x_0)+\partial_{\tilde x_{j_1}\cdots\tilde x_{j_{k_0}}}\varrho^{-}(t_0,\tilde x_0)}{k_0!}-\tilde a^{-}_{j_1\cdots j_{k_0}}.\tag{6.54}$$

Then, one finds

$$\begin{aligned}&\partial_{\tilde x_{j_1}\cdots\tilde x_{j_{k_0}}}\tilde a^{+}(t_0,\tilde x_0)+\partial_{\tilde x_{j_1}\cdots\tilde x_{j_{k_0}}}\tilde a^{-}(t_0,\tilde x_0)=0,\\&\qquad\qquad\forall\ j_1,j_2,\cdots,j_{k_0}\in\{2,3,\cdots,n\}.\end{aligned}\tag{6.55}$$

Similarly, from (6.51) and noting (6.18) and (6.19), we get

$$\begin{aligned}&\partial_{t\tilde x_{j_2}\cdots\tilde x_{j_{k_0}}}\tilde a^{\pm}(t_0,\tilde x_0)\\&=\partial_{t\tilde x_{j_2}\cdots\tilde x_{j_{k_0}}}\varrho^{\pm}(t_0,\tilde x_0)\\&\quad-2(k_0!)\sum_{i_1,\cdots,i_{k_0}=1}^{n}a^{\pm}_{i_1\cdots i_{k_0}}\xi^{\pm}_{i_1}(t_0)g_{i_2j_2}(\tilde x_0)\cdots g_{i_{k_0}j_{k_0}}(\tilde x_0),\end{aligned}\tag{6.56}$$

where $j_2,\cdots,j_{k_0}\in\{2,3,\cdots,n\}$. From (6.25) and (6.26), one has

$$\xi^{\pm}_{i}(t_0)=\sum_{\ell=1}^{n}g_{i\ell}(\tilde x_0)\eta^{\pm}_{\ell}.\tag{6.57}$$

Therefore, by (6.56) and (6.57), and noting (6.52), we get

$$\begin{aligned}
&\partial_{t\tilde{x}_{j_2}\cdots\tilde{x}_{j_{k_0}}}\tilde{a}^{\pm}(t_0,\tilde{x}_0)\\
&=\partial_{t\tilde{x}_{j_2}\cdots\tilde{x}_{j_{k_0}}}\varrho^{\pm}(t_0,\tilde{x}_0)\\
&\quad -2(k_0!)\sum_{i_1,\cdots,i_{k_0},\ell=1}^{n} a^{\pm}_{i_1\cdots i_{k_0}} g_{i_1\ell}(\tilde{x}_0)g_{i_2j_2}(\tilde{x}_0)\cdots g_{i_{k_0}j_{k_0}}(\tilde{x}_0)\eta^{\pm}_{\ell}\\
&=\partial_{t\tilde{x}_{j_2}\cdots\tilde{x}_{j_{k_0}}}\varrho^{\pm}(t_0,\tilde{x}_0)-2(k_0!)\sum_{\ell=1}^{n}\tilde{a}^{\pm}_{\ell j_2\cdots j_{k_0}}\eta^{\pm}_{\ell}\\
&=\partial_{t\tilde{x}_{j_2}\cdots\tilde{x}_{j_{k_0}}}\varrho^{\pm}(t_0,\tilde{x}_0)-2(k_0!)\sum_{\ell=2}^{n}\tilde{a}^{\pm}_{\ell j_2\cdots j_{k_0}}\eta^{\pm}_{\ell}-2(k_0!)\tilde{a}^{\pm}_{1j_2\cdots j_{k_0}}\eta^{\pm}_{1}.
\end{aligned} \tag{6.58}$$

Noting $\eta_1^+\neq 0$ (*see* Proposition 1), this gives

$$\begin{aligned}
\tilde{a}^{+}_{1j_2\cdots j_{k_0}}=\frac{1}{\eta_1^+}\Bigg[&\frac{\partial_{t\tilde{x}_{j_2}\cdots\tilde{x}_{j_{k_0}}}\varrho^{+}(t_0,\tilde{x}_0)+\partial_{t\tilde{x}_{j_2}\cdots\tilde{x}_{j_{k_0}}}\varrho^{-}(t_0,\tilde{x}_0)}{2(k_0!)}\\
&-\sum_{\ell=2}^{n}\Big(\tilde{a}^{+}_{\ell j_2\cdots j_{k_0}}\eta^{+}_{\ell}+\tilde{a}^{-}_{\ell j_2\cdots j_{k_0}}\eta^{-}_{\ell}\Big)-\eta_1^{-}\tilde{a}^{-}_{1j_2\cdots j_{k_0}}\Bigg],\\
&\forall\, j_2,\cdots,j_{k_0}\in\{2,3,\cdots,n\}.
\end{aligned} \tag{6.59}$$

Then, one finds

$$\begin{aligned}
\partial_{t\tilde{x}_{j_2}\cdots\tilde{x}_{j_{k_0}}}\tilde{a}^{+}(t_0,\tilde{x}_0)+\partial_{t\tilde{x}_{j_2}\cdots\tilde{x}_{j_{k_0}}}\tilde{a}^{-}(t_0,\tilde{x}_0)=0,&\\
\forall\, j_2,\cdots,j_{k_0}\in\{2,3,\cdots,n\}.&
\end{aligned} \tag{6.60}$$

This procedure determines all $\tilde{a}^{+}_{i_1i_2\cdots i_{k_0}}$ for $j_1,\cdots,j_{k_0}\in\{1,2,\cdots,n\}$ so that

$$\partial^{k_0}_{t,\tilde{x}'}\tilde{a}^{+}(t_0,\tilde{x}_0)+\partial^{k_0}_{t,\tilde{x}'}\tilde{a}^{-}(t_0,\tilde{x}_0)=0. \tag{6.61}$$

This completes the proof of Proposition 3. □

6.6. Highly concentrated approximate solutions for the heat equation with curved wavefronts

We now analyze what happens when the incoming ray $(t,x^-(t))$ for $\Box$ arrives at the interface γ at time t_0, i.e., $x_0\in\gamma$.

We still use the geodesic normal coordinates near the reflection point $x_0\in\gamma$, denoted as $\tilde{x}=(\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_n)\equiv(\tilde{x}_1,\tilde{x}')$, centered at the reflection point $\tilde{x}_0$, i.e., the new coordinate of x_0, such that:

i) γ is flat near $\tilde{x}_0$;

ii) Ω_1 is locally given by $\tilde{x}_1 < 0$, and the heat operator is locally given as follows:

$$\widetilde{H} = \partial_t - \sum_{i,j=1}^{n} \tilde{\alpha}_{ij}(\tilde{x})\partial_{\tilde{x}_i}\partial_{\tilde{x}_j} - \sum_{i=1}^{n} \tilde{\beta}_i(\tilde{x})\partial_{\tilde{x}_i}; \tag{6.62}$$

iii) Ω_2 is locally given by $\tilde{x}_1 > 0$, and the wave operator $\Box$ takes the following form

$$\widetilde{W} = \partial_{tt} - \sum_{j,k=1}^{n} \tilde{\alpha}_{jk}(\tilde{x})\partial_{\tilde{x}_j}\partial_{\tilde{x}_k} - \sum_{j=1}^{n} \tilde{\beta}_j(\tilde{x})\partial_{\tilde{x}_j}; \tag{6.63}$$

iv) For $\tilde{x}$ belonging to a small neighborhood of $\tilde{x}_0$, $(\tilde{\alpha}_{ij})_{n\times n} \in C^2$ is strictly positive definite and symmetric and $\tilde{\beta}_j$ are C^1 functions. Moreover one has

$$\tilde{\alpha}_{11}(\tilde{x}) \equiv 1, \qquad \tilde{\alpha}_{1k}(\tilde{x}) \equiv 0 \text{ for any } k = 2, \cdots, n. \tag{6.64}$$

Assume $\tilde{z}_\varepsilon^-(t,\tilde{x})$ to be the $\tilde{x}$-coordinate expression of the incoming wave $z_\varepsilon^-(t,x)$ given in (6.28), i.e.,

$$\tilde{z}_\varepsilon^-(t,\tilde{x}) = \varepsilon^{1-n/4}\tilde{a}^-(t,\tilde{x})e^{i\tilde{\phi}^-(t,\tilde{x})/\varepsilon} \tag{6.65}$$

where $\tilde{a}^-(t,\tilde{x})$ and $\tilde{\phi}^-(t,\tilde{x})$ are as in (6.43) and (6.37), respectively. As in [40], in order to construct approximate solutions of system (1.1), we will seek reflected waves $z_\varepsilon^+(t,x)$ similar to the previous subsection and approximate solutions of the form

$$\tilde{y}_\varepsilon = \tilde{y}_\varepsilon(t,\tilde{x}) = \varepsilon^{1-n/4}\widetilde{B}(t,\tilde{x})e^{i\tilde{\psi}(t,\tilde{x})/\varepsilon} \tag{6.66}$$

of the heat equation

$$\widetilde{H}\tilde{y} = 0. \tag{6.67}$$

In this subsection we recall the construction of the parabolic approximate solutions $\tilde{y}_\varepsilon$ in [40], which will be glued with the hyperbolic approximate solutions $z_\varepsilon^\pm(t,x)$ to produce approximate solutions of our transmission problem (1.1) in the next subsection.

First, put

$$\begin{aligned}
&f_0 \equiv f_0(t,0,\tilde{x}') \overset{\triangle}{=} \tilde{\phi}^-(t,0,\tilde{x}'),\\
&f_1 \equiv f_1(t,0,\tilde{x}') \overset{\triangle}{=} -i\sqrt{\sum_{i,j=2}^{n} \tilde{\alpha}_{ij}(0,\tilde{x}')\partial_{\tilde{x}_i}\tilde{\phi}^-(t,0,\tilde{x}')\partial_{\tilde{x}_j}\tilde{\phi}^-(t,0,\tilde{x}')}\,,\\
&f_2 \equiv f_2(t,0,\tilde{x}') \overset{\triangle}{=} -\frac{1}{2f_1}\sum_{i,j=2}^{n}\Big(2\tilde{\alpha}_{ij}(0,\tilde{x}')\partial_{\tilde{x}_i}f_0\partial_{\tilde{x}_j}f_1\\
&\qquad\qquad\qquad + \partial_{\tilde{x}_1}\tilde{\alpha}_{ij}(t,0,\tilde{x}')\partial_{\tilde{x}_i}f_0\partial_{\tilde{x}_j}f_0\Big).
\end{aligned} \tag{6.68}$$

We choose

$$\tilde{\psi}(t, \tilde{x}_1, \tilde{x}') \triangleq \sum_{j=0}^{2} \frac{f_j(t, 0, \tilde{x}')}{j!} \tilde{x}_1^j. \tag{6.69}$$

It is obvious that

$$\tilde{\psi}(t, 0, \tilde{x}') = \tilde{\phi}^-(t, 0, \tilde{x}'). \tag{6.70}$$

Also, based on Propositions 1 and 2, one has (recall (6.24) for the definition of $\tilde{x}_0$ and $\tilde{x}_0'$)

Proposition 4. ([40]) *Let assumption (6.23) hold. Then the function* $\tilde{\psi}$ *satisfies*

$$\operatorname{Im} \tilde{\psi}(t, \tilde{x}_1, \tilde{x}') \geq c(|t - t_0|^2 + |\tilde{x}_1| + |\tilde{x}' - \tilde{x}_0'|^2) \tag{6.71}$$

as $(t, \tilde{x}') \to (t_0, \tilde{x}_0')$ *and* $\tilde{x}_1 \to 0^-$*, and*

$$\operatorname{Im} \partial_{\tilde{x}_1} \tilde{\psi}(t_0, \tilde{x}_0) < 0. \tag{6.72}$$

Remark 12. From Proposition 4, it is easy to see that the factor $e^{i\tilde{\psi}/\varepsilon}$ localizes $\tilde{y}_\varepsilon$ in (6.66) in the region

$$|t - t_0|^2 + |\tilde{x}_1| + |\tilde{x}' - \tilde{x}_0'|^2 = O(\varepsilon). \tag{6.73}$$

Further, this proposition says that the energy of the parabolic component $\tilde{y}_\varepsilon$ is localized in a small neighborhood of the reflection point $(t_0, 0, \tilde{x}_0')$, with size $O(\varepsilon)$ in the $\tilde{x}_1$ direction but $O(\sqrt{\varepsilon})$ in t and $\tilde{x}'$.

In order to match the heat solution with the incoming wave we need to impose the initial condition for $\widetilde{B}$ at $\tilde{x}_1 = 0$:

$$\widetilde{B}(t, 0, \tilde{x}') = b_0(t, \tilde{x}'). \tag{6.74}$$

We will determine b_0 in the next subsection. Once this is done, we set

$$\begin{aligned} b_1(t, \tilde{x}') \triangleq \frac{1}{2\partial_{\tilde{x}_1} \tilde{\psi}(t, 0, \tilde{x}')} \Bigg[& -2 \sum_{i,j=2}^{n} \tilde{\alpha}_{ij}(0, \tilde{x}') \partial_{\tilde{x}_i} \tilde{\psi}(t, 0, \tilde{x}') \partial_{\tilde{x}_j} b_0 \\ & + \Bigg(\partial_t \tilde{\psi}(t, 0, \tilde{x}') - \sum_{i,j=1}^{n} \tilde{\alpha}_{ij}(0, \tilde{x}') \partial^2_{\tilde{x}_i \tilde{x}_j} \tilde{\psi}(t, 0, \tilde{x}') \\ & - \sum_{j=1}^{n} \tilde{\beta}_j(0, \tilde{x}') \partial_{\tilde{x}_j} \tilde{\psi}(t, 0, \tilde{x}') \Bigg) b_0 \Bigg]. \end{aligned} \tag{6.75}$$

We then choose

$$\widetilde{B}(t, \tilde{x}) \triangleq b_0(t, \tilde{x}') + b_1(t, \tilde{x}') \tilde{x}_1. \tag{6.76}$$

By [40, Proposition 4.11], we arrive at the following result:

Proposition 5. *For any $T > t_0$ and any $b_0 \in C^2$, the approximate solutions $\tilde{y}_\varepsilon$ defined by (6.66), with $\tilde{\psi}$ and $\widetilde{B}$ given by (6.69) and (6.76) respectively, satisfy*

$$\int_0^T \int_{\{\tilde{x}_1<0\}} |\widetilde{H}\tilde{y}_\varepsilon|^2 dtd\tilde{x} = O(\varepsilon^3). \tag{6.77}$$

Remark 13. As in [40], by (6.73) in Remark 12, one sees that the functions $\tilde{y}_\varepsilon$ are concentrated in a small neighborhood of the reflection point $(t_0, \tilde{x}_0)$. Fix any two small enough neighborhoods $\mathcal{O}_j$ (j=1,2) of $(t_0, \tilde{x}_0)$ in the half space $\left\{(t,\tilde{x}) \,\middle|\, \tilde{x}_1 < 0\right\}$ such that estimate (6.71) holds and $\mathcal{O}_1$ is a proper subset of $\mathcal{O}_2$. For any given C^2 function $\theta = \theta(t,\tilde{x})$, with $\theta \equiv 1$ in $\mathcal{O}_1$ and $\theta \equiv 0$ in $\left\{(t,\tilde{x}) \,\middle|\, \tilde{x}_1 < 0\right\} \setminus \mathcal{O}_2$, using Proposition 4 and noting (6.73) in Remark 12 again, one finds that Proposition 5 remains true for $\theta\tilde{y}_\varepsilon$. In this way, $\theta\tilde{y}_\varepsilon$ is a family of approximate solutions of the heat equation (6.67), which are supported in a small neighborhood of $(t_0, \tilde{x}_0)$. Since θ vanishes in $\left\{(t,\tilde{x}) \,\middle|\, \tilde{x}_1 < 0\right\} \setminus \mathcal{O}_2$, we may extend $\theta\tilde{y}_\varepsilon$ globally to $\{\tilde{x}_1 \le 0\}$ even if, as we do in the next subsection, $\tilde{y}_\varepsilon$ is only defined in a small neighborhood of the reflection point $(t_0, \tilde{x}_0)$.

6.7. Highly concentrated solutions of the transmission problem

Assume $z_\varepsilon^-(t,x)$ given by (6.28) to be the incoming wave. First, we seek approximate solutions $(\tilde{y}_\varepsilon, \tilde{z}_\varepsilon)$ of the transmission problem (1.1) but in the new coordinates $(\tilde{x}_1, \cdots, \tilde{x}_n)$, in which $\tilde{y}_\varepsilon$ is an approximate heat solution given by (6.66), and $\tilde{z}_\varepsilon$ is an approximate wave solution of the form

$$\begin{aligned} \tilde{z}_\varepsilon &= \tilde{z}_\varepsilon^-(t,\tilde{x}) + \tilde{z}_\varepsilon^+(t,\tilde{x}) \\ &= \varepsilon^{1-n/4}\left\{\tilde{A}^-(t,\tilde{x})e^{i\tilde{\phi}^-(t,\tilde{x})/\varepsilon} + \tilde{A}^+(t,\tilde{x})e^{i\tilde{\phi}^+(t,\tilde{x})/\varepsilon}\right\}, \end{aligned} \tag{6.78}$$

where $\tilde{z}^\pm(t,\tilde{x})$ and $\tilde{A}^\pm(t,\tilde{x})$ are the new coordinate expressions of $z^\pm(t,x)$ and $A^\pm(t,x)$ given by (6.28) and (6.29), respectively.

According to the Gaussian beam construction for approximate solutions of the wave equation in Subsection 6.4, and the construction of approximate solutions for the heat equation in Subsection 6.6, one needs to determine $a^+(t_0)$ and $\widetilde{B}(t,0,\tilde{x}')$, the initial value of $a^+(t)$ at $t = t_0$ and $\widetilde{B}(t,\tilde{x}_1,\tilde{x}')$ at $\tilde{x}_1 = 0$, respectively.

From the transmission condition on the reflection point $(t_0, \tilde{x}_0)$, one concludes that

$$\begin{cases} 0 = a^-(t_0) + a^+(t_0), \\ \partial_{\tilde{x}_1}\tilde{\psi}(t_0,\tilde{x}_0)\widetilde{B}(t_0,\tilde{x}_0) = \partial_{\tilde{x}_1}\tilde{\phi}^-(t_0,\tilde{x}_0)a^-(t_0) + \partial_{\tilde{x}_1}\tilde{\phi}^+(t_0,\tilde{x}_0)a^+(t_0). \end{cases} \tag{6.79}$$

However, by (6.72) in Proposition 4, we see that $\partial_{\tilde{x}_1}\tilde{\psi}(t_0,\tilde{x}_0) \neq 0$. Hence, we get

$$\begin{cases} a^+(t_0) = -a^-(t_0), \\ \widetilde{B}(t_0,\tilde{x}_0) = \dfrac{[\partial_{\tilde{x}_1}\tilde{\phi}^-(t_0,\tilde{x}_0) - \partial_{\tilde{x}_1}\tilde{\phi}^+(t_0,\tilde{x}_0)]a^-(t_0)}{\partial_{\tilde{x}_1}\tilde{\psi}(t_0,\tilde{x}_0)}. \end{cases} \tag{6.80}$$

These formulas extend to curved interfaces the reflection law (5.10) we found in the previous section for flat interfaces.

Choosing the initial data

$$\widetilde{B}(t,0,\tilde{x}') \equiv \widetilde{B}(t_0,\tilde{x}_0), \tag{6.81}$$

one determines $\widetilde{B}(t,\tilde{x})$ according to (6.76).

This completes the construction of the (local) approximate solutions $(\tilde{y}_\varepsilon, \tilde{z}_\varepsilon)$ of the transmission problem (1.1) (in the new coordinates).

Now, putting

$$\hat{z}_\varepsilon \equiv \hat{z}_\varepsilon \overset{\triangle}{=} z^-(t,x) + z^+(t,x),$$

and by Remark 13, returning the approximate solutions $\theta\tilde{y}_\varepsilon$ to the original coordinates (for a suitable cut-off function θ), called henceforth $\hat{y}_\varepsilon$, we actually obtain global approximate solutions $(\hat{y}_\varepsilon, \hat{z}_\varepsilon)$ of the transmission problem (1.1).

