Exact Controllability and Asymptotic Limit for Thin Plates

Isabel Figueiredo * Enrique Zuazua †

Abstract

We consider the exact controllability problem for a three-dimensional linear elastic thin plate, with thickness 2ε and a polygonal middle surface. Controls are imposed on the lateral surface and at the top and bottom of the plate. The asymptotic limit when $\varepsilon \to 0$ is computed. We obtain that the displacements converge to a controlled Kirchhoff-Love displacement, where the normal displacement satisfies the usual two-dimensional evolution equation for a linear plate, with controls on the boundary and in the interior of the plate.

1 Notations and main results

Consider a plate $\overline{\Omega}^\epsilon = \overline{\omega} \times [-\varepsilon, \varepsilon]$, where ω is a polygonal domain in $\mathbb{R}^2$ and $\varepsilon > 0$ a small parameter. Let $\Gamma^\varepsilon_\pm = \partial\omega \times \{-\varepsilon, \varepsilon\}$ be the upper and lower faces of the plate and $\Gamma^\varepsilon_0 = \partial\omega \times (-\varepsilon, \varepsilon)$ the lateral surface.

In the following greek indices $\alpha, \beta, ...$ will belong to the set $\{1, 2\}$, latin indices $i, j, ...$ will belong to the set $\{1, 2, 3\}$ and the usual summation convention on the repeated index will be adopted.

We consider the 3D (three-dimensional) elasticity problem for the plate Ω^ε, that is

$$\begin{cases} \rho^\varepsilon \ddot{\psi}^\varepsilon_i - \partial^\varepsilon_j \sigma^\varepsilon_{ij}(\boldsymbol{\psi}^\varepsilon) = 0, & \text{in} \quad Q^\varepsilon = \Omega^\varepsilon \times (0, T), \\ \psi^\varepsilon_i = u^\varepsilon_i, & \text{on} \quad \Sigma^\varepsilon_0 = \Gamma^\varepsilon_0 \times (0, T), \\ \sigma^\varepsilon_{ij}(\boldsymbol{\psi}^\varepsilon)\nu^\varepsilon_j = v^\varepsilon_i, & \text{on} \quad \Sigma^\varepsilon_\pm = \Gamma^\varepsilon_\pm \times (0, T), \\ \boldsymbol{\psi}^\varepsilon(0) = \boldsymbol{\psi}^\varepsilon_0, \quad \dot{\boldsymbol{\psi}}^\varepsilon(0) = \boldsymbol{\psi}^\varepsilon_1, & \text{in} \quad \Omega^\varepsilon. \end{cases} \tag{1.1}$$

The vector $\boldsymbol{\nu}^\varepsilon = (\nu^\varepsilon_j)$ is the unit outer normal vector along $\partial\Omega^\varepsilon$, ρ^ε is the density of mass and σ^ε_{ij} are the components of the stress tensor, that is

$$\sigma^\varepsilon_{ij}(\boldsymbol{\psi}^\varepsilon) = \lambda^\varepsilon e^\varepsilon_{pp}(\boldsymbol{\psi}^\varepsilon)\delta_{ij} + 2\mu^\varepsilon e^\varepsilon_{ij}(\boldsymbol{\psi}^\varepsilon), \tag{1.2}$$

where $e^\varepsilon_{ij}(\boldsymbol{\psi}^\varepsilon) = \frac{1}{2}(\partial^\varepsilon_j \psi^\varepsilon_i + \partial^\varepsilon_i \psi^\varepsilon_j)$ are the components of the stress tensor and λ^ε, μ^ε are the Lamé constants of the material. In addition, we assume that the plates are homogeneous. More precisely, λ^ε, μ^ε and ρ^ε do not depend on $\boldsymbol{x}^\varepsilon$ and verify

$$\lambda^\varepsilon = \lambda, \qquad \mu^\varepsilon = \mu, \qquad \rho^\varepsilon = \varepsilon^2 \rho, \tag{1.3}$$

where λ, μ and ρ are positive constants independent of ε.

*Departamento de Matemática, Universidade de Coimbra, Apartado 3008, 3000 Coimbra, Portugal.
†Departamento de Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, España.

The upper dot $\cdot$ (e.g. $\dot{\boldsymbol{\psi}}^\varepsilon(0)$) means time derivative and ∂_j^ε space derivative, with respect to x_j^ε. On the other hand, $\boldsymbol{\psi}_0^\varepsilon, \boldsymbol{\psi}_1^\varepsilon$ are the initial data and u_i^ε, v_i^ε are the controls that act on the system through the lateral boundary Γ_0^ε and the top and bottom of the plate $\Gamma_\pm^\varepsilon$, respectively. In order to simplify the notations we will omit often the dependence on t ; for instance in (1.1) it is understood that $\boldsymbol{\psi}^\varepsilon = (\psi_i^\varepsilon)$ and u_i^ε, v_i^ε depend on t.

The exact controllability problem for (1.1) may be formulated as follows : given initial data $\{\boldsymbol{\psi}_0^\varepsilon, \boldsymbol{\psi}_1^\varepsilon\}$ in a suitable energy space, does there exist a time $T > 0$ and controls $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ on Γ_0^ε and $\boldsymbol{v}^\varepsilon = (v_i^\varepsilon)$ on $\Gamma_\pm^\varepsilon$, such that the unique solution $\boldsymbol{\psi}^\varepsilon$ of problem (1.1) reaches the equilibrium at the time T, that is $\boldsymbol{\psi}^\varepsilon(T) = \dot{\boldsymbol{\psi}}^\varepsilon(T) = 0$?

This problem may be solved by the methods developed by Lions [7] and the result is established in theorem 1.1. However, we are also concerned with the behaviour of controls and controlled solutions as $\varepsilon \to 0$. Our goal is to construct controls $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{v}^\varepsilon$ such that, as $\varepsilon \to 0$, the controlled solution $\boldsymbol{\psi}^\varepsilon$ (satisfying $\boldsymbol{\psi}^\varepsilon(T) = \dot{\boldsymbol{\psi}}^\varepsilon(T) = 0$) converges in a suitable weak topology to the controlled solution of the corresponding limiting plate model. In order to do that, we will combine in a suitable way, the results of the asymptotical analysis for the plate problem, developed by Ciarlet [2] and his school, with the controllability methods introduced by Lions [7]. The main result is presented in theorem 1.2.

We will now describe some other notations and the principal results of this work.

To carry out the asymptotical analysis it is convenient to formulate problem (1.1) in a set independent of ε, by making some appropriate scalings in the unknowns and data - we refer in special to Ciarlet [2] for the justification of these scalings.

More specifically, we set $\Omega = \omega \times (-1, 1)$, $\gamma = \partial\omega$, $\Gamma_\pm = \partial\omega \times \{-1, 1\}$ and $\Gamma_0 = \partial\omega \times (-1, 1)$. We will denote by $d\Omega$ the element of volume in Ω, by $d\gamma$, $d\Gamma_\pm$, $d\Gamma_0$ the elements of surface in γ, $\Gamma_\pm$, Γ_0 respectively, and by dt the element of time in the interval $[0, T]$.

We introduce the scaled displacement $\boldsymbol{\psi}(\varepsilon) = (\psi_i(\varepsilon)) : \overline{\Omega} \times (0, T) \to R^3$ and the scaled initial data $\boldsymbol{\psi}_0(\varepsilon) = (\psi_{0i}(\varepsilon))$ and $\boldsymbol{\psi}_1(\varepsilon) = (\psi_{1i}(\varepsilon))$: $\overline{\Omega} \to R^3$ by letting

$$\begin{aligned} \psi_\alpha^\varepsilon(\boldsymbol{x}^\varepsilon, t) &= \varepsilon^2 \psi_\alpha(\varepsilon)(\boldsymbol{x}, t), & \psi_3^\varepsilon(\boldsymbol{x}^\varepsilon, t) &= \varepsilon \psi_3(\varepsilon)(\boldsymbol{x}, t), \\ \psi_{0\alpha}^\varepsilon(\boldsymbol{x}^\varepsilon, t) &= \varepsilon^2 \psi_{0\alpha}(\varepsilon)(\boldsymbol{x}, t), & \psi_{03}^\varepsilon(\boldsymbol{x}^\varepsilon, t) &= \varepsilon \psi_{03}(\varepsilon)(\boldsymbol{x}, t), \\ \psi_{1\alpha}^\varepsilon(\boldsymbol{x}^\varepsilon, t) &= \varepsilon^2 \psi_{1\alpha}(\varepsilon)(\boldsymbol{x}, t), & \psi_{13}^\varepsilon(\boldsymbol{x}^\varepsilon, t) &= \varepsilon \psi_{13}(\varepsilon)(\boldsymbol{x}, t), \end{aligned} \tag{1.4}$$

for all $\boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \overline{\Omega}^\varepsilon$ and all $t \geq 0$, where $\pi^\varepsilon(\boldsymbol{x}) = \pi^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, \varepsilon x_3)$.

We define the spaces

$$\begin{aligned} H^1_{|\Gamma_0}(\Omega) = &\quad \{v \in H^1(\Omega) : \ v_{|\Gamma_0} = 0\}, \qquad \boldsymbol{V}(\Omega) = \left[H^1_{|\Gamma_0}(\Omega)\right]^3 \\ X = &\quad \left\{g \in L^1(0, T; L^2(\Omega)) : \quad \dot{g} \in L^1(0, T; [H^1_{|\Gamma_0}(\Omega)]')\right\}. \end{aligned} \tag{1.5}$$

The spaces $\boldsymbol{V}(\Omega)$ and $H^1_{|\Gamma_0}(\Omega)$ are endowed with the usual norms. On the other hand X is a Banach space with the norm

$$||g||_X = \left[||g||^2_{L^1(0,T;L^2(\Omega))} + ||\dot{g}||^2_{L^1(0,T;[H^1_{|\Gamma_0}(\Omega)]')}\right]^{\frac{1}{2}}.$$

Now, the scaled elasticity problem corresponding to (1.1) is the following

$$\begin{cases} \varepsilon^2\rho\ddot{\psi}_\alpha(\varepsilon) - \partial_j\sigma_{\alpha j}(\varepsilon)(\boldsymbol{\psi}(\varepsilon)) = 0, & \text{in} \quad Q = \Omega \times (0,T), \\ \rho\ddot{\psi}_3(\varepsilon) - \partial_j\sigma_{3j}(\varepsilon)(\boldsymbol{\psi}(\varepsilon)) = 0, & \text{in} \quad Q = \Omega \times (0,T), \\ \psi_i(\varepsilon) = u_i(\varepsilon), & \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,T), \\ \sigma_{ij}(\varepsilon)(\boldsymbol{\psi}(\varepsilon))\nu_j = v_i(\varepsilon), & \text{on} \quad \Sigma_\pm = \Gamma_\pm \times (0,T), \\ \boldsymbol{\psi}(\varepsilon)(0) = \boldsymbol{\psi}_0(\varepsilon), \quad \dot{\boldsymbol{\psi}}(\varepsilon)(0) = \boldsymbol{\psi}_1(\varepsilon), & \text{in} \quad \Omega, \end{cases} \tag{1.6}$$

where ∂_j is the partial derivative with respect to x_j, $\sigma_{ij}(\varepsilon)(.)$ are the scaled components of the stress tensor, that is

$$\begin{aligned} \sigma_{\alpha\beta}(\varepsilon)(\boldsymbol{\psi}(\varepsilon)) &= \lambda\Big[e_{\zeta\zeta}(\boldsymbol{\psi}(\varepsilon)) + \frac{1}{\varepsilon^2}e_{33}(\boldsymbol{\psi}(\varepsilon))\Big]\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(\boldsymbol{\psi}(\varepsilon)), \\ \sigma_{\alpha 3}(\varepsilon)(\boldsymbol{\psi}(\varepsilon)) &= 2\mu\frac{1}{\varepsilon^2}e_{\alpha 3}(\boldsymbol{\psi}(\varepsilon)), \\ \sigma_{33}(\varepsilon)(\boldsymbol{\psi}(\varepsilon)) &= \frac{1}{\varepsilon^2}\Big[\lambda\Big\{e_{\zeta\zeta}(\boldsymbol{\psi}(\varepsilon)) + \frac{1}{\varepsilon^2}e_{33}(\boldsymbol{\psi}(\varepsilon))\Big\} + 2\mu\frac{1}{\varepsilon^2}e_{33}(\boldsymbol{\psi}(\varepsilon))\Big], \end{aligned} \tag{1.7}$$

and the vector $\boldsymbol{\nu} = (\nu_i)$ is the unit outer normal along $\partial\Omega$.

We introduce the function $\boldsymbol{q} = (q_j)$ by

$$\boldsymbol{q}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{x}_0, \qquad \forall \boldsymbol{x} \in \overline{\Omega}, \tag{1.8}$$

where $\boldsymbol{x}_0 = (x_{01}, x_{02}, 0)$ is a fixed point in ω, and the constants $R(\boldsymbol{x}_0)$ and $T(\varepsilon)$ by

$$R(\boldsymbol{x}_0) = ||\boldsymbol{x} - \boldsymbol{x}_0||_{L^\infty(\Omega)}, \qquad T(\varepsilon) = \varepsilon\frac{2\sqrt{\rho}}{\sqrt{\mu}}\max\Big\{R(\boldsymbol{x}_0), \frac{C(\Omega)}{R(\boldsymbol{x}_0)}\Big\}, \tag{1.9}$$

where $C(\Omega)$ is the constant of continuity of the trace map $tr\colon \boldsymbol{V}(\Omega) \to [L^2(\partial\Omega)]^3$. By τ_j we will denote the j-th component of the tangential gradient on $\partial\Omega$.

We have the following exact controllability result for the 3D plate Ω.

Theorem 1.1 *Let $0 < \varepsilon \leq 1$ be fixed. If $T > T(\varepsilon)$, the elasticity system (1.6) is exactly controllable. More precisely, if $\{\boldsymbol{\psi}_0(\varepsilon), \boldsymbol{\psi}_1(\varepsilon)\} \in [L^2(\Omega)]^3 \times [\boldsymbol{V}(\Omega)]'$ there exist controls of the form*

$$\begin{aligned} u_i(\varepsilon) &= (q_j\nu_j)\tfrac{\partial\phi_i(\varepsilon)}{\partial\nu}, \quad \textit{on} \quad \Gamma_0 \times (0,T), \\ v_\alpha(\varepsilon) &= (q_j\nu_j)\Big[\varepsilon^2\rho\ddot{\phi}_\alpha(\varepsilon) - \tau_j\sigma_{\alpha j}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\Big], \quad \textit{on} \quad \Gamma_\pm \times (0,T), \\ v_3(\varepsilon) &= (q_j\nu_j)\Big[\rho\ddot{\phi}_3(\varepsilon) - \tau_j\sigma_{3j}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\Big], \quad \textit{on} \quad \Gamma_\pm \times (0,T), \end{aligned} \tag{1.10}$$

where $\boldsymbol{\phi}(\varepsilon) = (\phi_i(\varepsilon))$ *is the solution of the following homogeneous 3D elasticity problem*

$$\begin{cases} \varepsilon^2 \rho \ddot{\phi}_\alpha(\varepsilon) - \partial_j \sigma_{\alpha j}(\varepsilon)(\boldsymbol{\phi}(\varepsilon)) = 0, & in \quad Q = \Omega \times (0,T), \\ \rho \ddot{\phi}_3(\varepsilon) - \partial_j \sigma_{3j}(\varepsilon)(\boldsymbol{\phi}(\varepsilon)) = 0, & in \quad Q = \Omega \times (0,T), \\ \phi_i(\varepsilon) = 0, & on \quad \Sigma_0 = \Gamma_0 \times (0,T), \\ \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\nu_j = 0, & on \quad \Sigma_\pm = \Gamma_\pm \times (0,T), \\ \boldsymbol{\phi}(\varepsilon)(0) = \boldsymbol{\phi}_0(\varepsilon), \quad \dot{\boldsymbol{\phi}}(\varepsilon)(0) = \boldsymbol{\phi}_1(\varepsilon), & in \quad \Omega, \end{cases} \tag{1.11}$$

such that the solution $\boldsymbol{\psi}(\varepsilon)$ *of (1.6) satisfies*

$$\boldsymbol{\psi}(\varepsilon)(T) = \dot{\boldsymbol{\psi}}(\varepsilon)(T) = 0.$$

Moreover, the initial control data $\{\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_1(\varepsilon)\}$ *belong to* $\boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$ *and verify*

$$\Lambda(\varepsilon)(\{\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_1(\varepsilon)\}) = \{\boldsymbol{\psi}_1(\varepsilon), -\boldsymbol{\psi}_0(\varepsilon)\}, \tag{1.12}$$

where $\Lambda(\varepsilon) : \boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3 \to [\boldsymbol{V}(\Omega)]' \times [L^2(\Omega)]^3$ *is the isomorphism given by HUM (Hilbert Uniqueness Method).* ∎

The theorem 1.1 may be proved by the methods developed in Lions [7], combining HUM (Hilbert Uniqueness Method) and multiplier techniques.

We notice that the optimal controllability time is $T^*(\varepsilon) = \varepsilon \frac{2\sqrt{\rho}}{\sqrt{\mu}} R(\boldsymbol{x}_0)$ and not the time $T(\varepsilon)$, indicated in (1.9). But we are forced to take the time $T(\varepsilon)$ and not $T^*(\varepsilon)$, in order to be able to compute the asymptotical limit when $\varepsilon \to 0$.

Remark 1.1 As it is seen in (1.10), the controls $\boldsymbol{u}(\varepsilon)$ and $\boldsymbol{v}(\varepsilon)$ acting on Γ_0 and $\Gamma_\pm$ have a different structure.

It is easy to see that

$$u_i(\varepsilon) = (q_j \nu_j) \frac{\partial \phi_i(\varepsilon)}{\partial \nu} \in L^2(\Gamma_0 \times (0,T)) \tag{1.13}$$

since $\frac{\partial \phi_i(\varepsilon)}{\partial \nu} \in L^2(0,T; L^2(\Gamma_0))$ for every solution of (1.11) with initial data in $\boldsymbol{V}(\Omega) \times (L^2(\Omega))^3$ (see corollary 2.2 below).

