On the lack of controllability of fractional in time ODE and PDE

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Abstract

This paper is devoted to the analysis of the problem of controllability of fractional (in time) Ordinary and Partial Differential Equations (ODE/PDE). The fractional time derivative introduces some memory effects on the system that need to be taken into account when defining the notion of control. In fact, in contrast with the classical ODE and PDE control theory, when driving these systems to rest, one is required not only to control the value of the state at the final time but also the memory accumulated by the long-tail effects that the fractional derivative introduces. As a consequence, the notion of null controllability to equilibrium needs to take into account both the state and the memory term. The existing literature so far is only concerned with the problem of partial controllability in which the state is controlled, but the behaviour of the memory term is ignored.

In the present paper we consider the full controllability problem and show that, due to the memory effects, even at the ODE level, controllability cannot be achieved in finite time. This negative result holds even for finite-dimensional systems in which the control is of full dimension. Consequently, the same negative results hold also for fractional PDE, regardless of whether they are of parabolic or hyperbolic nature.

This negative result exhibits a completely opposite behavior with respect to the existing literature on classical ODE and PDE control where sharp sufficient conditions for null controllability are well known.

Key Words. Fractional in time ODE, PDE, partial controllability, nul controllability, observability.

1 Introduction

This paper is devoted to the analysis of the problem of controllability of fractional (in time) Ordinary and Partial Differential Equations (ODE/PDE).

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These models arise, for instance, in the context of anomalous diffusion, taking the form:

$$\frac{\partial^\alpha y}{\partial t^\alpha} = -k(-\Delta)^\beta y, \tag{1.1}$$

where α and β are real numbers.

In this model, roughly speaking, the time derivative term corresponds to a long-time heavy tail decay and the spatial fractional derivative operator to non-local diffusion (see [28]).

Fractional calculus includes various extensions of the usual definition of derivative from integer to real order, including the Riemann-Liouville derivative, the Caputo derivative, the Riesz derivative, the Weyl derivative, etc. In this paper, we only consider the Caputo derivative.

These models are relevant, in particular, in the context of spatially disordered systems, porous media, fractal media, turbulent fluids and plasmas, biological media with traps, binding sites or macro-molecular crowding, stock price movements, etc. We refer the readers to [10, 9, 28, 29] and the rich references therein for the motivation and description of the model. On the other hand, we refer to [11, 12, 13, 14, 15, 19] and the rich references therein for mathematical analysis of these models.

Anomalous diffusion processes do not satisfy the Fick's law, according to which, for classical diffusion processes, the current V is proportional to the concentration gradient,

$$V(x,t) = -k\nabla y(x,t), \tag{1.2}$$

where k is the diffusion coefficient and y is the concentration. If, in addition, the material which diffuses is neither created nor destroyed, then the continuity equation leads to

$$\frac{\partial y(x,t)}{\partial t} = -\nabla\cdot V(x,t), \tag{1.3}$$

or to the classical diffusion model

$$\frac{\partial y(x,t)}{\partial t} = k\Delta y(x,t). \tag{1.4}$$

As mentioned above, the anomalous diffusion processes under consideration (1.1) do not fit in this frame and need to be formulated by means of fractional calculus and, as we shall see, the presence of the fractional time derivative has a great impact on the control properties of these models.

Here we are interested in the problem of controllability. More precisely, we address the problem of null controllability in which the objective is to drive the solution to rest, in other words, to the trivial null state, in finite time.

As we shall see, the presence of fractional derivatives in time forces us to revisit the concept of controllability analysed until now ([26]), to incorporate also memory effects due to the tail of the time fractional derivative. As a consequence, the notion of null controllability to equilibrium needs to take into account both the state and the memory term. The existing literature is only concerned with the problem of partial controllability in which the state is controlled, but the behaviour of the memory term is ignored.

In the present paper we consider the full controllability problem and show that, due to the memory effects, even at the ODE level controllability cannot be achieved.

This negative result holds even for finite-dimensional systems in which the control is of full dimension. Consequently, the same negative results hold also for fractional in time PDE, regardless of whether they are of parabolic or hyperbolic nature. This negative result exhibits a completely opposite behaviour with respect to the existing literature on classical ODE and PDE control.

So far the problem of controllability for fractional PDE has been considered in a number of articles (see [30], [32], [8]). But there, mainly, the goal was to analyse the impact of fractional diffusion, by keeping a classical differential behaviour in time. As shown in these articles, both for heat and wave-like processes, the controllability properties of the classical heat and wave equations are preserved depending on the amount of diffusion and propagation phenomena that the fractional Laplacians introduce.

The present paper is a first attempt to controllability in the context of fractional time derivative operators and the negative results we prove are mainly due to the long tail effect in time, and therefore are independent of the diffusion and propagation properties of the fractional laplacian. As we shall see, the property of controllability even fails for scalar fractional ODE or for systems of ODE in which the control has full range. We refer to [17] for an analysis of the optimal control problem of fractional diffusion processes and its finite element numerical approximation. Note however that, in that article, the pathological phenomena we observe here are not noticed since the goal there is to minimize a quadratic cost, without getting to fine controllability issues.