We arrive at the following conclusion:

Lemma 8. *Let $(t, x^-(t), \xi^-(t))$, with $x^-(0) \in \Omega_2$, be an incoming ray, which arrives at γ at time $t = t_0$ transversally and non-normally. Let $(t, x^+(t), \xi^+(t))$ be the reflected ray. Then $(\hat{y}_\varepsilon, \hat{z}_\varepsilon)$ constructed above satisfy (recall (6.22) for the definition of T^*):*

$$\begin{cases} (\partial_t - \Delta)\hat{y}_\varepsilon = r_1 & \text{in } (0,T^*)\times\Omega_1, \\ \Box\, \hat{z}_\varepsilon = r_2 & \text{in } (0,T^*)\times\Omega_2, \\ \hat{y}_\varepsilon = 0 & \text{on } (0,T^*)\times\Gamma_1, \\ \hat{z}_\varepsilon = r_3 & \text{on } (0,T^*)\times\Gamma_2, \\ \hat{y}_\varepsilon = \partial_t\hat{z}_\varepsilon + r_4, \quad \dfrac{\partial\hat{y}_\varepsilon}{\partial\nu_1} = -\dfrac{\partial\hat{z}_\varepsilon}{\partial\nu_2} + r_5 & \text{on } (0,T^*)\times\gamma, \end{cases} \tag{6.82}$$

with

$$\begin{aligned} &|r_1|_{L^2((0,T^*)\times\Omega_1)} = O(\varepsilon^{3/2}), \quad |r_2|_{L^2((0,T^*)\times\Omega_2)} = O(\varepsilon^{1/2}), \\ &|r_3|_{H^1((0,T^*)\times\Gamma_2)} = O(\varepsilon^{1/2}), \quad |r_4|_{H^2(0,T^*;H^{1/2}(\gamma))} = O(\varepsilon^{1/2}), \\ &|r_5|_{H^2(0,T^*;L^2(\gamma))} = O(\varepsilon^{1/2}), \\ &|\nabla\hat{y}_\varepsilon|^2_{(L^2((0,T^*)\times\Omega_1))^n} = O(\varepsilon), \quad E_\varepsilon(0) = E(\hat{y}_\varepsilon, \hat{z}_\varepsilon, \partial_t\hat{z}_\varepsilon)(0) \approx 1. \end{aligned} \tag{6.83}$$

Moreover, $\hat{y}_\varepsilon$ (resp. $\hat{z}_\varepsilon$) can be chosen so that its support is localized in any given neighborhood of the reflection point (resp. of the incoming and reflected rays). In particular, one may have

$$\operatorname{supp} r_j \subset (0,T^*) \times \operatorname{Int}\gamma, \qquad j = 4,5. \tag{6.84}$$

Here, $\operatorname{Int}\gamma$ *denotes the relative interior of* γ.

The proof of Lemma 8 is long and technical, hence we put it in Appendix B.

Remark 14. The previous higher order corrections on the amplitude and phase of the Gaussian beam for the wave equation have been added, precisely, to guarantee the error estimates in (6.83) to hold.

The above analysis yields the following result:

Lemma 9. *Let the assumptions in Lemma 8 hold. Then there is a family of solutions $\{(y_\varepsilon, z_\varepsilon)\}_{\varepsilon>0}$ of system (1.1) in $(0,T^*)$ (the initial conditions being excepted), such that*

$$|\nabla y_\varepsilon|^2_{(L^2((0,T^*)\times\Omega_1))^n} = O(\varepsilon), \quad E_\varepsilon(0) = E(y_\varepsilon, z_\varepsilon, \partial_t z_\varepsilon)(0) \approx 1. \tag{6.85}$$

We refer to Appendix C for the proof of Lemma 9. Now, from Lemma 9, we end up with the following crucial result:

Theorem 9. *Let $\operatorname{Cap}\Gamma_2 > 0$, and both Γ_2 and γ be of class C^4. For any $T > 0$, let $[0,T] \ni t \mapsto (t, x(t), \xi(t)) \in [0,T] \times \overline{\Omega_2} \times \mathbb{R}^n$ be a multiply reflected ray for the operator $\Box$ in Ω_2, which meets Γ_2 or γ transversally and non-normally. Then there is a family of solutions $\{(y_\varepsilon, z_\varepsilon)\}_{\varepsilon>0}$ of system (1.1) in $(0,T)$ (the initial conditions being excepted), such that*

$$|\nabla y_\varepsilon|^2_{(L^2((0,T)\times\Omega_1))^n} = O(\varepsilon), \qquad E_\varepsilon(0) = E(y_\varepsilon, z_\varepsilon, \partial_t z_\varepsilon)(0) \approx 1. \tag{6.86}$$

With numerous but small changes, similar to Theorem 9, one deduces that

Theorem 10. *Let $\operatorname{Cap}\Gamma_2 = 0$ and γ be of class C^4. For any $T > 0$, let $[0,T] \ni t \mapsto (t, x(t), \xi(t)) \in [0,T] \times \overline{\Omega_2} \times \mathbb{R}^n$ be a multiply reflected ray for the operator $\Box$ in Ω_2, which meets γ transversally and non-normally. Then there is a family of solutions $\{(y_\varepsilon, z_\varepsilon)\}_{\varepsilon>0}$ of system (1.1) in $(0,T)$ (the initial conditions being excepted), such that*

$$\begin{aligned} &|\nabla y_\varepsilon|^2_{(L^2((0,T)\times\Omega_1))^n} + |\partial_t \nabla y_\varepsilon|^2_{(L^2((0,T)\times\Omega_1))^n} = O(\varepsilon^{-1}), \\ &E(y_\varepsilon, z_\varepsilon, \partial_t z_\varepsilon)(0) + E(\partial_t y_\varepsilon, \partial_t z_\varepsilon, \partial_{tt} z_\varepsilon)(0) \geq c_0 \varepsilon^{-2}, \end{aligned} \tag{6.87}$$

where $c_0 > 0$ is a constant, independent of ε.

7. Polynomial decay rate and a weakened observability inequality

This section is devoted to prove the polynomial decay of smooth solutions of system (1.1) in several space dimensions under the GCC.

7.1. Statement of the main results

To begin with, consider the following wave equation:

$$\begin{cases} \Box\zeta = 0 & \text{in } (0,T)\times\Omega, \\ \zeta = 0 & \text{on } (0,T)\times\Gamma, \\ \zeta(0)=\zeta_0, \quad \zeta_t(0)=\zeta_1 & \text{in } \Omega. \end{cases} \tag{7.1}$$

We introduce the following internal observability assumption for the wave equation in Ω from the subdomain Ω_1:

(H) *There exist two constants $T_0>0$ and $C>0$ such that, for any $T\geq T_0$, all solutions of system (7.1) satisfy*

$$|\zeta_0|^2_{H_0^1(\Omega)} + |\zeta_1|^2_{L^2(\Omega)} \leq C\int_0^T\int_{\Omega_1}|\zeta_t|^2dxdt, \qquad \forall\, (\zeta_0,\zeta_1)\in H_0^1(\Omega)\times L^2(\Omega). \tag{7.2}$$

This condition asserts that the total energy of any solution to the wave equation in Ω can be observed in terms of the energy concentrated in Ω_1. This inequality holds if Ω_1 satisfies the *Geometric Optics Condition* (GCC) in Ω ([2]) when Ω is smooth enough (We refer to [6] for less regularity condition on the boundary of Ω). The GCC asserts that all rays of Geometric Optics in Ω enter the subdomain Ω_1 in an uniform time $T_0>0$. A particular example of subdomain for which this condition is satisfied is when Ω_1 is a neighborhood of a sufficiently large subset of the boundary Γ satisfying the multiplier condition ([30]) for $C^{1,1}$ domain Ω.

Now, we may state our polynomial decay result for system (1.1) as follows (recall (3.2) for the definition of $D(\mathcal{A})$).

Theorem 11. *i) Let Ω_1 satisfy* **(H)**, $\operatorname{Cap}\Gamma_1>0$ *and* $\operatorname{Cap}\Gamma_2>0$. *Then there is a constant $C>0$ such that for any $(y_0,z_0,z_1)\in D(\mathcal{A})$, the solution of (1.1) satisfies*

$$|(y(t),z(t),z_t(t))|_H \leq \frac{C}{t^{1/6}}|(y_0,z_0,z_1)|_{D(\mathcal{A})}, \qquad \forall\, t>0. \tag{7.3}$$

ii) Let Ω_1 satisfy **(H)**, *and* $\operatorname{Cap}\Gamma_2=0$. *Then there is a constant $C>0$ such that for any $(y_0,z_0,z_1)\in D(\mathcal{A})$, the solution of (1.1) satisfies*

$$||(y(t),z(t),z_t(t))-(0,c_1,0)||_H \leq \frac{C}{t^{1/6}}||(y_0,z_0,z_1)||_{D(\mathcal{A})}, \quad \forall\, t>0, \tag{7.4}$$

where $c_1=c_1(y_0,z_0,z_1)$ is the constant given in (3.6).

Remark 15. In Theorem 11 (and also Theorem 12 below), we assume $\operatorname{Cap} \Gamma_1 > 0$ because we need to use Poincaré inequality to bound the L^2-norm of a function in Ω_1 by its gradient (*see* (7.19)). It is not clear whether this condition is really necessary. Note that the assumption **(H)** by itself does not guarantee $\operatorname{Cap} \Gamma_1 > 0$. But the further condition $\operatorname{Cap} \Gamma_2 = 0$ does imply it. The case where $\operatorname{Cap} \Gamma_1 = 0$ is an interesting open problem.

Remark 16. Theorem 11 is not sharp for $n = 1$ since in [55] we have proved that the decay rate is $1/t^2$. However, similar to [40], the WKB asymptotic expansion for the flat interface developed in Section 5 allows to show that it is impossible to expect the same decay rate in several space dimensions. According to Remark 17 and the possible sharp weakened observability inequality (7.7) below, it is natural to expect $1/t$ to be the sharp polynomial decay rate for smooth solutions of (1.1) with initial data in $D(\mathcal{A})$. We refer to [15] for an interesting partial solution to this problem with a decay rate of the order of $1/t^{1-\delta}$ for all $\delta > 0$ but under stronger assumptions on the geometry that Ω is of C^∞ and $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$.

We refer to Subsection 7.3 for the proof of Theorem 11, which is based on the following key weakened observability inequality for system (1.1):

Theorem 12. *i) Let Ω_1 satisfy* **(H)**, $\operatorname{Cap} \Gamma_1 > 0$ *and* $\operatorname{Cap} \Gamma_2 > 0$. *Then there exists a constant $C > 0$ such that for any $(y_0, z_0, z_1) \in D(\mathcal{A}^3)$, and any $T \geq T_0$ (Recall* **(H)** *for T_0), the solution of (1.1) satisfies*

$$|(y_0, z_0, z_1)|_H \leq C|\nabla y|_{H^3(0,T;\,(L^2(\Omega_1))^n)}. \tag{7.5}$$

ii) Let Ω_1 satisfy **(H)**, *and* $\operatorname{Cap} \Gamma_2 = 0$. *Then there exists a constant $C > 0$ such that for any $(y_0, z_0, z_1) \in D(\mathcal{A}^3) \cap \mathcal{R}(\mathcal{A})$, and any $T \geq T_0$, the solution of (1.1) satisfies*

$$||(y_0, z_0, z_1)||_H \leq C\Big[|\nabla Y|_{(L^2((0,T)\times\Omega_1))^n)} + |\nabla y|_{H^3(0,T;\,(L^2(\Omega_1))^n)}\Big], \tag{7.6}$$

where Y is given by $(Y, Z, Z_t)(t) = \displaystyle\int_0^t (y, z, z_t)(s)ds + \mathcal{A}^{-1}(y_0, z_0, z_1)$.

The proof of Theorem 12 will be given in the next Subsection.

Remark 17. Note that (7.5) is, indeed, a weakened version of (1.13), in which we do not only use the norm of ∇y in $L^2((0,T)\times\Omega_1)$ to bound the total energy of solutions but the stronger norm in $H^3(0,T;(L^2(\Omega_1))^n)$. Nevertheless, inequality (7.5) is very likely not sharp. Indeed, under assumption **(H)**, it is reasonable to expect the following stronger inequality to hold:

$$|(y_0, z_0, z_1)|_H \leq C|\nabla y|_{H^{1/2}(0,T;L^2(\Omega_1))}. \tag{7.7}$$

But this is also an open problem. Note that, the WKB asymptotic expansion for the flat interface developed in Section 5 supports the possible validity of inequality (7.7). Indeed, from Section 5 we see that, in the case of non-normal incidence, as shown in (5.13), the energy $|\nabla y^\varepsilon|^2_{H^{1/2}(0,T;L^2(\Omega_1))}$ dissipated between $t=0$ and $t=T$ is of the order of $1/\varepsilon^2$, the same as the total energy of $(y^\varepsilon, z^\varepsilon)$. The Gaussian beam construction of approximate solutions developed in Section 6 also supports the possible validity of inequality (7.7). In the very recent paper [15], it is showed that the above conjecture is true under the further assumptions that Ω is of C^∞ and $\overline{\Gamma_1}\cap\overline{\Gamma_2}=\emptyset$.

Remark 18. In (7.6), there is an extra term $|\nabla Y|_{(L^2((0,T)\times\Omega_1))^n)}$. It is not clear whether this term can be dropped. Note however that this term does not affect the polynomial decay result in Theorem 11 ii). On the other hand, (7.6) is also a weakened version of (6.2), in which we do not only use the norm of ∇y and ∇Y in $L^2((0,T)\times\Omega_1)$ to bound the total energy of solutions but their stronger norms in $H^3(0,T;(L^2(\Omega_1))^n)$.

7.2. Proof of the weakened observability inequality

This subsection is devoted to the proof of Theorem 12. We distinguish two different cases.

i) **The case** $\mathrm{Cap}\,\Gamma_1>0$ **and** $\mathrm{Cap}\,\Gamma_2>0$**.** Set $w=y\chi_{\Omega_1}+z_t\chi_{\Omega_2}$. Then, by (1.1) and noting that $\partial z_t/\partial\nu_2=-\partial y_t/\partial\nu_1$, and by $(y_0,z_0,z_1)\in D(\mathcal{A}^2)$, it is easy to see that $w\in C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$ solves

$$\begin{cases}\Box w=(y_{tt}-y_t)\chi_{\Omega_1}+\left(\frac{\partial y}{\partial\nu_1}-\frac{\partial y_t}{\partial\nu_1}\right)\delta_\gamma & \text{in } (0,T)\times\Omega,\\ w=0 & \text{on } (0,T)\times\Gamma,\\ w(0)=y_0\chi_{\Omega_1}+z_1\chi_{\Omega_2},\quad w_t(0)=(\Delta y_0)\chi_{\Omega_1}+(\Delta z_0)\chi_{\Omega_2} & \text{in } \Omega,\end{cases}\tag{7.8}$$

in the following sense: $w(0)=y_0\chi_{\Omega_1}+z_1\chi_{\Omega_2}$ in Ω, and for any $s\in(0,T]$ and $\eta\in C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$, it holds

$$\begin{aligned}&\int_\Omega w_t(s)\eta(s)dx-\int_\Omega[(\Delta y_0)\chi_{\Omega_1}+(\Delta z_0)\chi_{\Omega_2}]\eta(0)dx\\&\quad+\int_0^s\int_\Omega(-w_t\eta_t+\nabla w\cdot\nabla\eta)dxdt\\&=\int_0^s\int_{\Omega_1}(y_{tt}-y_t)\eta dxdt+\int_0^s\int_\gamma\left(\frac{\partial y}{\partial\nu_1}-\frac{\partial y_t}{\partial\nu_1}\right)\eta d\gamma dt.\end{aligned}\tag{7.9}$$

Note that the last integral in (7.9) is concentrated on γ rather than on $\partial\Omega_1$ because η vanishes on the exterior boundary of Ω_1. Hence, it is equal to $\int_0^s\int_{\partial\Omega_1}(\frac{\partial y}{\partial\nu_1}-\frac{\partial y_t}{\partial\nu_1})\eta d\partial\Omega_1 dt$. The integral $\int_{\partial\Omega_1}(\frac{\partial y}{\partial\nu_1}-\frac{\partial y_t}{\partial\nu_1})\eta d\partial\Omega_1$ may be interpreted in the sense of duality pairing between $H^{1/2}(\partial\Omega_1)$ and

$H^{-1/2}(\partial\Omega_1)$. Indeed, by the standard semigroup theory, $(y_0, z_0, z_1) \in D(\mathcal{A}^2)$ implies that $(y, z, z_t) \in C^1([0,T]; D(\mathcal{A}))$. Hence by Lemma 1 and noting the structure of $D(\mathcal{A})$ in (3.2), $\frac{\partial y}{\partial \nu_1} - \frac{\partial y_t}{\partial \nu_1} \in C([0,T]; H^{-1/2}(\partial\Omega_1))$. Also, the trace theorem guarantees that $\eta \in C([0,T]; H^{1/2}(\partial\Omega_1))$.

We decompose w as $w = p + q$, where p and q are respectively solutions of

$$\begin{cases} \Box p = 0 & \text{in } (0,T) \times \Omega, \\ p = 0 & \text{on } (0,T) \times \Gamma, \\ p(0) = y_0\chi_{\Omega_1} + z_1\chi_{\Omega_2}, \quad p_t(0) = (\Delta y_0)\chi_{\Omega_1} + (\Delta z_0)\chi_{\Omega_2} & \text{in } \Omega \end{cases} \tag{7.10}$$

and

$$\begin{cases} \Box q = (y_{tt} - y_t)\chi_{\Omega_1} + \left(\frac{\partial y}{\partial \nu_1} - \frac{\partial y_t}{\partial \nu_1}\right)\delta_\gamma & \text{in } (0,T) \times \Omega, \\ q = 0 & \text{on } (0,T) \times \Gamma, \\ q(0) = q_t(0) = 0 & \text{in } \Omega. \end{cases} \tag{7.11}$$

By Assumption **(H)**, we conclude that for $T \geq T_0$, solutions of system (7.10) satisfy

$$\begin{aligned} &|y_0\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{H_0^1(\Omega)} + |(\Delta y_0)\chi_{\Omega_1} + (\Delta z_0)\chi_{\Omega_2}|_{L^2(\Omega)} \\ &\leq C|p_t|_{L^2((0,T)\times\Omega_1)}. \end{aligned} \tag{7.12}$$

We claim that

$$\begin{aligned} &|(y_0, z_0, z_1)|_H \\ &\leq C[|y_0\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{H_0^1(\Omega)} + |(\Delta y_0)\chi_{\Omega_1} + (\Delta z_0)\chi_{\Omega_2}|_{L^2(\Omega)}]. \end{aligned} \tag{7.13}$$

Indeed, by (1.7), we have

$$\begin{aligned} |(y_0, z_0, z_1)|_H &\leq C[|y_0|_{L^2(\Omega_1)} + |z_1|_{L^2(\Omega_2)} + |z_0|_{H^1_{\Gamma_2}(\Omega_2)}] \\ &\leq C[|y_0\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{L^2(\Omega)} + |z_0|_{H^1_{\Gamma_2}(\Omega_2)}] \\ &\leq C[|y_0\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{H_0^1(\Omega)} + |z_0|_{H^1_{\Gamma_2}(\Omega_2)}]. \end{aligned}$$

Denote by $\mathcal{D}_{\Gamma_2}(\Omega_2)$ all $C_0^\infty(\mathbb{R}^n)$ functions supported in $\Omega_2 \cup \gamma$. From the definition of $H^1_{\Gamma_2}(\Omega_2)$ in (1.3), it is easy to see that $\mathcal{D}_{\Gamma_2}(\Omega_2)$ is dense in $H^1_{\Gamma_2}(\Omega_2)$. Using integration by parts, noting $\frac{\partial z_0}{\partial \nu_2} = -\frac{\partial y_0}{\partial \nu_1}$, recalling the definition of $H_\gamma^{-1/2}$ in (2.3) and (2.2), using (2.4) and Lemma 1, one has

$$\begin{aligned}
|z_0|_{H^1_{\Gamma_2}(\Omega_2)} &= \sup_{f\in\mathcal{D}_{\Gamma_2}(\Omega_2)\setminus\{0\}} \frac{\int_{\Omega_2}\nabla z_0\cdot\nabla f dx}{|f|_{H^1_{\Gamma_2}(\Omega_2)}}\\
&= \sup_{f\in\mathcal{D}_{\Gamma_2}(\Omega_2)\setminus\{0\}} \frac{\int_\gamma \frac{\partial z_0}{\partial\nu_2} f d\gamma - \int_{\Omega_2}(\Delta z_0) f dx}{|f|_{H^1_{\Gamma_2}(\Omega_2)}}\\
&= \sup_{f\in\mathcal{D}_{\Gamma_2}(\Omega_2)\setminus\{0\}} \frac{-\int_\gamma \frac{\partial y_0}{\partial\nu_1} f d\gamma - \int_{\Omega_2}(\Delta z_0) f dx}{|f|_{H^1_{\Gamma_2}(\Omega_2)}}\\
&\le \sup_{f\in\mathcal{D}_{\Gamma_2}(\Omega_2)\setminus\{0\}} \frac{|\frac{\partial y_0}{\partial\nu_1}|_{H_\gamma^{-1/2}}|f|_{H^{1/2}(\partial\Omega_2)} + |\Delta z_0|_{L^2(\Omega_2)}|f|_{L^2(\Omega_2)}}{|f|_{H^1_{\Gamma_2}(\Omega_2)}}\\
&\le C\left[\Big|\frac{\partial y_0}{\partial\nu_1}\Big|_{H^{-1/2}(\partial\Omega_1)} + |\Delta z_0|_{L^2(\Omega_2)}\right]\\
&\le C\left[|\nabla y_0|_{L^2(\Omega_1)} + |\Delta y_0|_{L^2(\Omega_1)} + |\Delta z_0|_{L^2(\Omega_2)}\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
&|(y_0,z_0,z_1)|_H\\
&\le C\left[|y_0\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{H^1_0(\Omega)} + |\nabla y_0|_{L^2(\Omega_1)} + |\Delta y_0|_{L^2(\Omega_1)} + |\Delta z_0|_{L^2(\Omega_2)}\right]\\
&\le C[|y_0\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{H^1_0(\Omega)} + |(\Delta y_0)\chi_{\Omega_1} + (\Delta z_0)\chi_{\Omega_2}|_{L^2(\Omega)}],
\end{aligned}$$

which yields (7.13).