The control $\boldsymbol{v}(\varepsilon)$ has to be understood in the sense of the following duality formula

$$< \boldsymbol{v}(\varepsilon), \boldsymbol{\theta} >= \int_{\Sigma_\pm} (q_j \nu_j) \Big[\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon)) \partial_j \theta_i - \varepsilon^2 \rho \dot{\phi}_\alpha(\varepsilon) \dot{\theta}_\alpha - \rho \dot{\phi}_3(\varepsilon) \dot{\theta}_3 \Big] d\Gamma_\pm dt. \tag{1.14}$$

for all $\boldsymbol{\theta} = (\theta_i)$ regular enough. As we will see in corollary 2.1 this integral makes sense for every solution of system (1.11) with data in $\boldsymbol{V}(\Omega) \times (L^2(\Omega))^3$.

Each of the components $\ddot{\phi}(\varepsilon)$ and $\tau_j \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))$ has a trace on $\Gamma_\pm \times (0,T)$ but separately they are not smooth enough to be applied to finite energy solutions of (1.11). A similar phenomenon arises in the wave equations with Neumann boundary conditions (cf. Lions [7], chapter 3). ∎

Concerning the limit behaviour of the solution $\boldsymbol{\psi}(\varepsilon)$ of problem (1.6) as $\varepsilon \to 0$ with the controls defined by (1.10) for $T > 0$ fixed, our main result is the following.

Theorem 1.2 *Assume that $T > 0$. Let $\{\boldsymbol{\psi}(\varepsilon)\}_{\varepsilon>0}$ be the sequence of controlled solutions of problem (1.6) (for $\varepsilon > 0$ small enough, such that $T(\varepsilon) < T$) with controls (1.10). Suppose that the corresponding sequence of initial data $\{\boldsymbol{\psi}_0(\varepsilon), \boldsymbol{\psi}_1(\varepsilon)\}_{\varepsilon>0}$ verifies*

$$\exists C > 0 \text{ (independent of } \varepsilon) \quad : \quad ||\{\boldsymbol{\psi}_1(\varepsilon), -\boldsymbol{\psi}_0(\varepsilon)\}||_{[V(\Omega)]' \times [L^2(\Omega)]^3} \leq C. \tag{1.15}$$

Then, there exists a subsequence $\{\boldsymbol{\psi}(\varepsilon_k)\}_{\varepsilon_k>0}$ of $\{\boldsymbol{\psi}(\varepsilon)\}_{\varepsilon>0}$ and functions $\{\psi_\alpha, \psi_3\}$, in the space $X'^2 \times L^\infty(0,T; L^2(\Omega))$ such that, for any $\{f_1, f_2, f_3\} \in X^2 \times L^1(0,T; L^2(\Omega))$

$$< \psi_i(\varepsilon_k), f_i > \longrightarrow < \psi_i, f_i >, \qquad as \quad \varepsilon_k \to 0.$$

Moreover, the limit function $\boldsymbol{\psi} = (\psi_i)$ satisfies :

i) *$\boldsymbol{\psi} = (\psi_i)$ is a Kirchhoff–Love displacement, that is, ψ_3 is independent of x_3 and $\psi_\alpha = \hat{\psi}_\alpha - x_3 \partial_\alpha \psi_3$, where $\hat{\psi}_\alpha$ is independent of x_3.*

ii) *ψ_3 is the solution (in the transposition sense) of the following 2D plate problem*

$$\begin{cases} 2\rho\ddot{\psi}_3 + \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\psi_3 = 2\rho\ddot{\phi}_3 + \frac{8\mu(\lambda+\mu)}{\lambda+2\mu}\Delta^2\phi_3, & in \quad \omega \times (0,T), \\ \psi_3 = 0, & on \quad \partial\omega \times (0,T), \\ \frac{\partial\psi_3}{\partial\nu} = (q_\alpha \nu_\alpha)\Delta\phi_3, & on \quad \partial\omega \times (0,T), \\ \psi_3(0) = \frac{1}{2}\int_{-1}^{+1}\psi_{03}dx_3, \quad \dot{\psi}_3(0) = \frac{1}{2}\int_{-1}^{+1}\psi_{13}dx_3, & in \quad \omega, \\ \psi_3(T) = \dot{\psi}_3(T) = 0, & in \quad \omega. \end{cases} \tag{1.16}$$

The couple $\{\psi_{03}, \psi_{13}\}$ is the weak limit in the space $L^2(\Omega) \times [H^1_{|\Gamma_0}(\Omega)]'$ of the sequence $\{\psi_{03}(\varepsilon_k), \psi_{13}(\varepsilon_k)\}_{\varepsilon_k>0}$ and the function ϕ_3, defining the controls in problem (1.16), is the unique solution of

$$\begin{cases} 2\rho\ddot{\phi}_3 + \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\phi_3 = 0, & in \quad \omega \times (0,T), \\ \phi_3 = \frac{\partial\phi_3}{\partial\nu} = 0, & on \quad \partial\omega \times (0,T), \\ \phi_3(0) = \frac{1}{2}\int_{-1}^{+1}\phi_{03}dx_3, \quad \dot{\phi}_3(0) = \frac{1}{2}\int_{-1}^{+1}\phi_{13}dx_3, & in \quad \omega, \end{cases} \tag{1.17}$$

where $\{\phi_{03}, \phi_{13}\}$ is the weak limit in $H^1_{|\Gamma_0}(\Omega) \times L^2(\Omega)$ of the sequence $\{\phi_{03}(\varepsilon_k), \phi_{13}(\varepsilon_k)\}_{\varepsilon_k>0}$, with $\{\boldsymbol{\phi}_0(\varepsilon_k), \boldsymbol{\phi}_1(\varepsilon_k)\} = \Lambda^{-1}(\varepsilon_k)(\{\boldsymbol{\psi}_1(\varepsilon_k), -\boldsymbol{\psi}_0(\varepsilon_k)\})$. Moreover $\frac{1}{2}\int_{-1}^{+1}\phi_{03}dx_3 \in H^2_0(\omega)$.

iii) *The displacements $\hat{\psi}_1$, $\hat{\psi}_2$ are equal to zero in $\omega \times (0,T)$.* ∎

So we conclude that, at the limit, the exactly controlled 3D-elasticity problem (1.6), with controls (1.10), becomes a 2D plate problem (as given for instance in Duvaut-Lions [4]), which is exactly controllable at the time $T > 0$, with controls on the boundary and in the interior of the plate.

Observe that the limit problem (1.16) presents to different types of control. First, an internal control acting as a volume force, which, roughly, is the limit of the controls imposed at the bottom and lower faces of the 3D plate. Second, a boundary control which is the contribution of the controls on the lateral boundary of the plate.

The limit system (1.16) is exactly controllable in an arbitrary small time with controls supported in the boundary (see Appendix 1 in Lions [7] and also Niane [9]). It would be interesting to see if that boundary control can be obtained as the limit of the boundary controls for the 3D plate. In this respect the work by Yan [16] is worth mentionning. In [16], the boundary control for a n–dimensional wave equation is obtained as the limit of the boundary controls of the wave equation in $(n+1)$–dimensional thin domains.

Remark 1.2 We observe that in (1.16) the internal control is such that

$$\ddot{\phi}_3 \in [H^1(0,T;L^2(\omega))]', \qquad \Delta^2\phi_3 \in L^2(0,T;H^{-2}(\omega))]. \tag{1.18}$$

By definition

$$< \ddot{\phi}_3, \theta_3 >= -\int_{\omega\times(0,T)} \dot{\phi}_3\dot{\theta}_3 d\omega dt, \qquad \forall \theta_3 \in H^1(0,T;L^2(\omega)). \tag{1.19}$$

On the other hand

$$< \Delta^2\phi_3, \theta_3 >= \int_{\omega\times(0,T)} \Delta\phi_3\Delta\theta_3 d\omega dt, \qquad \forall \theta_3 \in L^2(0,T;H_0^2(\omega)). \tag{1.20}$$

We also notice that the controls corresponding to the 2D plate problem (1.16) can also be obtained by HUM, as it is explained in subsection 4.4. ∎

To interpret the limit problem (1.16) with respect to the original plate $\overline{\Omega}^\varepsilon = \overline{\omega}\times[-\varepsilon,\varepsilon]$ it is convenient to formulate (1.16) in $\overline{\Omega}^\varepsilon$. In order to do that, we define the functions ξ_3^ε, η_3^ε and the initial data $\{\xi_{03}^\varepsilon, \xi_{13}^\varepsilon\}$, $\{\eta_{03}^\varepsilon, \eta_{13}^\varepsilon\}$ by the de-scalings

$$\begin{aligned}
\xi_3^\varepsilon(x_1,x_2,t) &= \varepsilon\psi_3(x_1,x_2,t), & \eta_3^\varepsilon(x_1,x_2,t) &= \varepsilon\phi_3(x_1,x_2,t),\\
\xi_{03}^\varepsilon(x_1,x_2,t) &= \varepsilon\psi_{03}(x_1,x_2,t), & \eta_{03}^\varepsilon(x_1,x_2,t) &= \varepsilon\phi_{03}(x_1,x_2,t),\\
\xi_{13}^\varepsilon(x_1,x_2,t) &= \varepsilon\psi_{13}(x_1,x_2,t), & \eta_{13}^\varepsilon(x_1,x_2,t) &= \varepsilon\phi_{13}(x_1,x_2,t),
\end{aligned} \tag{1.21}$$

for all $(x_1,x_2)\in\overline{\omega}$ and all $t\in[0,T]$. Then, problem (1.16) is equivalent to the following one (see subsection 4.3) :

Theorem 1.3 *The de-scaled function ξ_3^ε is the solution (in the transposition sense) of the problem*

$$\left\{\begin{aligned}
&2\rho^\varepsilon\varepsilon\ddot{\xi}_3^\varepsilon + \varepsilon^3\tfrac{8\mu^\varepsilon(\lambda^\varepsilon+\mu^\varepsilon)}{3(\lambda^\varepsilon+2\mu^\varepsilon)}\Delta^2\xi_3^\varepsilon = 2\rho^\varepsilon\varepsilon\ddot{\eta}_3^\varepsilon + \varepsilon^3\tfrac{8\mu^\varepsilon(\lambda^\varepsilon+\mu^\varepsilon)}{\lambda^\varepsilon+2\mu^\varepsilon}\Delta^2\eta_3^\varepsilon, \quad in \quad \omega\times(0,T),\\
&\xi_3^\varepsilon = 0, \quad on \quad \partial\omega\times(0,T),\\
&\tfrac{\partial\xi_3^\varepsilon}{\partial\nu} = \varepsilon^3(q_\alpha\nu_\alpha)\Delta\eta_3^\varepsilon, \quad on \quad \partial\omega\times(0,T),\\
&\xi_3^\varepsilon(0) = \tfrac{1}{2\varepsilon}\textstyle\int_{-\varepsilon}^{+\varepsilon}\xi_{03}^\varepsilon dx_3^\varepsilon, \quad \dot{\xi}_3^\varepsilon(0) = \tfrac{1}{2\varepsilon}\textstyle\int_{-\varepsilon}^{+\varepsilon}\xi_{13}^\varepsilon dx_3^\varepsilon, \quad in \quad \omega,\\
&\xi_3^\varepsilon(T) = \dot{\xi}_3^\varepsilon(T) = 0, \quad in \quad \omega,
\end{aligned}\right. \tag{1.22}$$

where η_3^ε is the solution of the equations

$$\begin{cases} 2\rho^\varepsilon \varepsilon \ddot{\eta}_3^\varepsilon + \varepsilon^3 \frac{8\mu^\varepsilon(\lambda^\varepsilon+\mu^\varepsilon)}{3(\lambda^\varepsilon+2\mu^\varepsilon)} \Delta^2 \eta_3^\varepsilon = 0, \quad in \quad \omega \times (0,T), \\ \eta_3^\varepsilon = \frac{\partial \eta_3^\varepsilon}{\partial \nu} = 0, \quad on \quad \partial\omega \times (0,T), \\ \eta_3^\varepsilon(0) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{+\varepsilon} \eta_{03}^\varepsilon dx_3^\varepsilon, \quad \dot{\eta}_3^\varepsilon(0) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{+\varepsilon} \eta_{13}^\varepsilon dx_3^\varepsilon, \quad in \quad \omega. \blacksquare \end{cases} \tag{1.23}$$

We observe that equations (1.22) are the standard time–dependent 2D equations, for the linearly elastic plate $\overline{\Omega}^\varepsilon = \overline{\omega} \times [-\varepsilon, \varepsilon]$ as given for instance in Duvaut-Lions [4], with controls on the boundary and in the interior of the plate.

We remark that a similar problem to the one that is presented here, has already been treated by Paulin-Vanninathan [13] and Yan [16], for the case of the wave equation, posed in 3 and $(n+1)$-dimensional thin domains, respectively.

The analysis of the 3D elasticity system that has been carried out in this work is technically more difficult than the one is required to analyse wave equations in thin domains. The methods we developped are different than those of [13] and [16].

Inspired by [3] and based on the identities that classical multipliers provide (that establish relations between boundary and internal energies of the plate) we have constructed controls that have a complex structure but that allow to prove boundedness of the solutions and to identify the limit problem as well.

The rest of the paper is organized as follows :

2 Preliminary results

In this section we will enumerate some results that will be needed in sections 3 and 4. First of all we will recall some existence and regularity results for problem (1.6). Then we will deduce some energy estimates. Finally, we will establish some identities for the 3D and 2D plate problems, by using multiplier techniques.

2.1 Existence and regularity for the 3D plate problem

Let us introduce the operator $A(\varepsilon)$, associated to the scaled and static 3D elasticity plate problem that is

$$\left\{\begin{aligned} -\partial_j\sigma_{ij}(\varepsilon)(\boldsymbol{u}(\varepsilon)) = f_i(\varepsilon), &\quad \text{in} \quad \Omega,\\ u_i(\varepsilon) = 0, &\quad \text{on} \quad \Gamma_0,\\ \sigma_{ij}(\varepsilon)(\boldsymbol{u}(\varepsilon))\nu_j = 0, &\quad \text{on} \quad \Gamma_\pm. \end{aligned}\right. \tag{2.1}$$

We recall that $A(\varepsilon) : \boldsymbol{V}(\Omega) \to [\boldsymbol{V}(\Omega)]'$ is an isomorphism verifying

$$< A(\varepsilon)(\boldsymbol{u}), \boldsymbol{v} >= a(\varepsilon)(\boldsymbol{u}, \boldsymbol{v}), \qquad \forall (\boldsymbol{u}, \boldsymbol{v}) \in [\boldsymbol{V}(\Omega)]^2,$$

where $a(\varepsilon)(.,.)\colon \boldsymbol{V}(\Omega) \times \boldsymbol{V}(\Omega) \to R$ is the bilinear form defined by

$$\left\{\begin{aligned} a(\varepsilon)(\boldsymbol{u},\boldsymbol{v}) &= \textstyle\int_\Omega \sigma_{ij}(\varepsilon)(\boldsymbol{u})\partial_j v_i d\Omega = \int_\Omega \sigma_{ij}(\varepsilon)(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) d\Omega\\ &= \textstyle\int_\Omega \Big[\lambda\Big(e_{\alpha\alpha}(\boldsymbol{u}) + \frac{1}{\varepsilon^2}e_{33}(\boldsymbol{u})\Big)\Big(e_{\beta\beta}(\boldsymbol{v}) + \frac{1}{\varepsilon^2}e_{33}(\boldsymbol{v})\Big)\\ &\quad +2\mu\Big(e_{\alpha\beta}(\boldsymbol{u})e_{\alpha\beta}(\boldsymbol{v}) + 2\tfrac{1}{\varepsilon}e_{\alpha 3}(\boldsymbol{u})\tfrac{1}{\varepsilon}e_{\alpha 3}(\boldsymbol{v}) + \tfrac{1}{\varepsilon^2}e_{33}(\boldsymbol{u})\tfrac{1}{\varepsilon^2}e_{33}(\boldsymbol{v})\Big)\Big]d\Omega, \end{aligned}\right. \tag{2.2}$$

or equivalently

$$\left\{\begin{aligned} a(\varepsilon)(\boldsymbol{u},\boldsymbol{v}) &= \textstyle\int_\Omega \Big[(\lambda+\mu)\Big(\partial_\alpha u_\alpha + \frac{1}{\varepsilon^2}\partial_3 u_3\Big)\Big(\partial_\beta v_\beta + \frac{1}{\varepsilon^2}\partial_3 v_3\Big)\\ &\quad +\mu\Big(\partial_\alpha u_\beta\partial_\alpha v_\beta + \tfrac{1}{\varepsilon^2}(\partial_3 u_\alpha\partial_3 v_\alpha + \partial_\alpha u_3\partial_\alpha v_3) + \tfrac{1}{\varepsilon^2}\partial_3 u_3\tfrac{1}{\varepsilon^2}\partial_3 v_3\Big)\Big]d\Omega. \end{aligned}\right. \tag{2.3}$$

The bilinear form $a(\varepsilon)(.,.)$ is continuous and $\boldsymbol{V}(\Omega)$-elliptic, that is, for any $\boldsymbol{u}$, $\boldsymbol{v}$ in $\boldsymbol{V}(\Omega)$

$$\begin{aligned} &\exists C_1 > 0 \quad \text{(independent of} \quad \varepsilon\text{)}: \quad a(\varepsilon)(\boldsymbol{u},\boldsymbol{v}) \le C_1\tfrac{1}{\varepsilon^4}||\boldsymbol{u}||_{V(\Omega)}||\boldsymbol{v}||_{V(\Omega)},\\ &\exists C_2 > 0 \quad \text{(independent of} \quad \varepsilon\text{)}: \quad a(\varepsilon)(\boldsymbol{v},\boldsymbol{v}) \ge C_2||\boldsymbol{v}||^2_{V(\Omega)}. \end{aligned} \tag{2.4}$$

The assumption on the geometry of the plate, imposing that the middle surface ω is a polygonal domain in $\mathbb{R}^2$, implies that the domain $D_{A(\varepsilon)}$ of the operator $A(\varepsilon)$ verifies (cf. Nicaise [11])

$$D_{A(\varepsilon)} = \Big\{\boldsymbol{v} \in \boldsymbol{V}(\Omega) : A(\varepsilon)(\boldsymbol{v}) \in [L^2(\Omega)]^3\Big\} \subset [H^{\frac{3}{2}+\delta}(\Omega)]^3 \cap \boldsymbol{V}(\Omega), \quad \text{for some} \quad \delta > 0. \tag{2.5}$$

So the unique solution $\boldsymbol{u}(\varepsilon)$ of (2.1) has the regularity $[H^{\frac{3}{2}+\delta}(\Omega)]^3$, if $f_i(\varepsilon) \in L^2(\Omega)$.