2 Formulation of the null controllability problem

In this section, we give a precise formulation for systems of the form (1.1) and its variants.

We first give the definition of the fractional time derivative.

Given a Banach space X, $a \in \mathbb{R}$, and $f \in C^1([a,+\infty);X)$ and $\alpha \in (0,1)$, we introduce the left-hand side Caputo derivative of f at a as follows:

$$\partial_{t,a+}^{\alpha} f \triangleq \int_a^t \frac{f'(s)}{(t-s)^{\alpha}} ds. \tag{2.1}$$

Remark 2.1 *In the above definition, we only consider the case that $\alpha \in (0,1)$ whereas the Caputo derivative can be defined for any complex number α with a nonnegative real part (see [4, 34]).*

Let H be a Hilbert space and A a linear unbounded operator on H, which generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on H. Let U be another Hilbert space and $B \in \mathcal{L}(U,H)$ the control operator.

We consider the following control system:

$$\begin{cases} \partial_{t,0+}^{\alpha} y - Ay = Bu & \text{in } (0,+\infty), \\ y(0) = y_0, \end{cases} \tag{2.2}$$

with $\alpha \in (0,1)$, $y_0 \in H$ and $u \in L^2(0,T;U)$.

Here we do not discuss the well-posedness of the system (2.2), which is a wide topic, out of the scope of this paper. The well-posedness of the system for the models we shall consider can be easily achieved. Actually, proceeding as in [17], provided A is self-adjoint and of compact inverse, employing the spectral basis associated to the eigenvectors of A, solutions of (2.2) can be represented in an unique manner using separation of variables and Fourier series representations. We refer the readers to [4, 31, 34] for a more systematic discussion of these solvability issues.

Thus, in what follows, we always assume that there is a unique solution

$$y(\cdot) \in C([0, +\infty); H) \cap C^1([0, +\infty); D(A^*)')$$

to the system (2.2) such that for any $\varphi \in D(A^*)$ and $t \in (0, T]$,

$$\int_0^t \langle \partial_t^\alpha y(s), \varphi \rangle_{D(A^*)', D(A^*)} ds - \int_0^t \langle y(s), D^* \varphi \rangle_H ds = \int_0^t \langle Bv, \varphi \rangle_H.$$

Before giving the definition of the property of null controllability of (2.2), let us first recall that, for classical abstract differential equations of the form

$$\begin{cases} \partial_t \hat{y} - A\hat{y} = Bv & \text{in } (0, +\infty), \\ \hat{y}(0) = \hat{y}_0, \end{cases} \tag{2.3}$$

where $\hat{y}_0 \in H$ and $v \in L^2(0, T; U)$, the system (2.3) is said to be null controllable at time $T > 0$ if for any $\hat{y}_0 \in H$, there is a control $v \in L^2(0, T; U)$ such that the corresponding solution $\hat{y}(\cdot)$ satisfies that $\hat{y}(T) = 0$.

Remark 2.2 *System* (2.3) *is a classical linear control system, which has been studied extensively in the literature. When A is a fractional spatial diffusion operator, it describes some special anomalous diffusion processes and its null controllability has been analysed in [24, 30, 32, 8].*

The emphasis on the present paper is precisely on the dramatic impact that fractional in time derivatives have on the controllability properties of the system.

The above definition, when translated as such for fractional systems of the form (2.2), would lead to a notion of partial but not of full null controllability, in the sense that, for fractional in time derivatives, due to memory effects induced by the integral term, the fact that the solution y reaches the null value at time $t = T$ does not guarantee that the solution stays at rest for $t \geq T$ when the control action stops.

The partial null controllability problem, for the system (2.2), in which one is merely interested in reaching the state 0 at time $t = T$, was studied by some authors (see [7, 23, 26] and the references therein). Similar to the classical controllability result for ODE, when $H = \mathbb{R}^n$, A is a n-dimensional matrix and B is a $n \times m$ matrix, the authors show that the Kalman rank condition of (A, B) is a sufficient and necessary condition for the partial null controllability of the system (2.2). More precisely, they proved that (2.2) is partial null controllable if and only if

$$\text{rank}(B, AB, \cdots, A^{n-1}B) = n.$$

Here we consider the following rather stronger notion of null controllability:

Definition 2.1 *System* (2.2) *is null controllable at time* $T > 0$ *if for any* $y_0 \in H$*, there is a control* $u \in L^2(0,T;U)$ *such that the corresponding solution* $y(\cdot)$ *satisfies that* $y(t) = 0$ *for all* $t \geq T$*.*

In the previous definition we implicitly assume that the control u that has its support in $t \in [0,T]$, vanishes afterwards, i. e. $u(t) \equiv 0$ for all $t \geq T$. A similar notion was consider in [1] for scalar ODE with Riemann-Liouville fractional derivative.

Of course, for classical differential equations (both ODE and PDE), both notions of null controllability coincide since the uniqueness of solutions for the Cauchy problem in that setting ensures that, once the solution reaches the equilibrium 0 at time $t = T$, it remains there $y(t) \equiv 0$ for all $t \geq T$ if no added control is implemented after $t \geq T$.