Combining (7.12) and (7.13), we conclude that

$$|(y_0,z_0,z_1)|_H \le C|p_t|_{L^2((0,T)\times\Omega_1)}. \tag{7.14}$$

Now, by (7.14) and recalling that $p = w - q$ and $w|_{\Omega_1} = y$, it is easy to see that

$$|(y_0,z_0,z_1)|_H \le C(|q_t|_{L^2((0,T)\times\Omega_1)} + |y_t|_{L^2((0,T)\times\Omega_1)}). \tag{7.15}$$

Note however that the initial data (y_0,z_0,z_1) are assumed to belong to $D(\mathcal{A}^3)$. Hence, one has $w_t, p_t \in C([0,T];H^1_0(\Omega))\cap C^1([0,T];L^2(\Omega))$. Therefore, $q_t = w_t - p_t \in C([0,T];H^1_0(\Omega))\cap C^1([0,T];L^2(\Omega))$. On the other hand, the solution $q\in C([0,T];H^1_0(\Omega))\cap C^1([0,T];\ L^2(\Omega))$ of (7.11) can be defined as in (7.9). Hence, using q_t as a test function, and noting $q(0) = q_t(0) = 0$ in Ω and $q_t = 0$ on $(0,T)\times\Gamma$, one finds that

$$\begin{aligned}
&\int_\Omega [|q_t(s,x)|^2 + |\nabla q(s,x)|^2]dx\\
&= 2\int_0^s\int_{\Omega_1}(y_{tt} - y_t)q_t dxdt + 2\int_0^s\int_{\partial\Omega_1}\left(\frac{\partial y}{\partial\nu_1} - \frac{\partial y_t}{\partial\nu_1}\right) q_t d\partial\Omega_1 dt.
\end{aligned} \tag{7.16}$$

Now, integrating (7.16) with respect to s from 0 to T, exchanging the order of integration, using integration by parts and noting $q(0) = 0$, and the first equation of (1.1), we get

$$\begin{aligned}\int_0^T \int_{\Omega_1} [q_t^2 + |\nabla q|^2] dxdt &\leq C \int_0^T \int_{\Omega_1} |y_{tt} - y_t||q_t| dxdt \\ &\quad +2 \int_0^T \int_{\partial\Omega_1} (T-t) \left(\frac{\partial y}{\partial \nu_1} - \frac{\partial y_t}{\partial \nu_1} \right) q_t d\partial\Omega_1 dt \\ &= C \int_0^T \int_{\Omega_1} |y_{tt} - y_t||q_t| dxdt \\ &\quad -2 \int_0^T \int_{\partial\Omega_1} \left[(T-t) \left(\frac{\partial y}{\partial \nu_1} - \frac{\partial y_t}{\partial \nu_1} \right) \right]_t q d\partial\Omega_1 dt.\end{aligned} \tag{7.17}$$

On the other hand, using Lemma 1 and the trace theorem, one concludes that

$$\begin{aligned}&-2 \int_0^T \int_{\partial\Omega_1} \left[(T-t) \left(\frac{\partial y}{\partial \nu_1} - \frac{\partial y_t}{\partial \nu_1} \right) \right]_t q d\partial\Omega_1 dt \\ &\leq C \int_0^T \left(\left| \frac{\partial y}{\partial \nu_1} - \frac{\partial y_t}{\partial \nu_1} \right|_{H^{-1/2}(\partial\Omega_1)} \right. \\ &\qquad \left. + \left| \frac{\partial y_t}{\partial \nu_1} - \frac{\partial y_{tt}}{\partial \nu_1} \right|_{H^{-1/2}(\partial\Omega_1)} \right) |q|_{H^{1/2}(\partial\Omega_1)} dt \\ &\leq C \left| \frac{\partial y}{\partial \nu_1} \right|_{H^2(0,T;H^{-1/2}(\partial\Omega_1))} |q|_{L^2(0,T;H^{1/2}(\partial\Omega_1))}.\end{aligned} \tag{7.18}$$

Now, combining (7.17) and (7.18), we get

$$\begin{aligned}&\int_0^T \int_{\Omega_1} [q_t^2 + |\nabla q|^2] dxdt \\ &\leq C \left[|y_t|^2_{H^1(0,T;L^2(\Omega_1))} + \left| \frac{\partial y}{\partial \nu_1} \right|^2_{H^2(0,T;H^{-1/2}(\partial\Omega_1))} \right] \\ &\quad + \frac{1}{2} \int_0^T \int_{\Omega_1} [q_t^2 + |\nabla q|^2] dxdt \\ &\leq C \left[|y_t|^2_{H^1(0,T;L^2(\Omega_1))} + |\nabla y|^2_{H^2(0,T;(L^2(\Omega_1))^n)} \right. \\ &\qquad \left. + |\Delta y|^2_{H^2(0,T;L^2(\Omega_1))} \right] + \frac{1}{2} \int_0^T \int_{\Omega_1} [q_t^2 + |\nabla q|^2] dxdt \\ &\leq C \left[|y_t|^2_{H^2(0,T;L^2(\Omega_1))} + |\nabla y|^2_{H^2(0,T;(L^2(\Omega_1))^n)} \right] \\ &\quad + \frac{1}{2} \int_0^T \int_{\Omega_1} [q_t^2 + |\nabla q|^2] dxdt.\end{aligned} \tag{7.19}$$

Consequently, combining (7.15) and (7.19), recalling $\mathrm{Cap}\,\Gamma_1 > 0$ and using Poincaré inequality, we arrive at the desired estimate (7.5).

ii) **The case** $\mathrm{Cap}\,\Gamma_2 = 0$**.** The proof is very close to that of the first case and we only give the sketch. Note that, in this case, it is easy to see that $\mathrm{Cap}\,\Gamma_1 > 0$ (otherwise $\mathrm{Cap}\,\Gamma = 0$, which is impossible). Similar to (7.5), one deduces that

$$|(y_0, z_0, z_1)|_H \le C|\nabla y|_{H^3(0,T;\,(L^2(\Omega_1))^n)}, \quad \forall\, (y_0, z_0, z_1) \in D(\mathcal{A}^3). \tag{7.20}$$

Note that we also assume $(y_0, z_0, z_1) \in \mathcal{R}(\mathcal{A})$. Applying the same argument in the first case to $(Y, Z, Z_t)(t)$ as in (4.5), we conclude that

$$\begin{aligned} |\mathcal{A}^{-1}(y_0, z_0, z_1)|_H &\le C|\nabla Y|_{H^3(0,T;\,(L^2(\Omega_1))^n)}, \\ &\forall\, (y_0, z_0, z_1) \in D(\mathcal{A}^2) \cap \mathcal{R}(\mathcal{A}). \end{aligned} \tag{7.21}$$

However, from the definition of $\mathcal{A}$, one has $|z_0|_{L^2(\Omega_2)} \le |\mathcal{A}^{-1}(y_0, z_0, z_1)|_H$. This fact, combined with (7.20)–(7.21) and $Y_t = y$ in $(0,T) \times \Omega_1$, yields the desired estimate (7.6). □

7.3. Proof of the polynomial decay result

This subsection is devoted to the proof of Theorem 11. For this purpose, we recall the following known technical result (*see* [1, Lemma 5.2]):

Proposition 6. *Let* $\{a_k\}_{k=1}^{\infty}$ *be a sequence of positive numbers satisfying*

$$a_{k+1} \le a_k - C a_{k+1}^{2+\alpha}, \qquad \forall\, k \ge 1$$

for some constants $C > 0$ *and* $\alpha > -1$*. Then there is a constant* $M = M(C, \alpha) > 0$ *such that*

$$a_k \le \frac{M}{(k+1)^{\frac{1}{1+\alpha}}}, \qquad \forall\, k \ge 1.$$

We distinguish the proof of Theorem 11 into two different cases.

i) **The case** $\mathrm{Cap}\,\Gamma_1 > 0$ **and** $\mathrm{Cap}\,\Gamma_2 > 0$**.** The proof is divided into two steps.

Step 1. Let us derive some dissipative laws for system (1.1). First of all, from (1.10), for any $T \ge S \ge 0$, we get

$$E(T) - E(S) = -\int_S^T \int_{\Omega_1} |\nabla y|^2 dx dt, \tag{7.22}$$

where $E(\cdot)$ is defined by (1.8). Next, put

$$\begin{aligned} &E_j(t) = E(\mathcal{A}^j(y,z,z_t))(t),\\ &(Y_j, Z_j, \partial_t Z_j) = \mathcal{A}^j(y,z,z_t), \qquad j=-3,-2,-1,1,\\ &\mathcal{E}(t) = E_{-3}(t)+E_{-2}(t)+E_{-1}(t)+E(t). \end{aligned} \tag{7.23}$$

As a consequence of (7.22), we have

$$E_j(T)-E_j(S) = -\int_S^T\int_{\Omega_1}|\nabla Y_j|^2 dxdt, \quad j=-3,-2,-1. \tag{7.24}$$

Therefore, by (7.22) and (7.24), we conclude that for any $0\le S\le T<\infty$, it holds

$$\mathcal{E}(S)-\mathcal{E}(T) = \int_S^T\int_{\Omega_1}\Big(\sum_{j=-3}^{-1}|\nabla Y_j|^2+|\nabla y|^2\Big)dxdt. \tag{7.25}$$

Step 2. Applying the first conclusion in Theorem 12 to $(Y_{-3}, Z_{-3}, \partial_t Z_{-3})$, and noting that $\partial_t Y_{-1}=y$, $\partial_t Y_{-2}=Y_{-1}$ and $\partial_t Y_{-3}=Y_{-2}$ in $(0,T)\times\Omega_1$, we get

$$E_{-3}(0)\le C\int_0^T\int_{\Omega_1}\Big(\sum_{j=-3}^{-1}|\nabla Y_j|^2+|\nabla y|^2\Big)dxdt. \tag{7.26}$$

Combining (7.25) and (7.26), we see that

$$E_{-3}(0)\le C\Big(\mathcal{E}(0)-\mathcal{E}(T)\Big). \tag{7.27}$$

Applying (2.10) in Corollary 1 to $\mathcal{A}$ defined by (3.1)–(3.2), one deduces that

$$\mathcal{E}(t)\le CE(t)\le C|E_{-3}(t)|^{1/4}|E_1(t)|^{3/4}, \qquad \forall\, t\ge 0. \tag{7.28}$$

Consequently, by (7.27) and (7.28), we arrive at

$$\frac{|\mathcal{E}(0)|^4}{|E_1(0)|^3}\le C\Big(\mathcal{E}(0)-\mathcal{E}(T)\Big). \tag{7.29}$$

Now, let us fix $T>T_0$. We set

$$\alpha_m = \mathcal{E}(mT), \qquad m=0,1,2,\cdots.$$

Then

$$\frac{\alpha_{1+m}^4}{|E_1(0)|^3}\le C(\alpha_m-\alpha_{1+m}), \tag{7.30}$$

where $C>0$ is a constant, independent of m and (y_0,z_0,z_1). Without loss of generality, we assume that $E_1(0)=1$. Hence

$$\alpha_{m+1}^4\le C(\alpha_m-\alpha_{m+1}) \tag{7.31}$$

for some constant C which is independent of m and (y_0, z_0, z_1). Finally, applying Proposition 6 to (7.31), one gets (7.3) immediately.

ii) **The case** $\mathrm{Cap}\,\varGamma_2 = 0$**.** As in the first case, the proof is divided into two steps.

Step 1′. For any given $(y_0, z_0, z_1) \in H$, by Theorem 1 ii), there is a constant $c_1 = c_1(y_0, z_0, z_1)$ defined by (3.6) such that $(y_0, z_0 - c_1, z_1) \in \mathcal{R}(\mathcal{A})$. Obviously,

$$\begin{aligned} e^{\mathcal{A}t}(y_0, z_0, z_1) &= e^{\mathcal{A}t}(y_0, z_0 - c_1, z_1) + e^{\mathcal{A}t}(0, c_1, 0) \\ &= e^{\mathcal{A}t}(y_0, z_0 - c_1, z_1) + (0, c_1, 0), \qquad \forall\, t \geq 0. \end{aligned}$$

Therefore, it suffices to show that for all $(y_0, z_0, z_1) \in D(\mathcal{A}) \cap \mathcal{R}(\mathcal{A})$, the solution of (1.1) satisfies

$$||(y(t), z(t), z_t(t))||_H \leq \frac{C}{t^{1/6}} ||(y_0, z_0, z_1)||_{D(\mathcal{A})}, \qquad \forall\, t > 0. \tag{7.32}$$

Let us derive some dissipative laws for system (1.1). First of all, from (1.10), for any $T \geq S \geq 0$, we get (Recall (1.8) for the definition of $E(\cdot)$)

$$E(T) - E(S) = -\int_S^T \int_{\Omega_1} |\nabla y|^2 dx dt. \tag{7.33}$$

Put

$$(\widetilde{Y}_{-1}, \widetilde{Z}_{-1}, \partial_t \widetilde{Z}_{-1})(t) = \int_0^t (y, z, z_t)(s) ds + \mathcal{A}^{-1}(y_0, z_0, z_1). \tag{7.34}$$

Then $(\widetilde{Y}_{-1}, \widetilde{Z}_{-1}, \partial_t \widetilde{Z}_{-1})$ satisfies (1.1) with initial data $(\widetilde{Y}_{-1}, \widetilde{Z}_{-1}, \partial_t \widetilde{Z}_{-1})(0) = \mathcal{A}^{-1}(y_0, z_0, z_1)$.

Note that $\mathcal{A}^{-1}(y_0, z_0, z_1)$ does not need to belong to $\mathcal{R}(\mathcal{A})$. But, as shown in the last assertion in Theorem 1 ii), one can find a constant $c_2 \in \mathbb{C}$ so that

$$\mathcal{A}^{-1}(y_0, z_0, z_1) - (0, c_2, 0) \in \mathcal{R}(\mathcal{A}).$$

Put

$$\begin{aligned} &(\widetilde{Y}_{-2}, \widetilde{Z}_{-2}, \partial_t \widetilde{Z}_{-2})(t) \\ &= \int_0^t (\widetilde{Y}_{-1}, \widetilde{Z}_{-1}, \partial_t \widetilde{Z}_{-1})(s) ds + \mathcal{A}^{-1}\Big(\mathcal{A}^{-1}(y_0, z_0, z_1) - (0, c_2, 0)\Big). \end{aligned} \tag{7.35}$$

Then $(\widetilde{Y}_{-2}, \widetilde{Z}_{-2}, \partial_t \widetilde{Z}_{-2})$ satisfies (1.1) with initial data

$$(\widetilde{Y}_{-2}, \widetilde{Z}_{-2}, \partial_t \widetilde{Z}_{-2})(0) = \mathcal{A}^{-1}\Big(\mathcal{A}^{-1}(y_0, z_0, z_1) - (0, c_2, 0)\Big).$$

Similarly, one can find a constant $c_3 \in \mathbb{C}$ so that

$$\mathcal{A}^{-1}\Big(\mathcal{A}^{-1}(y_0, z_0, z_1) - (0, c_2, 0)\Big) - (0, c_3, 0) \in \mathcal{R}(\mathcal{A}).$$

Put

$$\begin{aligned}&(\widetilde Y_{-3},\widetilde Z_{-3},\partial_t\widetilde Z_{-3})(t)\\&=\int_0^t(\widetilde Y_{-2},\widetilde Z_{-2},\partial_t\widetilde Z_{-2})(s)ds\\&\quad+\mathcal{A}^{-1}\Big[\mathcal{A}^{-1}\Big(\mathcal{A}^{-1}(y_0,z_0,z_1)-(0,c_2,0)\Big)-(0,c_3,0)\Big].\end{aligned}\tag{7.36}$$

Then $(\widetilde Y_{-3},\widetilde Z_{-3},\partial_t\widetilde Z_{-3})$ satisfies (1.1) with initial data

$$(\widetilde Y_{-3},\widetilde Z_{-3},\partial_t\widetilde Z_{-3})(0)=\mathcal{A}^{-1}\Big[\mathcal{A}^{-1}\Big(\mathcal{A}^{-1}(y_0,z_0,z_1)-(0,c_2,0)\Big)-(0,c_3,0)\Big].$$

Once more, we can find a constant $c_4\in\mathbb{C}$ so that

$$\mathcal{A}^{-1}\Big[\mathcal{A}^{-1}\Big(\mathcal{A}^{-1}(y_0,z_0,z_1)-(0,c_2,0)\Big)-(0,c_3,0)\Big]-(0,c_4,0)\in\mathcal{R}(\mathcal{A}).$$

Put

$$\begin{aligned}&(\widetilde Y_{-4},\widetilde Z_{-4},\partial_t\widetilde Z_{-4})(t)\\&=\int_0^t(\widetilde Y_{-3},\widetilde Z_{-3},\partial_t\widetilde Z_{-3})(s)ds\\&\quad+\mathcal{A}^{-1}\Big\{\mathcal{A}^{-1}\Big[\mathcal{A}^{-1}\Big(\mathcal{A}^{-1}(y_0,z_0,z_1)-(0,c_2,0)\Big)-(0,c_3,0)\Big]-(0,c_4,0)\Big\}.\end{aligned}\tag{7.37}$$

Then $(\widetilde Y_{-4},\widetilde Z_{-4},\partial_t\widetilde Z_{-4})$ satisfies (1.1) with initial data

$$\begin{aligned}&(\widetilde Y_{-4},\widetilde Z_{-4},\partial_t\widetilde Z_{-4})(0)\\&=\mathcal{A}^{-1}\Big\{\mathcal{A}^{-1}\Big[\mathcal{A}^{-1}\Big(\mathcal{A}^{-1}(y_0,z_0,z_1)-(0,c_2,0)\Big)-(0,c_3,0)\Big]-(0,c_4,0)\Big\}.\end{aligned}$$

It is easy to see that

$$\begin{aligned}&\frac{d}{dt}(\widetilde Y_j,\widetilde Z_j,\partial_t\widetilde Z_j)(t)=\mathcal{A}(\widetilde Y_j,\widetilde Z_j,\partial_t\widetilde Z_j)(t)=(\widetilde Y_{1+j},\widetilde Z_{1+j},\partial_t\widetilde Z_{1+j})(t),\\&\qquad\qquad j=-4,-3,-2.\\&\frac{d}{dt}(\widetilde Y_{-1},\widetilde Z_{-1},\partial_t\widetilde Z_{-1})(t)=\mathcal{A}(\widetilde Y_{-1},\widetilde Z_{-1},\partial_t\widetilde Z_{-1})(t)=(y,z,z_t)(t).\end{aligned}\tag{7.38}$$

Next, put

$$\begin{aligned}&E_j(t)=E((\widetilde Y_j,\widetilde Z_j,\partial_t\widetilde Z_j))(t),\qquad j=-4,-3,-2,-1,\\&\mathcal{F}(t)=E_{-4}(t)+E_{-3}(t)+E_{-2}(t)+E_{-1}(t)+E(t).\end{aligned}\tag{7.39}$$

As a consequence of (7.33), we conclude that for ant $0\le S\le T<\infty$, it holds

$$\mathcal{F}(S)-\mathcal{F}(T)=\int_S^T\int_{\Omega_1}\Big(\sum_{j=-4}^{-1}|\nabla\widetilde Y_j|^2+|\nabla y|^2\Big)dxdt.\tag{7.40}$$

Step 2′. Applying the second conclusion in Theorem 12 to $(\widetilde{Y}_{-3}, \widetilde{Z}_{-3}, \partial_t\widetilde{Z}_{-3})$, and noting (7.38), we get

$$||(\widetilde{Y}_{-3}, \widetilde{Z}_{-3}, \partial_t\widetilde{Z}_{-3})(0)||_H^2 \le C\int_0^T\int_{\Omega_1}\Big(\sum_{j=-4}^{-1}|\nabla\widetilde{Y}_j|^2 + |\nabla y|^2\Big)dxdt. \quad (7.41)$$

Combining (7.40) and (7.41), we see that

$$||(\widetilde{Y}_{-3}, \widetilde{Z}_{-3}, \partial_t\widetilde{Z}_{-3})(0)||_H^2 \le C\Big[\mathcal{F}(0) - \mathcal{F}(T)\Big]. \quad (7.42)$$

Applying the second assertion in Corollary 1 to $\mathcal{A}$ defined by (3.1)–(3.2) and u given by $(\widetilde{Y}_{-3}, \widetilde{Z}_{-3}, \partial_t\widetilde{Z}_{-3})(t)$, and noting (7.38), we conclude that

$$\begin{aligned}\mathcal{F}(t) &\le C||(\widetilde{Y}_{-3}, \widetilde{Z}_{-3}, \partial_t\widetilde{Z}_{-3})(t)||_{D(\mathcal{A}^3)}\\ &\le C||(\widetilde{Y}_{-3}, \widetilde{Z}_{-3}, \partial_t\widetilde{Z}_{-3})(t)||_H^{1/4}||(\widetilde{Y}_{-3}, \widetilde{Z}_{-3}, \partial_t\widetilde{Z}_{-3})(t)||_{D(\mathcal{A}^4)}^{3/4}\\ &\le C||(\widetilde{Y}_{-3}, \widetilde{Z}_{-3}, \partial_t\widetilde{Z}_{-3})(t)||_H^{1/4}||(y, z, z_t)(t)||_{D(\mathcal{A})}^{3/4}, \qquad \forall\, t \ge 0.\end{aligned} \quad (7.43)$$

Consequently, by (7.42) and (7.43), we arrive at

$$\frac{|\mathcal{F}(0)|^4}{||(y(0), z(0), z_t(0))||_{D(\mathcal{A})}^3} \le C\Big[\mathcal{F}(0) - \mathcal{F}(T)\Big]. \quad (7.44)$$

It is easy to see that (7.44) is the counterpart of (7.29) in the present case. Therefore, proceeding as in Step 1, and noting that $||(y(t), z(t), z_t(t))||_H \le \mathcal{F}(t)$ for any $t \ge 0$, one obtains (7.32). □

8. Logarithmic decay without the GCC

In the previous section we have shown that, when the heat subdomain Ω_1 satisfies the GCC in Ω, the energy of solutions with data in $D(\mathcal{A})$ decays polynomially. This section is devoted to analyze the problem of the decay rate without the GCC. However, as we shall see, our analysis does not suffice to conclude any decay rate for system (1.1) under consideration. Therefore, we consider the simpler model (1.2) and show the logarithmic decay without the GCC. We also discuss the difficulty that arises when analyzing the logarithmic decay for system (1.1).