We consider now the scaled evolution 3D-elasticity system

$$\begin{cases} \varepsilon^2\rho\ddot{\theta}_\alpha(\varepsilon) - \partial_j\sigma_{\alpha j}(\varepsilon)(\boldsymbol{\theta}(\varepsilon)) = f_\alpha(\varepsilon), & \text{in} \quad Q = \Omega \times (0,T), \\ \rho\ddot{\theta}_3(\varepsilon) - \partial_j\sigma_{3j}(\varepsilon)(\boldsymbol{\theta}(\varepsilon)) = f_3(\varepsilon), & \text{in} \quad Q = \Omega \times (0,T), \\ \theta_i(\varepsilon) = 0, & \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,T), \\ \sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon))\nu_j = 0, & \text{on} \quad \Sigma_\pm = \Gamma_\pm \times (0,T), \\ \boldsymbol{\theta}(\varepsilon)(0) = \boldsymbol{\theta}_0(\varepsilon), \quad \dot{\boldsymbol{\theta}}(\varepsilon)(0) = \boldsymbol{\theta}_1(\varepsilon), & \text{in} \quad \Omega. \end{cases} \tag{2.6}$$

We recall the following existence and regularity result (cf. Nicaise [11], Grisvard [5]).

Theorem 2.1 *Assume that $\boldsymbol{\theta}_0(\varepsilon) \in \boldsymbol{V}(\Omega)$, $\boldsymbol{\theta}_1(\varepsilon) \in [L^2(\Omega)]^3$ and $f_i(\varepsilon) \in L^1(0,T;L^2(\Omega))$. Then, there exists a unique solution $\boldsymbol{\theta}(\varepsilon)$ of problem (2.6) with the regularity*

$$\boldsymbol{\theta}(\varepsilon) \in C^0([0,T];\boldsymbol{V}(\Omega)) \cap C^1([0,T];[L^2(\Omega)]^3) \cap W^{2,1}(0,T;[\boldsymbol{V}(\Omega)]^{'}). \tag{2.7}$$

Moreover if $\boldsymbol{\theta}_0(\varepsilon) \in D_{A(\varepsilon)}$, $\boldsymbol{\theta}_1(\varepsilon) \in \boldsymbol{V}(\Omega)$ and $\boldsymbol{f}(\varepsilon) \in L^1(0,T;\boldsymbol{V}(\Omega))$ the unique solution $\boldsymbol{\theta}(\varepsilon)$ of (2.6) has the regularity

$$\boldsymbol{\theta}(\varepsilon) \in C^0([0,T];[H^{\frac{3}{2}+\delta}(\Omega)]^3 \cap \boldsymbol{V}(\Omega)) \cap C^1([0,T];\boldsymbol{V}(\Omega)) \cap W^{2,1}(0,T;[L^2(\Omega)]^3). \blacksquare \tag{2.8}$$

We also recall the following density result which will be used several times (cf. Nicaise [10]).

Theorem 2.2 *Let $\boldsymbol{\theta}_0(\varepsilon) \in \boldsymbol{V}(\Omega)$, $\boldsymbol{\theta}_1(\varepsilon) \in [L^2(\Omega)]^3$ and $\boldsymbol{f}(\varepsilon) \in L^1(0,T;[L^2(\Omega)]^3)$ and let $\boldsymbol{\theta}(\varepsilon)$ be the unique solution of (2.6). Then, there exist sequences $\boldsymbol{\theta}_{0m}(\varepsilon) \in D_{A(\varepsilon)}$, $\boldsymbol{\theta}_{1m}(\varepsilon) \in \boldsymbol{V}(\Omega)$ and $\boldsymbol{f}_m(\varepsilon) \in C^0([0,T];\boldsymbol{V}(\Omega))$, such that as $m \to \infty$*

$$\begin{aligned} \boldsymbol{\theta}_{0m}(\varepsilon) &\longrightarrow \boldsymbol{\theta}_0(\varepsilon), && \textit{strongly in} && \boldsymbol{V}(\Omega), \\ \boldsymbol{\theta}_{1m}(\varepsilon) &\longrightarrow \boldsymbol{\theta}_1(\varepsilon), && \textit{strongly in} && [L^2(\Omega)]^3, \\ \boldsymbol{f}_m(\varepsilon) &\longrightarrow \boldsymbol{f}(\varepsilon), && \textit{strongly in} && L^1(0,T;[L^2(\Omega)]^3). \end{aligned} \tag{2.9}$$

Moreover, the solution $\boldsymbol{\theta}_m(\varepsilon)$ of (2.6), with data $\boldsymbol{\theta}_{0m}(\varepsilon)$, $\boldsymbol{\theta}_{1m}(\varepsilon)$ and $\boldsymbol{f}_m(\varepsilon)$ fulfils

$$\boldsymbol{\theta}_m(\varepsilon) \in C^0([0,T];D_{A(\varepsilon)}) \cap C^1([0,T];\boldsymbol{V}(\Omega)) \cap C^2([0,T];[L^2(\Omega)]^3),$$

and

$$\boldsymbol{\theta}_m(\varepsilon) \longrightarrow \boldsymbol{\theta}(\varepsilon) \quad \textit{in} \quad C^0([0,T];\boldsymbol{V}(\Omega)) \cap C^1([0,T];[L^2(\Omega)]^3), \quad \textit{as} \quad m \to \infty. \blacksquare \tag{2.10}$$

2.2 Estimates on the energy of the 3D plate problem

Let us denote by $E^{\theta(\varepsilon)}(t)$ the energy of $\boldsymbol{\theta}(\varepsilon)(t)$, solution of (2.6), at the time $t \in [0,T]$ that is

$$E^{\theta(\varepsilon)}(t) = \frac{1}{2}\int_\Omega \rho\Big[\varepsilon^2\sum_{\alpha=1}^2|\dot{\theta}_\alpha(\varepsilon)(t)|^2 + |\dot{\theta}_3(\varepsilon)(t)|^2\Big]d\Omega + \frac{1}{2}a(\varepsilon)(\boldsymbol{\theta}(\varepsilon)(t),\boldsymbol{\theta}(\varepsilon)(t)). \tag{2.11}$$

We remark that if $t = 0$,

$$E^{\theta(\varepsilon)}(0) = \frac{1}{2}\Big[\rho\varepsilon^2\sum_{\alpha=1}^2||\theta_{1\alpha}(\varepsilon)||^2_{L^2(\Omega)} + \rho||\theta_{13}(\varepsilon)||^2_{L^2(\Omega)} + a(\varepsilon)(\boldsymbol{\theta}_0(\varepsilon),\boldsymbol{\theta}_0(\varepsilon))\Big].$$

Lemma 2.1 *Assume that $\boldsymbol{\theta}_0(\varepsilon) \in \boldsymbol{V}(\Omega)$, $\boldsymbol{\theta}_1(\varepsilon) \in [L^2(\Omega)]^3$ and $\boldsymbol{f}(\varepsilon) \in L^1(0,T;[L^2(\Omega)]^3)$. Then we have the following energy estimate*

$$E^{\theta(\varepsilon)}(t) \leq C_1\Big\{E^{\theta(\varepsilon)}(0) + \frac{1}{2\varepsilon^2\rho}\sum_{\alpha=1}^{2}\Big[\int_0^T ||f_\alpha(\varepsilon)||_{L^2(\Omega)}dt\Big]^2 + \frac{1}{2\rho}\Big[\int_0^T ||f_3(\varepsilon)||_{L^2(\Omega)}dt\Big]^2\Big\}, \quad (2.12)$$

whith C_1 a positive constant independent of ε. Moreover, if $f_\alpha(\varepsilon) \in X$, $f_3(\varepsilon) \in L^1(0,T;L^2(\Omega))$, (cf. (1.5) for the definition of X), then

$$E^{\theta(\varepsilon)}(t) \leq C_2\Big\{E^{\theta(\varepsilon)}(0) + \sum_{\alpha=1}^{2}||f_\alpha(\varepsilon)||_X^2 + \Big[\int_0^T ||f_3(\varepsilon)||_{L^2(\Omega)}dt\Big]^2\Big\}, \quad (2.13)$$

where C_2 is a positive constant independent of ε.

Proof : In view of theorem 2.2 it is enough to prove (2.12) for

$$\boldsymbol{\theta}_0(\varepsilon) \in D_{A(\varepsilon)}, \quad \boldsymbol{\theta}_1(\varepsilon) \in \boldsymbol{V}(\Omega), \quad \boldsymbol{f}(\varepsilon) \in C^0([0,T];\boldsymbol{V}(\Omega)). \quad (2.14)$$

On the other hand, in view of theorem 2.2 and taking into account that $C^1([0,T];H^1_{|\Gamma_0}(\Omega))$ is dense in X, it is enough to prove (2.13) for

$$\begin{array}{ll} & \boldsymbol{\theta}_0(\varepsilon) \in D_{A(\varepsilon)}, \qquad \boldsymbol{\theta}_1(\varepsilon) \in \boldsymbol{V}(\Omega), \\ f_\alpha(\varepsilon) \in C^1([0,T];H^1_{|\Gamma_0}(\Omega)), & \qquad f_3(\varepsilon) \in C^0([0,T];H^1_{|\Gamma_0}(\Omega)). \end{array} \quad (2.15)$$

In each case (2.14) or (2.15) the solution $\boldsymbol{\theta}(\varepsilon)$ of (2.6) has the regularity (2.8). So in particular we have

$$\left\{\begin{array}{ll} \frac{dE^{\theta(\varepsilon)}(t)}{dt} & =< \varepsilon^2\rho\ddot{\theta}_\alpha(\varepsilon)(t), \dot{\theta}_\alpha(\varepsilon)(t) > + < \rho\ddot{\theta}_3(\varepsilon)(t), \dot{\theta}_3(\varepsilon)(t) > \\ & +a(\varepsilon)(\dot{\boldsymbol{\theta}}(\varepsilon)(t), \dot{\boldsymbol{\theta}}(\varepsilon)(t)) = \int_\Omega f_i(\varepsilon)(t)\dot{\theta}_i(\varepsilon)(t)d\Omega. \end{array}\right. \quad (2.16)$$

So integrating on time we get

$$E^{\theta(\varepsilon)}(t) - E^{\theta(\varepsilon)}(0) = \int_0^t\int_\Omega f_i(\varepsilon)(s)\dot{\theta}_i(\varepsilon)(s)d\Omega ds. \quad (2.17)$$

With the hypothesis (2.14) we easily deduce estimate (2.12) from (2.17), because

$$\left\{\begin{array}{l} \Big|\int_0^T\int_\Omega f_i(\varepsilon)(t)\dot{\theta}_i(\varepsilon)(t)d\Omega dt\Big| \leq \int_0^T ||f_i(\varepsilon)(t)||_{L^2(\Omega)}dt\ \max_{t\in[0,T]}||\dot{\theta}_i(\varepsilon)(t)||_{L^2(\Omega)} \\ \leq \sum_{\alpha=1}^2 \frac{1}{2\varepsilon^2\rho}\Big[\int_0^T ||f_\alpha(\varepsilon)(t)||_{L^2(\Omega)}dt\Big]^2 + \sum_{\alpha=1}^2 \frac{\varepsilon^2\rho}{2}\max_{t\in[0,T]}||\dot{\theta}_\alpha(\varepsilon)(t)||^2_{L^2(\Omega)} \\ +\frac{1}{2\rho}\Big[\int_0^T ||f_3(\varepsilon)(t)||_{L^2(\Omega)}dt\Big]^2 + \frac{\rho}{2}\max_{t\in[0,T]}||\dot{\theta}_3(\varepsilon)(t)||^2_{L^2(\Omega)}. \end{array}\right. \quad (2.18)$$

With the hypothesis (2.15) we remark that, integrating by parts with respect to time the terms $\int_0^t\int_\Omega f_\alpha(\varepsilon)(s)\dot{\theta}_\alpha(\varepsilon)(s)d\Omega ds$, we get

$$\left\{\begin{array}{ll} \int_0^t\int_\Omega f_\alpha(\varepsilon)(s)\dot{\theta}_\alpha(\varepsilon)(s)d\Omega ds & = \int_\Omega f_\alpha(\varepsilon)(t)\theta_\alpha(\varepsilon)(t)d\Omega - \int_\Omega f_\alpha(\varepsilon)(0)\theta_\alpha(\varepsilon)(0)d\Omega \\ & -\int_0^t < \dot{f}_\alpha(\varepsilon)(s), \theta_\alpha(\varepsilon)(s) > ds, \end{array}\right. \quad (2.19)$$

where $< .,. >$ denotes the duality between $H^1_{|\Gamma_0}(\Omega)$ and its dual. So we obtain, by Young's inequality

$$\left\{\begin{array}{l} \left|\int_0^t \int_\Omega \Big[f_\alpha(\varepsilon)(s)\dot\theta_\alpha(\varepsilon)(s) + f_3(\varepsilon)(s)\dot\theta_3(\varepsilon)(s)\Big] d\Omega ds\right| \\ \leq \sum_{\alpha=1}^2 \left(\frac{a}{2\rho}||f_\alpha(\varepsilon)(t)||^2_X + \frac{\rho}{2a}||\theta_\alpha(\varepsilon)(t)||^2_{L^2(\Omega)}\right) \\ + \sum_{\alpha=1}^2 \left(\frac{a}{2\rho}||f_\alpha(\varepsilon)(0)||^2_{L^2(\Omega)} + \frac{\rho}{2a}||\theta_\alpha(\varepsilon)(0)||^2_{L^2(\Omega)}\right) \\ + \sum_{\alpha=1}^2 \left(\frac{a}{2\rho}||f_\alpha(\varepsilon)(t)||^2_X + \frac{\rho}{2a}\max_{t\in[0,T]}||\theta_\alpha(\varepsilon)(t)||^2_{H^1_{|\Gamma_0}(\Omega)}\right) \\ + \frac{1}{2\rho}\Big[\int_0^T ||f_3(\varepsilon)(t)||_{L^2(\Omega)}dt\Big]^2 + \frac{\rho}{2}\max_{t\in[0,T]}||\dot\theta_3(\varepsilon)(t)||^2_{L^2(\Omega)}, \qquad \forall a \in \mathbb{R}^+. \end{array}\right. \tag{2.20}$$

So from (2.17) and (2.20) we obtain estimate (2.13), after choosing the constant a such that $\frac{3\rho}{aC_2} < 1$, where C_2 is the ellipticity constant defined on (2.4). ∎

2.3 Identities related to the 3D plate problem

Let us consider the homogeneous 3D elasticity problem (1.11) whose solution is $\boldsymbol{\phi}(\varepsilon)$. Then we have the following result.

Lemma 2.2 *Let $\boldsymbol{\theta}(\varepsilon)$ be the solution of (2.6) with data $\boldsymbol{\theta}_0(\varepsilon) \in D_{A(\varepsilon)}$, $\boldsymbol{\theta}_1(\varepsilon) \in \boldsymbol{V}(\Omega)$, $\boldsymbol{f}(\varepsilon) \in L^1(0,T;\boldsymbol{V}(\Omega))$ and $\boldsymbol{\phi}(\varepsilon)$ the solution of the homogeneous system (1.11) with data $\boldsymbol{\phi}_0(\varepsilon) \in D_{A(\varepsilon)}$, $\boldsymbol{\phi}_1(\varepsilon) \in \boldsymbol{V}(\Omega)$. Then the following identity holds*

$$\left\{\begin{array}{l} \int_{\sum_0}(\boldsymbol{q}\cdot\boldsymbol{\nu})\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)d\Gamma_0 dt \\ + \int_{\sum_\pm}(\boldsymbol{q}\cdot\boldsymbol{\nu})\Big[\varepsilon^2\rho\dot\theta_\alpha(\varepsilon)\dot\phi_\alpha(\varepsilon) + \rho\dot\theta_3(\varepsilon)\dot\phi_3(\varepsilon) - \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)\Big]d\Gamma_\pm dt \\ = \Big[\int_\Omega \rho\Big(\varepsilon^2\dot\theta_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon)) + \dot\theta_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_3(\varepsilon))\Big)d\Omega\Big]_0^T \\ + \Big[\int_\Omega \rho\Big(\varepsilon^2\dot\phi_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon)) + \dot\phi_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\theta_3(\varepsilon))\Big)d\Omega\Big]_0^T \\ + \int_0^T\int_\Omega 3\rho\Big[\varepsilon^2\dot\theta_\alpha(\varepsilon)\dot\phi_\alpha(\varepsilon) + \dot\theta_3(\varepsilon)\dot\phi_3(\varepsilon)\Big]d\Omega dt \\ - \int_0^T\int_\Omega \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)d\Omega dt - \int_0^T\int_\Omega f_i(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_i(\varepsilon))d\Omega dt. \end{array}\right. \tag{2.21}$$

Proof : We first remark that the assumptions on the data insure that the solutions $\boldsymbol{\theta}(\varepsilon)$ and $\boldsymbol{\phi}(\varepsilon)$ belong to the space (cf. theorem 2.1)

$$C^0([0,T];D_{A(\varepsilon)}) \cap C^1([0,T];\boldsymbol{V}(\Omega)) \cap W^{2,1}(0,T;[L^2(\Omega)]^3). \tag{2.22}$$

Multiplying the first and second equations of (2.6) by $\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon)$ and $\boldsymbol{q}\cdot\nabla\phi_3(\varepsilon)$ respectively and integrating by parts on Q we get

$$\left\{\begin{aligned}
&\Big[\int_\Omega \rho\Big(\varepsilon^2\dot\theta_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon))+\dot\theta_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_3(\varepsilon))\Big)d\Omega\Big]_0^T\\
&-\int_0^T\int_\Omega \rho\Big[\varepsilon^2\dot\theta_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\dot\phi_\alpha(\varepsilon))+\dot\theta_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\dot\phi_3(\varepsilon))\Big]d\Omega dt\\
&+\int_0^T\int_\Omega \sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon))\partial_j\phi_i(\varepsilon)d\Omega dt+\int_0^T\int_\Omega \sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon))q_k\partial_{jk}\phi_i(\varepsilon)d\Omega dt\\
&=\int_0^T\int_\Omega f_i(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_i(\varepsilon))d\Omega dt+\int_{\sum_0}(\boldsymbol{q}\cdot\boldsymbol{\nu})\sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon))\partial_j\phi_i(\varepsilon)d\Gamma_0 dt.
\end{aligned}\right. \tag{2.23}$$