The main result of this article shows that, for fractional derivative models, the null controllability property in the strict sense of Definition 2.1 fails systematically due to the memory effects induced by the integral entering in the fractional in time derivative.

This negative result is relevant not only from the perspective of null controllability but in a more general control theoretical context. Indeed, let us for instance consider the classical linear quadratic optimal control problem on infinite time horizon ([18]), in which the goal is to minimize the cost

$$\mathcal{J}(u) = \int_0^\infty \big(|y(t)|_H^2 + |u(t)|_U^2\big)dt, \tag{2.4}$$

where u is a control and y is the corresponding solution.

More precisely, consider the optimal control problem (**Problem (LQ)):** For each $y_0 \in H$, find a $\overline{u}(\cdot) \in L^2(0,+\infty;U)$ such that

$$\mathcal{J}(\overline{u}(\cdot)) = \inf_{u(\cdot)\in L^2(0,+\infty;U)} \mathcal{J}(u(\cdot)). \tag{2.5}$$

Obviously, the first issue that arises when considering this type of optimal control problems is whether the set of feasible controls is non void since, otherwise, the minimization problem does not make sense. Obviously, the feasible set contains all null controls in finite time, and this is one of the first consequences of null controllability. Indeed, if we can choose a control $u \in L^2(0,T;U)$ such that the state $y(t) \equiv 0$ for all $t \geq T$, then the integral on the cost is limited to the finite time interval $[0,T]$ and is obviously finite.

As we shall see, however, these fractional in time systems fail to be null controllable, and therefore, whether the set of feasible controls is non-empty or not will depend on the decay properties of solutions of the free dynamics without control. Thus, Problem (LQ) in itself is an interesting issue that requires further investigation.

3 A negative result on null controllability

The main result of this paper is of negative nature:

Theorem 3.1 *System* (2.2) *is not null controllable for any* $T > 0$ *in the sense of Definition 2.1.*

Note that this negative result is independent of the functional setting, the nature of the generator A and the control operator B. In fact the same negative result holds for finite-dimensional systems even when B is surjective. . In particular the first order scalar fractional DE fails to be null controllable as well.

To prove Theorem 3.1, let us first recall the following classical Müntz theorem (see [16] for a proof).

Lemma 3.1 *Let* $\{\sigma_k\}_{k=0}^{\infty}$ *with* $0 = \sigma_0 < \sigma_1 < \cdots$ *be an increasing sequence of non-negative real numbers. Then the set* span $\{s^{\sigma_k}\}_{k=0}^{\infty}$ *is dense in* $C([0, L])$ *for any* $L > 0$ *if and only if*

$$\sum_{k=1}^{\infty} \frac{1}{\sigma_k} = +\infty. \tag{3.1}$$

Proof of Theorem 3.1: Let us assume that the system (2.2) is null controllable for a $T > 0$ in the sense of Definition 2.1. Then, for any $t \geq T$, we have that

$$\int_0^T \frac{y'(s)}{(t-s)^{\alpha}} ds = \int_0^t \frac{y'(s)}{(t-s)^{\alpha}} ds = 0, \quad \forall\, t \geq T.$$

Thus, for any $\varphi \in D(A^*)$,

$$\int_0^T \frac{\langle y'(s), \varphi \rangle_{D(A^*)', D(A^*)}}{(t-s)^{\alpha}} ds = 0, \quad \forall\, t > T,$$

which implies that

$$\int_0^T \frac{\langle y'(s), \varphi \rangle_{D(A^*)', D(A^*)}}{(t-s)^{\alpha+j}} ds = 0, \quad \forall\, t > T,\ j \in \{0\} \cup \mathbb{N}. \tag{3.2}$$

Let $\sigma = (t-s)^{-1}$. From (3.2), we have that

$$\int_{\frac{1}{t}}^{\frac{1}{t-T}} \Big\langle y'\Big(t - \frac{1}{\sigma}\Big), \varphi \Big\rangle_{D(A^*)', D(A^*)} \sigma^{\alpha+j-2} d\sigma = 0, \quad \forall\, t > T,\ j \in \{0\} \cup \mathbb{N}. \tag{3.3}$$

Let

$$f(\sigma) = \Big\langle y'\Big(t - \frac{1}{\sigma}\Big), \varphi \Big\rangle_{D(A^*)', D(A^*)} \sigma^{\alpha-2}.$$

From (3.3), we get that

$$\int_{\frac{1}{t}}^{\frac{1}{t-T}} f(\sigma)\sigma^j d\sigma = 0, \quad \forall\, t > T,\ j \in \{0\} \cup \mathbb{N}. \tag{3.4}$$

This, together with Lemma 3.1, implies that

$$f(\cdot) \equiv 0 \text{ in } \Big[\frac{1}{t}, \frac{1}{t-T}\Big).$$

Hence, we find that

$$y'(\cdot) \equiv 0 \text{ in } [0, T).$$

Taking into account that $y(T) = 0$ this also implies that $y \equiv 0$ in $[0, T]$. □

4 The dual observation problem

It is by now well known in classical ODE and PDE control theory, that the controllability and observability properties are in duality. Thus, it is natural to analyze the signification of the negative result on null controllability in what concerns the dual observability property.