8.1. The main result

The main result in this section reads as follows (Recall Remark 6 i) for the definition of $\widetilde{\mathcal{A}}$).

Theorem 13. *Let $\Gamma_1 \neq \emptyset$ be a $n-1$ dimensional topological manifold. Then, for any $s \in (0, 1/16)$, there is a constant $C_s > 0$ such that for any $(y_0, z_0, z_1) \in D(\widetilde{\mathcal{A}})$, the solution of (1.2) satisfies*

$$|(y(t), z(t), z_t(t))|_{\widetilde{H}} \leq \frac{C_s |(y_0, z_0, z_1)|_{D(\widetilde{\mathcal{A}})}}{\ln^s(1+t)}, \qquad \forall\, t > 0. \tag{8.1}$$

Remark 19. This decay rate has been improved in [15] by getting (8.1) for $s = 1$ under the additional assumptions that Ω is of C^∞ and $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$

The proof of Theorem 13 will be given in Subsection 8.3. Note however that the method we employ to prove (8.1) does not seem to apply to system (1.1). Indeed, as we will see later, the key point to prove Theorem 13 is to derive a (weak) logarithmic observability estimate for system (1.2) (*see* Lemma 11). This uses in an essential way the key fact that $\widetilde{\mathcal{A}}^{-1}(y_0, z_0, z_1)$ is more regular than (y_0, z_0, z_1). But, as shown in Remark 4, the second component of $\mathcal{A}^{-1}(y_0, z_0, z_1)$ has the same regularity as z_0. Therefore this argument does not apply to system (1.1).

Remark 20. By using the result in an unpublished note ([37]), one can drop the assumption in Theorem 13 that $\Gamma_1 \neq \emptyset$ is a $n-1$ dimensional topological manifold (See also Remark 22 below).

8.2. A logarithmic observability inequality

As mentioned before, in order to prove Theorem 13, we need to derive a (weak) logarithmic observability inequality for the solutions of system (1.2). For this purpose, we recall the following known result:

Lemma 10. ([41]) *Let Γ_0 be a nonempty open subset of Γ. Then there is a constant $T_1 = T_1(\Omega, \Gamma_0) > 0$ such that, for any $T > T_1$, one can find a constant $C = C(T, \Gamma_0, \Omega) > 0$ so that solutions of (7.1) satisfy*

$$\begin{aligned} &|(\zeta_0, \zeta_1)|_{L^2(\Omega)\times H^{-1}(\Omega)} \\ &\leq C\left[\ln\left(2 + \frac{|(\zeta_0, \zeta_1)|_{H_0^1(\Omega)\times L^2(\Omega)}}{|\frac{\partial \zeta}{\partial \nu}|_{L^2((0,T)\times\Gamma_0)}}\right)\right]^{-1/2} |(\zeta_0, \zeta_1)|_{H_0^1(\Omega)\times L^2(\Omega)}, \\ &\qquad\qquad \forall\, (\zeta_0, \zeta_1) \in H_0^1(\Omega) \times L^2(\Omega). \end{aligned} \tag{8.2}$$

Remark 21. Note that (8.2) holds without any geometric condition on Γ_0 (other than being a non-empty open subset of Γ). In particular, Γ_0 is not required to satisfy the GCC.

As a direct consequence of Lemma 10, we have

Corollary 2. *Let ω be an open subset of Ω so that $\partial\omega$ contains a nonempty open subset of Γ, and T_1 given as in Lemma 10. Then, for any $T > T_1$ and $s \in [0,1/2) \cup (1/2,1]$, one can find a constant $C = C(T,s,\omega,\Omega) > 0$ so that solutions of (7.1) satisfy*

$$\begin{aligned}&|(\zeta_0,\zeta_1)|_{H_0^{1-s}(\Omega)\times H^{-s}(\Omega)}\\ &\le C\left[\ln\left(2+\frac{|(\zeta_0,\zeta_1)|_{H_0^1(\Omega)\times L^2(\Omega)}}{|\zeta|_{H^1(0,T;L^2(\omega))}}\right)\right]^{-s/2}|(\zeta_0,\zeta_1)|_{H_0^1(\Omega)\times L^2(\Omega)}, \quad (8.3)\\ &\qquad\forall\,(\zeta_0,\zeta_1)\in H_0^1(\Omega)\times L^2(\Omega).\end{aligned}$$

Remark 22. The key point to prove Corollary 2 is to derive an internal counterpart of (8.2), i.e., inequality (8.7) below. As it was communicated to us by K. D. Phung ([37]), this inequality also holds when the subdomain ω is any nonempty open subset of Ω without the condition that $\partial\omega\cap\Gamma\ne\emptyset$, and can be proved using the approach developed in [41] without going through (8.2). For this reason, the assumption that $\Gamma_1\ne\emptyset$ is a $n-1$ dimensional topological manifold is, very likely, not needed, but this remains to be proved rigorously.

Proof of Corollary 2. Fix $T > T_1$, and choose ε_0 small enough so that $T-2\varepsilon_0 > T_1$. Applying Lemma 10 to system (7.1) with $(0,T)$ replaced by $(\varepsilon_0, T-\varepsilon_0)$, noting that $|(\zeta(t),\zeta_t(t))|_{L^2(\Omega)\times H^{-1}(\Omega)} = |(\zeta_0,\zeta_1)|_{L^2(\Omega)\times H^{-1}(\Omega)}$ and $|(\zeta(t),\zeta_t(t))|_{H_0^1(\Omega)\times L^2(\Omega)} = |(\zeta_0,\zeta_1)|_{H_0^1(\Omega)\times L^2(\Omega)}$ for all t, we conclude that

$$\begin{aligned}&|(\zeta_0,\zeta_1)|_{L^2(\Omega)\times H^{-1}(\Omega)}\\ &\le C\left[\ln\left(2+\frac{|(\zeta_0,\zeta_1)|_{H_0^1(\Omega)\times L^2(\Omega)}}{|\partial\zeta/\partial\nu|_{L^2((\varepsilon_0,T-\varepsilon_0)\times\Gamma_0)}}\right)\right]^{-1/2}|(\zeta_0,\zeta_1)|_{H_0^1(\Omega)\times L^2(\Omega)}, \quad (8.4)\\ &\qquad\forall\,(\zeta_0,\zeta_1)\in H_0^1(\Omega)\times L^2(\Omega).\end{aligned}$$

Assume Γ_0 to be any nonempty open subset of Γ so that $\Gamma_0\subset\partial\omega_0$ for some subdomain ω_0 of ω such that $\overline{\omega_0}\subset\omega$. Noting that ζ satisfies (7.1), it is easy to show that (*see* for example, [30])

$$|\partial\zeta/\partial\nu|_{L^2((\varepsilon_0,T-\varepsilon_0)\times\Gamma_0)} \le C|\zeta|_{H^1((\varepsilon_0/2,T-\varepsilon_0/2)\times\omega_0)}. \quad (8.5)$$

Noting again that ζ satisfies (7.1), one finds

$$|\nabla\zeta|_{L^2((\varepsilon_0/2,T-\varepsilon_0/2)\times\omega_0)} \le C|\zeta|_{H^1(0,T;L^2(\omega))}. \quad (8.6)$$

Finally, combining (8.4), (8.5) and (8.6), we arrive at

$$\begin{aligned}&|(\zeta_0,\zeta_1)|_{L^2(\Omega)\times H^{-1}(\Omega)}\\ &\le C\left[\ln\left(2+\frac{|(\zeta_0,\zeta_1)|_{H_0^1(\Omega)\times L^2(\Omega)}}{|\zeta|_{H^1(0,T;L^2(\omega))}}\right)\right]^{-1/2}|(\zeta_0,\zeta_1)|_{H_0^1(\Omega)\times L^2(\Omega)}, \quad (8.7)\\ &\qquad\forall\,(\zeta_0,\zeta_1)\in H_0^1(\Omega)\times L^2(\Omega).\end{aligned}$$

On the other hand, by interpolation ([31]), one has

$$\begin{aligned}
&|(\zeta_0,\zeta_1)|_{H_0^{1-s}(\Omega)\times H^{-s}(\Omega)}\\
&=|(\zeta_0,\zeta_1)|_{[H_0^1(\Omega)\times L^2(\Omega),L^2(\Omega)\times H^{-1}(\Omega)]_s}\\
&\le C|(\zeta_0,\zeta_1)|^{1-s}_{H_0^1(\Omega)\times L^2(\Omega)}|(\zeta_0,\zeta_1)|^s_{L^2(\Omega)\times H^{-1}(\Omega)},
\end{aligned}\tag{8.8}$$

where $[\cdot,\cdot]_s$ denotes the interpolation space with exponent s. Combining (8.8) and (8.7), we arrive at the desired inequality (8.3). □

We now state the following logarithmic observability inequality for the solutions of system (1.2).

Lemma 11. *Let $\Gamma_1\neq\emptyset$ be a $n-1$ dimensional topological manifold, and T_1 given as in Lemma 10. Then, for any $T>T_1$ and $s\in[0,1/2)$, there exists a constant $C>0$ such that, for all $(y_0,z_0,z_1)\in D(\widetilde{\mathcal{A}}^2)$, solutions of system (1.2) satisfy*

$$|(y_0,z_0,z_1)|_{\widetilde H}\le C\left[\ln\left(2+\frac{|(y_0,z_0,z_1)|_{D(\widetilde A)}}{|y|_{H^3(0,T;L^2(\Omega_1))}}\right)\right]^{-s/2}|(y_0,z_0,z_1)|_{D(\widetilde{A}^2)}. \tag{8.9}$$

The proof of Lemma 11 is given in Appendix D.

8.3. Proof of the logarithmic decay

In this subsection, we prove the logarithmic decay result in Theorem 13.

Proof of Theorem 13. The proof is divided into several steps.

Step 1. Similar to the first step in the proof of Theorem 11, we derive first some dissipative law for system (1.2). Define the energy of system (1.2) by

$$\widetilde E(t)\triangleq\widetilde E(y,z,z_t)(t)=\frac12|(y(t),z(t),z_t(t))|^2_{\widetilde H}. \tag{8.10}$$

By means of the classical energy method, we have:

$$\frac{d}{dt}\widetilde E(t)=-\int_{\Omega_1}|y_t|^2dx. \tag{8.11}$$

Define

$$\begin{aligned}
&\widetilde E_j(t)\triangleq\widetilde E(\widetilde{\mathcal{A}}^j(y,z,z_t))(t),\qquad j=-3,-2,-1,1,2,\\
&\widetilde{\mathcal{E}}_2(t)\triangleq\widetilde E_{-1}(t)+\widetilde E(t)+\widetilde E_1(t)+\widetilde E_2(t),\\
&\widetilde{\mathcal{E}}(t)\triangleq\widetilde E_{-3}(t)+\widetilde E_{-2}(t)+\widetilde E_{-1}(t)+\widetilde E(t).
\end{aligned}\tag{8.12}$$

Then, as a consequence of (8.11), and noting that the time derivative of the first component of $\widetilde{\mathcal{A}}^{-1}(y,z,z_t)$ is equal to y, one has

$$\begin{aligned} &\frac{d}{dt}\widetilde{E}_{-1}(t) = -\int_{\Omega_1}|y|^2dx, \quad \frac{d}{dt}\widetilde{E}_1(t) = -\int_{\Omega_1}|y_{tt}|^2dx, \\ &\frac{d}{dt}\widetilde{E}_2(t) = -\int_{\Omega_1}|y_{ttt}|^2dx. \end{aligned} \tag{8.13}$$

Hence, for any given $T>0$, integrating and adding the identities in (8.11) and (8.13) with respect to t from 0 to T, we get

$$\widetilde{\mathcal{E}}_2(0) - \widetilde{\mathcal{E}}_2(T) = |y|^2_{H^3(0,T;L^2(\Omega_2))}. \tag{8.14}$$

Similarly, in view of the dissipative nature of system (1.2), one deduces that

$$\frac{d\widetilde{\mathcal{E}}(t)}{dt} \le 0. \tag{8.15}$$

Step 2. By Lemma 11, noting that $|(y_0,z_0,z_1)|^2_{\widetilde{H}} = \widetilde{E}(0)$ and $|(y_0,z_0,z_1)|^2_{D(\widetilde{A}^i)} = \widetilde{E}_i(0)$ for $i=1,2$, we see that, for any $s\in(0,1/4)$, there exists a constant $C=C(s)>0$ such that, for all $(y_0,z_0,z_1)\in D(\widetilde{\mathcal{A}}^2)$, solutions of system (1.2) satisfy

$$\widetilde{E}_1(0)e^{-\left(C\widetilde{E}_2(0)/\widetilde{E}(0)\right)^{1/s}} \le |y|^2_{H^3(0,T;L^2(\Omega_1))}. \tag{8.16}$$

Combining (8.14) and (8.16), we find

$$\widetilde{E}_1(0)e^{-\left(C\widetilde{E}_2(0)/\widetilde{E}(0)\right)^{1/s}} \le \widetilde{\mathcal{E}}_2(0) - \widetilde{\mathcal{E}}_2(T), \qquad \forall\, s\in(0,1/4). \tag{8.17}$$

Applying (8.17) to system (1.2) with initial data (y_0,z_0,z_1) replaced by $\widetilde{\mathcal{A}}^{-2}(y_0,z_0,z_1)$, and noting that $e^{\widetilde{\mathcal{A}}t}$ and $\widetilde{\mathcal{A}}^{-2}$ commute for each $t\ge 0$, we conclude that

$$\widetilde{E}_{-1}(0)e^{-\left(C\widetilde{E}(0)/\widetilde{E}_{-2}(0)\right)^{1/s}} \le \widetilde{\mathcal{E}}(0) - \widetilde{\mathcal{E}}(T), \qquad \forall\, s\in(0,1/4). \tag{8.18}$$

On the other hand, one has

$$\widetilde{E}(t) \le \widetilde{\mathcal{E}}(t) \le C\widetilde{E}(t), \qquad \forall\, t\ge 0. \tag{8.19}$$

Hence, by Lemma 4, we conclude that

$$\widetilde{E}_{-1}(0) \ge \frac{|\widetilde{\mathcal{E}}(0)|^2}{C\widetilde{E}_1(0)}, \qquad \widetilde{E}_{-2}(0) \ge \frac{|\widetilde{\mathcal{E}}(0)|^3}{C|\widetilde{E}_1(0)|^2}. \tag{8.20}$$

Combining (8.18)–(8.20), we get

$$\frac{|\widetilde{\mathcal{E}}(0)|^2}{\widetilde{E}_1(0)}e^{-\left(C|\widetilde{E}_1(0)|/|\widetilde{\mathcal{E}}(0)|\right)^{1/s}} \le C\big(\widetilde{\mathcal{E}}(0) - \widetilde{\mathcal{E}}(T)\big), \quad \forall\, s\in(0,1/8). \tag{8.21}$$

Step 3. Now, put

$$\alpha_m = \widetilde{\mathcal{E}}(mT), \qquad m = 0, 1, 2, \cdots.$$

Then, noting that $\widetilde{E}_1(t)$ is decreasing with respect to t, it follows from (8.21) that

$$\begin{aligned} &\frac{\alpha_m^2}{\widetilde{E}_1(0)} e^{-\left(C|\widetilde{E}_1(0)|/\alpha_m\right)^{1/s}} \\ &\leq \frac{\alpha_m^2}{\widetilde{E}_1(mT)} e^{-\left(C|\widetilde{E}_1(mT)|/\alpha_m\right)^{1/s}} \leq C(\alpha_m - \alpha_{1+m}), \end{aligned} \tag{8.22}$$

where $s \in (0, 1/8)$, $C = C(s) > 0$ is a constant, independent of m and (y_0, z_0, z_1). Without loss of generality, we assume that $\widetilde{E}_1(0) = 1$. Hence

$$\alpha_m^2 e^{-\left(C/\alpha_m\right)^{1/s}} \leq C(\alpha_m - \alpha_{1+m}), \qquad \forall\, s \in (0, 1/8), \tag{8.23}$$

with $C = C(s) > 0$, independent of m and (y_0, z_0, z_1).

By (8.15), it is easy to show that $\{\alpha_m\}_{m=1}^{\infty}$ is a decreasing sequence. Hence, it follows from (8.23) that, for any $k \in \mathbb{N}$,

$$\begin{aligned} k\alpha_k^2 e^{-\left(C/\alpha_k\right)^{1/s}} &\leq \sum_{m=1}^{k} \alpha_m^2 e^{-\left(C/\alpha_m\right)^{1/s}} \\ &\leq C(\alpha_1 - \alpha_{1+k}) \leq C\alpha_1, \qquad \forall\, s \in (0, 1/8), \end{aligned} \tag{8.24}$$

for a constant $C = C(s) > 0$, independent of k. However, since

$$\min_{x \in (0, \alpha_1)} x^2 e^{\left(C/x\right)^{1/s}} > 0,$$

it is easy to check that, for some constant $C_s > 0$, it holds

$$C_s \alpha_k^2 \geq e^{-\left(C/\alpha_k\right)^{1/s}}. \tag{8.25}$$

Combining (8.24) and (8.25), we end up with

$$k e^{-2\left(C/\alpha_k\right)^{1/s}} \leq C_s \alpha_1, \qquad \forall\, s \in (0, 1/8). \tag{8.26}$$

Hence, for sufficiently large $k \in \mathbb{N}$,

$$\alpha_k \leq C \left(\frac{\ln k - \ln(C_s \alpha_1)}{2} \right)^{-s}, \qquad \forall\, s \in (0, 1/8),$$

which gives the desired result (8.1). $\square$

9. Appendix A: Proof of Theorem 1

This appendix is devoted to prove Theorem 1. We distinguish two different cases.

i) **The case** $\mathrm{Cap}\,\Gamma_2 > 0$**.** In this case, the proof is standard. For completeness, we give here the sketch. We use the equivalent norm $|\cdot|_H$ in (1.7) for H, and denote by $(\cdot\,,\cdot)_H$ the corresponding scalar product.