In a similar way, we multiply the first and second equations of (1.11) by $\boldsymbol{q}\cdot\nabla\theta_\alpha(\varepsilon)$ and $\boldsymbol{q}\cdot\nabla\theta_3(\varepsilon)$ respectively and integrate by parts on Q. We get

$$\left\{\begin{aligned}
&\Big[\int_\Omega \rho\Big(\varepsilon^2\dot\phi_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\theta_\alpha(\varepsilon))+\dot\phi_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\theta_3(\varepsilon))\Big)d\Omega\Big]_0^T\\
&-\int_0^T\int_\Omega \rho\Big[\varepsilon^2\dot\phi_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\dot\theta_\alpha(\varepsilon))+\dot\phi_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\dot\theta_3(\varepsilon))\Big]d\Omega dt\\
&+\int_0^T\int_\Omega \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)d\Omega dt+\int_0^T\int_\Omega \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))q_k\partial_{jk}\theta_i(\varepsilon)d\Omega dt\\
&=\int_{\sum_0}(\boldsymbol{q}\cdot\boldsymbol{\nu})\sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon))\partial_j\phi_i(\varepsilon)d\Gamma_0 dt.
\end{aligned}\right. \tag{2.24}$$

Adding the two identities above and remarking that

$$\left\{\begin{aligned}
&-\int_0^T\int_\Omega \rho\Big[\varepsilon^2\dot\theta_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\dot\phi_\alpha(\varepsilon))+\dot\theta_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\dot\phi_3(\varepsilon))\Big]d\Omega dt\\
&-\int_0^T\int_\Omega \rho\Big[\varepsilon^2\dot\phi_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\dot\theta_\alpha(\varepsilon))+\dot\phi_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\dot\theta_3(\varepsilon))\Big]d\Omega dt\\
&=-\int_0^T\int_\Omega \rho\boldsymbol{q}\cdot\nabla\Big[\varepsilon^2\dot\theta_\alpha(\varepsilon)\dot\phi_\alpha(\varepsilon)+\dot\theta_3(\varepsilon)\dot\phi_3(\varepsilon)\Big]d\Omega dt\\
&=\int_0^T\int_\Omega 3\rho\Big[\varepsilon^2\dot\theta_\alpha(\varepsilon)\dot\phi_\alpha(\varepsilon)+\dot\theta_3(\varepsilon)\dot\phi_3(\varepsilon)\Big]d\Omega dt\\
&-\int_{\sum_\pm}(\boldsymbol{q}\cdot\boldsymbol{\nu})\rho\Big[\varepsilon^2\dot\theta_\alpha(\varepsilon)\dot\phi_\alpha(\varepsilon)+\dot\theta_3(\varepsilon)\dot\phi_3(\varepsilon)\Big]d\Gamma_\pm dt,
\end{aligned}\right. \tag{2.25}$$

and

$$\left\{\begin{aligned}
&\int_0^T\int_\Omega \Big[\sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon))q_k\partial_{jk}\phi_i(\varepsilon)+\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))q_k\partial_{jk}\theta_i(\varepsilon)\Big]d\Omega dt\\
&=\int_0^T\int_\Omega q_k\partial_k\Big[\sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon))\partial_j\phi_i(\varepsilon)\Big]d\Omega dt=-3\int_0^T\int_\Omega \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)d\Omega dt\\
&+\int_{\sum_0}(\boldsymbol{q}\cdot\boldsymbol{\nu})\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)d\Gamma_0 dt+\int_{\sum_\pm}(\boldsymbol{q}\cdot\boldsymbol{\nu})\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)d\Gamma_\pm dt,
\end{aligned}\right.$$

we obtain the identity (2.21). Note that the integrations by parts above can be carried out because of the regularity of $\boldsymbol{\theta}(\varepsilon)$ and $\boldsymbol{\phi}(\varepsilon)$. In particular, the fact that $\boldsymbol{\theta}(\varepsilon)(t)$ and $\boldsymbol{\phi}(\varepsilon)(t)$ belong to $\left(H^{\frac{3}{2}+\delta}(\Omega)\right)^3$, for all $0\le t\le T$, plays a crucial role. ∎

The identity (2.21) can be written in a different way. In fact we notice that

$$\begin{cases} \Big[\int_\Omega \rho\Big[\varepsilon^2\theta_\alpha(\varepsilon)\dot{\phi}_\alpha(\varepsilon)+\theta_3(\varepsilon)\dot{\phi}_3(\varepsilon)\Big]d\Omega\Big]_0^T & = \int_0^T\int_\Omega \rho\Big[\varepsilon^2\dot{\theta}_\alpha(\varepsilon)\dot{\phi}_\alpha(\varepsilon)+\dot{\theta}_3(\varepsilon)\dot{\phi}_3(\varepsilon)\Big]d\Omega dt \\ & -\int_0^T\int_\Omega \sigma_{ij}(\varepsilon)(\phi(\varepsilon))\partial_j\theta_i(\varepsilon)d\Omega dt. \end{cases}$$

This formula is deduced directly, multiplying the first two equations of (1.11) by $\theta_\alpha(\varepsilon)$ and $\theta_3(\varepsilon)$ respectively, and integrating on Q. So identity (2.21) becomes

$$\begin{cases} \int_{\sum_0}(\boldsymbol{q}\cdot\boldsymbol{\nu})\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)d\Gamma_0 dt \\ +\int_{\sum_\pm}(\boldsymbol{q}\cdot\boldsymbol{\nu})\Big[\varepsilon^2\rho\dot{\theta}_\alpha(\varepsilon)\dot{\phi}_\alpha(\varepsilon)+\rho\dot{\theta}_3(\varepsilon)\dot{\phi}_3(\varepsilon)-\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)\Big]d\Gamma_\pm dt \\ =\Big[\int_\Omega \rho\Big(\varepsilon^2\dot{\theta}_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon))+\dot{\theta}_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_3(\varepsilon))\Big)d\Omega\Big]_0^T \\ +\Big[\int_\Omega \rho\Big(\varepsilon^2\dot{\phi}_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon))+\dot{\phi}_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\theta_3(\varepsilon))\Big)d\Omega\Big]_0^T \\ +2\Big[\int_\Omega \rho\Big(\varepsilon^2\theta_\alpha(\varepsilon)\dot{\phi}_\alpha(\varepsilon)+\theta_3(\varepsilon)\dot{\phi}_3(\varepsilon)\Big)d\Omega\Big]_0^T-\int_0^T\int_\Omega f_i(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon))d\Omega dt \\ +\int_0^T\int_\Omega \sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon))\partial_j\phi_i(\varepsilon)d\Omega dt+\int_0^T\int_\Omega \rho\Big[\varepsilon^2\dot{\theta}_\alpha(\varepsilon)\dot{\phi}_\alpha(\varepsilon)+\dot{\theta}_3(\varepsilon)\dot{\phi}_3(\varepsilon)\Big]d\Omega dt. \end{cases} \tag{2.26}$$

The case where $\boldsymbol{\theta}(\varepsilon)=\boldsymbol{\phi}(\varepsilon)$ is particularly revelant :

Corollary 2.1 *Let $\boldsymbol{\phi}(\varepsilon)$ be the solution of (1.11), with initial data in $D_{A(\varepsilon)}\times\boldsymbol{V}(\Omega)$. Then the following identity holds*

$$\begin{cases} \frac{1}{2}\int_{\sum_0}(\boldsymbol{q}\cdot\boldsymbol{\nu})\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\phi_i(\varepsilon)d\Gamma_0 dt \\ +\frac{1}{2}\int_{\sum_\pm}(\boldsymbol{q}\cdot\boldsymbol{\nu})\Big[\varepsilon^2\rho\sum_{\alpha=1,2}\dot{\phi}_\alpha^2(\varepsilon)+\rho\dot{\phi}_3^2(\varepsilon)-\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\phi_i(\varepsilon)\Big]d\Gamma_\pm dt \\ =\Big[\int_\Omega \rho\Big[\varepsilon^2\dot{\phi}_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon)+\phi_\alpha(\varepsilon))+\dot{\phi}_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_3(\varepsilon)+\phi_3(\varepsilon))\Big]d\Omega\Big]_0^T \\ +\int_0^T E^{\phi(\varepsilon)}(0)dt. \end{cases} \tag{2.27}$$

Proof : In order to show (2.27) it is enough to apply identity (2.26) with $f_i(\varepsilon)=0$ and $\boldsymbol{\theta}(\varepsilon)=\boldsymbol{\phi}(\varepsilon)$, and to remark that the energy of the elasticity system (1.11) is conserved, that is

$$E^{\phi(\varepsilon)}(t)=E^{\phi(\varepsilon)}(0), \qquad \forall t\in[0,T]. \blacksquare$$

Let us now prove that $\frac{\partial\phi_i(\varepsilon)}{\partial\nu}$ makes sense in $L^2(0,T;L^2(\Gamma_0))$ for every solution of (1.11) with initial data in $\boldsymbol{V}(\Omega)\times[L^2(\Omega)]^3$. We define a function $\boldsymbol{h}=(h_1,h_2,h_3)\in[W^{1,\infty}(\Omega)]^3$, such that

$$\begin{cases} h_1,\ h_2 & \text{are independent of } x_3, \\ h_3=0, & \\ h_\alpha=\nu_\alpha, & \text{on } \Gamma_0. \end{cases} \tag{2.28}$$

Lemma 2.3 *Let $\phi(\varepsilon)$ be the solution of (1.11), with initial data $\{\phi_0(\varepsilon), \phi_1(\varepsilon)\} \in D_{A(\varepsilon)} \times \boldsymbol{V}(\Omega)$. Then the following identity holds :*

$$\left\{\begin{array}{l} \frac{1}{2}\int_{\sum_0} \sigma_{ij}(\varepsilon)(\phi(\varepsilon))\partial_j\phi_i(\varepsilon)d\Gamma_0 dt \\ = \Big[\int_\Omega \rho\Big\{\varepsilon^2\dot{\phi}_\alpha(\varepsilon)h_k\partial_k\phi_\alpha(\varepsilon) + \dot{\phi}_3(\varepsilon)h_k\partial_k\phi_3(\varepsilon)\Big\}d\Omega\Big]_0^T \\ +\frac{1}{2}\int_Q \partial_k h_k\Big[\varepsilon^2\rho|\dot{\phi}_\alpha(\varepsilon)|^2 + \rho|\dot{\phi}_3(\varepsilon)|^2 - \sigma_{ij}(\varepsilon)(\phi(\varepsilon))\partial_j\phi_i(\varepsilon)\Big]d\Omega dt \\ + \int_Q \sigma_{ij}(\varepsilon)(\phi(\varepsilon))\partial_j h_k\partial_k\phi_i(\varepsilon)d\Omega dt. \end{array}\right. \tag{2.29}$$

Proof : We multiply the first and second equations of (1.11) by $h_k\partial_k\phi_\alpha(\varepsilon)$ and $h_k\partial_k\phi_3(\varepsilon)$, respectively. Then (2.29) follows by integration by parts on Q. ∎

We now deduce the following corollary :

Corollary 2.2 *For $i = 1, 2, 3$ the fonction $\frac{\partial\phi_i(\varepsilon)}{\partial\nu}$ makes sense in $L^2(0, T; L^2(\Gamma_0))$ for every solution of (1.11) with initial data in $\boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$.*

Proof : Taking into account that $\phi_i(\varepsilon) = 0$ in Γ_0, then $\partial_j\phi_i(\varepsilon) = \nu_j\frac{\partial\phi_i(\varepsilon)}{\partial\nu}$ in Γ_0, and in particular $\partial_3\phi_i(\varepsilon) = 0$ because $\nu_3 = 0$ in Γ_0. So, in view of the definition of $\sigma_{ij}(\varepsilon)(\phi(\varepsilon))\partial_j\phi_i(\varepsilon)$ (cf. (2.3)) we have in Γ_0

$$\left\{\begin{array}{l} \sigma_{ij}(\varepsilon)(\phi(\varepsilon))\partial_j\phi_i(\varepsilon) \\ = (\lambda+\mu)\Big|\sum_{\alpha=1,2}\nu_\alpha\frac{\partial\phi_\alpha(\varepsilon)}{\partial\nu}\Big|^2 + \mu\Big[\sum_{\alpha=1,2}\Big|\frac{\partial\phi_\alpha(\varepsilon)}{\partial\nu}\Big|^2 + \Big|\frac{1}{\varepsilon}\frac{\partial\phi_3(\varepsilon)}{\partial\nu}\Big|^2\Big]. \end{array}\right. \tag{2.30}$$

So from (2.29), (2.30) and the density of $D_{A(\varepsilon)} \times \boldsymbol{V}(\Omega)$ in $\boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$ we deduce that $\frac{\partial\phi_i(\varepsilon)}{\partial\nu}$ makes sense in $L^2(0, T; L^2(\Gamma_0))$. On the other hand, the map

$$\{\phi_0(\varepsilon), \phi_1(\varepsilon)\} \in \boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3 \longrightarrow \frac{\partial\phi_i(\varepsilon)}{\partial\nu} \in L^2(0, T; L^2(\Gamma_0)),$$

is continuous. ∎

2.4 Identities related with the 2D plate problem

We consider the 2D plate equation

$$\left\{\begin{array}{ll} 2\rho\ddot{\theta}_3 - \partial_{\alpha\beta}m_{\alpha\beta}(\theta_3) = f_3, & \text{in} \quad \omega \times (0, T), \\ \theta_3 = \frac{\partial\theta_3}{\partial\nu} = 0, & \text{on} \quad \partial\omega \times (0, T), \\ \theta_3(0) = \theta_{03}, \quad \dot{\theta}_3(0) = \theta_{13}, & \text{in} \quad \omega, \end{array}\right. \tag{2.31}$$

where

$$\begin{aligned} m_{\alpha\beta}(\theta_3) &= -\Big\{\frac{4\mu\lambda}{3(\lambda+2\mu)}\Delta\theta_3\delta_{\alpha\beta} + \frac{4}{3}\mu\partial_{\alpha\beta}\theta_3\Big\} \\ \partial_{\alpha\beta}m_{\alpha\beta}(\theta_3) &= -\frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\theta_3. \end{aligned} \tag{2.32}$$

Since ω is a polygonal domain in $\mathbb{R}^2$, then the domain of the bilaplacian operator with Dirichlet boundary conditions satifies (cf. Niane [9])

$$D_{\Delta^2} = \Big\{\boldsymbol{v} \in H_0^2(\omega) : \Delta^2\boldsymbol{v} \in L^2(\omega)\Big\} \subset H^{\frac{5}{2}+\delta}(\omega) \cap H_0^2(\omega), \quad \text{for some} \quad \delta \in]0, \frac{1}{2}[. \tag{2.33}$$

We have the following classical result, which provides the existence and uniqueness of solution to (2.31).

Theorem 2.3 *Assume that $\theta_{03} \in H_0^2(\omega)$, $\theta_{13} \in L^2(\omega)$ and $f_3 \in L^1(0,T;L^2(\omega))$. Then there exists a unique solution of the equation (2.31) such that*

$$\theta_3 \in C^0([0,T];H_0^2(\omega)) \cap C^1([0,T];L^2(\omega)) \cap W^{2,1}(0,T;H^{-2}(\omega)) \tag{2.34}$$

Moreover, if $\theta_{03} \in D_{\Delta^2}$, $\theta_{13} \in H_0^2(\omega)$ and $f_3 \in L^1(0,T;H_0^2(\omega))$, then the unique solution of (2.31) satisfies

$$\theta_3 \in C^0([0,T];D_{\Delta^2}) \cap C^1([0,T];H_0^2(\omega)) \cap W^{2,1}(0,T;L^2(\omega)). \blacksquare \tag{2.35}$$

Let ϕ_3 be the solution of the 2D homogeneous plate problem

$$\begin{cases} 2\rho\ddot{\phi}_3 - \partial_{\alpha\beta}m_{\alpha\beta}(\phi_3) = 0, \quad \text{in} \quad \omega\times(0,T), \\ \phi_3 = \frac{\partial\phi_3}{\partial\nu} = 0, \quad \text{on} \quad \partial\omega\times(0,T), \\ \phi_3(0) = \phi_{03}, \quad \dot{\phi}_3(0) = \phi_{13}, \quad \text{in} \quad \omega. \end{cases} \tag{2.36}$$

We have the following result, which is the counterpart of lemma 2.2 :

Lemma 2.4 *Assume that $f_3 \in L^1(0,T;H_0^2(\omega))$ and $\{\theta_{03},\theta_{13}\}$, $\{\phi_{03},\phi_{13}\} \in D_{\Delta^2}\times H_0^2(\omega)$. Then, the following identity holds :*

$$\begin{cases} -\int_{\partial\omega\times(0,T)}(q_\zeta\nu_\zeta)m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\gamma dt = \Big[\int_\omega 2\rho\Big[\dot{\theta}_3(q_\alpha\partial_\alpha\phi_3) + \dot{\phi}_3(q_\alpha\partial_\alpha\theta_3)\Big]d\omega\Big]_0^T \\ +2\int_{\omega\times(0,T)} 2\rho\dot{\theta}_3\dot{\phi}_3 d\omega dt - 2\int_{\omega\times(0,T)} m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\omega dt - \int_{\omega\times(0,T)} f_3(q_\alpha\partial_\alpha\phi_3)d\omega dt. \end{cases} \tag{2.37}$$

Proof : We multiply the first equation of (2.31) by $q_\alpha\partial_\alpha\phi_3$, where ϕ_3 is the solution of (2.36) and then, we multiply the first equation of (2.36) by $q_\alpha\partial_\alpha\theta_3$, where θ_3 is the solution of (2.31). Then we integrate by parts on $\omega\times(0,T)$. The validity of the integration by parts is assured by the results of Niane [9], and in particular, by the fact that $\phi_3(t)$ and $\theta_3(t)$ belong to $H^{\frac{5}{2}+\delta}(\omega)$ for all $t \in [0,T]$. Addding the two equations we obtain formula (2.37). $\blacksquare$

3 The exact controllability problem for the 3D plate

The aim of this section is to prove the exact controllability result for the 3D plate Ω, which is established in theorem 1.1. In order to do that we first deduce some a priori estimates for the energy $E^{\phi(\varepsilon)}(t)$ of the homogeneous 3D problem (1.11). These estimates are obtained almost directly from formula (2.27) and they are the key to prove by HUM (Hilbert Uniqueness Method, cf. Lions [7]), the exact controllability result. We also introduce the transposition formulation of the 3D problem (3.8). Finally, we define the HUM operator $\Lambda(\varepsilon)$, and we show that it is an isomorphism between $\boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$ and its dual, which enables us to prove theorem 1.1.