To this end, it is necessary to introduce the adjoint system and, for this, we need to define the right-sided Caputo fractional derivative at $b \in \mathbb{R}$ as follows:

$$(\partial_{t,b-} f)(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b \frac{f'(s)}{(s-t)^\alpha} ds, \quad \forall f \in C^1((-\infty, b]; X). \tag{4.1}$$

Consider the following system:

$$\begin{cases} \partial_{t,T-}^\alpha z^\tau - A^* z^\tau = \dfrac{\alpha z_0}{\Gamma(1-\alpha)(\tau - t)^{1+\alpha}} & \text{in } (0,T), \\ z^\tau(T) = 0, \end{cases} \tag{4.2}$$

with $z_0 \in D(A^*)$.

The following holds:

Proposition 4.1 *System (2.2) is null controllable in time T if and only if the system (4.2) is initially observable in the sense that there is a constant C such that for any $\tau \in (T, +\infty)$ and $z_0 \in D(A^*)$, it holds*

$$\Big| \int_0^T \frac{t^{1-\alpha} z^\tau(t)}{\Gamma(2-\alpha)} + \frac{z_0}{\Gamma(1-\alpha)(\tau-t)^\alpha} dt \Big|_H \le C |B^* z^\tau|_{L^2(0,T;U)}. \tag{4.3}$$

Proof: The "if" part. Let

$$\mathcal{X} \stackrel{\triangle}{=} \{ B^* z^\tau \,|\, z \text{ solves (4.2) with a } z_0 \in D(A^*) \ \}.$$

Clearly, $\mathcal{X}$ is a linear subspace of $L^2(0,T;U)$. Define a linear functional $\mathcal{F}$ on $\mathcal{X}$ as follows:

$$\mathcal{F}(B^* z^\tau) = -\frac{1}{\Gamma(1-\alpha)} \int_0^T \Big\langle \frac{z^\tau(t)}{t^\alpha} - \frac{z_0}{(\tau-t)^\alpha}, y_0 \Big\rangle_H dt.$$

From (4.3), we have that

$$\big| \mathcal{F}(B^* z^\tau) \big| \le C |y_0|_H |B^* z^\tau|_{L^2(0,T;U)}.$$

This implies that $\mathcal{F}$ is a bounded linear functional on $\mathcal{X}$. By the Hahn-Banach theorem, $\mathcal{F}$ can be extended to be a bounded linear functional on $L^2(0,T;U)$. We still denote the extension by $\mathcal{F}$ if there is no confusion. By Riesz representation theorem, there is a $u \in L^2(0,T;U)$ such that

$$\mathcal{F}(B^* z^\tau) = \int_0^T \langle B^* z^\tau, u \rangle_U.$$

We claim this $u(\cdot)$ is the desired control. Indeed, by integrating by parts, one has that

$$\begin{aligned}
&\int_0^T \langle (z^\tau(t), \partial_t^\alpha y)(t)\rangle_{D(A^*),D(A^*)'} dt \\
&= \Big\langle z^\tau(t), \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} y'(s) ds \Big\rangle_{D(A^*),D(A^*)'} \Big|_0^T \\
&\quad - \int_0^T \Big\langle \partial_t z^\tau(t), \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} y'(s) ds \Big\rangle_{D(A^*),D(A^*)'} \\
&= - \int_0^T \Big\langle \partial_t z^\tau(t), \frac{1}{\Gamma(2-\alpha)} (t-s)^{1-\alpha} y(s) \Big|_0^t + \frac{1-\alpha}{\Gamma(2-\alpha)} \int_0^t \frac{y(s)}{(t-s)^\alpha} ds \Big\rangle_H \\
&= \int_0^T \Big\langle \partial_t z^\tau(t), \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} y_0 \Big\rangle_H dt - \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^t \Big\langle \partial_t z^\tau(t), \frac{y(s)}{(t-s)^\alpha} ds \Big\rangle_H \\
&= -\frac{1}{\Gamma(1-\alpha)} \int_0^T \Big\langle \frac{z^\tau(t)}{t^\alpha}, y_0 \Big\rangle_H dt + \int_0^T \Big\langle \partial_{t,T-}^\alpha z^\tau(t), y(t) \Big\rangle_H dt.
\end{aligned} \tag{4.4}$$

Clearly,

$$\langle Ay, z^\tau \rangle_H = \langle y, A^* z^\tau \rangle_H \tag{4.5}$$

and

$$\begin{aligned}
&\int_0^T \Big\langle y(t), \frac{\alpha z_0}{\Gamma(1-\alpha)(\tau-t)^{1+\alpha}} \Big\rangle_H dt \\
&= -\frac{1}{\Gamma(1-\alpha)} \Big\langle y(T), \frac{z_0}{(\tau-T)^\alpha} \Big\rangle_H + \frac{1}{\Gamma(1-\alpha)} \Big\langle y_0, \frac{z_0}{(\tau-T)^\alpha} \Big\rangle_H \\
&\quad + \Big\langle \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{y'(t)}{(\tau-t)^\alpha} dt, z_0 \Big\rangle_H
\end{aligned} \tag{4.6}$$