It is easy to show that

$$\mathrm{Re}\,(\mathcal{A}Y, Y)_H = -\int_{\Omega_1} |\nabla Y_1|^2 dx \leq 0, \quad \forall\, Y = (Y_1, Y_2, Y_3) \in D(\mathcal{A}). \tag{9.1}$$

It remains to show that $0 \in \rho(\mathcal{A})$. For this purpose, we fix any $F = (F_1, F_2, F_3) \in H$ and we look for $Y = (Y_1, Y_2, Y_3) \in D(\mathcal{A})$ such that

$$\mathcal{A}Y = F. \tag{9.2}$$

By (3.2) and (1.4), it follows from (9.2) that $Y_3 = F_2 \in H^1_{\Gamma_2}(\Omega_2)$. We solve the following system

$$\begin{cases} \Delta Y_1 = F_1 & \text{in } \Omega_1, \\ Y_1 = 0 & \text{on } \Gamma_1, \\ Y_1 = F_2 & \text{on } \gamma. \end{cases} \tag{9.3}$$

Obviously, $Y_1 \in H^1_{\Gamma_1}(\Omega_1)$ and by Lemma 1, $\frac{\partial Y_1}{\partial \nu_1} \in H^{-1/2}(\partial\Omega_1)$. Further, by Lax-Milgram Theorem, we solve the following system

$$\begin{cases} \Delta Y_2 = F_3 & \text{in } \Omega_2, \\ Y_2 = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial Y_2}{\partial \nu_2} = -\dfrac{\partial Y_1}{\partial \nu_1} & \text{on } \gamma \end{cases}$$

and get $Y_2 \in H^1_{\Gamma_2}(\Omega_2)$. It is easy to see that $(Y_1, Y_2, Y_3) \in D(\mathcal{A})$. Therefore $\mathcal{A}^{-1}$ exists and it is a bounded operator in H. Therefore, $\mathcal{A}$ generates a contractive C_0-semigroup in H.

ii) **The case** $\mathrm{Cap}\,\Gamma_2 = 0$**.** In this case, we use the norm $||\cdot||_H$ in (1.5) for H, and denote by $[\cdot\,,\cdot]_H$ its scalar product. First, one has

$$\begin{aligned}
&\mathrm{Re}\,[(\mathcal{A}-I)Z, Z]_H \\
&= -\int_{\Omega_1} |\nabla Z_1|^2 dx - \int_{\Omega_1} |Z_1|^2 dx + \int_{\Omega_2} Z_3 \overline{Z_2} dx \\
&\quad - \int_{\Omega_2} |Z_2|^2 dx - \int_{\Omega_2} |\nabla Z_2|^2 dx - \int_{\Omega_2} |Z_3|^2 dx \\
&\le - \int_{\Omega_1} |\nabla Z_1|^2 dx - \int_{\Omega_1} |Z_1|^2 dx - \frac{1}{2}\int_{\Omega_2} |Z_2|^2 dx \\
&\quad - \int_{\Omega_2} |\nabla Z_2|^2 dx - \frac{1}{2}\int_{\Omega_2} |Z_3|^2 dx \\
&\le 0, \qquad \forall\, Z = (Z_1, Z_2, Z_3) \in D(\mathcal{A}),
\end{aligned} \tag{9.4}$$

where I denotes the identity operator in H.

Next, we show that $0 \in \rho(\mathcal{A} - I)$. For this purpose, we fix any $F = (F_1, F_2, F_3) \in H$ and we look for $Z = (Z_1, Z_2, Z_3) \in D(\mathcal{A})$ such that $(\mathcal{A} - I)Z = F$, which is equivalent to

$$\begin{cases}
\Delta Z_1 - Z_1 = F_1 & \text{in } \Omega_1, \\
\Delta Z_2 - Z_2 = F_2 + F_3 & \text{in } \Omega_2, \\
Z_1 = 0 & \text{on } \Gamma_1, \\
Z_1 = Z_2 + F_2, \quad \dfrac{\partial Z_2}{\partial \nu_2} = -\dfrac{\partial Z_1}{\partial \nu_1} & \text{on } \gamma,
\end{cases} \tag{9.5}$$

and

$$Z_3 = Z_2 + F_2, \qquad \text{in } \Omega_2. \tag{9.6}$$

By the classical transposition method ([31]), one can show that system (9.5) admits a unique weak solution $(Z_1, Z_2) \in H^1_{\Gamma_1}(\Omega_1) \times H^1(\Omega_2)$. Then, by (9.6), we get $Z_3 \in H^1(\Omega_2)$. One can further check that $(Z_1, Z_2, Z_3) \in D(\mathcal{A})$. Therefore, $(\mathcal{A}-I)^{-1}$ is well-defined and it is a bounded operator in H. Thus, $\mathcal{A}-I$ generates a contractive C_0-semigroup in H. It is obvious that $0 \notin \rho(\mathcal{A})$ since for any constant $c \in \mathbb{C} \setminus \{0\}$, $(0, c, 0) \in D(\mathcal{A})$ and $\mathcal{A}(0, c, 0) = 0$.

Let us show (3.3). Assume $(f_1, f_2, f_3) \in \mathcal{R}(\mathcal{A})$. Then there is a $(G, Y_2, Y_3) \in D(\mathcal{A})$ such that $\mathcal{A}(G, Y_2, Y_3) = (f_1, f_2, f_3)$. As in Case 1, we conclude that $Y_3 = f_2$, and G and Y_2 solve respectively systems (3.4) and

$$\begin{cases}
\Delta Y_2 = f_3 & \text{in } \Omega_2 \\
\dfrac{\partial Y_2}{\partial \nu_2} = -\dfrac{\partial G}{\partial \nu_1} & \text{on } \gamma.
\end{cases} \tag{9.7}$$

Now, by integrating the first equation of (9.7) in Ω_2, using its second equation, we get

$$\int_{\Omega_2} f_3 dx = \int_\gamma \frac{\partial Y_2}{\partial \nu_1} d\gamma = -\int_\gamma \frac{\partial G(f_1, f_2)}{\partial \nu_1} d\gamma.$$

Therefore,

$$\mathcal{R}(\mathcal{A}) \subset \Big\{ (f_1, f_2, f_3) \in H \;\Big|\; \int_{\Omega_2} f_3 dx + \int_\gamma \frac{G(f_1, f_2)}{\partial \nu_1} d\gamma = 0 \Big\}.$$

Conversely, assume $(f_1, f_2, f_3) \in H$ satisfying

$$\int_{\Omega_2} f_3 dx + \int_\gamma \frac{\partial G(f_1, f_2)}{\partial \nu_1} d\gamma = 0. \tag{9.8}$$

By Lax-Milgram's lemma there exists a unique solution Y_2 of the Neumann problem (9.7) in the the space $\Big\{ f \in H^1(\Omega_2) \;\Big|\; \int_{\Omega_2} f dx = 0 \Big\}$. Indeed $G = G(f_1, f_2)$ and f_3 satisfy the compatibility condition (9.8)). The corresponding solution Y_2 is such that $(G, Y_2, f_2) \in D(\mathcal{A})$ and $\mathcal{A}(G, Y_2, f_2) = (f_1, f_2, f_3)$. Therefore, we have proved (3.3).

The closeness of $\mathcal{R}(\mathcal{A})$ in H follows from Lemma 1. Note that the above argument also yields that $\mathcal{A} : \overset{\circ}{H} \bigcap D(\mathcal{A}) \to \mathcal{R}(\mathcal{A})$ is a one to one and onto linear continuous map.

Further, we show that (3.5) holds. We claim that, for any $(f_1, f_2, f_3) \equiv (y_0, z_0, z_1) \in H$, there is a constant $c_1 \in \mathbb{C}$ such that $(f_1, f_2 - c_1, f_3) \in \mathcal{R}(\mathcal{A})$. Indeed, by (3.3)–(3.4) and (2.5) (with $A_0 = 1$), and recalling Remark 1, we see that c_1 is uniquely determined by

$$\int_{\Omega_2} f_3 dx + \int_\gamma \frac{\partial G(f_1, f_2)}{\partial \nu_1} d\gamma - \frac{c_1}{c_0} = 0.$$

This also yields (3.6). On the other hand, by Lemma 2 and Remark 1, it is easy to show that $\mathcal{R}(\mathcal{A}) \bigcap \Big\{ (0, c, 0) \;\Big|\; c \in \mathbb{C} \Big\} = \{0\}$.

Finally, let us show that the semigroup $\{e^{\mathcal{A}t}\}_{t \geq 0}$ is bounded in H. For this, for any $(y_0, z_0, z_1) \in H$, we choose c_1 as in (3.6) so that $(y_0, z_0 - c_1, z_1) \in \mathcal{R}(\mathcal{A}) \subset H$, and $(0, c_1, 0)$ is a static solution to (1.1). Clearly,

$$\begin{aligned} e^{\mathcal{A}t}(y_0, z_0, z_1) &= e^{\mathcal{A}t}(y_0, z_0 - c_1, z_1) + e^{\mathcal{A}t}(0, c_1, 0) \\ &= e^{\mathcal{A}t}(y_0, z_0 - c_1, z_1) + (0, c_1, 0). \end{aligned} \tag{9.9}$$

By (3.6), we see that

$$|c_1| \leq C \left[|z_1|_{L^2(\Omega_2)} + \int_\gamma \frac{\partial G(y_0, z_0)}{\partial \nu_1} d\gamma \right]. \tag{9.10}$$

Recalling that $G = G(y_0, z_0) \in H^1(\Omega_1)$ solves system (3.4) with (f_1, f_2) replaced by (y_0, z_0), by Lemma 1 and the standard regularity theory for elliptic equations, we get

$$\begin{aligned} &\left| \int_\gamma \frac{\partial G(y_0, z_0)}{\partial \nu_1} d\gamma \right| \leq C \left| \frac{\partial G(y_0, z_0)}{\partial \nu_1} \right|_{H^{-1/2}(\gamma)} \\ &\leq C \Big[|\nabla G(y_0, z_0)|_{(L^2(\Omega_1))^n} + |\Delta G(y_0, z_0)|_{(L^2(\Omega_1))^n} \Big] \\ &\leq C \Big[|y_0|_{L^2(\Omega_1)} + \Big| z_0|_\gamma \Big|_{H^{1/2}(\gamma)} \Big] \leq C(|y_0|_{L^2(\Omega_1)} + |z_0|_{H^1(\Omega_2)}). \end{aligned} \tag{9.11}$$

Combining (9.10) and (9.11), and noting the definition of $||\cdot||_H$, it follows

$$|c_1| \le C||(y_0, z_0, z_1)||_H. \tag{9.12}$$

On the other hand, put

$$(y, z, z_t)(t) = e^{\mathcal{A}t}(y_0, z_0 - c_1, z_1). \tag{9.13}$$

It is easy to see that (y, z, z_t) solves (1.1) with initial data $(y_0, z_0 - c_1, z_1)$. In view of (1.8)–(1.10), we deduce that

$$\begin{aligned} &|y(t)|^2_{L^2(\Omega_1)} + |\nabla z(t)|^2_{(L^2(\Omega_2))^n} + |z_t(t)|^2_{L^2(\Omega_2)} \\ &\le |y_0|^2_{L^2(\Omega_1)} + |\nabla(z_0 - c_1)|^2_{(L^2(\Omega_2))^n} + |z_1|^2_{L^2(\Omega_2)} \le ||(y_0, z_0, z_1)||^2_H. \end{aligned} \tag{9.14}$$

Next we have to estimate $z(t)$ in $L^2(\Omega_2)$ since, in the case $\mathrm{Cap}\,\Gamma_2 = 0$, $|\cdot|_H$ is only a semi-norm on H. Recall that $\mathcal{A} : \overset{\circ}{H} \bigcap D(\mathcal{A}) \to \mathcal{R}(\mathcal{A})$ is a one to one and onto linear continuous operator. Hence, $\mathcal{A}$ is well-defined on $\mathcal{R}(\mathcal{A})$, and by Banach's Inverse Operator Theorem, $\mathcal{A}^{-1} : \mathcal{R}(\mathcal{A}) \to \overset{\circ}{H} \bigcap D(\mathcal{A})$ is a bounded linear operator. Therefore,

$$\begin{aligned} |\mathcal{A}^{-1}(y_0, z_0 - c_1, z_1)|_H &\le C|(y_0, z_0 - c_1, z_1)|_H = C|(y_0, z_0, z_1)|_H \\ &\le C||(y_0, z_0, z_1)||_H. \end{aligned} \tag{9.15}$$

Denote

$$(Y, Z, Z_t)(t) = \int_0^t (y, z, z_t)(s)ds + \mathcal{A}^{-1}(y_0, z_0 - c_1, z_1). \tag{9.16}$$

It is easy to see that

$$Z_t = z, \qquad \text{in } \Omega_2. \tag{9.17}$$

Also, similar to the first inequality in (9.14), one has

$$\begin{aligned} &|Y(t)|^2_{L^2(\Omega_1)} + |\nabla Z(t)|^2_{(L^2(\Omega_2))^n} + |Z_t(t)|^2_{L^2(\Omega_2)} \\ &\le |\mathcal{A}^{-1}(y_0, z_0 - c_1, z_1)|^2_H. \end{aligned} \tag{9.18}$$

Combining (9.13), (9.14), (9.15), (9.17) and (9.18), we end up with

$$\begin{aligned} &||e^{\mathcal{A}t}(y_0, z_0 - c_1, z_1)||^2_H \\ &= |y(t)|^2_{L^2(\Omega_1)} + |z(t)|^2_{L^2(\Omega_2)} + |\nabla z(t)|^2_{(L^2(\Omega_2))^n} + |z_t(t)|^2_{L^2(\Omega_2)} \\ &\le C||(y_0, z_0, z_1)||^2_H. \end{aligned} \tag{9.19}$$

Now, combining (9.9), (9.12) and (9.19), we conclude that there is a constant $C > 0$, such that for all $(y_0, z_0, z_1) \in H$, it holds

$$||e^{\mathcal{A}t}(y_0, z_0, z_1)||_H \le C||(y_0, z_0, z_1)||_H, \qquad \forall\, t \ge 0,$$

which yields the boundedness of the semigroup $\{e^{\mathcal{A}t}\}_{t\ge 0}$ in H. This completes the proof of Theorem 1. □

10. Appendix B: Proof of Lemma 8

This appendix is devoted to prove Lemma 8. We only prove the estimate on $|r_4|_{H^2(0,T^*;H^{1/2}(\gamma))}$ (The estimate on $|r_5|_{H^2(0,T^*;L^2(\gamma))}$ can be proved similarly). The other estimates in (6.83) follow from the analysis in Subsections 6.4 and 6.6.

Let $\tilde r_4 = \tilde r_4(t,\tilde x)$ and $\tilde a^k_\pm = \tilde a^k_\pm(t,\tilde x)$ be respectively the expressions of r_4 and $a^k_\pm$ in the new coordinate $\tilde x$ $(k=1,2)$. To simplify the notations, we write $\tilde r_j(t,0,\tilde x')$, $\tilde a^\pm(t,0,\tilde x')$, $\tilde a^k_\pm(t,0,\tilde x')$ and $\tilde\phi^\pm(t,0,\tilde x')$ simply as $\tilde r_j$, $\tilde a^\pm$, $\tilde a^k_\pm$ and $\tilde\phi^\pm$, respectively.

Step 1. First, by (6.66), (6.70), (6.78) and (6.81), we see that

$$\begin{aligned}\tilde r_4 = &-i\varepsilon^{-n/4}\Big(\tilde a^-\tilde\phi^-_t e^{i\tilde\phi^-/\varepsilon}+\tilde a^+\tilde\phi^+_t e^{i\tilde\phi^+/\varepsilon}\Big)\\ &+\varepsilon^{1-n/4}\Big\{\Big[\widetilde B(t_0,\tilde x_0)-\tilde a^-_t - i\tilde a^1_-\tilde\phi^-_t-\varepsilon(\partial_t\tilde a^1_-+i\tilde a^2_-\tilde\phi^-_t)+O(\varepsilon^2)\Big]e^{i\tilde\phi^-/\varepsilon}\\ &\quad-\Big[\tilde a^+_t+i\tilde a^1_+\tilde\phi^+_t+\varepsilon(\partial_t\tilde a^1_+ + i\tilde a^2_+\tilde\phi^+_t)+O(\varepsilon^2)\Big]e^{i\tilde\phi^+/\varepsilon}\Big\}.\end{aligned}\tag{10.1}$$

By Proposition 3 and noting the first equation in (6.80), we may suitably choose $a^+_{i_1\cdots i_k}\in\mathbb C$ and $b^+_{i_1\cdots i_k}\in\mathbb C$, $i_1,\cdots,i_k=1,2,\cdots,n$ and $k=1,\cdots,7$, such that

$$\tilde a^+ + \tilde a^- \equiv \tilde a^+(t,0,\tilde x')+\tilde a^-(t,0,\tilde x') = O(|t-t_0|^8+|\tilde x'-\tilde x'_0|^8),\tag{10.2}$$

and

$$\tilde\phi^+ - \tilde\phi^- \equiv \tilde\phi^+(t,0,\tilde x')-\tilde\phi^-(t,0,\tilde x') = O(|t-t_0|^8+|\tilde x'-\tilde x'_0|^8)\tag{10.3}$$

as $(t,\tilde x')\to(t_0,\tilde x'_0)$. Therefore,

$$\begin{aligned}e^{i\tilde\phi^-/\varepsilon}-e^{i\tilde\phi^+/\varepsilon} &= \frac{i(\tilde\phi^- - \tilde\phi^+)}{\varepsilon}\int_0^1 e^{i\tilde\phi^- + is(\tilde\phi^- - \tilde\phi^+)/\varepsilon}ds\\ &= \frac{i}{\varepsilon}\int_0^1 e^{i\tilde\phi^- + is(\tilde\phi^- - \tilde\phi^+)/\varepsilon}ds\, O(|t-t_0|^8+|\tilde x'-\tilde x'_0|^8).\end{aligned}\tag{10.4}$$

Hence, by (10.2)–(10.4), for any $s\in\mathbb N$, we have

$$\begin{aligned}&\tilde a^-(\tilde\phi^-_t)^s e^{i\tilde\phi^-/\varepsilon}+\tilde a^+(\tilde\phi^+_t)^s e^{i\tilde\phi^+/\varepsilon}\\ &=\tilde a^-(\tilde\phi^-_t)^s(e^{i\tilde\phi^-/\varepsilon}-e^{i\tilde\phi^+/\varepsilon})+(\tilde a^-(\tilde\phi^-_t)^s+\tilde a^+(\tilde\phi^+_t)^s)e^{i\tilde\phi^+/\varepsilon}\\ &=\frac{i}{\varepsilon}\int_0^1 e^{i\tilde\phi^- + is(\tilde\phi^- - \tilde\phi^+)/\varepsilon}ds\, O(|t-t_0|^8+|\tilde x'-\tilde x'_0|^8)\\ &\quad+e^{i\tilde\phi^+/\varepsilon}O(|t-t_0|^8+|\tilde x'-\tilde x'_0|^8).\end{aligned}\tag{10.5}$$

According to Remarks 9 and 13, the supports of $\hat z_\varepsilon|_{(0,T^*)\times\partial\Omega_2}$ and $\hat y_\varepsilon|_{(0,T^*)\times\partial\Omega_2}$ can be chosen to be very small. Now, for any $a^k_\pm(t,x)\in$

$C^7([0,T]\times\overline{\Omega_2};\ \mathbb{C})$ $(k=1,2)$, by (10.1) and (10.5), and noting Remark 11, we can use the change of variable $x\mapsto\tilde x$ to get

$$|r_4|^2_{L^2((0,T^*)\times\gamma)}\le C\int_0^{T^*}\int_{\mathbb{R}^{n-1}}|\tilde r_4|^2dtd\tilde x'=O(\varepsilon^2). \tag{10.6}$$