3.1 Estimates for the energy of the homogeneous 3D plate problem

Theorem 3.1 (Direct Inequality) *Let $0 < \varepsilon \leq 1$ and $T > 0$ be fixed. Assume that $\phi_0 \in D_{A(\varepsilon)}$, $\phi_1 \in \boldsymbol{V}(\Omega)$. Then the solution $\boldsymbol{\phi}(\varepsilon)$ of (1.11), with initial data $\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\}$ satisfies the following estimate.*

$$\left\{ \begin{array}{l} \Big| \int_{\sum_0} (\boldsymbol{q} \cdot \boldsymbol{\nu}) \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon)) \partial_j \phi_i(\varepsilon) d\Gamma_0 dt \\ + \int_{\sum_\pm} (\boldsymbol{q} \cdot \boldsymbol{\nu}) \Big[\varepsilon^2 \rho \sum_{\alpha=1,2} \dot{\phi}_\alpha^2(\varepsilon) + \rho \dot{\phi}_3^2(\varepsilon) - \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon)) \partial_j \phi_i(\varepsilon) \Big] d\Gamma_\pm dt \Big| \\ \leq C E^{\phi(\varepsilon)}(0), \end{array} \right. \tag{3.1}$$

where C is a constant independent of ε and of the solution $\boldsymbol{\phi}(\varepsilon)$, but depending on ρ, μ, Ω and T.

Proof : From formula (2.27) we directly obtain that the first two integrals in (3.1) are bounded from above by $CE^{\phi(\varepsilon)}(0)$. ∎

Theorem 3.2 (Inverse Inequality) *Let us fix $0 < \varepsilon \leq 1$ and $T > T(\varepsilon)$. Then, for every solution $\boldsymbol{\phi}(\varepsilon)$ of (1.11) with initial data $\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\} \in D_{A(\varepsilon)} \times \boldsymbol{V}(\Omega)$, the following estimate holds*

$$\left\{ \begin{array}{l} [T - T(\varepsilon)] E^{\phi(\varepsilon)}(0) \leq C(\boldsymbol{x}_0, \rho, \mu, \Omega) \Big\{ \int_{\sum_0} \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon)) \partial_j \phi_i(\varepsilon) d\Gamma_0 dt \\ + \int_{\sum_\pm} (\boldsymbol{q} \cdot \boldsymbol{\nu}) \Big[\varepsilon^2 \rho \sum_{\alpha=1,2} \dot{\phi}_\alpha^2(\varepsilon) + \rho \dot{\phi}_3^2(\varepsilon) - \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon)) \partial_j \phi_i(\varepsilon) \Big] d\Gamma_\pm dt \Big\}, \end{array} \right. \tag{3.2}$$

where $C(\boldsymbol{x}_0, \rho, \mu, \Omega)$ is a positive constant depending on the arguments, but does not depend on ε and on the solution $\boldsymbol{\phi}(\varepsilon)$.

Proof : Let us denote by $X(\varepsilon)(t)$ the quantity

$$X(\varepsilon)(t) = \int_\Omega \rho \Big\{ \varepsilon^2 \dot{\phi}_\alpha(\varepsilon) \Big[\boldsymbol{q} \cdot \nabla \phi_\alpha(\varepsilon) + \phi_\alpha(\varepsilon) \Big] + \dot{\phi}_3(\varepsilon) \Big[\boldsymbol{q} \cdot \nabla \phi_3(\varepsilon) + \phi_3(\varepsilon) \Big] \Big\} d\Omega.$$

Then for any constant $a > 0$ and any $t \in [0, T]$ we have by Young's inequality

$$\left\{ \begin{array}{ll} |X(\varepsilon)(t)| & \leq \frac{a}{2} \int_\Omega \Big[\varepsilon^2 \rho \sum_{\alpha=1,2} |\dot{\phi}_\alpha(\varepsilon)|^2 + \rho |\dot{\phi}_3(\varepsilon)|^2 \Big] d\Omega \\ & + \frac{1}{2a} \int_\Omega \Big[\varepsilon^2 \rho \sum_{\alpha=1,2} |\boldsymbol{q} \cdot \nabla \phi_\alpha(\varepsilon) + \phi_\alpha(\varepsilon)|^2 + \rho |\boldsymbol{q} \cdot \nabla \phi_3(\varepsilon) + \phi_3(\varepsilon)|^2 \Big] d\Omega. \end{array} \right. \tag{3.3}$$

We remark that an integration by parts on Ω and the expression of the bilinear form $a(\varepsilon)(.,.)$ (cf. (2.3)) give

$$\left\{\begin{array}{l} \frac{1}{2a}\int_\Omega \Big[\varepsilon^2\rho\sum_{\alpha=1,2}|\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon)+\phi_\alpha(\varepsilon)|^2+\rho|\boldsymbol{q}\cdot\nabla\phi_3(\varepsilon)+\phi_3(\varepsilon)|^2\Big]d\Omega \\ = \frac{1}{2a}\int_\Omega \Big[\varepsilon^2\rho\sum_{\alpha=1,2}|\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon)|^2-2\varepsilon^2\rho\sum_{\alpha=1,2}|\phi_\alpha(\varepsilon)|^2+\rho|\boldsymbol{q}\cdot\nabla\phi_3(\varepsilon)|^2-2\rho|\phi_3(\varepsilon)|^2\Big]d\Omega \\ +\frac{1}{2a}\int_{\Gamma_\pm}\Big[\varepsilon^2\rho(\boldsymbol{q}\cdot\boldsymbol{\nu})\sum_{\alpha=1,2}|\phi_\alpha(\varepsilon)|^2+\rho(\boldsymbol{q}\cdot\boldsymbol{\nu})|\phi_3(\varepsilon)|^2\Big]d\Gamma_\pm \\ \le \frac{1}{2a}\Big[\frac{R^2(\boldsymbol{x}_0)\rho\varepsilon^2}{\mu}a(\varepsilon)(\boldsymbol{\phi}(\varepsilon),\boldsymbol{\phi}(\varepsilon))+\int_{\Gamma_\pm}(\boldsymbol{q}\cdot\boldsymbol{\nu})\Big(\varepsilon^2\rho\sum_{\alpha=1,2}|\phi_\alpha(\varepsilon)|^2+\rho|\phi_3(\varepsilon)|^2\Big)d\Gamma_\pm\Big]. \end{array}\right. \tag{3.4}$$

But due to the continuity of the trace map $tr\colon \boldsymbol{V}(\Omega)\to[L^2(\partial\Omega)]^3$ and again the expression of the bilinear form $a(\varepsilon)(.,.)$ (cf. (2.3))

$$\left\{\begin{array}{l} \frac{1}{2a}\int_{\Gamma_\pm}\Big[\varepsilon^2\rho(\boldsymbol{q}\cdot\boldsymbol{\nu})\sum_{\alpha=1,2}|\phi_\alpha(\varepsilon)|^2+\rho(\boldsymbol{q}\cdot\boldsymbol{\nu})|\phi_3(\varepsilon)|^2\Big]d\Gamma_\pm \\ = \frac{\varepsilon^2\rho}{2a}\int_{\Gamma_\pm}\Big[\sum_{\alpha=1,2}|\phi_\alpha(\varepsilon)|^2+|\frac{1}{\varepsilon}\phi_3(\varepsilon)|^2\Big]d\Gamma_\pm \\ \le \frac{\varepsilon^2\rho}{2a}\frac{C(\Omega)}{\mu}a(\varepsilon)(\boldsymbol{\phi}(\varepsilon),\boldsymbol{\phi}(\varepsilon)), \end{array}\right. \tag{3.5}$$

where $C(\Omega)$ is the constant of continuity of the trace map tr .

Choosing $a=\varepsilon\sqrt{\frac{\rho}{\mu}}R(\boldsymbol{x}_0)$, from (3.3)– (3.5) we get

$$\left\{\begin{array}{ll} |X(\varepsilon)| & \le \varepsilon\sqrt{\frac{\rho}{\mu}}R(\boldsymbol{x}_0)\frac{1}{2}\int_\Omega\Big(\varepsilon^2\rho\sum_{\alpha=1,2}|\dot\phi_\alpha(\varepsilon)|^2+\rho|\dot\phi_3(\varepsilon)|^2\Big)d\Omega+\varepsilon\sqrt{\frac{\rho}{\mu}}R(\boldsymbol{x}_0)\frac{1}{2}a(\varepsilon)(\boldsymbol{\phi}(\varepsilon),\boldsymbol{\phi}(\varepsilon)) \\ & +\frac{1}{2}\varepsilon\sqrt{\frac{\rho}{\mu}}\frac{C(\Omega)}{R(\boldsymbol{x}_0)}a(\varepsilon)(\boldsymbol{\phi}(\varepsilon),\boldsymbol{\phi}(\varepsilon)) \\ & \le \frac{\varepsilon\sqrt{\rho}}{\sqrt{\mu}}\max\Big\{R(\boldsymbol{x}_0),\frac{C(\Omega)}{R(\boldsymbol{x}_0)}\Big\}E^{\phi(\varepsilon)}(0). \end{array}\right. \tag{3.6}$$

So from equality (2.27) and the estimate (3.6) we immediatly deduce (3.2). ∎

Let us now fix ε and T, with $0<\varepsilon\le 1$ and $T>T(\varepsilon)$. From theorems 3.1 and 3.2 we deduce that the application

$$\begin{array}{rcl} D_{A(\varepsilon)}\times\boldsymbol{V}(\Omega) & \longrightarrow & \mathbb{R} \\ \{\boldsymbol{\phi}_0,\boldsymbol{\phi}_1\} & \longrightarrow & |||\{\boldsymbol{\phi}_0,\boldsymbol{\phi}_1\}|||_\varepsilon \end{array}$$

where

$$\left\{\begin{array}{ll} |||\{\boldsymbol{\phi}_0,\boldsymbol{\phi}_1\}|||_\varepsilon & = \Big\{\int_{\sum_0}(\boldsymbol{q}\cdot\boldsymbol{\nu})\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\boldsymbol{\phi}_i(\varepsilon)d\Gamma_0dt \\ & +\int_{\sum_\pm}(\boldsymbol{q}\cdot\boldsymbol{\nu})\Big[\varepsilon^2\rho\sum_{\alpha=1,2}\dot\phi^2_\alpha(\varepsilon)+\rho\dot\phi^2_3(\varepsilon)-\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\phi_i(\varepsilon)\Big]d\Gamma_\pm dt\Big\}^{\frac{1}{2}} \end{array}\right. \tag{3.7}$$

is a norm in $D_{A(\varepsilon)}\times\boldsymbol{V}(\Omega)$, equivalent to the usual norm in $\boldsymbol{V}(\Omega)\times[L^2(\Omega)]^3$. We recall that in (3.7) $\boldsymbol{\phi}(\varepsilon)$ is the solution of the homogeneous 3D system (1.11) with initial data $\{\boldsymbol{\phi}_0,\boldsymbol{\phi}_1\}$.

But as $D_{A(\varepsilon)} \times \boldsymbol{V}(\Omega)$ is dense in $\boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$, for the usual norm in $\boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$, we conclude that the closure of $D_{A(\varepsilon)} \times \boldsymbol{V}(\Omega)$ with respect to the norm $|||.|||_\varepsilon$ is exactly the space $\boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$.

3.2 Transposition formulation

We consider now the controlled 3D elasticity problem

$$\begin{cases} \varepsilon^2 \rho \ddot{\psi}_\alpha(\varepsilon) - \partial_j \sigma_{\alpha j}(\varepsilon)(\boldsymbol{\psi}(\varepsilon)) = 0, & \text{in} \quad Q = \Omega \times (0,T), \\ \rho \ddot{\psi}_3(\varepsilon) - \partial_j \sigma_{3j}(\varepsilon)(\boldsymbol{\psi}(\varepsilon)) = 0, & \text{in} \quad Q = \Omega \times (0,T), \\ \psi_i(\varepsilon) = (q_j \nu_j) \frac{\partial \phi_i(\varepsilon)}{\partial \nu}, & \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,T), \\ \sigma_{\alpha j}(\varepsilon)(\boldsymbol{\psi}(\varepsilon))\nu_j = (q_j \nu_j)\Big[\varepsilon^2 \rho \ddot{\phi}_\alpha(\varepsilon) - \tau_j \sigma_{\alpha j}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\Big], & \text{on} \quad \Sigma_\pm = \Gamma_\pm \times (0,T), \\ \sigma_{3j}(\varepsilon)(\boldsymbol{\psi}(\varepsilon))\nu_j = (q_j \nu_j)\Big[\rho \ddot{\phi}_3(\varepsilon) - \tau_j \sigma_{3j}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\Big], & \text{on} \quad \Sigma_\pm = \Gamma_\pm \times (0,T), \\ \boldsymbol{\psi}(\varepsilon)(T) = 0, \quad \dot{\boldsymbol{\psi}}(\varepsilon)(T) = 0, & \text{in} \quad \Omega, \end{cases} \tag{3.8}$$

where $\boldsymbol{\phi}(\varepsilon)$ is the solution of the 3D elasticity problem (1.11) with initial data $\{\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_1(\varepsilon)\}$ in the space $\boldsymbol{V}(\Omega) \times (L^2(\Omega))^3$, and the controls are defined as in remark 1.1.

The solution $\boldsymbol{\psi}(\varepsilon)$ of (3.8) has to be understood in the sense of transposition. The transposition formulation of (3.8) can be obtained as follows :

i) Consider $\boldsymbol{\theta}(\varepsilon)$, the solution of the 3D elasticity problem (2.6), with applied body forces $f_i(\varepsilon)$ and initial data $\{\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon)\}$.

ii) We multiply the first two equations of (3.8) by $\theta_\alpha(\varepsilon)$ and $\theta_3(\varepsilon)$ respectively and integrate by parts on Q, assuming that $\boldsymbol{\psi}(\varepsilon)$ is smooth enough. Using in particular (1.14) we obtain formally the following identity

$$\begin{cases} \int_\Omega \Big[\varepsilon^2 \rho \dot{\psi}_\alpha(\varepsilon)(0)\theta_{0\alpha}(\varepsilon) + \rho \dot{\psi}_3(\varepsilon)(0)\theta_{03}(\varepsilon)\Big] d\Omega \\ - \int_\Omega \Big[\varepsilon^2 \rho \psi_\alpha(\varepsilon)(0)\theta_{1\alpha}(\varepsilon) + \rho \psi_3(\varepsilon)(0)\theta_{13}(\varepsilon)\Big] d\Omega \\ = \int_Q \psi_i(\varepsilon) f_i(\varepsilon) d\Omega + \int_{\sum_0} (\boldsymbol{q} \cdot \boldsymbol{\nu}) \sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon)) \partial_j \boldsymbol{\phi}_i(\varepsilon) d\Gamma_0 dt \\ + \int_{\sum_\pm} (\boldsymbol{q} \cdot \boldsymbol{\nu}) \Big[\varepsilon^2 \rho \dot{\phi}_\alpha(\varepsilon) \dot{\theta}_\alpha(\varepsilon) + \rho \dot{\phi}_3(\varepsilon) \dot{\theta}_3(\varepsilon) - \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon)) \partial_j \theta_i(\varepsilon)\Big] d\Gamma_\pm dt. \end{cases} \tag{3.9}$$

iii) Based on the formal equation (3.9) we introduce the following definition of solution of (3.8) in the sense of transposition.

Definition 3.1 *The function $\boldsymbol{\psi}(\varepsilon)$ is a solution of the 3D elasticity problem (3.8) in the sense of transposition, if $\psi_i(\varepsilon) \in L^\infty(0,T;L^2(\Omega))$, the traces $\{\boldsymbol{\psi}(\varepsilon)(0), \dot{\boldsymbol{\psi}}(\varepsilon)(0)\}$ make sense in $[L^2(\Omega)]^3 \times$*

$[\boldsymbol{V}(\Omega)]'$ *and* $\boldsymbol{\psi}(\varepsilon)$ *verifies*

$$\left\{\begin{array}{l} <\{\dot{\boldsymbol{\psi}}(\varepsilon)(0),-\boldsymbol{\psi}(\varepsilon)(0)\},\{\boldsymbol{\theta}_0(\varepsilon),\boldsymbol{\theta}_1(\varepsilon)\}>_\varepsilon - \int_Q \psi_i(\varepsilon)f_i(\varepsilon)d\Omega dt \\ = \int_{\sum_0}(\boldsymbol{q}\cdot\boldsymbol{\nu})\sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon))\partial_j\boldsymbol{\phi}_i(\varepsilon)d\Gamma_0 dt \\ + \int_{\sum_\pm}(\boldsymbol{q}\cdot\boldsymbol{\nu})\Big[\varepsilon^2\rho\dot{\phi}_\alpha(\varepsilon)\dot{\theta}_\alpha(\varepsilon)+\rho\dot{\phi}_3(\varepsilon)\dot{\theta}_3(\varepsilon)-\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)\Big]d\Gamma_\pm dt, \end{array}\right. \tag{3.10}$$

for any $\boldsymbol{f}(\varepsilon)=(f_i(\varepsilon))\in L^1(0,T;[L^2(\Omega)]^3)$ *and any* $\{\boldsymbol{\theta}_0(\varepsilon),\boldsymbol{\theta}_1(\varepsilon)\}\in\boldsymbol{V}(\Omega)\times[L^2(\Omega)]^3$*, with*

$$\left\{\begin{array}{l} <\{\dot{\boldsymbol{\psi}}(\varepsilon)(0),-\boldsymbol{\psi}(\varepsilon)(0)\},\{\boldsymbol{\theta}_0(\varepsilon),\boldsymbol{\theta}_1(\varepsilon)\}>_\varepsilon= \\ <\Big\{(\varepsilon^2\rho\dot{\psi}_\alpha(\varepsilon)(0),\rho\dot{\psi}_3(\varepsilon)(0)),-(\varepsilon^2\rho\psi_\alpha(\varepsilon)(0),\rho\psi_3(\varepsilon)(0))\Big\},\Big\{(\theta_{0\alpha}(\varepsilon),\theta_{03}(\varepsilon)),(\theta_{1\alpha}(\varepsilon),\theta_{13}(\varepsilon))\Big\}> \end{array}\right. \tag{3.11}$$

where $<.,.>$ *is the duality between* $\boldsymbol{V}(\Omega)\times[L^2(\Omega)]^3$ *and its dual.* ∎

We recall that in (3.10), $\boldsymbol{\theta}(\varepsilon)$ is the solution of the 3D elasticity problem (2.6) with data $\boldsymbol{f}(\varepsilon)$, $\boldsymbol{\theta}_0(\varepsilon)$, $\boldsymbol{\theta}_1(\varepsilon)$ and $\boldsymbol{\phi}(\varepsilon)$ is the solution of the homogeneous 3D elasticity problem (1.11) with initial data $\{\boldsymbol{\phi}_0(\varepsilon),\boldsymbol{\phi}_1(\varepsilon)\}\in\boldsymbol{V}(\Omega)\times[L^2(\Omega)]^3$.