From (4.4) to (4.6), we have that

$$\begin{aligned}
&-\frac{1}{\Gamma(1-\alpha)} \int_0^T \Big\langle \frac{z^\tau(t)}{t^\alpha} - \frac{z_0}{(\tau-t)^\alpha}, y_0 \Big\rangle_H dt \\
&+ \frac{1}{\Gamma(1-\alpha)(\tau-T)^\alpha} \langle z_0, y(T) \rangle_H - \Big\langle z_0, \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{y'(t)}{(\tau-t)^\alpha} dt \Big\rangle_H = \int_0^T \langle u, B^* z^\tau \rangle_H dt.
\end{aligned} \tag{4.7}$$

Therefore, for any $\tau \in (T, +\infty)$ and $z_0 \in D(A^*)$, it holds that

$$\frac{1}{\Gamma(1-\alpha)(\tau-T)^\alpha} \langle z_0, y(T) \rangle_H - \Big\langle z_0, \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{y'(t)}{(\tau-t)^\alpha} dt \Big\rangle_{D(A^*),D(A^*)'} = 0. \tag{4.8}$$

Since

$$\begin{aligned}
&\lim_{\tau \to T^+} \Big\langle z_0, \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{y'(t)}{(\tau-t)^\alpha} dt \Big\rangle_{D(A^*),D(A^*)'} \\
&= \Big\langle z_0, \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{y'(t)}{(T-t)^\alpha} dt \Big\rangle_{D(A^*),D(A^*)'} = \langle z_0, (\partial_t^\alpha y)(T) \rangle_{D(A^*),D(A^*)'} < \infty,
\end{aligned}$$

it holds that

$$\lim_{\tau \to T^+} \frac{1}{\Gamma(1-\alpha)(\tau-T)^\alpha} \langle z_0, y(T) \rangle_H < \infty,$$

which implies that for all $z_0 \in D(A^*)$, $\langle z_0, y(T)\rangle_H = 0$. Hence, we get that $y(T) = 0$. From (4.8), we find that for any $z_0 \in D(A^*)$,

$$\Big\langle z_0, \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{y'(t)}{(\tau - t)^\alpha} dt \Big\rangle_{D(A^*),D(A^*)'} = 0.$$

This implies that for any $\tau \in (T, +\infty)$,

$$\frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{y'(t)}{(\tau - t)^\alpha} dt = 0. \tag{4.9}$$

Hence, for any $\tau > T$,

$$(\partial_{t,0+}^\alpha y)(\tau) = \frac{1}{\Gamma(1-\alpha)} \int_0^\tau \frac{y'(t)}{(\tau - t)^\alpha} dt + \frac{1}{\Gamma(1-\alpha)} \int_T^\tau \frac{y'(t)}{(\tau - t)^\alpha} dt = (\partial_{t,T+}^\alpha y)(\tau). \tag{4.10}$$

From (2.2) and (4.10), for $\tau > T$, $y(\cdot)$ is the solution to

$$\begin{cases} (\partial_{t,T+}^\alpha y)(\tau) + Ay(\tau) = 0 \quad \text{in } (T, +\infty), \\ y(T) = 0. \end{cases} \tag{4.11}$$

Hence, we get that $y(\tau) = 0$ for all $\tau \geq T$.

The "only if" part. Define a bounded linear operator $\mathcal{L} : H \to H$ as follows:

$$\mathcal{L}(z_0) = -\frac{1}{\Gamma(1-\alpha)} \int_0^T \Big\langle \frac{z^\tau(t)}{t^\alpha} - \frac{z_0}{(\tau - t)^\alpha}, y_0 \Big\rangle_H dt,$$

where $z^\tau(\cdot)$ is the solution to (4.2). If (4.3) does not hold, then, one can find a sequence $\{z_0^k\}_{k=1}^\infty \subset H$ with $z_0^k \neq 0$ for all $k \in \mathbb{N}$, such that the corresponding solutions $z^{\tau,k}$ to (4.2) (with z_0 replaced by z_0^k) satisfy that

$$|B^* z^{\tau,k}|_{L^2(0,T;U)} \leq \frac{1}{k} \Big| \frac{1}{\Gamma(1-\alpha)} \int_0^T \Big\langle \frac{z^\tau(t)}{t^\alpha} - \frac{z_0}{(\tau - t)^\alpha}, y_0 \Big\rangle_H dt \Big|_H. \tag{4.12}$$