Step 2. From (10.1), we get

$$\begin{aligned}\partial_t\tilde r_4=&\ \varepsilon^{-1-n/4}\Big(\tilde a^-(\tilde\phi_t^-)^2e^{i\tilde\phi^-/\varepsilon}+\tilde a^+(\tilde\phi_t^+)^2e^{i\tilde\phi^+/\varepsilon}\Big)\\&+i\varepsilon^{-n/4}\Big[\Big((\widetilde B(t_0,\tilde x_0)-\tilde a_t^--i\tilde a_-^1\tilde\phi_t^-)\tilde\phi_t^--(\tilde a^-\tilde\phi_t^-)_t\Big)e^{i\tilde\phi^-/\varepsilon}\\&\qquad-\Big((\tilde a_t^++i\tilde a_+^1\tilde\phi_t^+)\tilde\phi_t^++(\tilde a^+\tilde\phi_t^+)_t\Big)e^{i\tilde\phi^+/\varepsilon}\Big]\\&-\varepsilon^{1-n/4}\Big[\Big((\tilde a_t^-+i\tilde a_-^1\tilde\phi_t^-)_t+i(\partial_t\tilde a_-^1+i\tilde a_-^2\tilde\phi_t^-)\tilde\phi_t^-+O(\varepsilon)\Big)e^{i\tilde\phi^-/\varepsilon}\\&\qquad+\Big((\tilde a_t^++i\tilde a_+^1\tilde\phi_t^+)_t+i(\partial_t\tilde a_+^1+i\tilde a_+^2\tilde\phi_t^+)\tilde\phi_t^++O(\varepsilon)\Big)e^{i\tilde\phi^+/\varepsilon}\Big],\end{aligned} \tag{10.7}$$

and

$$\begin{aligned}\partial_{\tilde x_j}\tilde r_4=&\ \varepsilon^{-1-n/4}\Big(\tilde a^-\tilde\phi_t^-\tilde\phi_{\tilde x_j}^-e^{i\tilde\phi^-/\varepsilon}+\tilde a^+\tilde\phi_t^+\tilde\phi_{\tilde x_j}^+e^{i\tilde\phi^+/\varepsilon}\Big)\\&+i\varepsilon^{-n/4}\Big[\Big((\widetilde B(t_0,\tilde x_0)-\tilde a_t^--i\tilde a_-^1\tilde\phi_t^-)\tilde\phi_{\tilde x_j}^--(\tilde a^-\tilde\phi_t^-)_{\tilde x_j}\Big)e^{i\tilde\phi^-/\varepsilon}\\&\qquad-\Big((\tilde a_t^++i\tilde a_+^1\tilde\phi_t^+)\tilde\phi_{\tilde x_j}^++(\tilde a^+\tilde\phi_t^+)_{\tilde x_j}\Big)e^{i\tilde\phi^+/\varepsilon}\Big]\\&+e^{i\tilde\phi^-/\varepsilon}O(\varepsilon^{1-n/4})+e^{i\tilde\phi^+/\varepsilon}O(\varepsilon^{1-n/4}).\end{aligned} \tag{10.8}$$

Now, we choose $a_\pm^k(t,x)$ to the following form $(k=1,2)$:

$$a_\pm^k(t,x)=a_\pm^k+\sum_{s=1}^7\sum_{i_1,\cdots,i_s=1}^n a_{i_1\cdots i_s}^{\pm k}(x_{i_1}-x_{i_1}^\pm(t))\cdots(x_{i_s}-x_{i_s}^\pm(t)), \tag{10.9}$$

where a_-^k and $a_{i_1\cdots i_s}^{-k}$ are constants given arbitrarily, while the constants a_+^k and $a_{i_1\cdots i_s}^{+k}$ are determined below. Then

$$\begin{aligned}&\tilde a_\pm^k(t,\tilde x)\\&=a_\pm^k+\sum_{s=1}^7\sum_{i_1,\cdots,i_s=1}^n a_{i_1\cdots i_s}^{\pm k}(x_{i_1}(\tilde x)-x_{i_1}^\pm(t))\cdots(x_{i_s}(\tilde x)-x_{i_s}^\pm(t)).\end{aligned} \tag{10.10}$$

Put

$$\begin{aligned}\Psi_1\equiv\Psi_1(t,\tilde x')&\stackrel{\triangle}{=}\widetilde B(t_0,\tilde x_0)-\tilde a_t^--i\tilde a_-^1\tilde\phi_t^-,\\ \Phi_1\equiv\Phi_1(t,\tilde x')&\stackrel{\triangle}{=}\tilde a_t^++i\tilde a_+^1\tilde\phi_t^+.\end{aligned} \tag{10.11}$$

Similar to Proposition 3, and noting that $\partial_t\tilde\phi^\pm(t_0,\tilde x_0)\neq 0$ (*see* (6.38) in Proposition 2), it is easy to show that there exist constants a_+^1 and $a_{i_1\cdots i_s}^{+1}$ such that

$$\Psi_1-\Phi_1=O(|t-t_0|^8+|\tilde x'-\tilde x_0'|^8),\qquad\text{as }(t,\tilde x')\to(t_0,\tilde x_0'). \tag{10.12}$$

Hence, by (10.2)–(10.4) and (10.12), we get

$$
\begin{aligned}
&\Big((\widetilde{B}(t_0,\tilde{x}_0)-\tilde{a}_t^- - i\tilde{a}_-^1\tilde{\phi}_t^-)\tilde{\phi}_t^- - (\tilde{a}^-\tilde{\phi}_t^-)_t\Big)e^{i\tilde{\phi}^-/\varepsilon}\\
&\quad -\Big((\tilde{a}_t^+ + i\tilde{a}_+^1\tilde{\phi}_t^+)\tilde{\phi}_t^+ + (\tilde{a}^+\tilde{\phi}_t^+)_t\Big)e^{i\tilde{\phi}^+/\varepsilon}\\
&=(\Psi_1\tilde{\phi}_t^- - (\tilde{a}^-\tilde{\phi}_t^-)_t)e^{i\tilde{\phi}^-/\varepsilon} - (\Phi_1\tilde{\phi}_t^+ + (\tilde{a}^+\tilde{\phi}_t^+)_t)e^{i\tilde{\phi}^+/\varepsilon}\\
&=(\Psi_1\tilde{\phi}_t^- - (\tilde{a}^-\tilde{\phi}_t^-)_t)\Big(e^{i\tilde{\phi}^-/\varepsilon} - e^{i\tilde{\phi}^+/\varepsilon}\Big)\\
&\quad +\Big((\Psi_1-\Phi_1)\tilde{\phi}_t^+ - (\tilde{a}^-\tilde{\phi}_t^-)_t - (\tilde{a}^+\tilde{\phi}_t^+)_t\Big)e^{i\tilde{\phi}^+/\varepsilon}\\
&=\frac{i}{\varepsilon}\int_0^1 e^{i\tilde{\phi}^- + is(\tilde{\phi}^- - \tilde{\phi}^+)/\varepsilon}ds\,O(|t-t_0|^8 + |\tilde{x}'-\tilde{x}_0'|^8)\\
&\quad +e^{i\tilde{\phi}^+/\varepsilon}O(|t-t_0|^7 + |\tilde{x}'-\tilde{x}_0'|^7).
\end{aligned}
\tag{10.13}
$$

Therefore, for any $a_\pm^2(t,x) \in C^7([0,T]\times\overline{\Omega_2};\ \mathbb{C})$, by (10.7), (10.5) and (10.13), and noting Remark 11, one finds

$$
|\partial_t r_4|^2_{L^2((0,T^*)\times\gamma)} \le C\int_0^{T^*}\int_{\mathbb{R}^{n-1}} |\partial_t\tilde{r}_4|^2 dt d\tilde{x}' = O(\varepsilon^2). \tag{10.14}
$$

Similarly, one gets

$$
|r_4|^2_{L^2(0,T^*;H^1(\gamma))} \le C|\tilde{r}_4|^2_{L^2(0,T^*;H^1(\mathbb{R}^{n-1}))} \le C. \tag{10.15}
$$

Interpolating (10.6) and (10.15), we find

$$
|r_4|_{L^2(0,T^*;H^{1/2}(\gamma))} = O(\varepsilon^{1/2}). \tag{10.16}
$$

Step 3. From (10.7), we see that

$$
\begin{aligned}
\partial_{tt}\tilde{r}_4 &= i\varepsilon^{-2-n/4}\Big(\tilde{a}^-(\tilde{\phi}_t^-)^3 e^{i\tilde{\phi}^-/\varepsilon} + \tilde{a}^+(\tilde{\phi}_t^+)^3 e^{i\tilde{\phi}^+/\varepsilon}\Big)\\
&\quad +\varepsilon^{-1-n/4}\Big\{\Big[\Big(\tilde{a}^-(\tilde{\phi}_t^-)^2\Big)_t\\
&\qquad -\Big((\widetilde{B}(t_0,\tilde{x}_0)-\tilde{a}_t^- - i\tilde{a}_-^1\tilde{\phi}_t^-)\tilde{\phi}_t^- - (\tilde{a}^-\tilde{\phi}_t^-)_t\Big)\tilde{\phi}_t^-\Big]e^{i\tilde{\phi}^-/\varepsilon}\\
&\qquad +\Big[\Big(\tilde{a}^+(\tilde{\phi}_t^+)^2\Big)_t + \Big((\tilde{a}_t^+ + i\tilde{a}_+^1\tilde{\phi}_t^+)\tilde{\phi}_t^+ + (\tilde{a}^+\tilde{\phi}_t^+)_t\Big)\tilde{\phi}_t^+\Big]e^{i\tilde{\phi}^+/\varepsilon}\Big\}\\
&\quad +i\varepsilon^{-n/4}\Big\{\Big[\Big((\widetilde{B}(t_0,\tilde{x}_0)-\tilde{a}_t^- - i\tilde{a}_-^1\tilde{\phi}_t^-)\tilde{\phi}_t^- - (\tilde{a}^-\tilde{\phi}_t^-)_t\Big)_t\\
&\qquad -\Big((\tilde{a}_t^- + i\tilde{a}_-^1\tilde{\phi}_t^-)_t + i(\partial_t\tilde{a}_-^1 + i\tilde{a}_-^2\tilde{\phi}_t^-)\tilde{\phi}_t^-\Big)\tilde{\phi}_t^-\Big]e^{i\tilde{\phi}^-/\varepsilon}\\
&\qquad -\Big[\Big((\tilde{a}_t^+ + i\tilde{a}_+^1\tilde{\phi}_t^+)\tilde{\phi}_t^+ + (\tilde{a}^+\tilde{\phi}_t^+)_t\Big)_t\\
&\qquad +\Big((\tilde{a}_t^+ + i\tilde{a}_+^1\tilde{\phi}_t^+)_t + i(\partial_t\tilde{a}_+^1 + i\tilde{a}_+^2\tilde{\phi}_t^+)\tilde{\phi}_t^+\Big)\tilde{\phi}_t^+\Big]e^{i\tilde{\phi}^+/\varepsilon}\Big\}\\
&\quad +e^{i\tilde{\phi}^-/\varepsilon}O(\varepsilon^{1-n/4}) + e^{i\tilde{\phi}^+/\varepsilon}O(\varepsilon^{1-n/4}).
\end{aligned}
\tag{10.17}
$$

Put

$$
\begin{aligned}
&\Psi_2 \equiv \Psi_2(t,\tilde{x}') \stackrel{\triangle}{=} (\tilde{a}_t^- + i\tilde{a}_-^1\tilde{\phi}_t^-)_t + i(\partial_t\tilde{a}_-^1 + i\tilde{a}_-^2\tilde{\phi}_t^-)\tilde{\phi}_t^-,\\
&\Phi_2 \equiv \Phi_2(t,\tilde{x}') \stackrel{\triangle}{=} (\tilde{a}_t^+ + i\tilde{a}_+^1\tilde{\phi}_t^+)_t + i(\partial_t\tilde{a}_+^1 + i\tilde{a}_+^2\tilde{\phi}_t^+)\tilde{\phi}_t^+.
\end{aligned}
\tag{10.18}
$$

Similar to (10.12), one can find constants a_+^2 and $a_{i_1\cdots i_s}^{+2}$ such that

$$
\Psi_2 + \Phi_2 = O(|t-t_0|^8 + |\tilde{x}' - \tilde{x}_0'|^8), \qquad \text{as } (t,\tilde{x}') \to (t_0,\tilde{x}_0'). \tag{10.19}
$$

Hence, by (10.11), (10.12), (10.18) and (10.19), noting (10.2)–(10.4), we get

$$
\begin{aligned}
&\Big[\Big((\widetilde{B}(t_0,\tilde{x}_0) - \tilde{a}_t^- - i\tilde{a}_-^1\tilde{\phi}_t^-)\tilde{\phi}_t^- - (\tilde{a}^-\tilde{\phi}_t^-)_t\Big)_t\\
&\quad -\Big((\tilde{a}_t^- + i\tilde{a}_-^1\tilde{\phi}_t^-)_t + i(\partial_t\tilde{a}_-^1 + i\tilde{a}_-^2\tilde{\phi}_t^-)\tilde{\phi}_t^-\Big)\tilde{\phi}_t^-\Big]e^{i\tilde{\phi}^-/\varepsilon}\\
&\quad -\Big[\Big((\tilde{a}_t^+ + i\tilde{a}_+^1\tilde{\phi}_t^+)\tilde{\phi}_t^+ + (\tilde{a}^+\tilde{\phi}_t^+)_t\Big)_t\\
&\quad +\Big((\tilde{a}_t^+ + i\tilde{a}_+^1\tilde{\phi}_t^+)_t + i(\partial_t\tilde{a}_+^1 + i\tilde{a}_+^2\tilde{\phi}_t^+)\tilde{\phi}_t^+\Big)\tilde{\phi}_t^+\Big]e^{i\tilde{\phi}^+/\varepsilon}\\
&= \Big[\Big(\Psi_1\tilde{\phi}_t^- - (\tilde{a}^-\tilde{\phi}_t^-)_t\Big)_t - \Psi_2\tilde{\phi}_t^-\Big]e^{i\tilde{\phi}^-/\varepsilon}\\
&\quad -\Big[\Big(\Phi_1\tilde{\phi}_t^+ + (\tilde{a}^+\tilde{\phi}_t^+)_t\Big)_t + \Phi_2\tilde{\phi}_t^+\Big]e^{i\tilde{\phi}^+/\varepsilon}\\
&= \Big[\Big(\Psi_1\tilde{\phi}_t^- - (\tilde{a}^-\tilde{\phi}_t^-)_t\Big)_t - \Psi_2\tilde{\phi}_t^-\Big]\Big(e^{i\tilde{\phi}^-/\varepsilon} - e^{i\tilde{\phi}^+/\varepsilon}\Big)\\
&\quad +\Big[\Big(\Psi_1\tilde{\phi}_t^- - (\tilde{a}^-\tilde{\phi}_t^-)_t\Big)_t - \Big(\Phi_1\tilde{\phi}_t^+ + (\tilde{a}^+\tilde{\phi}_t^+)_t\Big)_t\\
&\quad -\Psi_2\tilde{\phi}_t^- - \Phi_2\tilde{\phi}_t^+\Big]e^{i\tilde{\phi}^+/\varepsilon}\\
&= \frac{i}{\varepsilon}\int_0^1 e^{i\tilde{\phi}^- + is(\tilde{\phi}^- - \tilde{\phi}^+)/\varepsilon}ds\, O(|t-t_0|^8 + |\tilde{x}' - \tilde{x}_0'|^8)\\
&\quad +e^{i\tilde{\phi}^+/\varepsilon}O(|t-t_0|^6 + |\tilde{x}' - \tilde{x}_0'|^6).
\end{aligned}
\tag{10.20}
$$

Similarly, by (10.11) and (10.12), and noting (10.2)–(10.4), we have

$$\begin{aligned}
&\Big[\Big(\tilde{a}^-(\tilde{\phi}_t^-)^2\Big)_t-\Big((\widetilde{B}(t_0,\tilde{x}_0)-\tilde{a}_t^--i\tilde{a}_-^1\tilde{\phi}_t^-)\tilde{\phi}_t^--(\tilde{a}^-\tilde{\phi}_t^-)_t\Big)\tilde{\phi}_t^-\Big]e^{i\tilde{\phi}^-/\varepsilon}\\
&\quad+\Big[\Big(\tilde{a}^+(\tilde{\phi}_t^+)^2\Big)_t+\Big((\tilde{a}_t^++i\tilde{a}_+^1\tilde{\phi}_t^+)\tilde{\phi}_t^++(\tilde{a}^+\tilde{\phi}_t^+)_t\Big)\tilde{\phi}_t^+\Big]e^{i\tilde{\phi}^+/\varepsilon}\\
&=\Big[\Big(\tilde{a}^-(\tilde{\phi}_t^-)^2\Big)_t-\Big(\Psi_1\tilde{\phi}_t^--(\tilde{a}^-\tilde{\phi}_t^-)_t\Big)\tilde{\phi}_t^-\Big]e^{i\tilde{\phi}^-/\varepsilon}\\
&\quad+\Big[\Big(\tilde{a}^+(\tilde{\phi}_t^+)^2\Big)_t+\Big(\Phi_1\tilde{\phi}_t^++(\tilde{a}^+\tilde{\phi}_t^+)_t\Big)\tilde{\phi}_t^+\Big]e^{i\tilde{\phi}^+/\varepsilon}\\
&=\Big[\Big(\tilde{a}^-(\tilde{\phi}_t^-)^2\Big)_t-\Big(\Psi_1\tilde{\phi}_t^--(\tilde{a}^-\tilde{\phi}_t^-)_t\Big)\tilde{\phi}_t^-\Big]\Big(e^{i\tilde{\phi}^-/\varepsilon}-e^{i\tilde{\phi}^+/\varepsilon}\Big)\\
&\quad+\Big[\Big(\tilde{a}^+(\tilde{\phi}_t^+)^2+\tilde{a}^-(\tilde{\phi}_t^-)^2\Big)_t+(\tilde{a}^-\tilde{\phi}_t^-)_t\tilde{\phi}_t^-+(\tilde{a}^+\tilde{\phi}_t^+)_t\tilde{\phi}_t^+\\
&\quad+\Phi_1(\tilde{\phi}_t^+)^2-\Psi_1(\tilde{\phi}_t^-)^2\Big]e^{i\tilde{\phi}^+/\varepsilon}\\
&=\frac{i}{\varepsilon}\int_0^1 e^{i\tilde{\phi}^-+is(\tilde{\phi}^--\tilde{\phi}^+)/\varepsilon}ds\,O(|t-t_0|^8+|\tilde{x}'-\tilde{x}_0'|^8)\\
&\quad+e^{i\tilde{\phi}^+/\varepsilon}O(|t-t_0|^7+|\tilde{x}'-\tilde{x}_0'|^7).
\end{aligned}\tag{10.21}$$

Therefore, by (10.5), (10.17), (10.20) and (10.21), and noting Remark 11, one finds

$$|\partial_{tt}r_4|^2_{L^2((0,T^*)\times\gamma)}\le C\int_0^{T^*}\int_{\mathbb{R}^{n-1}}|\partial_{tt}\tilde{r}_4|^2dtd\tilde{x}'=O(\varepsilon^2).\tag{10.22}$$

Similarly, one has

$$|\partial_t r_4|^2_{L^2(0,T^*;H^1(\gamma))}\le C|\partial_t\tilde{r}_4|^2_{L^2(0,T^*;H^1(\mathbb{R}^{n-1}))}\le C.\tag{10.23}$$

Interpolating (10.14) and (10.23), we find

$$|\partial_t r_4|_{L^2(0,T^*;H^{1/2}(\gamma))}=O(\varepsilon^{1/2}).\tag{10.24}$$