The next theorem, whose proof is similar to that of lemma 1.5, p.151 of Lions [7], establishes the existence of a solution $\boldsymbol{\psi}(\varepsilon)$ of (3.8), in the transposition sense.

Theorem 3.3 *Let* ε *and* T *be fixed, with* $0<\varepsilon\le 1$ *and* $T>0$*. Let* $\boldsymbol{\phi}(\varepsilon)$ *be the solution of the homogeneous 3D elasticity system (1.11) with initial data* $\{\boldsymbol{\phi}_0(\varepsilon),\boldsymbol{\phi}_1(\varepsilon)\}\in\boldsymbol{V}(\Omega)\times[L^2(\Omega)]^3$*. Then there exists a unique solution* $\boldsymbol{\psi}(\varepsilon)$ *in* $L^\infty(0,T;[L^2(\Omega)]^3)$ *of problem (3.8) in the transposition sense.*

Sketch of the proof : The existence of $\boldsymbol{\psi}(\varepsilon)$ satisfying definition 3.1, follows from duality arguments. Let us define the application

$$\begin{array}{rcl} \Upsilon & : & D_{A(\varepsilon)}\times\boldsymbol{V}(\Omega)\times L^1(0,T;\boldsymbol{V}(\Omega))\to\mathbb{R} \\ & & \Big(\boldsymbol{\theta}_0(\varepsilon),\boldsymbol{\theta}_1(\varepsilon),\boldsymbol{f}(\varepsilon)\Big)\to\Upsilon\Big(\boldsymbol{\theta}_0(\varepsilon),\boldsymbol{\theta}_1(\varepsilon),\boldsymbol{f}(\varepsilon)\Big) \end{array}$$

where

$$\left\{\begin{array}{l} \Upsilon\Big(\boldsymbol{\theta}_0(\varepsilon),\boldsymbol{\theta}_1(\varepsilon),\boldsymbol{f}(\varepsilon)\Big)=\int_{\sum_0}(\boldsymbol{q}\cdot\boldsymbol{\nu})\sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon))\partial_j\boldsymbol{\phi}_i(\varepsilon)d\Gamma_0 dt \\ + \int_{\sum_\pm}(\boldsymbol{q}\cdot\boldsymbol{\nu})\Big[\varepsilon^2\rho\dot{\phi}_\alpha(\varepsilon)\dot{\theta}_\alpha(\varepsilon)+\rho\dot{\phi}_3(\varepsilon)\dot{\theta}_3(\varepsilon)-\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)\Big]d\Gamma_\pm dt. \end{array}\right. \tag{3.12}$$

By formula (2.21), the application (3.12) is well defined, is linear and continuous with respect to the norm

$$\Big[||.||^2_{V(\Omega)}+||.||^2_{[L^2(\Omega)]^3}+||.||^2_{[L^1(0,T;V(\Omega))]^3}\Big]^{\frac{1}{2}}. \tag{3.13}$$

More precisely

$$
\left\{
\begin{array}{l}
\Big|\Upsilon\Big(\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon), \boldsymbol{f}(\varepsilon)\Big)\Big| \leq C \max_{0\leq t\leq T} \Big[E^{\phi(\varepsilon)}(t)\Big]^{\frac{1}{2}} \Big[E^{\theta(\varepsilon)}(t)\Big]^{\frac{1}{2}} \\
\leq C_1 \Big\{||\boldsymbol{\phi}_1(\varepsilon)||^2_{[L^2(\Omega)]^3} + a(\varepsilon)(\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_0(\varepsilon))\Big\}^{1/2} \\
\Big\{||\boldsymbol{\theta}_1(\varepsilon)||^2_{[L^2(\Omega)]^3} + a(\varepsilon)(\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_0(\varepsilon)) \\
+\frac{1}{2\varepsilon^2\rho} \sum_{\alpha=1}^2 \Big[\int_0^T ||f_\alpha(\varepsilon)||_{L^2(\Omega)} dt\Big]^2 + \frac{1}{2\rho}\Big[\int_0^T ||f_3(\varepsilon)||_{L^2(\Omega)} dt\Big]^2\Big\}^{1/2},
\end{array}
\right. \tag{3.14}
$$

where C, C_1 denote different positive constants independent of ε. So (3.14) proves the continuity of the application (3.12) with respect to the norm (3.13). Because of the continuity of $a(\varepsilon)(.,.)$, cf. (2.4), and since $D_{A(\varepsilon)}$ is dense in $\boldsymbol{V}(\Omega)$, $\boldsymbol{V}(\Omega)$ is dense in $[L^2(\Omega)]^3$ and $L^1(0,T;\boldsymbol{V}(\Omega))$ is dense in $L^1(0,T;[L^2(\Omega)]^3)$, the map (3.12) can be extended to a linear and continuous map defined in

$$
\boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3 \times L^1(0,T;[L^2(\Omega)]^3). \tag{3.15}
$$

Thus there exists a unique function $\boldsymbol{\psi}(\varepsilon) = \psi_i(\varepsilon) \in L^\infty(0,T;[L^2(\Omega)]^3)$ and a unique pair of functions $\{\vartheta_1(\varepsilon), -\vartheta_0(\varepsilon)\}$ in $[\boldsymbol{V}(\Omega)]' \times [L^2(\Omega)]^3$ satisfying the equations (3.10) with $\dot{\boldsymbol{\psi}}(\varepsilon)(0)$ replaced by $\vartheta_1(\varepsilon)$ and $\boldsymbol{\psi}(\varepsilon)(0)$ replaced by $\vartheta_0(\varepsilon)$.

The remaining part of the proof consists in showing that $\boldsymbol{\psi}(\varepsilon)(0) = \vartheta_0(\varepsilon)$, $\dot{\boldsymbol{\psi}}(\varepsilon)(0) = \vartheta_1(\varepsilon)$. But as the reasoning follows the same steps as in Lions [7], p.151, lemma 1.5, we will omit it. ∎

Remark 3.1 We remark that because of formula (2.21) and the density result of theorem 2.2 formula (3.10) is equivalent to

$$
\left\{
\begin{array}{l}
< \{\dot{\boldsymbol{\psi}}(\varepsilon)(0), -\boldsymbol{\psi}(\varepsilon)(0)\}, \{\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon)\} >_\varepsilon - \int_Q \psi_i(\varepsilon) f_i(\varepsilon) d\Omega dt \\
= \Big[\int_\Omega \rho\Big(\varepsilon^2 \dot{\theta}_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon)) + \dot{\theta}_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_3(\varepsilon))\Big) d\Omega\Big]_0^T \\
+\Big[\int_\Omega \rho\Big(\varepsilon^2 \dot{\phi}_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon)) + \dot{\phi}_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\theta_3(\varepsilon))\Big) d\Omega\Big]_0^T \\
+ \int_0^T \int_\Omega 3\rho\Big[\varepsilon^2 \dot{\theta}_\alpha(\varepsilon)\dot{\phi}_\alpha(\varepsilon) + \dot{\theta}_3(\varepsilon)\dot{\phi}_3(\varepsilon)\Big] d\Omega dt \\
- \int_0^T \int_\Omega \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon) d\Omega dt - \int_0^T \int_\Omega f_i(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon)) d\Omega dt. \quad \blacksquare
\end{array}
\right. \tag{3.16}
$$

3.3 The HUM operator

We introduce the HUM operator $\Lambda(\varepsilon) : \boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3 \rightarrow [\boldsymbol{V}(\Omega)]' \times [L^2(\Omega)]^3$. Let ε be fixed, with $0 < \varepsilon \leq 1$ and $\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\} \in \boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$. First we solve the homogeneous 3D elasticity problem (1.11) with initial data $\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\}$ and then we solve problem (3.8) in the transposition sense (that is, we solve the equation (3.10)). Let $\boldsymbol{\psi}(\varepsilon)$ be the solution of (3.10) and let $\boldsymbol{\psi}_0(\varepsilon) = \boldsymbol{\psi}(\varepsilon)(0)$, $\boldsymbol{\psi}_1(\varepsilon) = \dot{\boldsymbol{\psi}}(\varepsilon)(0)$. Then we define

$$
\Lambda(\varepsilon)(\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\}) = \{\boldsymbol{\psi}_1(\varepsilon), -\boldsymbol{\psi}_0(\varepsilon)\}, \tag{3.17}
$$

that is,

$$< \Lambda(\varepsilon)(\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\}), \{\boldsymbol{\theta}_0, \boldsymbol{\theta}_1\} >_\varepsilon = < \{\boldsymbol{\psi}_1(\varepsilon), -\boldsymbol{\psi}_0(\varepsilon)\}, \{\boldsymbol{\theta}_0, \boldsymbol{\theta}_1\} >_\varepsilon,$$

for any $\{\boldsymbol{\theta}_0, \boldsymbol{\theta}_1\} \in \boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$. The next theorem describes properties of the HUM operator $\Lambda(\varepsilon)$.

Theorem 3.4 *Let $0 < \varepsilon \leq 1$ and $T > T(\varepsilon)$ be fixed. The operator $\Lambda(\varepsilon)$ is a continuous isomorphism between $\boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$ and its dual $[\boldsymbol{V}(\Omega)]' \times [L^2(\Omega)]^3$. Moreover, for any $T > 0$ fixed, if $0 < \varepsilon$ with $T > T(\varepsilon)$, then if $\{\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_1(\varepsilon)\} = \Lambda^{-1}(\varepsilon)(\{\boldsymbol{\psi}_1, -\boldsymbol{\psi}_0\})$ where $\{\boldsymbol{\psi}_1, -\boldsymbol{\psi}_0\} \in [\boldsymbol{V}(\Omega)]' \times [L^2(\Omega)]^3$, we have*

$$\left\{ \begin{array}{l} \Big\{ ||(\varepsilon\sqrt{\rho}\phi_{11}(\varepsilon), \varepsilon\sqrt{\rho}\phi_{12}(\varepsilon), \sqrt{\rho}\phi_{13}(\varepsilon))||^2_{[L^2(\Omega)]^3} + a(\varepsilon)(\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_0(\varepsilon)) \Big\}^{\frac{1}{2}} \\ \leq \frac{C}{T-T(\varepsilon)} ||\{\boldsymbol{\psi}_1, -\boldsymbol{\psi}_0\}||_{[V(\Omega)]' \times [L^2(\Omega)]^3}, \end{array} \right. \quad (3.18)$$

where C is a constant independent of ε.

Proof : Let us take $\boldsymbol{f}(\varepsilon) = 0$ in the transposition formula (3.10). Then by definition of $\Lambda(\varepsilon)$, we have

$$\left\{ \begin{array}{l} < \Lambda(\varepsilon)(\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\}), \{\boldsymbol{\theta}_0, \boldsymbol{\theta}_1\} >_\varepsilon = \int_{\sum_0} (\boldsymbol{q} \cdot \boldsymbol{\nu}) \sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon)) \partial_j \boldsymbol{\phi}_i(\varepsilon) d\Gamma_0 dt \\ + \int_{\sum_\pm} (\boldsymbol{q} \cdot \boldsymbol{\nu}) \Big[\varepsilon^2 \rho \dot{\phi}_\alpha(\varepsilon) \dot{\theta}_\alpha(\varepsilon) + \rho \dot{\phi}_3(\varepsilon) \dot{\theta}_3(\varepsilon) - \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon)) \partial_j \theta_i(\varepsilon) \Big] d\Gamma_\pm dt, \end{array} \right. \quad (3.19)$$

for any $\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\} \in \boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$ and any $\{\boldsymbol{\theta}_0, \boldsymbol{\theta}_1\} \in \boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$.

Because $0 < \varepsilon \leq 1$ and by formula (3.14) we conclude that

$$\left\{ \begin{array}{l} < \Lambda(\varepsilon)(\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\}), \{\boldsymbol{\theta}_0, \boldsymbol{\theta}_1\} >_\varepsilon \\ \leq C_0 \Big\{ ||\boldsymbol{\phi}_1(\varepsilon)||^2_{[L^2(\Omega)]^3} + a(\varepsilon)(\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_0(\varepsilon)) \Big\}^{1/2} \Big\{ ||\boldsymbol{\theta}_1(\varepsilon)||^2_{[L^2(\Omega)]^3} + a(\varepsilon)(\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_0(\varepsilon)) \Big\}^{1/2}, \end{array} \right.$$

where C_0 is a positive constant independent of ε. Consequently, by the continuity of $a(\varepsilon)(.,.)$, cf. (2.4), the operator $\Lambda(\varepsilon)$ is continuous for each ε.

If we take now $\{\boldsymbol{\theta}_0, \boldsymbol{\theta}_1\} = \{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\}$, we obtain

$$\left\{ \begin{array}{l} < \Lambda(\varepsilon)(\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\}), \{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\} >_\varepsilon = |||\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\}|||^2_\varepsilon \\ \geq \frac{T-T(\varepsilon)}{C_1} ||\{(\phi_{01}, \phi_{02}, \phi_{03}), (\varepsilon\sqrt{\rho}\phi_{11}, \varepsilon\sqrt{\rho}\phi_{12}, \phi_{13})\}||_{V(\Omega) \times [L^2(\Omega)]^3}, \end{array} \right. \quad (3.20)$$

with $C_1 > 0$ independent of ε, because of the inverse inequality of theorem 3.2 and the uniform ellipticity of $a(\varepsilon)(.,.)$, cf. (2.4). So (3.20) means that, for each ε, the bilinear form associated to $\Lambda(\varepsilon)$ is $\boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$ –elliptic with respect to ε. Thus $\Lambda(\varepsilon)$ is an isomorphism.

Let us now consider $T > 0$ fixed, $0 < \varepsilon$, $T > T(\varepsilon)$ and $\{\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_1(\varepsilon)\} = \Lambda^{-1}(\varepsilon)(\{\boldsymbol{\psi}_1, -\boldsymbol{\psi}_0\})$ where $\{\boldsymbol{\psi}_1, -\boldsymbol{\psi}_0\} \in [\boldsymbol{V}(\Omega)]' \times [L^2(\Omega)]^3$. Then

$$|||\{\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_1(\varepsilon)\}|||^2_\varepsilon = < \{\boldsymbol{\psi}_1, -\boldsymbol{\psi}_0\}, \{\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_1(\varepsilon)\} >_\varepsilon . \quad (3.21)$$

And again by the inverse inequality of theorem 3.2 and the uniform ellipticity of $a(\varepsilon)(.,.)$ we immediately deduce that

$$\left\{\begin{array}{l} ||(\varepsilon\sqrt{\rho}\phi_{11}(\varepsilon),\varepsilon\sqrt{\rho}\phi_{12}(\varepsilon),\sqrt{\rho}\phi_{13}(\varepsilon))||^2_{[L^2(\Omega)]^3}+a(\varepsilon)(\boldsymbol{\phi}_0(\varepsilon),\boldsymbol{\phi}_0(\varepsilon)) \\ \leq \frac{C_3}{T-T(\varepsilon)}|||\{\boldsymbol{\phi}_0(\varepsilon),\boldsymbol{\phi}_1(\varepsilon)\}|||^2_\varepsilon = \frac{C_3}{T-T(\varepsilon)} < \{\boldsymbol{\psi}_1,-\boldsymbol{\psi}_0\},\{\boldsymbol{\phi}_0(\varepsilon),\boldsymbol{\phi}_1(\varepsilon)\} >_\varepsilon \\ \leq \frac{C_3}{T-T(\varepsilon)}||\{\boldsymbol{\psi}_1,-\boldsymbol{\psi}_0\}||_{[V(\Omega)]'\times[L^2(\Omega)]^3} \\ ||\{(\varepsilon\sqrt{\rho}\phi_{01}(\varepsilon),\varepsilon\sqrt{\rho}\phi_{02}(\varepsilon),\sqrt{\rho}\phi_{03}(\varepsilon)),(\varepsilon\sqrt{\rho}\phi_{11}(\varepsilon),\varepsilon\sqrt{\rho}\phi_{12}(\varepsilon),\sqrt{\rho}\phi_{13}(\varepsilon))\}||_{V(\Omega)\times[L^2(\Omega)]^3} \\ \leq \frac{C}{T-T(\varepsilon)}||\{\boldsymbol{\psi}_1,-\boldsymbol{\psi}_0\}||_{[V(\Omega)]'\times[L^2(\Omega)]^3} \\ \Big\{||(\varepsilon\sqrt{\rho}\phi_{11}(\varepsilon),\varepsilon\sqrt{\rho}\phi_{12}(\varepsilon),\sqrt{\rho}\phi_{13}(\varepsilon))||^2_{[L^2(\Omega)]^3}+a(\varepsilon)(\boldsymbol{\phi}_0(\varepsilon),\boldsymbol{\phi}_0(\varepsilon))\Big\}^{\frac{1}{2}}, \end{array}\right. \tag{3.22}$$

where C_3 and C are a positive constants independent of ε, so we derive (3.18). ∎

3.4 Proof of theorem 1.1 (Controllability result)

Let us see that the fact that $\Lambda(\varepsilon)$ is an isomorphism from $\boldsymbol{V}(\Omega)\times[L^2(\Omega)]^3$ into its dual immediatly implies theorem 1.1. Let $0<\varepsilon\leq 1$ and $T>T(\varepsilon)$ be fixed. We take $\{\boldsymbol{\psi}_1,-\boldsymbol{\psi}_0\}\in[\boldsymbol{V}(\Omega)]'\times[L^2(\Omega)]^3$ and we find $\{\boldsymbol{\phi}_0(\varepsilon),\boldsymbol{\phi}_1(\varepsilon)\}\in\boldsymbol{V}(\Omega)\times[L^2(\Omega)]^3$ solution of the problem

$$\Lambda(\varepsilon)(\{\boldsymbol{\phi}_0(\varepsilon),\boldsymbol{\phi}_1(\varepsilon)\})=\{\boldsymbol{\psi}_1,-\boldsymbol{\psi}_0\}. \tag{3.23}$$

Then, we solve the homogeneous 3D elasticity problem (1.11), with initial data $\{\boldsymbol{\phi}_0(\varepsilon),\boldsymbol{\phi}_1(\varepsilon)\}$ and we obtain the function $\boldsymbol{\phi}(\varepsilon)$. Next, with this function we define the controls $u_i(\varepsilon)$ and $v_i(\varepsilon)$ as described in (1.10). In view of (3.23) the solution $\boldsymbol{\psi}(\varepsilon)$ of (3.8) satisfies

$$\boldsymbol{\psi}(\varepsilon)(0)=\boldsymbol{\psi}_0, \qquad \dot{\boldsymbol{\psi}}(\varepsilon)=\boldsymbol{\psi}_1,$$

or, equivalently, the solution $\boldsymbol{\psi}(\varepsilon)$ of (1.6) satisfies

$$\boldsymbol{\psi}(\varepsilon)(T)=\dot{\boldsymbol{\psi}}(\varepsilon)(T)=0.$$

4 Asymptotic Limit

In this section we will prove theorems 1.2 and 1.3. In order to do that, we first recall two results from asymptotical analysis (theorems 4.1 and 4.2 below). Then, in lemma 4.2, we analyse the limit behaviour, as $\varepsilon\to 0$, of the transposition formula (3.10), using the identity (2.21). In lemma 4.3 we conclude that at the limit the controlled displacement is of the Kirchhoff-Love type. These results and a suitable de-scaling (already indicated in (1.21)) are all we need to prove theorems 1.2 and 1.3.