Write

$$\tilde{z}_0^k = \frac{\sqrt{k} z_0^k}{\mathcal{L}(z_0^k)},$$

and denote by $\tilde{z}^{\tau,k}$ the corresponding solution to (4.2) (with z_0 replaced by $\tilde{z}_0^k$). Then, it follows from (4.12) that, for each $k \in \mathbb{N}$,

$$|B^* z^{\tau,k}|_{L^2(0,T;U)} \leq \frac{1}{\sqrt{k}} \tag{4.13}$$

and

$$|\mathcal{L}(\tilde{z}_0^k)|_H = \sqrt{k}. \tag{4.14}$$

On the other hand, since the system (2.2) is null controllable, for a given $y_0 \in H$, we have a control $u \in L^2(0,T;U)$ driving the corresponding solution to rest. Similar to the proof of (4.7), we have that

$$-\frac{1}{\Gamma(1-\alpha)}\int_0^T \Big\langle \frac{\tilde{z}^{\tau,k}(t)}{t^\alpha} - \frac{\tilde{z}_0^k}{(\tau-t)^\alpha}, y_0\Big\rangle_H dt = \int_0^T \langle u, B^*\tilde{z}^{\tau,k}\rangle_H dt. \tag{4.15}$$

By (4.13) and (4.15), we have that

$$\mathcal{L}(\tilde{z}_0^k) \text{ tends to 0 weakly in } H \text{ as } k \to +\infty$$

Hence, by the Principle of Uniform Boundedness, the sequence $\{\mathcal{L}(\tilde{z}_0^k)\}_{k=1}^\infty$ is uniformly bounded in H, which contradicts (4.14). This completes the proof of Proposition 4.1.

As an immediate corollary of Theorem 3.1 and Proposition 4.1, we have the following result.

Corollary 4.1 *The system* (4.2) *is not initially observable.*

5 Partial null controllability

In this section we analyse more closely the particular and relevant case of fractional diffusion models.

Let $\Omega \subset \mathbb{R}^d (d \in \mathbb{N})$ be a bounded domain with the C^∞ boundary $\partial\Omega$, and $\omega \subset \Omega$ be an open subset. Define an unbounded linear operator A_Ω on $L^2(\Omega)$ as follows:

$$\begin{cases} D(A_\Omega) = H^2(\Omega) \cap H_0^1(\Omega), \\ A_\Omega \varphi = -\Delta\varphi, \quad \forall \varphi \in D(A_\Omega). \end{cases}$$

Denote by $\{\lambda_j\}_{j=1}^\infty$ with $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ the eigenvalues of A_Ω and $\{e_j\}_{j=1}^\infty$ with $|e_j|_{L^2(\Omega)} = 1$ the corresponding eigenvectors. For $\beta \in [0,+\infty)$, we define A_Ω^β as follows:

$$\begin{cases} D(A_\Omega^\beta) = \Big\{ f \,\Big|\, f = \sum_{j=1}^\infty f_j e_j \text{ with } \sum_{j=1}^\infty \lambda_j^{2\beta} f_j^2 < \infty \Big\}, \\ A_\Omega^\beta f = \sum_{j=1}^\infty \lambda_j^\beta f_j e_j, \quad \forall\, f = \sum_{j=1}^\infty f_j e_j. \end{cases}$$

Let $H = U = L^2(\Omega)$, $A = A_\Omega^\beta$ and $Bu = \chi_\omega u$. Then the system (2.2) reads

$$\begin{cases} \partial_t^\alpha y - A_\Omega^\beta y = Bu \quad \text{in } (0,+\infty), \\ y(0) = y_0. \end{cases} \tag{5.1}$$

A possible strategy to establish the partial null controllability of (5.1) would be inspired in the iteration method for the null control of the heat equation introduced by Lebeau and Robbiano in [20] (see also [22]). It works well for $\alpha = 1$ (classical first order derivative term in time) and $\beta > \frac{1}{2}$, as described in [24], [30] and [32]. Let us explain its main idea here.

Let

$$T_k = \begin{cases} 0, & \text{if } k = 1, \\ T\sum_{i=1}^{k-1} 2^{-i}, & \text{if } k > 1, \end{cases} \tag{5.2}$$

and

$$\widetilde{T}_k = \begin{cases} \dfrac{T}{4}, & \text{if } k = 1, \\ T\Big(\sum_{i=1}^{k-1} 2^{-i} + 2^{-k-1}\Big), & \text{if } k > 1. \end{cases} \tag{5.3}$$

Put

$$I_k = [T_k, \widetilde{T}_k), \quad J_k = [\widetilde{T}_k, T_{k+1}), \quad r_k = \frac{16C_1^2}{(T_{k+1} - \widetilde{T}_k)^4}, \quad \text{for } k = 1, 2, \cdots$$

Clearly,

$$r_k \to +\infty \text{ as } k \to +\infty. \tag{5.4}$$

For each $k \in \mathbb{N}$, denote by P_k the orthogonal projection from $L^2(G)$ to $\mathrm{Span}_{\lambda_j \le r_k}\{e_j\}$. On each interval I_k, one can find a control $u^{(k)}(\cdot) \in L^2(I_k; L^2(\omega))$ such that the corresponding solution $y^{(k)}(\cdot)$ to (5.1) on I_k satisfies

$$P_k\big(y^{(k)}\big(\widetilde{T}_k\big)\big) = 0.$$

Furthermore, there is a constant C_1, independent of k and y, such that

$$|u|_{L^2(I_k;L^2(\omega))} \le C_1 e^{C_1\sqrt{r_k}} |y(T_k)|_{L^2(\Omega)}. \tag{5.5}$$