Step 4. From (10.17), we obtain

$$
\begin{aligned}
&\partial_{\tilde{x}_j}\partial_{tt}\tilde{r}_4\\
&=-\varepsilon^{-3-n/4}\Big(\tilde{a}^-(\tilde{\phi}_t^-)^3\tilde{\phi}_{\tilde{x}_j}^-e^{i\tilde{\phi}^-/\varepsilon}+\tilde{a}^+(\tilde{\phi}_t^+)^3\tilde{\phi}_{\tilde{x}_j}^+e^{i\tilde{\phi}^+/\varepsilon}\Big)\\
&\quad+i\varepsilon^{-2-n/4}\Big\{\Big[\Big(\tilde{a}^-(\tilde{\phi}_t^-)^3\Big)_{\tilde{x}_j}+\Big((\tilde{a}^-(\tilde{\phi}_t^-)^2)_t-((\widetilde{B}(t_0,\tilde{x}_0)-\tilde{a}_t^-\\
&\qquad-i\tilde{a}_-^1\tilde{\phi}_t^-)\tilde{\phi}_t^--(\tilde{a}^-\tilde{\phi}_t^-)_t)\tilde{\phi}_t^-\Big)\tilde{\phi}_{\tilde{x}_j}^-\Big]e^{i\tilde{\phi}^-/\varepsilon}+\Big[\Big(\tilde{a}^+(\tilde{\phi}_t^+)^3\Big)_{\tilde{x}_j}\\
&\qquad+\Big((\tilde{a}^+(\tilde{\phi}_t^+)^2)_t+((\tilde{a}_t^++i\tilde{a}_+^1\tilde{\phi}_t^+)\tilde{\phi}_t^++(\tilde{a}^+\tilde{\phi}_t^+)_t)\tilde{\phi}_t^+\Big)\tilde{\phi}_{\tilde{x}_j}^+\Big]e^{i\tilde{\phi}^+/\varepsilon}\Big\}\\
&\quad+\varepsilon^{-1-n/4}\Big\{\Big[\Big((\tilde{a}^-(\tilde{\phi}_t^-)^2)_t-((\widetilde{B}(t_0,\tilde{x}_0)-\tilde{a}_t^--i\tilde{a}_-^1\tilde{\phi}_t^-)\tilde{\phi}_t^-\\
&\qquad-(\tilde{a}^-\tilde{\phi}_t^-)_t)\tilde{\phi}_t^-\Big)_{\tilde{x}_j}-\Big(((\widetilde{B}(t_0,\tilde{x}_0)-\tilde{a}_t^--i\tilde{a}_-^1\tilde{\phi}_t^-)\tilde{\phi}_t^--(\tilde{a}^-\tilde{\phi}_t^-)_t)_t\\
&\qquad-((\tilde{a}_t^-+i\tilde{a}_-^1\tilde{\phi}_t^-)_t+i(\partial_t\tilde{a}_-^1+i\tilde{a}_-^2\tilde{\phi}_t^-)\tilde{\phi}_t^-)\tilde{\phi}_t^-\Big)\tilde{\phi}_{\tilde{x}_j}^-\Big]e^{i\tilde{\phi}^-/\varepsilon}\\
&\qquad+\Big[\Big((\tilde{a}^+(\tilde{\phi}_t^+)^2)_t+((\tilde{a}_t^++i\tilde{a}_+^1\tilde{\phi}_t^+)\tilde{\phi}_t^++(\tilde{a}^+\tilde{\phi}_t^+)_t)\tilde{\phi}_t^+\Big)_{\tilde{x}_j}\\
&\qquad+\Big(((\tilde{a}_t^++i\tilde{a}_+^1\tilde{\phi}_t^+)\tilde{\phi}_t^++(\tilde{a}^+\tilde{\phi}_t^+)_t)_t\\
&\qquad+((\tilde{a}_t^++i\tilde{a}_+^1\tilde{\phi}_t^+)_t+i(\partial_t\tilde{a}_+^1+i\tilde{a}_+^2\tilde{\phi}_t^+)\tilde{\phi}_t^+)\tilde{\phi}_t^+\Big)\tilde{\phi}_{\tilde{x}_j}^+\Big]e^{i\tilde{\phi}^+/\varepsilon}\Big\}\\
&\quad+e^{i\tilde{\phi}^-/\varepsilon}O(\varepsilon^{-n/4})+e^{i\tilde{\phi}^+/\varepsilon}O(\varepsilon^{-n/4}).
\end{aligned}
\tag{10.25}
$$

By (10.2)–(10.4), for $j=2,\cdots,n$, we have

$$
\begin{aligned}
&\tilde{a}^-(\tilde{\phi}_t^-)^3\tilde{\phi}_{\tilde{x}_j}^-e^{i\tilde{\phi}^-/\varepsilon}+\tilde{a}^+(\tilde{\phi}_t^+)^3\tilde{\phi}_{\tilde{x}_j}^+e^{i\tilde{\phi}^+/\varepsilon}\\
&=\frac{i}{\varepsilon}\int_0^1 e^{i\tilde{\phi}^-+is(\tilde{\phi}^--\tilde{\phi}^+)/\varepsilon}ds\,O(|t-t_0|^8+|\tilde{x}'-\tilde{x}_0'|^8)\\
&\quad+e^{i\tilde{\phi}^+/\varepsilon}O(|t-t_0|^8+|\tilde{x}'-\tilde{x}_0'|^8).
\end{aligned}
\tag{10.26}
$$

Further, by (10.11) and (10.12), we see that, for $j=2,\cdots,n$,

$$
\begin{aligned}
&\Big[\Big(\tilde{a}^-(\tilde{\phi}_t^-)^3\Big)_{\tilde{x}_j}+\Big((\tilde{a}^-(\tilde{\phi}_t^-)^2)_t-((\widetilde{B}(t_0,\tilde{x}_0)-\tilde{a}_t^--i\tilde{a}_-^1\tilde{\phi}_t^-)\tilde{\phi}_t^-\\
&\quad-(\tilde{a}^-\tilde{\phi}_t^-)_t)\tilde{\phi}_t^-\Big)\tilde{\phi}_{\tilde{x}_j}^-\Big]e^{i\tilde{\phi}^-/\varepsilon}+\Big[\Big(\tilde{a}^+(\tilde{\phi}_t^+)^3\Big)_{\tilde{x}_j}+\Big((\tilde{a}^+(\tilde{\phi}_t^+)^2)_t\\
&\quad+((\tilde{a}_t^++i\tilde{a}_+^1\tilde{\phi}_t^+)\tilde{\phi}_t^++(\tilde{a}^+\tilde{\phi}_t^+)_t)\tilde{\phi}_t^+\Big)\tilde{\phi}_{\tilde{x}_j}^+\Big]e^{i\tilde{\phi}^+/\varepsilon}\\
&=\Big[\Big(\tilde{a}^-(\tilde{\phi}_t^-)^3\Big)_{\tilde{x}_j}+\Big((\tilde{a}^-(\tilde{\phi}_t^-)^2)_t-(\Psi_1\tilde{\phi}_t^--(\tilde{a}^-\tilde{\phi}_t^-)_t)\tilde{\phi}_t^-\Big)\tilde{\phi}_{\tilde{x}_j}^-\Big]e^{i\tilde{\phi}^-/\varepsilon}\\
&\quad+\Big[\Big(\tilde{a}^+(\tilde{\phi}_t^+)^3\Big)_{\tilde{x}_j}+\Big((\tilde{a}^+(\tilde{\phi}_t^+)^2)_t+(\Phi_1\tilde{\phi}_t^++(\tilde{a}^+\tilde{\phi}_t^+)_t)\tilde{\phi}_t^+\Big)\tilde{\phi}_{\tilde{x}_j}^+\Big]e^{i\tilde{\phi}^+/\varepsilon}\\
&=\frac{i}{\varepsilon}\int_0^1 e^{i\tilde{\phi}^-+is(\tilde{\phi}^--\tilde{\phi}^+)/\varepsilon}ds\,O(|t-t_0|^8+|\tilde{x}'-\tilde{x}_0'|^8)\\
&\quad+e^{i\tilde{\phi}^+/\varepsilon}O(|t-t_0|^7+|\tilde{x}'-\tilde{x}_0'|^7).
\end{aligned}
\tag{10.27}
$$

Similarly, by (10.11), (10.12), (10.18) and (10.19), noting (10.2)–(10.4), for $j = 2, \cdots, n$, we get

$$
\begin{aligned}
&\Big[\Big((\tilde a^-(\tilde\phi_t^-)^2)_t - ((\widetilde B(t_0,\tilde x_0) - \tilde a_t^- - i\tilde a_-^1\tilde\phi_t^-)\tilde\phi_t^- - (\tilde a^-\tilde\phi_t^-)_t)\tilde\phi_t^-\Big)_{\tilde x_j} \\
&\quad -\Big(((\widetilde B(t_0,\tilde x_0) - \tilde a_t^- - i\tilde a_-^1\tilde\phi_t^-)\tilde\phi_t^- - (\tilde a^-\tilde\phi_t^-)_t)_t \\
&\quad -((\tilde a_t^- + i\tilde a_-^1\tilde\phi_t^-)_t + i(\partial_t\tilde a_-^1 + i\tilde a_-^2\tilde\phi_t^-)\tilde\phi_t^-)\tilde\phi_t^-\Big)\tilde\phi_{\tilde x_j}^-\Big]e^{i\tilde\phi^-/\varepsilon} \\
&\quad +\Big[\Big((\tilde a^+(\tilde\phi_t^+)^2)_t + ((\tilde a_t^+ + i\tilde a_+^1\tilde\phi_t^+)\tilde\phi_t^+ + (\tilde a^+\tilde\phi_t^+)_t)\tilde\phi_t^+\Big)_{\tilde x_j} \\
&\quad +\Big(((\tilde a_t^+ + i\tilde a_+^1\tilde\phi_t^+)\tilde\phi_t^+ + (\tilde a^+\tilde\phi_t^+)_t)_t \\
&\quad +((\tilde a_t^+ + i\tilde a_+^1\tilde\phi_t^+)_t + i(\partial_t\tilde a_+^1 + i\tilde a_+^2\tilde\phi_t^+)\tilde\phi_t^+)\tilde\phi_t^+\Big)\tilde\phi_{\tilde x_j}^+\Big]e^{i\tilde\phi^+/\varepsilon} \\
&= \Big[\Big((\tilde a^-(\tilde\phi_t^-)^2)_t - (\Psi_1\tilde\phi_t^- - (\tilde a^-\tilde\phi_t^-)_t)\tilde\phi_t^-\Big)_{\tilde x_j} \\
&\quad -\Big((\Psi_1\tilde\phi_t^- - (\tilde a^-\tilde\phi_t^-)_t)_t - \Psi_2\tilde\phi_t^-\Big)\tilde\phi_{\tilde x_j}^-\Big]e^{i\tilde\phi^-/\varepsilon} \\
&\quad +\Big[\Big((\tilde a^+(\tilde\phi_t^+)^2)_t + (\Phi_1\tilde\phi_t^+ + (\tilde a^+\tilde\phi_t^+)_t)\tilde\phi_t^+\Big)_{\tilde x_j} \\
&\quad +\Big((\Phi_1\tilde\phi_t^+ + (\tilde a^+\tilde\phi_t^+)_t)_t + \Phi_2\tilde\phi_t^+\Big)\tilde\phi_{\tilde x_j}^+\Big]e^{i\tilde\phi^+/\varepsilon} \\
&= \frac{i}{\varepsilon}\int_0^1 e^{i\tilde\phi^- + is(\tilde\phi^- - \tilde\phi^+)/\varepsilon}ds\, O(|t-t_0|^8 + |\tilde x' - \tilde x_0'|^8) \\
&\quad + e^{i\tilde\phi^+/\varepsilon}O(|t-t_0|^6 + |\tilde x' - \tilde x_0'|^6).
\end{aligned}
\tag{10.28}
$$

Therefore, by (10.25), (10.5), (10.26)–(10.28), and noting Remark 11, one finds

$$|\partial_{tt}r_4|^2_{L^2(0,T^*;H^1(\gamma))} \le C|\partial_{tt}\tilde r_4|^2_{L^2(0,T^*;H^1(\mathbb{R}^{n-1}))} \le C. \tag{10.29}$$

Interpolating (10.22) and (10.23), we find

$$|\partial_{tt}r_4|_{L^2(0,T^*;H^{1/2}(\gamma))} = O(\varepsilon^{1/2}). \tag{10.30}$$

Finally, combining (10.6), (10.16), (10.24) and (10.30), we arrive at the desired estimate on r_4 in (6.83). This completes the proof of Lemma 8.

□

11. Appendix C: Proof of Lemma 9

This appendix is devoted to prove Lemma 9. We need to correct the approximate solutions $\{(\hat y_\varepsilon, \hat z_\varepsilon)\}$ of system (1.1), given in Lemma 8, to become a family of exact solutions of system (1.1). For this, let

$$y_\varepsilon = \hat y_\varepsilon + v_\varepsilon^1 + v_\varepsilon^2, \qquad z_\varepsilon = \hat z_\varepsilon + w_\varepsilon^1 + w_\varepsilon^2, \tag{11.1}$$

where $(v_\varepsilon^i, w_\varepsilon^i)$ $(i = 1, 2)$ solve respectively (recall Lemma 8 for the definition of r_j, $j = 1, 2, 3, 4, 5$)

$$\begin{cases} \partial_t v_\varepsilon^1 - \Delta v_\varepsilon^1 = -r_1 & \text{in } (0, T^*) \times \Omega_1, \\ \Box\, w_\varepsilon^1 = -r_2 & \text{in } (0, T^*) \times \Omega_2, \\ v_\varepsilon^1 = 0 & \text{on } (0, T^*) \times \Gamma_1, \\ w_\varepsilon^1 = -r_3 & \text{on } (0, T^*) \times \Gamma_2, \\ v_\varepsilon^1 = \partial_t w_\varepsilon^1, \quad \dfrac{\partial v_\varepsilon^1}{\partial \nu_1} = -\dfrac{\partial w_\varepsilon^1}{\partial \nu_2} & \text{on } (0, T^*) \times \gamma, \\ v_\varepsilon^1(0) = 0 & \text{in } \Omega_1, \\ w_\varepsilon^1(0) = \partial_t w_\varepsilon^1(0) = 0 & \text{in } \Omega_2 \end{cases} \tag{11.2}$$

and

$$\begin{cases} \partial_t v_\varepsilon^2 - \Delta v_\varepsilon^2 = 0 & \text{in } (0, T^*) \times \Omega_1, \\ \Box\, w_\varepsilon^2 = 0 & \text{in } (0, T^*) \times \Omega_2, \\ v_\varepsilon^2 = 0 & \text{on } (0, T^*) \times \Gamma_1, \\ w_\varepsilon^2 = 0 & \text{on } (0, T^*) \times \Gamma_2, \\ v_\varepsilon^2 = \partial_t w_\varepsilon^2 - r_4, \quad \dfrac{\partial v_\varepsilon^2}{\partial \nu_1} = -\dfrac{\partial w_\varepsilon^2}{\partial \nu_2} - r_5 & \text{on } (0, T^*) \times \gamma, \\ v_\varepsilon^2(0) = 0 & \text{in } \Omega_1, \\ w_\varepsilon^2(0) = \partial_t w_\varepsilon^2(0) = 0 & \text{in } \Omega_2. \end{cases} \tag{11.3}$$

Similar to the proof of [32, Corollary 11], it is easy to show that

$$|\nabla v_\varepsilon^1|^2_{(L^2((0,T^*)\times\Omega_1))^n} = O(\varepsilon). \tag{11.4}$$

On the other hand, applying the classical energy method to system (11.3), we conclude that for any $s \in [0, T^*]$, it holds

$$\begin{aligned} &\int_0^s \int_{\Omega_1} |\nabla v_\varepsilon^2|^2 dx dt + \frac{1}{2}\int_{\Omega_1} |v_\varepsilon^2(s)|^2 dx \\ &\quad + \frac{1}{2}\int_{\Omega_2} [|\partial_t w_\varepsilon^2(s)|^2 + |\nabla w_\varepsilon^2(s)|^2] dx \\ &= -\int_0^s \int_\gamma \Big[\frac{\partial v_\varepsilon^2}{\partial \nu_1} r_4 + v_\varepsilon^2 r_5 + r_4 r_5\Big] d\gamma dt. \end{aligned} \tag{11.5}$$

From (11.3), one sees that $(\partial_t v_\varepsilon^2, \partial_t w_\varepsilon^2, \partial_{tt} w_\varepsilon^2)$ satisfies

$$\begin{cases} \partial_{tt} v_\varepsilon^2 - \Delta(\partial_t v_\varepsilon^2) = 0 & \text{in } (0,T^*)\times\Omega_1, \\ \Box\,(\partial_t w_\varepsilon^2) = 0 & \text{in } (0,T^*)\times\Omega_2, \\ \partial_t v_\varepsilon^2 = 0 & \text{on } (0,T^*)\times\Gamma_1, \\ \partial_t w_\varepsilon^2 = 0 & \text{on } (0,T^*)\times\Gamma_2, \\ \partial_t v_\varepsilon^2 = \partial_{tt} w_\varepsilon^2 - \partial_t r_4, \quad \dfrac{\partial(\partial_t v_\varepsilon^2)}{\partial\nu_1} = -\dfrac{\partial(\partial_t w_\varepsilon^2)}{\partial\nu_2} - \partial_t r_5 & \text{on } (0,T^*)\times\gamma, \\ \partial_t v_\varepsilon^2(0) = 0 & \text{in } \Omega_1, \\ \partial_t w_\varepsilon^2(0) = \partial_{tt} w_\varepsilon^2(0) = 0 & \text{in } \Omega_2. \end{cases} \tag{11.6}$$

Similarly, applying the classical energy method to system (11.6), and noting the first equation in (11.3), we conclude that for any $s\in[0,T^*]$, it holds

$$\begin{aligned} &\int_0^s\int_{\Omega_1}|\partial_t\nabla v_\varepsilon^2|^2dxdt + \frac12\int_{\Omega_1}|\Delta v_\varepsilon^2(s)|^2dx \\ &\quad+\frac12\int_{\Omega_2}(|\partial_{tt}w_\varepsilon^2(s)|^2 + |\partial_t\nabla w_\varepsilon^2(s)|^2)dx \\ &= -\int_0^s\int_\gamma\Big[\frac{\partial(\partial_t v_\varepsilon^2)}{\partial\nu_1}\partial_t r_4 + \partial_t v_\varepsilon^2\partial_t r_5 + \partial_t r_4\partial_t r_5\Big]d\gamma dt. \end{aligned} \tag{11.7}$$

Combining (11.5) and (11.7), for any $s'\in[0,T^*]$, we have

$$\begin{aligned} &\int_0^{s'}\int_{\Omega_1}|\nabla v_\varepsilon^2|^2dxdt + \frac12\int_{\Omega_1}[|v_\varepsilon^2(s',x)|^2 + |\Delta v_\varepsilon^2(s',x)|^2]dx \\ &\le -\int_0^{s'}\int_\gamma\Big[\frac{\partial(\partial_t v_\varepsilon^2)}{\partial\nu_1}\partial_t r_4 + \frac{\partial v_\varepsilon^2}{\partial\nu_1}r_4 + \partial_t v_\varepsilon^2\partial_t r_5 \\ &\qquad\qquad + v_\varepsilon^2 r_5 + \partial_t r_4\partial_t r_5 + r_4 r_5\Big]d\gamma dt. \end{aligned} \tag{11.8}$$

Now, for any $s\in[0,T^*]$, integrating (11.8) with respect to s' from 0 to s, exchanging the order of integration, using integration by parts and noting

(6.84), one arrives at

$$
\begin{aligned}
&\int_0^s \int_{\Omega_1} [|v_\varepsilon^2|^2 + |\Delta v_\varepsilon^2|^2] dxdt \\
&\le -2\int_0^s (s-t)\int_\gamma \Big[\frac{\partial(\partial_t v_\varepsilon^2)}{\partial\nu_1}\partial_t r_4 + \partial_t v_\varepsilon^2 \partial_t r_5\Big] d\gamma dt \\
&\quad -2\int_0^s \int_0^{s'} \int_\gamma \Big[\frac{\partial v_\varepsilon^2}{\partial\nu_1} r_4 + v_\varepsilon^2 r_5 + \partial_t r_4 \partial_t r_5 + r_4 r_5\Big] d\gamma dt ds' \\
&= -2\int_0^s \int_\gamma \Big[\frac{\partial(v_\varepsilon^2)}{\partial\nu_1}\partial_t\Big((s-t)\partial_t r_4\Big) + v_\varepsilon^2 \partial_t\Big((s-t)\partial_t r_4\Big)\Big] d\gamma dt \\
&\quad -2\int_0^s \int_0^{s'} \int_\gamma \Big[\frac{\partial v_\varepsilon^2}{\partial\nu_1} r_4 + v_\varepsilon^2 r_5 + \partial_t r_4 \partial_t r_5 + r_4 r_5\Big] d\gamma dt ds' \\
&\le C\int_0^s \int_\gamma \Big[\Big|\frac{\partial v_\varepsilon^2}{\partial\nu_1}\Big|(|r_4| + |\partial_t r_4| + |\partial_{tt} r_4|) \\
&\qquad\qquad + |v_\varepsilon^2|(|r_5| + |\partial_t r_5| + |\partial_{tt} r_5|)\Big] d\gamma dt.
\end{aligned}
\tag{11.9}
$$

Combining (11.8) and (11.9), we end up with

$$
\begin{aligned}
&\int_0^s \int_{\Omega_1} [|v_\varepsilon^2|^2 + |\nabla v_\varepsilon^2|^2 + |\Delta v_\varepsilon^2|^2] dxdt \\
&\le C\int_0^s \int_\gamma \Big[\Big|\frac{\partial v_\varepsilon^2}{\partial\nu_1}\Big|(|r_4| + |\partial_t r_4| + |\partial_{tt} r_4|) \\
&\qquad\qquad + |v_\varepsilon^2|(|r_5| + |\partial_t r_5| + |\partial_{tt} r_5|)\Big] d\gamma dt,
\end{aligned}
\tag{11.10}
$$

where $C > 0$ is a constant, independent of $s \in [0, T]$.