4.1 Some results from asymptotical analysis

Theorem 4.1 (Weak convergence of $\boldsymbol{\phi}(\varepsilon)$ solution of (1.11)) *Assume that the initial data $\{\boldsymbol{\phi}_0(\varepsilon),\boldsymbol{\phi}_1(\varepsilon)\}_{\varepsilon>0}\in\boldsymbol{V}(\Omega)\times[L^2(\Omega)]^3$ of the homogeneous problem (1.11) satisfy*

$$E^{\phi(\varepsilon)}(0)=\frac{1}{2}\Big\{||(\varepsilon\sqrt{\rho}\phi_{11}(\varepsilon),\varepsilon\sqrt{\rho}\phi_{12}(\varepsilon),\sqrt{\rho}\phi_{13}(\varepsilon))||^2_{[L^2(\Omega)]^3}+a(\varepsilon)(\boldsymbol{\phi}_0(\varepsilon),\boldsymbol{\phi}_0(\varepsilon))\Big\}\leq C, \tag{4.1}$$

where C is a positive constant independent of ε. Let $\boldsymbol{\phi}(\varepsilon)$ be the solution of (1.11), with initial data $\{\boldsymbol{\phi}_0(\varepsilon),\boldsymbol{\phi}_1(\varepsilon)\}$. Then there exists a subsequence $\{\boldsymbol{\phi}(\varepsilon_k)\}_{\varepsilon_k>0}$ satisfying the following

i) *There exists* $\boldsymbol{\phi} \in L^\infty(0,T;\boldsymbol{V}(\Omega)) \cap W^{1,\infty}(0,T;[L^2(\Omega)]^3)$ *such that, as* $\varepsilon_k \to 0$

$$\begin{array}{rll} \boldsymbol{\phi}(\varepsilon_k) \longrightarrow \boldsymbol{\phi}, & \textit{weakly* in} & L^\infty(0,T;\boldsymbol{V}(\Omega)), \\ \dot{\phi}_3(\varepsilon_k) \longrightarrow \dot{\phi}_3, & \textit{weakly* in} & L^\infty(0,T;L^2(\Omega)), \\ \varepsilon\dot{\phi}_\alpha(\varepsilon_k) \longrightarrow 0, & \textit{weakly* in} & L^\infty(0,T;L^2(\Omega)), \\ e_{\alpha\beta}(\boldsymbol{\phi}(\varepsilon_k)) \longrightarrow e_{\alpha\beta}(\boldsymbol{\phi}), & \textit{weakly* in} & L^\infty(0,T;L^2(\Omega)), \\ \frac{1}{\varepsilon}e_{\alpha 3}(\boldsymbol{\phi}(\varepsilon_k)) \longrightarrow 0, & \textit{weakly* in} & L^\infty(0,T;L^2(\Omega)), \\ \frac{1}{\varepsilon^2}e_{33}(\boldsymbol{\phi}(\varepsilon_k)) \longrightarrow -\frac{\lambda}{\lambda+2\mu}e_{\alpha\alpha}(\boldsymbol{\phi}), & \textit{weakly* in} & L^\infty(0,T;L^2(\Omega)). \end{array} \tag{4.2}$$

ii) *The limit function* $\boldsymbol{\phi} = (\phi_\alpha, \phi_3)$ *is a Kirchhoff-Love displacement, that is*

$$\begin{array}{lll} \phi_3 & \textit{is independent of} & x_3, \\ \phi_\alpha = \hat{\phi}_\alpha - x_3\partial_\alpha\phi_3, \quad \hat{\phi}_\alpha & \textit{is independent of} & x_3. \end{array} \tag{4.3}$$

Moreover $\hat{\phi}_\alpha = 0$ *and* $\phi_3 \in C^0([0,T];H_0^2(\omega) \cap C^1([0,T];L^2(\omega))$ *and* ϕ_3 *is the unique solution of the 2D plate problem*

$$\left\{ \begin{array}{r} 2\rho\ddot{\phi}_3 - \partial_{\alpha\beta}m_{\alpha\beta}(\phi_3) = 0, \quad \textit{in} \quad \omega \times (0,T), \\ \phi_3 = \frac{\partial\phi_3}{\partial\nu} = 0, \quad \textit{on} \quad \partial\omega \times (0,T), \\ \phi_3(0) = \frac{1}{2}\int_{-1}^{+1}\phi_{03}dx_3, \quad \dot{\phi}_3(0) = \frac{1}{2}\int_{-1}^{+1}\phi_{13}dx_3, \quad \textit{in} \quad \omega, \end{array} \right. \tag{4.4}$$

where ϕ_{03} *is the weak limit of* $\{\phi_{03}(\varepsilon_k)\}_{\varepsilon_k>0}$ *in* $H^1_{|\Gamma_0}(\Omega)$ *such that* $\frac{1}{2}\int_{-1}^{+1}\phi_{03}dx_3 \in H_0^2(\omega)$, *and* ϕ_{13} *is the weak limit of* $\{\phi_{13}(\varepsilon_k)\}_{\varepsilon_k>0}$ *in* $L^2(\Omega)$.

Proof : (cf. Raoult [14], [15] and Ciarlet [2], theorems 5.2-2 and 5.2-3, p.175-177). ∎

Theorem 4.2 (Strong convergence of $\boldsymbol{\theta}(\varepsilon)$ solution of (2.6)) *Suppose that the initial data* $\{\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon)\}_{\varepsilon>0} \in \boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$ *and the applied body forces* $\{\boldsymbol{f}(\varepsilon)\}_{\varepsilon>0}$ *of the 3D elasticity problem (2.6) satisfy the following :*

1) $f_\alpha(\varepsilon) \in X$, $f_3(\varepsilon) \in L^1(0,T,L^2(\Omega))$ *and as* $\varepsilon \to 0$

$$\begin{array}{rll} f_\alpha(\varepsilon) \longrightarrow f_\alpha, & \textit{strongly in} & X, \\ f_3(\varepsilon) \longrightarrow f_3, & \textit{strongly in} & L^1(0,T;L^2(\Omega)). \end{array} \tag{4.5}$$

2) *The sequence* $\{\boldsymbol{\theta}_0(\varepsilon)\}_{\varepsilon>0} \in \boldsymbol{V}(\Omega)$ *verifies, as* $\varepsilon \to 0$

$$\begin{array}{rll} \boldsymbol{\theta}_0(\varepsilon) \longrightarrow \boldsymbol{\theta}_0, & \textit{strongly in} & \boldsymbol{V}(\Omega), \\ \frac{1}{\varepsilon}e_{\alpha 3}(\boldsymbol{\theta}_0(\varepsilon)) \longrightarrow 0, & \textit{strongly in} & L^2(\Omega), \\ \frac{1}{\varepsilon^2}e_{33}(\boldsymbol{\theta}_0(\varepsilon)) \longrightarrow -\frac{\lambda}{\lambda+2\mu}e_{\alpha\alpha}(\boldsymbol{\theta}_0), & \textit{strongly in} & L^2(\Omega). \end{array} \tag{4.6}$$

The displacement $\boldsymbol{\theta}_0$ *is a Kirchhoff-Love displacement, that is* $\boldsymbol{\theta_0} = (\hat{\theta}_{0\alpha} - x_3\partial_\alpha\theta_{03}, \theta_{03})$, *where* $\hat{\theta}_{0\alpha}$, θ_{03} *are independent of* x_3 *and* $\hat{\boldsymbol{\theta}}_0 = (\hat{\theta}_{01}, \hat{\theta}_{02})$ *is the solution of the 2D problem*

$$\left\{ \begin{array}{l} -\partial_\beta n_{\alpha\beta}(\hat{\boldsymbol{\theta}}_0) = \int_{-1}^{+1} f_\alpha dx_3, \quad in \quad \omega \times (0,T), \\ \\ \hat{\theta}_{0\alpha} = 0, \quad on \quad \partial\omega, \end{array} \right. \tag{4.7}$$

where

$$n_{\alpha\beta}(\hat{\boldsymbol{\theta}}) = \frac{4\lambda\mu}{\lambda + 2\mu} e_{\zeta\zeta}(\hat{\boldsymbol{\theta}})\delta_{\alpha\beta} + 4\mu e_{\alpha\beta}(\hat{\boldsymbol{\theta}}). \tag{4.8}$$

3) *The sequence* $\{\boldsymbol{\theta}_1(\varepsilon)\}_{\varepsilon>0} \in [L^2(\Omega)]^3$ *satisfies, as* $\varepsilon \to 0$

$$\begin{array}{llll} \varepsilon\theta_{1\alpha}(\varepsilon) \longrightarrow 0, & \text{strongly in} & L^2(\Omega), & \\ \theta_{13}(\varepsilon) \longrightarrow \theta_{13}, & \text{strongly in} & L^2(\Omega), & \theta_{13} \in L^2(\omega). \end{array} \tag{4.9}$$

Then, if the hypotheses (4.5)–(4.9) are verified, the sequence of solutions $\{\boldsymbol{\theta}(\varepsilon)\}_{\varepsilon>0}$ *of (2.6) fulfils the following*

i) *There exists* $\boldsymbol{\theta} \in L^\infty(0,T;\boldsymbol{V}(\Omega)) \cap H^1(0,T;[L^2(\Omega)]^3)$, *such that as* $\varepsilon \to 0$

$$\begin{array}{rll} \boldsymbol{\theta}(\varepsilon) \longrightarrow \boldsymbol{\theta}, & \text{strongly in} & L^2(0,T;\boldsymbol{V}(\Omega)), \\ \dot{\theta}_3(\varepsilon) \longrightarrow \dot{\theta}_3, & \text{strongly in} & L^2(0,T;L^2(\Omega)), \\ \varepsilon\dot{\theta}_\alpha(\varepsilon) \longrightarrow 0, & \text{strongly in} & L^2(0,T;L^2(\Omega)), \\ e_{\alpha\beta}(\boldsymbol{\theta}(\varepsilon)) \longrightarrow e_{\alpha\beta}(\boldsymbol{\theta}), & \text{strongly in} & L^2(0,T;L^2(\Omega)), \\ \dfrac{1}{\varepsilon} e_{\alpha 3}(\boldsymbol{\theta}(\varepsilon)) \longrightarrow 0, & \text{strongly in} & L^2(0,T;L^2(\Omega)), \\ \dfrac{1}{\varepsilon^2} e_{33}(\boldsymbol{\theta}(\varepsilon)) \longrightarrow -\dfrac{\lambda}{\lambda + 2\mu} e_{\alpha\alpha}(\boldsymbol{\theta}), & \text{strongly in} & L^2(0,T;L^2(\Omega)). \end{array} \tag{4.10}$$

ii) *The limit* $\boldsymbol{\theta}$ *is a Kirchhoff-Love displacement, that is* $\boldsymbol{\theta} = (\hat{\theta}_\alpha - x_3\partial_\alpha\theta_3, \theta_3)$, *with* $\hat{\theta}_\alpha$ *and* θ_3 *independent of* x_3, $\hat{\theta}_\alpha \in L^2(0,T;H_0^1(\omega))$, $\theta_3 \in C^0([0,T];H_0^2(\omega)) \cap C^1([0,T];L^2(\omega))$ *:*

- θ_3 *is the solution of the following 2D plate problem*

$$\left\{ \begin{array}{ll} 2\rho\ddot{\theta}_3 - \partial_{\alpha\beta} m_{\alpha\beta}(\theta_3) = \int_{-1}^{+1} f_3 dx_3 + \partial_\alpha \int_{-1}^{+1} x_3 f_\alpha dx_3, & in \quad \omega \times (0,T), \\ \\ \theta_3 = \frac{\partial\theta_3}{\partial\nu} = 0, & on \quad \partial\omega \times (0,T), \\ \\ \theta_3(0) = \frac{1}{2}\int_{-1}^{+1} \theta_{03} dx_3, \quad \dot{\theta}_3(0) = \frac{1}{2}\int_{-1}^{+1} \theta_{13} dx_3, & in \quad \omega, \end{array} \right. \tag{4.11}$$

- $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2)$ *is the solution of the following 2D elasticity problem*

$$\left\{ \begin{array}{l} -\partial_\beta n_{\alpha\beta}(\hat{\boldsymbol{\theta}}) = \int_{-1}^{+1} f_\alpha dx_3, \quad in \quad \omega \times (0,T), \\ \\ \hat{\theta}_\alpha = 0, \quad on \quad \partial\omega. \end{array} \right. \tag{4.12}$$

Proof : (cf. Raoult [14], [15] and Ciarlet [2], theorems 5.2-2 and 5.2-3, p.175-177). ∎

Remark 4.1 It is important to remark, because it will be used in the sequel, that if $\boldsymbol{\theta}$ is a Kirchhoff-Love displacement that belongs to $\boldsymbol{V}(\Omega)$, then

$$\boldsymbol{\theta} = (\hat{\theta}_1 - x_3\partial_1\theta_3, \hat{\theta}_2 - x_3\partial_2\theta_3, \theta_3),$$

where the functions $\hat{\theta}_1$, $\hat{\theta}_2$ and θ_3 are independent of x_3 and satisfy

$$\theta_3 \in H_0^2(\omega), \qquad \hat{\theta}_\alpha \in H_0^1(\omega). \quad \blacksquare$$

4.2 Auxiliar lemmas

In this section, we will prove some asymptotic results that will be used to prove theorems 1.2 and 1.3.

With the results of theorems 4.1 and 4.2 we will now calculate the limit of the boundary integrals of (3.10), using identity (2.21).

Lemma 4.1 *Suppose that the sequence* $\{\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_1(\varepsilon)\}_{\varepsilon>0} \in D_{A(\varepsilon)} \times \boldsymbol{V}(\Omega)$ *satisfies the assumption of theorem 4.1, and let us denote by* $\{\boldsymbol{\phi}(\varepsilon)\}_{\varepsilon>0}$ *the corresponding subsequence of solutions of (1.11) verifying the condtions (4.2)–(4.4) of theorem 4.1. Assume also that the sequences* $\{\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon)\}_{\varepsilon>0} \in D_{A(\varepsilon)} \times \boldsymbol{V}(\Omega)$ *and* $\{\boldsymbol{f}(\varepsilon)\}_{\varepsilon>0} \in L^1(0,T;\boldsymbol{V}(\Omega))$ *satisfy the hypotheses of theorems 4.2 and let* $\{\boldsymbol{\theta}(\varepsilon)\}_{\varepsilon>0}$ *be the corresponding sequence of solutions of (2.6), verifying (4.10)–(4.12).*

Then

$$\left\{\begin{array}{l} \lim_{\varepsilon\to 0}\Big\{ \int_{\sum_0}(\boldsymbol{q}\cdot\boldsymbol{\nu})\sigma_{ij}(\varepsilon)(\boldsymbol{\theta}(\varepsilon))\partial_j\boldsymbol{\phi}_i(\varepsilon)d\Gamma_0 dt \\ \\ + \int_{\sum_\pm}(\boldsymbol{q}\cdot\boldsymbol{\nu})\Big[\varepsilon^2\rho\dot{\phi}_\alpha(\varepsilon)\dot{\theta}_\alpha(\varepsilon) + \rho\dot{\phi}_3(\varepsilon)\dot{\theta}_3(\varepsilon) - \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)\Big]d\Gamma_\pm dt\Big\} \\ \\ = \Big[\int_\omega 2\rho\Big(\dot{\theta}_3(q_\alpha\partial_\alpha\phi_3) + \dot{\phi}_3(q_\alpha\partial_\alpha\theta_3)\Big)d\omega\Big]_0^T + 3\int_{\omega\times(0,T)} 2\rho\dot{\theta}_3\dot{\phi}_3 d\omega dt \\ \\ + \int_{\omega\times(0,T)} m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\omega dt - < f_3 + x_3\partial_\beta f_\beta, q_\alpha\partial_\alpha\phi_3 > . \end{array}\right. \tag{4.13}$$

where θ_3 *and* ϕ_3 *are the solutions of problems (4.11) and (4.4), respectively, and*

$$< f_3 + x_3\partial_\beta f_\beta, q_\alpha\partial_\alpha\phi_3 >= \int_{\Omega\times(0,T)} f_3(q_\alpha\partial_\alpha\phi_3)d\Omega dt - \int_{\Omega\times(0,T)} x_3 f_\beta\partial_\beta(q_\alpha\partial_\alpha\phi_3)d\Omega dt. \tag{4.14}$$

Proof : By (2.21), we can compute the limit of the boundary integrals indicated in (4.13), if we calculate the limit of the right-hand side of (2.21). Due to (4.2) and (4.10) we easily obtain

$$\left\{\begin{array}{l} \lim_{\varepsilon\to 0}\Big\{\Big[\int_\Omega \rho\Big(\varepsilon^2\dot{\theta}_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon)) + \dot{\theta}_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_3(\varepsilon))\Big)d\Omega\Big]_0^T \\ \\ +\Big[\int_\Omega \rho\Big(\varepsilon^2\dot{\phi}_\alpha(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_\alpha(\varepsilon)) + \dot{\phi}_3(\varepsilon)(\boldsymbol{q}\cdot\nabla\theta_3(\varepsilon))\Big)d\Omega\Big]_0^T\Big\} \\ \\ = \Big[\int_\omega 2\rho\Big(\dot{\theta}_3(q_\alpha\partial_\alpha\phi_3) + \dot{\phi}_3(q_\alpha\partial_\alpha\theta_3)\Big)d\omega\Big]_0^T, \end{array}\right. \tag{4.15}$$

and

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega 3\rho\Big[\varepsilon^2\dot\theta_\alpha(\varepsilon)\dot\phi_\alpha(\varepsilon)+\dot\theta_3(\varepsilon)\dot\phi_3(\varepsilon)\Big]d\Omega dt = 3\int_{\omega\times(0,T)} 2\rho\dot\theta_3\dot\phi_3 d\omega dt. \tag{4.16}$$