On every interval J_k, we let the heat equation freely evolve. We start by having the initial datum for the equation on I_1 to be y_0. For the initial datum on I_k, $k = 2, 3, \cdots$, we define it the ending value of the solution to the equation on J_{k-1}. The initial datum of the equation on J_k, $k = 1, 2, \cdots$, is given by the ending value of the solution for the equation on I_k. If there is no eigenvalue of $-A$ in $(r_k, r_{k+1}]$, we simply set $u^{(k)}(\cdot) = 0$ on I_k. Thanks to the energy decay of (5.1), we can get that there is a constant C_2, independent of k and y, such that

$$|y(T_k)|_{L^2(\Omega)} \le Ce^{-Cr_{k-1}^{\beta}}. \tag{5.6}$$

The inequalities (5.5) and (5.6) yield that the control

$$|u|_{L^2(I_k;L^2(\omega))} \le (C_1 + C_2)e^{C_1\sqrt{r_k} - C_2 r_{k-1}^{\beta}} |y_0|_{L^2(\Omega)}. \tag{5.7}$$

Hence, we obtain that

$$u(\cdot) = \sum_{k=1}^{\infty} \chi_{I_k}(\cdot) u^{(k)} \in L^2(0, T; L^2(\omega)),$$

drives the solution of the system (5.1) to 0 at time T.

Clearly, that a key point to guarantee the above strategy works is the exponential decay of the solution, i.e., the Fourier coefficient corresponding to the j-th eigenfunction e_j of y

decays as $e^{-\lambda_j^\beta}$. Such kind of decay compensates the term $e^{C_1\sqrt{r_k}}$ if $\beta > \frac{1}{2}$. This also explains why the method fails when $\beta < \frac{1}{2}$. The reason why the method also fails in the critical case $\beta = 1/2$ is more subtle and is related with Müntz Lemma above (see [30]).

This iterative method also fails when considering fractional order in time derivative terms as in (5.1).

Indeed, let us recall that for $a, b \in \mathbb{C}$, $\operatorname{Re} a > 0$, the Mittag-Leffler function $E_{a,b}$ is defined by

$$E_{a,b}(s) = \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(ak+b)}, \qquad \forall s \in \mathbb{C}. \tag{5.8}$$

Let us recall some properties of Mittag-Leffler functions as follows ([6]):

$$E_{\alpha,1}(x) = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{a_k(\alpha)}{x^{k+1}}, \quad 0 \le \alpha < 1, \tag{5.9}$$

where $\{a_k(\alpha)\}_{k=0}^{\infty} \subset \mathbb{R}$ and $a_0(\alpha) \neq 0$ for $\alpha \in (0,1)$. This implies that

$$E_{\alpha,1}(-x) = O\Big(\frac{1}{x}\Big), \qquad \text{as } x \to +\infty. \tag{5.10}$$

Furthermore, the following recurrence relation holds:

$$E_{\alpha,1}(-x) = \frac{2x}{\pi} \int_0^\infty \frac{E_{2\alpha,1}(-\gamma^2)}{x^2+\gamma^2} d\gamma, \quad 0 \le \alpha \le 1. \tag{5.11}$$

Particularly,

$$E_{1,1}(-x) = \frac{2x}{\pi} \int_0^\infty \frac{\cosh(i\gamma)}{x^2+\gamma^2} d\gamma = e^{-x} \tag{5.12}$$

and

$$E_{\frac{1}{2},1}(-x) = \frac{2x}{\pi} \int_0^\infty \frac{e^{-\gamma^2}}{x^2+\gamma^2} d\gamma. \tag{5.13}$$

By (5.13), we have that

$$E_{\frac{1}{2},1}(-x) = \frac{2x}{\pi} \int_0^{x^2} \frac{e^{-\gamma^2}}{x^2+\gamma^2} d\gamma + \frac{2x}{\pi} \int_{x^2}^\infty \frac{e^{-\gamma^2}}{x^2+\gamma^2} d\gamma \le 2\Big(\frac{1}{x} + e^{-x^2}\Big). \tag{5.14}$$

This indicates that there are two components in $E_{\frac{1}{2},1}(\cdot)$. As $x \to -\infty$, one of them decays polynomially and the other one decays exponentially.

For $j \in \mathbb{N}$, put

$$y_{0,j} = \langle y_0, e_j \rangle_{L^2(\Omega)} \text{ and } u_j = \langle \chi_\omega u, e_j \rangle_{L^2(\Omega)}.$$

The solution to (5.1) reads

$$y(x,t) = \sum_{j=1}^{\infty} E_{\alpha,1}(-\lambda_j^\beta t^\alpha) y_{0,j} e_j(x) + \sum_{j=1}^{\infty} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda^\beta (t-s)^\alpha) u_j(s) e_j(x) ds. \tag{5.15}$$

If we take $u = 0$, then the solution is

$$y(x,t) = \sum_{j=1}^{\infty} E_{\alpha,1}(-\lambda_j^{\beta} t^{\alpha}) y_{0,j} e_j(x). \tag{5.16}$$

Hence, if $\alpha \in (0,1)$, it holds that

$$E_{\alpha,1}(-\lambda_j^{\beta} t^{\alpha}) = O\Big(\frac{1}{\lambda_j^{\beta} t^{\alpha}}\Big) \text{ as } j \to +\infty.$$

This concludes that the solution to (5.1) decays polynomially rather than exponentially.