However, noting (6.84) again, by Lemma 1 and the trace theorem, we see that

$$
\begin{aligned}
&\int_0^s \int_\gamma \Big[\Big|\frac{\partial v_\varepsilon^2}{\partial\nu_1}\Big|(|r_4| + |\partial_t r_4| + |\partial_{tt} r_4|) \\
&\qquad\qquad + |v_\varepsilon^2|(|r_5| + |\partial_t r_5| + |\partial_{tt} r_5|)\Big] d\gamma dt \\
&\le \Big\{\int_0^s \int_{\Omega_1} [|v_\varepsilon^2|^2 + |\nabla v_\varepsilon^2|^2 + |\Delta v_\varepsilon^2|^2] dxdt\Big\}^{1/2} \\
&\qquad \times [|r_4|_{H^2(0,T^*;H^{1/2}(\gamma))} + |r_5|_{H^2(0,T^*;L^2(\gamma))}].
\end{aligned}
\tag{11.11}
$$

From (11.10) and (11.11), we conclude that for any $s \in [0, T^*]$, it holds

$$
\begin{aligned}
&\int_0^s \int_{\Omega_1} [|v_\varepsilon^2|^2 + |\nabla v_\varepsilon^2|^2 + |\Delta v_\varepsilon^2|^2] dxdt \\
&\le C[|r_4|^2_{H^2(0,T^*;H^{1/2}(\gamma))} + |r_5|^2_{H^2(0,T^*;L^2(\gamma))}].
\end{aligned}
\tag{11.12}
$$

By (11.12) and noting (6.83), we end up with

$$
|\nabla v_\varepsilon^2|^2_{(L^2((0,T^*)\times\Omega_1))^n} = O(\varepsilon). \tag{11.13}
$$

Finally, combining (11.1), (11.4) and (11.13), it is easy to check that $(y_\varepsilon, z_\varepsilon)$ satisfy the conclusion of Lemma 9. $\square$

12. Appendix D: Proof of Lemma 11

This Appendix is devoted to prove Lemma 11. The proof is divided into several steps.

Step 1. Set $w = y\chi_{\Omega_1} + z\chi_{\Omega_2}$. Then, by system (1.2), it is easy to see that $w \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ satisfies

$$\begin{cases} \Box w = (y_{tt} - y_t)\chi_{\Omega_1} & \text{in } (0,T)\times\Omega, \\ w = 0 & \text{on } (0,T)\times\Gamma, \\ w(0) = y_0\chi_{\Omega_1} + z_0\chi_{\Omega_2}, \quad w_t(0) = (\Delta y_0)\chi_{\Omega_1} + z_1\chi_{\Omega_2} & \text{in } \Omega. \end{cases} \tag{12.1}$$

We decompose w as

$$w = p + q, \tag{12.2}$$

where p and q are respectively solutions of

$$\begin{cases} \Box p = 0 & \text{in } (0,T)\times\Omega, \\ p = 0 & \text{on } (0,T)\times\Gamma, \\ p(0) = y_0\chi_{\Omega_1} + z_0\chi_{\Omega_2}, \quad p_t(0) = (\Delta y_0)\chi_{\Omega_1} + z_1\chi_{\Omega_2} & \text{in } \Omega \end{cases} \tag{12.3}$$

and

$$\begin{cases} \Box q = (y_{tt} - y_t)\chi_{\Omega_1} & \text{in } (0,T)\times\Omega, \\ q = 0 & \text{on } (0,T)\times\Gamma, \\ q(0) = q_t(0) = 0 & \text{in } \Omega. \end{cases} \tag{12.4}$$

Applying Corollary 2 (with $\omega = \Omega_1$ and $s \in (0,1/2)$) to system (12.3), we conclude that for $T>0$ large enough, solutions of (12.3) satisfy

$$\begin{aligned} &|y_0\chi_{\Omega_1} + z_0\chi_{\Omega_2}|_{H_0^{1-s}(\Omega)} + |(\Delta y_0)\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{H^{-s}(\Omega)} \\ &\le C\left[\ln\left(2 + \frac{F(y_0,z_0,z_1)}{|p|_{H^1(0,T;L^2(\Omega_1))}}\right)\right]^{-s/2} F(y_0,z_0,z_1), \end{aligned} \tag{12.5}$$

where

$$F(y_0,z_0,z_1) \overset{\triangle}{=} \big|\big(y_0\chi_{\Omega_1} + z_0\chi_{\Omega_2}, (\Delta y_0)\chi_{\Omega_1} + z_1\chi_{\Omega_2}\big)\big|_{H_0^1(\Omega)\times L^2(\Omega)}. \tag{12.6}$$

Now, by (12.2) and (12.5), it is easy to see that

$$\begin{aligned} &|y_0\chi_{\Omega_1} + z_0\chi_{\Omega_2}|_{H_0^{1-s}(\Omega)} + |(\Delta y_0)\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{H^{-s}(\Omega)} \\ &\le C\left[\ln\left(2 + \frac{F(y_0,z_0,z_1)}{|q|_{H^1(0,T;L^2(\Omega_1))} + |y|_{H^1(0,T;L^2(\Omega_1))}}\right)\right]^{-s/2} F(y_0,z_0,z_1). \end{aligned} \tag{12.7}$$

However, applying the classical energy estimate to (12.4), we conclude that

$$\int_0^T \int_{\Omega_1} [q_t^2 + |\nabla q|^2] dxdt \le C|y|^2_{H^2(0,T;L^2(\Omega_1))}. \tag{12.8}$$

Consequently, combining (12.7) and (12.8), for $s \in (0, 1/2)$, one obtains

$$\begin{aligned} &|y_0\chi_{\Omega_1} + z_0\chi_{\Omega_2}|_{H_0^{1-s}(\Omega)} + |(\Delta y_0)\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{H^{-s}(\Omega)} \\ &\le C\left[\ln\left(2 + \frac{F(y_0, z_0, z_1)}{|y|_{H^2(0,T;L^2(\Omega_1))}}\right)\right]^{-s/2} F(y_0, z_0, z_1), \\ &\qquad \forall\, (y_0, z_0, z_1) \in D(\widetilde{\mathcal{A}}). \end{aligned} \tag{12.9}$$

Step 2. We claim that, for any $s \in [0, 1/2)$, it holds

$$\begin{aligned} &|\widetilde{\mathcal{A}}^{-1}(y_0, z_0, z_1)|_{\widetilde{H}} \\ &\le |y_0\chi_{\Omega_1} + z_0\chi_{\Omega_2}|_{H_0^{1-s}(\Omega)} + |(\Delta y_0)\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{H^{-s}(\Omega)}. \end{aligned} \tag{12.10}$$

Indeed, one has

$$\begin{aligned} &|y_0|_{L^2(\Omega_1)} + |z_0|_{L^2(\Omega_2)} + |\Delta y_0|_{H^{-s}(\Omega_1)} + |z_1|_{H^{-s}(\Omega_2)} \\ &\le C\Big[|y_0\chi_{\Omega_1} + z_0\chi_{\Omega_2}|_{H_0^{1-s}(\Omega)} + |(\Delta y_0)\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{H^{-s}(\Omega)}\Big]. \end{aligned} \tag{12.11}$$

In view of (12.11), it suffices to show that

$$\begin{aligned} &|\widetilde{\mathcal{A}}^{-1}(y_0, z_0, z_1)|_{\widetilde{H}} \le C[|y_0|_{L^2(\Omega_1)} + |z_0|_{L^2(\Omega_2)} + |z_1|_{H^{-s}(\Omega_2)}], \\ &\qquad \forall\, (y_0, z_0, z_1) \in L^2(\Omega_1) \times L^2(\Omega_2) \times H^{-s}(\Omega_2). \end{aligned} \tag{12.12}$$

For this purpose, put

$$(Y_0, Z_0, Z_1) \triangleq \widetilde{\mathcal{A}}^{-1}(y_0, z_0, z_1).$$

Then, by the definition of $\widetilde{\mathcal{A}}$ in (3.11)–(3.12), we see that

$$Z_1 = z_0; \tag{12.13}$$

and

$$(Y_0, Z_0) = -\mathcal{L}^{-1}\Big(y_0\chi_{\Omega_1} + z_1\chi_{\Omega_2}\Big), \tag{12.14}$$

where $\mathcal{L}$ denotes the Laplacian $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions.

Now, from (12.14) we get

$$\begin{aligned} |(Y_0, Z_0)|_{H_0^1(\Omega)} &\le C|y_0\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{H^{-s}(\Omega)} \\ &\le C[|y_0|_{L^2(\Omega_1)} + |z_1|_{H^{-s}(\Omega_2)}]. \end{aligned} \tag{12.15}$$

Here we have used the fact that $|y_0\chi_{\Omega_1} + z_1\chi_{\Omega_2}|_{H^{-s}(\Omega)} \leq |y_0|_{H^{-s}(\Omega_1)} + |z_1|_{H^{-s}(\Omega_2)}$ because $H_0^s(\Omega_1) = H^s(\Omega_1)$ and $H_0^s(\Omega_2) = H^s(\Omega_2)$ for $s \in [0, 1/2)$. Hence, (12.12) holds.

On the other hand, by (12.6), (1.6) and recalling again the definition of $\widetilde{\mathcal{A}}$ in (3.11)–(3.12), it is easy to check that

$$|(y_0, z_0, z_1)|_{\widetilde{H}} \leq F(y_0, z_0, z_1) \leq |(y_0, z_0, z_1)|_{D(\widetilde{A})}. \tag{12.16}$$

Therefore, combining (12.9), (12.10) and (12.16), we conclude that, for any $s \in (0, 1/2)$ and for all $(y_0, z_0, z_1) \in D(\widetilde{\mathcal{A}})$, solutions of system (1.2) satisfy

$$\begin{aligned} &|\widetilde{\mathcal{A}}^{-1}(y_0, z_0, z_1)|_{\widetilde{H}} \\ &\leq C\left[\ln\left(2 + \frac{|(y_0, z_0, z_1)|_{\widetilde{H}}}{|y|_{H^2(0,T;L^2(\Omega_1))}}\right)\right]^{-s/2} |(y_0, z_0, z_1)|_{D(\widetilde{A})}. \end{aligned} \tag{12.17}$$

Finally, applying estimate (12.17) to $(y, z, z_t)_t = \widetilde{A}(y, z, z_t)$ (which solves also system (1.2), but with initial data (y_0, z_0, z_1) replaced by $\widetilde{A}(y_0, z_0, z_1)$), we obtain (8.9). □

Acknowledgement. The work is supported by the Grant MTM2005-00714 of the Spanish MEC, the DOMINO Project CIT-370200-2005-10 in the PROFIT program of the MEC (Spain), the SIMUMAT projet of the CAM (Spain), the FANEDD of China (Project No: 200119), the EU TMR Project "Smart Systems", the NCET of China under grant NCET-04-0882, and the NSF of China under grants 10371084 and 10525105. The authors are grateful to Dr. Thomas Duyckaerts for fruitful discussions, and to the anonymous referees for invaluable comments.

References

1. K. Ammari and M. Tucsnak, *Stabilization of Bernoulli-Euler beams by means of a pointwise feedback force, SIAM J. Control Optim.,* **39** (2000), 1160–1181.
2. C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, SIAM J. Control Optim.,* **30** (1992), 1024–1065.
3. H. Beirão da Veiga, *On the existence of strong solutions to a coupled fluid-structure evolution problem, J. Math. Fluid Mech.,* **6** (2004), 21–52.
4. F. J. Blom, *A monolithical fluid-structure interaction algorithm applied to the piston problem, Comput. Methods Appl. Mech. Engrg.,* **167** (1998), 369–391.
5. M. Boulakia, *Existence of weak solutions for an interaction problem between an elastic structure and a compressible viscous fluid, C. R. Math. Acad. Sci. Paris,* **340** (2005), 113–118.
6. N. Burq, *Contrôlabilité exacte des ondes dans des ouverts peu réguliers, Asymptot. Anal.,* **14** (1997), 157–191.

7. P. Causin, J. F. Gerbeau and F. Nobile, *Added-mass effect in the design of partitioned algorithms for fluid-structure problems, Comput. Methods Appl. Mech. Engrg.,* **194** (2005), 4506–4527.
8. A. Chambolle *et al.*, *Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate, J. Math. Fluid Mech.,* **7** (2005), 368–404.
9. C. Conca, J. San Martín and M. Tucsnak, *Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid, Comm. Partial Differential Equations,* **25** (2000), 1019–1042.
10. D. Coutand and S. Shkoller, *Motion of an elastic solid inside an incompressible viscous fluid, Arch. Ration. Mech. Anal.,* **176** (2005), 25–102.
11. R. Dautray and J.-L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques. Tome 3,* Collection du Commissariat à l'Énergie Atomique: Série Scientifique. [Collection of the Atomic Energy Commission: Science Series], Masson, Paris, 1985, With the collaboration of M. Artola, C. Bardos, M. Cessenat, A. Kavenoky, H. Lanchon, P. Lascaux, B. Mercier, O. Pironneau, B. Scheurer and R. Sentis.
12. B. Desjardins and M. J. Esteban, *On weak solutions for fluid-rigid structure interaction: compressible and incompressible models, Commun. Partial Differential Equations,* **25** (2000), 1399–1413.
13. B. Desjardins *et al.*, *Weak solutions for a fluid-elastic structure interaction model, Rev. Mat. Complut.,* **14** (2001), 523–538.
14. Q. Du *et al.*, *Semidiscrete finite element approximations of a linear fluid-structure interaction problem, SIAM J. Numer. Anal.,* **42** (2004), 1–29.
15. T. Duyckaerts, *Optimal decay rates of the solutions of hyperbolic-parabolic systems coupled by an interface,* Preprint, 2005.
16. T. Duyckaerts, private communication.
17. E. Feireisl, *On the motion of rigid bodies in a viscous compressible fluid, Arch. Ration. Mech. Anal.,* **167** (2003), 281–308.
18. J. F. Gerbeau and M. Vidrascu, *A quasi-Newton algorithm based on a reduced model for fluid-structure interaction problems in blood flows, M2AN Math. Model. Numer. Anal.,* **37** (2003), 631–647.
19. C. Grandmont and Y. Maday, *Existence for an unsteady fluid-structure interaction problem, M2AN Math. Model. Numer. Anal.,* **34** (2000), 609–636.
20. M. Gunzburger, H.-C. Lee and G. A. Seregin, *Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions, J. Math. Fluid Mech.,* **2** (2000), 219–266.
21. M. Heil, *An efficient solver for the fully coupled solution of large-displacement fluid-structure interaction problems, Comput. Methods Appl. Mech. Engrg.,* **193** (2004), 1–23.
22. K.-H. Hoffmann and V. N. Starovoitov, *On a motion of a solid body in a viscous fluid. Two-dimensional case, Adv. Math. Sci. Appl.,* **9** (1999), 633–648.
23. L. Hörmander, *The Analysis of Linear Partial Differential Operators III,* Springer, Berlin, 1985.
24. M. S. Howe, *Acoustics of fluid-structure interactions,* Cambridge Monographs on Mechanics, Cambridge University Press, Cambridge, 1998.
25. O. Yu. Imanuvilov and M. Yamamoto, *Remarks on Carleman estimates and exact controllability of the Lamé system, Journées "Équations aux Dérivées Partielles" (Forges-les-Eaux, 2002),* Exp. No. V, 19 pp., Univ. Nantes, Nantes, 2002.

26. D. Ishihara and S. Yoshimura, *A monolithic approach for interaction of incompressible viscous fluid and an elastic body based on fluid pressure Poisson equation, Int. J. Numer. Meth. Engng,* **64** (2005), 167–203.
27. P. le Tallec and S. Mani, *Numerical analysis of a linearised fluid-structure interaction problem, Numer. Math.,* **87** (2000), 317–354.
28. G. Lebeau, *Contrôle analytique I: Estimations a priori, Duke Math. J.,* **68** (1992), 1–30.
29. T. T. Li *et al.*, *Boundary value problems with equivalued surface boundary conditions for self-adjoint elliptic differential equations (I), J. Fudan Univ. (Nat. Sci.),* **1** (1976), 61–71 (In Chinese).
30. J. L. Lions, *Contrôlabilité Exacte, Stabilisation et Perturbations de Systèmes Distribués. Tome 1: Contrôlabilité Exacte*, RMA, **8**, Masson, Paris, 1988.
31. J. L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications (Vol. I),* Springer-Verlag, New York, 1972.
32. F. Macià and E. Zuazua, *On the lack of observability for wave equations: a Gaussian beam approach, Asymptot. Anal.,* **32** (2002), 1–26.
33. I. C. Mathews, *Numerical techniques for three-dimensional steady-state fluid-structure interaction, J. Acoust. Soc. Amer.,* **79** (1986), 1317–1325.
34. H. Morand and R. Ohayon, *Fluid Structure Interaction: Applied Numerical Methods,* Wiley, New York, 1995.
35. R. Ohayon and C. Felippa, eds., *Special Issue on Fluid-Structure Interactions, Comput. Methods Appl. Mech. Engrg.,* **190** (2001), 2977–3292.
36. A. Pazy, *Semigroups of linear operators and applications to partial differential equations,* Applied Mathematical Sciences, **44**, Springer-Verlag, New York, 1983.
37. K. D. Phung, *Quantification of unique continuation for the wave equation with Dirichlet boundary condition,* unpublished manuscript, 2004.
38. J. Ralston, *Gaussian beams and the propagation of singularities,* in *Studies in Partial Differential Equations*, Edited by W. Littman, MAA Studies in Mathematics, **23**, Washington, DC, 1982, 206–248.
39. J. Rauch, *Qualitative behavior of dissipative wave equations on bounded domains, Arch. Rational Mech. Anal.,* **62** (1976), 77–85.
40. J. Rauch, X. Zhang and E. Zuazua, *Polynomial decay for a hyperbolic-parabolic coupled system, J. Math. Pures Appl.,* **84** (2005), 407–470.
41. L. Robbiano, *Fonction de coût et contrôle des solutions des équations hyperboliques, Asymptotic Anal.,* **10** (1995), 95–115.
42. E. H. Rogers, *A minimax theory for overdamped systems, Arch. Rational Mech. Anal.,* **16** (1964), 89–96.
43. J. San Martín, V. Starovoitov and M. Tucsnak, *Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid, Arch. Ration. Mech. Anal.,* **161** (2002), 113–147.
44. D. Serre, *Chute libre d'un solide dans un fluide visqueux incompressible. Existence, Japan J. Appl. Math.,* **4** (1987), 99–110.
45. M. Slemrod, *Weak asymptotic decay via a "relaxed invariance principle" for a wave equation with nonlinear, nonmonotone damping, Proc. Roy. Soc. Edinburgh Sect. A,* **113** (1989), 87–97.
46. K. Stein *et al.*, *Fluid-structure interaction modelling of parachute soft-landing dynamics,* Internat. J. Numer. Methods Fluids, **47** (2005), 619–631.
47. S. Tabachnikov, *Billiards,* Panor. Synth., No. 1, 1995.

48. T. Takahashi, *Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain, Adv. Differential Equations,* **8** (2003), 1499–1532.
49. T. Takahashi and M. Tucsnak, *Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid, J. Math. Fluid Mech.,* **6** (2004), 53–77.
50. M. E. Taylor, *Pseudodifferential Operators,* Princeton Mathematical Series, **34**, Princeton University Press, Princeton, N.J., 1981.
51. R. van Loon *et al.*, *A combined fictitious domain/adaptive meshing method for fluidCstructure interaction in heart valves, Int. J. Numer. Meth. Fluids,* **46** (2004), 533–544.
52. J.-L. Vázquez and E. Zuazua, *Large time behavior for a simplified 1D model of fluid-solid interaction, Comm. Partial Differential Equations,* **28** (2003), 1705–1738.
53. D. Xia *et al.*, *Theory of Real Functions and Functional Analysis, Vol. 2 (Second version),* Gaodeng Jiaoyu Chubanshe (Higher Education Press), Beijing, 1985. (In Chinese).
54. X. Zhang and F. Li, *Existence, uniqueness and limit behavior of solutions to a nonlinear boundary-value problem with equivalued surface, Nonlinear Anal.,* **34** (1998), 525–536.
55. X. Zhang and E. Zuazua, *Control, observation and polynomial decay for a coupled heat-wave system, C. R. Acad. Sci. Paris Série I,* **336** (2003), 823–828.
56. W. P. Ziemer, *Weakly Differentiable Functions, Sobolev Spaces and Functions of Bounded Variation,* Graduate Texts in Mathematics, **120**, Springer-Verlag, Berlin etc., 1989.
57. E. Zuazua, *Controllability and observability of partial differential equations: some results and open problems,* in *Handbook of Differential Equations: Evolutionary Differential Equations, vol. 3,* Elsevier Science, to appear.

School of Mathematics, Sichuan University,
Chengdu 610064, China; and
Departamento de Matemáticas, Facultad de Ciencias,
Universidad Autónoma deMadrid,
28049 Madrid, Spain.
email:xu.zhang@uam.es

and

Departamento de Matemáticas, Facultadde Ciencias,
Universidad Autónoma de Madrid,
28049 Madrid, Spain.
email:enrique.zuazua@uam.es