On the other hand, by (5.12), we have that y decays exponentially if $\alpha = 1$.

6 Further comments

- **Existence of feasible controls for Problem (LQ).**

  We have proved in Section 3 that it is impossible to prove the existence of a feasible control for Problem (LQ) by establishing the null controllability of (2.2). There is however another possibility to achieve this goal, without requiring the null controllability property that fails. indeed, note that in order to prove that the class of feasible controls is non-empty it is sufficient to show the existence of controls such that the solution to (2.2) decays fast enough so that $\mathcal{J}(u) < +\infty$. One could try to achieve this property as in the classical ODE and PDE control theory by means of a suitable feedback operator,

  Let us comment on the stabilization problem for a special case of (2.2), i.e., (5.1).

  We first consider the simplest case that $\omega = \Omega$. If $\alpha = 1$, to guarantee that the solution to (2.3) decays exponentially with a decay rate λ, it suffices to take the feedback control $u = -\lambda \hat{y}$. However, if $\alpha \neq 1$ with the same feedback $u = -\lambda y$ in (5.1), we have that

$$\begin{cases} \partial_t^{\alpha} y - A_{\Omega}^{\beta} y = -\lambda y \quad \text{in } (0,+\infty), \\ y(0) = y_0. \end{cases} \tag{6.1}$$

  The solution to (6.1) is

$$y(x,t) = \sum_{j=1}^{\infty} E_{\alpha,1}\big(-(\lambda_j^{\beta} + \lambda) t^{\alpha}\big) y_{0,j} e_j(x). \tag{6.2}$$

  From (5.9), it holds that

$$E_{\alpha,1}\big(-(\lambda_j^{\beta} + \lambda) t^{\alpha}\big) = O\Big(\frac{1}{(\lambda_j^{\beta} + \lambda) t^{\alpha}}\Big) \text{ as } j \to +\infty.$$

  Hence, such kind of feedback does not modify the decay rate of y as t tends infinity.

  Obviouslyy if $\alpha > \frac{1}{2}$, the null control $u \equiv 0$ is feasible, since the free solution decays sufficiently has to ensure that the cost functional J is finite. Whether Problem (LQ) has a feasible control for the case $\alpha < \frac{1}{2}$ is unknown.

- **Stabilization.**

  The stabilization problem for the system (2.2) consists precisely on finding a feedback control $u = F(y)$, with a suitable linear map F, to accelerate the speed of the decay of solutions of the free system as $t \to \infty$.

  The stabilization problem for the system (2.3) involving classical first order in time derivatives, has been studied extensively. Usually, one can use an instantaneous feedback operator, acting in any give time instance t, out of partial measurements of the state at time t, without employing the past memory of the system, $F \in \mathcal{L}(H, U)$, to stabilize the system (2.3). One typically seeks for exponential decay properties although in some particular cases of infinite-dimensional conservative systems, the decay achieved can be slow, either polynomial or logarithmic in time(see [3, 10, 21, 33] and the references therein).

  However, in the present context of fractional in time models, instantaneous feedback operators $F \in \mathcal{L}(H, U)$, may not suffice to achieve the exponential decay of the system (2.2). This can be seen, for instance, on the following toy model:

$$\begin{cases} \partial_t^\alpha y = u & \text{in } (0, +\infty), \\ y(0) = y_0. \end{cases} \tag{6.3}$$

  In this case, $F \in \mathcal{L}(H, U) = \mathcal{L}(\mathbb{R}, \mathbb{R})$ means that $Fy = cy$ for some $c \in \mathbb{R}$. Furthermore, to stabilize the system, we should let $c < 0$. Then the solution to (6.3) reads

$$y(t) = E_{\alpha,1}(ct^\alpha)y_0 = O\left(\frac{1}{ct^\alpha}\right) \text{ as } t \to +\infty.$$

  Clearly, no matter what the value of c is, the solution decays polynomially. This example also shows that if $F \in \mathcal{L}(H, U)$, the rate of the polynomial decay depends only on α, which cannot be improved by the choice of F.

  According to this example, to stabilize (2.2), one should use other forms of feedback operators, possibly including the effect of the past memory of the system.

- **Higher order fractional in time derivatives.**

  We have studied the case that $0 < \alpha < 1$ and proved that (2.2) is not null controllable. By a similar argument, one can prove that for any $\alpha > 0$, $\alpha \notin \mathbb{N}$, the system (2.2) is not null controllable. We also can show that a linear bounded feedback cannot be used to stabilize the system (2.2) for any $\alpha > 0$, $\alpha \notin \mathbb{N}$.

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