Let us compute now the limit of $\int_Q f_i(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_i(\varepsilon))d\Omega dt$, when $\varepsilon\to 0$. Since $\partial_3\phi_3=0$, we directly obtain (cf. (4.3))

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega f_i(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_i(\varepsilon))d\Omega dt = \int_0^T\int_\Omega\Big[f_i(q_\alpha\partial_\alpha\phi_i)+f_\beta(q_3\partial_3\phi_\beta)\Big]d\Omega dt \tag{4.17}$$

But, as

$$\begin{aligned}&q_3(\boldsymbol{x})=x_3,\quad \partial_3\phi_\beta=-\partial_\beta\phi_3\\ &\partial_\alpha\phi_\beta=\partial_\alpha\hat\phi_\beta-x_3\partial_{\alpha\beta}\phi_3=-x_3\partial_{\alpha\beta}\phi_3,\qquad\text{since}\quad \hat\phi_\beta=0,\end{aligned} \tag{4.18}$$

we get

$$\left\{\begin{aligned}&\textstyle\int_0^T\int_\Omega\Big[f_i(q_\alpha\partial_\alpha\phi_i)+f_\beta(q_3\partial_3\phi_\beta)\Big]d\Omega dt\\ &=\textstyle\int_0^T\int_\Omega\Big[f_\beta q_\alpha(-x_3\partial_{\alpha\beta}\phi_3)+f_3(q_\alpha\partial_\alpha\phi_3)-f_\beta x_3\partial_\beta\phi_3\Big]d\Omega dt\\ &=-<\textstyle\int_{-1}^{+1} f_\beta q_\alpha x_3dx_3,\partial_{\alpha\beta}\phi_3>+<\int_{-1}^{+1}f_3dx_3,q_\alpha\partial_\alpha\phi_3>-<\int_{-1}^{+1}x_3f_\beta dx_3,\partial_\beta\phi_3>\\ &=<\textstyle\int_{-1}^{+1}\partial_\beta(f_\beta q_\alpha)x_3dx_3,\partial_\alpha\phi_3>+<\int_{-1}^{+1}f_3dx_3,q_\alpha\partial_\alpha\phi_3>-<\int_{-1}^{+1}x_3f_\beta dx_3,\partial_\beta\phi_3>\\ &=<\textstyle\int_{-1}^{+1}\partial_\beta f_\beta q_\alpha x_3dx_3+\int_{-1}^{+1}f_\alpha x_3dx_3,\partial_\alpha\phi_3>+<\int_{-1}^{+1}f_3dx_3,q_\alpha\partial_\alpha\phi_3>\\ &-<\textstyle\int_{-1}^{+1}x_3f_\beta dx_3,\partial_\beta\phi_3>=<\int_{-1}^{+1}(f_3+\partial_\beta f_\beta x_3)dx_3,q_\alpha\partial_\alpha\phi_3>\\ &=\textstyle\int_{\omega\times(0,T)}\Big(\int_{-1}^{+1}f_3dx_3\Big)(q_\alpha\partial_\alpha\phi_3)d\omega dt-\int_{\omega\times(0,T)}\Big(\int_{-1}^{+1}x_3f_\beta dx_3\Big)\partial_\beta(q_\alpha\partial_\alpha\phi_3)d\omega dt,\end{aligned}\right. \tag{4.19}$$

where the duality is understood in the sense of $L^1(0,T;L^2(\omega))$ and its dual $L^\infty(0,T;L^2(\omega))$. So from (4.17) and (4.19) we obtain

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega f_i(\varepsilon)(\boldsymbol{q}\cdot\nabla\phi_i(\varepsilon))d\Omega dt =<\int_{-1}^{+1}\Big(f_3+x_3\partial_\beta f_\beta\Big)dx_3,q_\alpha\partial_\alpha\phi_3>. \tag{4.20}$$

Finally it remains to obtain the limit of $\int_Q\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)d\Omega dt$ when $\varepsilon\to 0$. We have directly from the definition of $\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)$ (cf. (2.2)) and the convergences (4.2) and (4.10)

$$\left\{\begin{aligned}&\textstyle\lim_{\varepsilon\to 0}\int_Q\sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)d\Omega dt\\ &=\textstyle\lim_{\varepsilon\to 0}\Big\{\int_Q\Big[\lambda e_{\alpha\alpha}(\boldsymbol{\phi}(\varepsilon))e_{\beta\beta}(\boldsymbol{\theta}(\varepsilon))+2\mu e_{\alpha\beta}(\boldsymbol{\phi}(\varepsilon))e_{\alpha\beta}(\boldsymbol{\theta}(\varepsilon))\Big]d\Omega dt\\ &+\textstyle\int_Q\lambda\Big[e_{\alpha\alpha}(\boldsymbol{\phi}(\varepsilon))\frac{1}{\varepsilon^2}e_{33}(\boldsymbol{\theta}(\varepsilon))+\frac{1}{\varepsilon^2}e_{33}(\boldsymbol{\phi}(\varepsilon))e_{\beta\beta}(\boldsymbol{\theta}(\varepsilon))\Big]d\Omega dt\\ &+\textstyle\int_Q 4\mu\frac{1}{\varepsilon}e_{\alpha3}(\boldsymbol{\phi}(\varepsilon))\frac{1}{\varepsilon}e_{\alpha3}(\boldsymbol{\theta}(\varepsilon))d\Omega dt+\int_Q(\lambda+2\mu)\frac{1}{\varepsilon^2}e_{33}(\boldsymbol{\phi}(\varepsilon))\frac{1}{\varepsilon^2}e_{33}(\boldsymbol{\theta}(\varepsilon))d\Omega dt\Big\}\\ &=\textstyle\int_Q\Big[\frac{2\mu\lambda}{\lambda+2\mu}e_{\alpha\alpha}(\boldsymbol{\phi})e_{\beta\beta}(\boldsymbol{\theta})+2\mu e_{\alpha\beta}(\boldsymbol{\phi})e_{\alpha\beta}(\boldsymbol{\theta})\Big]d\Omega dt.\end{aligned}\right. \tag{4.21}$$

But, taking into account that

$$\begin{aligned} e_{\alpha\beta}(\boldsymbol{\theta}) &= \frac{1}{2}\Big(\partial_\alpha \hat{\theta}_\beta + \partial_\beta \hat{\theta}_\alpha\Big) - x_3 \partial_{\alpha\beta}\theta_3 \\ e_{\alpha\beta}(\boldsymbol{\phi}) &= -x_3 \partial_{\alpha\beta}\phi_3 \end{aligned}$$

we obtain that

$$\lim_{\varepsilon\to 0}\int_Q \sigma_{ij}(\varepsilon)(\boldsymbol{\phi}(\varepsilon))\partial_j\theta_i(\varepsilon)d\Omega dt = \int_{\omega\times(0,T)} -m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\omega dt, \tag{4.22}$$

where $m_{\alpha\beta}(\theta_3)$ is defined in (2.32). So from (4.15), (4.16), (4.20) and (4.22) we deduce the result. ∎

We can now study the limit behaviour of the transposition formula (3.10).

Lemma 4.2 *Let $T > 0$ and $\{\boldsymbol{\psi}(\varepsilon)\}_{\varepsilon>0}$ (with ε small enough and such that $T(\varepsilon) < T$) be the sequence of solutions of the exact controllability problems (3.10) with initial data $\{\boldsymbol{\psi}_0(\varepsilon), \boldsymbol{\psi}_1(\varepsilon)\}_{\varepsilon>0}$ satisfying*

$$\exists C > 0 \quad (\textit{independent of} \quad \varepsilon) \quad : \quad ||\{\boldsymbol{\psi}_1(\varepsilon), -\boldsymbol{\psi}_0(\varepsilon)\}||_{[V(\Omega)]'\times[L^2(\Omega)]^3} \leq C. \tag{4.23}$$

In addition we suppose that the sequences of functions $\{\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon)\}_{\varepsilon>0}$ and $\{f_\alpha(\varepsilon), f_3(\varepsilon)\}_{\varepsilon>0}$ of (3.10) verify

$$\{\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon)\}_{\varepsilon>0} \in \boldsymbol{V}(\Omega)\times[L^2(\Omega)]^3, \qquad \{f_\alpha(\varepsilon), f_3(\varepsilon)\}_{\varepsilon>0} \in X^2 \times L^1(0,T;L^2(\Omega)), \tag{4.24}$$

and satisfy the assumptions of theorems 4.2.

Then, there exists $\psi_\alpha \in X'$ and $\psi_3 \in L^\infty(0,T;L^2(\Omega))$ such that

$$\left\{ \begin{array}{l} <\{\rho\psi_{13}, -\rho\psi_{03}\}, \{\theta_{03}, \theta_{13}\}> = <\psi_i, f_i> + \int_{\omega\times(0,T)} \Big[2\rho\dot{\theta}_3\dot{\phi}_3 + 3m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3\Big] d\omega dt \\ \\ -\int_{\partial\omega\times(0,T)} (q_\zeta \nu_\zeta) m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\gamma dt, \end{array} \right. \tag{4.25}$$

where

- *$\{\psi_{13}, -\psi_{03}\}$ is the weak limit in $[H^1_{|\Gamma_0}(\Omega)]' \times L^2(\Omega)$, of a subsequence of $\{\psi_{13}(\varepsilon), -\psi_{03}(\varepsilon)\}_{\varepsilon>0}$ denoted by $\{\psi_{13}(\varepsilon_k), -\psi_{03}(\varepsilon_k)\}_{\varepsilon_k>0}$,*
- *ψ_α is the weak limit of the subsequence $\{\psi_\alpha(\varepsilon_k)\}_{\varepsilon_k>0}$ in X', ψ_3 is the weak* limit of the subsequence $\{\psi_3(\varepsilon_k)\}_{\varepsilon_k>0}$ in $L^\infty(0,T;L^2(\Omega))$,*
- *$\{\theta_{03}, \theta_{13}\}$ is the strong limit in $H^1_{|\Gamma_0}(\Omega) \times L^2(\Omega)$ of the sequence $\{\theta_{03}(\varepsilon), \theta_{13}(\varepsilon)\}_{\varepsilon>0}$, and such that θ_{03}, θ_{13} are independent of x_3 and $\theta_{03} \in H^2_0(\omega)$,*
- *$\{f_\alpha, f_3\}$ is the strong limit in $X^2 \times L^1(0,T;L^2(\Omega))$, of $\{f_\alpha(\varepsilon), f_3(\varepsilon)\}_{\varepsilon>0}$,*
- *θ_3 is the solution of (4.11),*

- ϕ_3 *is the solution of (4.4) with initial data* $\{1/2\int_{+1}^{-1}\phi_{03}dx_3, 1/2\int_{+1}^{-1}\phi_{13}dx_3\}$*, such that* $1/2\int_{+1}^{-1}\phi_{03}dx_3 \in H_0^2(\omega)$*, and where* $\{\phi_{03}, \phi_{13}\}$ *is the weak limit of* $\{\phi_{03}(\varepsilon_k), \phi_{13}(\varepsilon_k)\}_{\varepsilon_k>0}$ *in* $H^1_{|\Gamma_0}(\Omega) \times L^2(\Omega)$ *and*

$$\{\boldsymbol{\phi}_0(\varepsilon_k), \boldsymbol{\phi}_1(\varepsilon_k)\} = \Lambda^{-1}(\varepsilon_k)(\{\boldsymbol{\psi}_1(\varepsilon_k), -\boldsymbol{\psi}_0(\varepsilon_k)\}).$$

Proof : In order to obtain (4.25) we will compute the limit of the transposition formula (3.10) when $\varepsilon \to 0$. The controlled solution $\boldsymbol{\psi}(\varepsilon)$ of the 3D elasticity problem (3.8) with initial data $\{\boldsymbol{\psi}_0(\varepsilon), \boldsymbol{\psi}_1(\varepsilon)\}$ satisfies formula (3.16). We now want to compute the limit of (3.16) as $\varepsilon \to 0$. We first remark that, because of (4.23), there exists a subsequence of $\{\boldsymbol{\psi}_0(\varepsilon), \boldsymbol{\psi}_1(\varepsilon)\}$, that we still denote by $\{\boldsymbol{\psi}_0(\varepsilon), \boldsymbol{\psi}_1(\varepsilon)\}_{\varepsilon>0}$, that is weakly convergent in $[L^2(\Omega)]^3 \times [\boldsymbol{V}(\Omega)]'$, and so

$$< \{\boldsymbol{\psi}_1(\varepsilon), -\boldsymbol{\psi}_0(\varepsilon)\}, \{\theta_0(\varepsilon), \theta_1(\varepsilon)\} >_\varepsilon \longrightarrow < \{\rho\psi_{13}, -\rho\psi_{03}\}, \{\theta_{03}, \theta_{13}\} >, \quad \text{as} \quad \varepsilon \to 0, \tag{4.26}$$

since $\{\theta_0(\varepsilon), \theta_1(\varepsilon)\}_{\varepsilon>0}$ is strongly convergent in $\boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$.

We also remark that the initial data $\{\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_1(\varepsilon)\} = \Lambda^{-1}(\varepsilon)(\{\boldsymbol{\psi}_1(\varepsilon), -\boldsymbol{\psi}_0(\varepsilon)\})$ satisfy

$$\begin{cases} \exists C > 0 \quad (\text{independent of} \quad \varepsilon) \quad : \\ ||(\varepsilon\sqrt{\rho}\phi_{11}(\varepsilon), \varepsilon\sqrt{\rho}\phi_{12}(\varepsilon), \sqrt{\rho}\phi_{13}(\varepsilon))||^2_{[L^2(\Omega)]^3} + a(\varepsilon)(\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_0(\varepsilon)) \le C, \end{cases} \tag{4.27}$$

because of the assumption (4.23) and estimate (3.18). So $\{\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_1(\varepsilon)\}_{\varepsilon>0}$ satifies the assumption (4.1). Let us denote by $\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1\}$ the weak limit of a subsequence of $\{\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_1(\varepsilon)\}_{\varepsilon>0}$, still denoted by $\{\boldsymbol{\phi}_0(\varepsilon), \boldsymbol{\phi}_1(\varepsilon)\}_{\varepsilon>0}$, in the space $\boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$.

Let us fix ε, and choose now $\{\theta_0(\varepsilon), \theta_1(\varepsilon)\} = \{0, 0\}$. From (3.14), (2.13) and (4.27) we immediately obtain that

i) if $f_\beta(\varepsilon) = 0$, $\beta = 1, 2$

$$| < \psi_3(\varepsilon), f_3(\varepsilon) > | \le C \int_0^T ||f_3(\varepsilon)||_{L^2(\Omega)} dt, \tag{4.28}$$

ii) if $f_3(\varepsilon) = 0$, $f_\alpha(\varepsilon) = 0$

$$| < \psi_\beta(\varepsilon), f_\beta(\varepsilon) > | \le C \int_0^T ||f_\beta(\varepsilon)||_X dt, \quad \text{for} \quad \beta \ne \alpha, \tag{4.29}$$

where C denotes a positive constant, independent of ε.

So there exist $\psi_\alpha \in X'$, $\alpha = 1, 2$ and $\psi_3 \in L^\infty(0, T; L^2(\Omega))$, such that

$$\sum_i < \psi_i(\varepsilon), f_i(\varepsilon) > \longrightarrow \sum_i < \psi_i, f_i >, \qquad \text{when} \quad \varepsilon \to 0. \tag{4.30}$$

From (4.26), (4.30), (4.13), the density result of theorem 2.2 and finally the identity (2.37), in the limit we obtain the identity (4.25). ∎

Remark 4.2 We remark that in (4.25), $\theta_{03} \in H_0^2(\omega)$ (cf. (4.6)–(4.7)) and $\theta_{13} \in L^2(\omega)$ (cf. (4.9)), so the term $< \{\rho\psi_{13}, -\rho\psi_{03}\}, \{\theta_{03}, \theta_{13}\} >$ can be written in the form

$$< \{\rho \int_{-1}^{+1} \psi_{13} dx_3, -\rho \int_{-1}^{+1} \psi_{03} dx_3\}, \{\theta_{03}, \theta_{13}\} >$$

where the duality is taken in $H^{-2}(\omega) \times L^2(\omega)$ (we recall that $\psi_{13} \in [H^1_{|\Gamma_0}(\Omega)]'$ and $\psi_{03} \in L^2(\Omega)$, as stated in lemma 4.2).∎

Our aim now is to prove that the limit displacements are of the Kirchhoff-Love type.

Lemma 4.3 *The limit displacements $\boldsymbol{\psi} = (\psi_i)$ of the controlled displacements $\boldsymbol{\psi}(\varepsilon)$ is a Kirchhoff -Love displacement, that is*

$$\begin{aligned} &\psi_3 && \text{is independent of} \quad x_3, \\ &\psi_\alpha = \hat{\psi}_\alpha - x_3 \partial_\alpha \psi_3, \qquad \hat{\psi}_\alpha && \text{is independent of} \quad x_3. \end{aligned} \tag{4.31}$$

Moreover $\hat{\psi}_\alpha = 0$ in $\omega \times (0,T)$.

Proof :

i) First we prove that ψ_3 is independent of x_3, that is $\partial_3 \psi_3 = 0$ in $\mathcal{D}'(\Omega \times (0,T))$, which means

$$< \psi_3, -\partial_3 f >= 0, \qquad \forall f \in \mathcal{D}(\Omega \times (0,T)). \tag{4.32}$$

In the transposition formula (3.10), we consider sequences $\{\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon)\}_{\varepsilon>0}$, $\{\boldsymbol{f}(\varepsilon)\}_{\varepsilon>0}$ of test functions such that, as $\varepsilon \to 0$

$$\begin{aligned} \boldsymbol{\theta}_0(\varepsilon) &\longrightarrow 0, && \text{strongly in} && \boldsymbol{V}(\Omega), \\ \boldsymbol{\theta}_1(\varepsilon) &\longrightarrow 0, && \text{strongly in} && [L^2(\Omega)]^3, \\ (f_1(\varepsilon), f_2(\varepsilon), f_3(\varepsilon)) &\longrightarrow (0,0,-\partial_3 f), && \text{strongly in} && X^2 \times L^1(0,T;L^2(\Omega)). \end{aligned}$$

We remark that such sequences $\{\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon)\}_{\varepsilon>0}$ and $\{\boldsymbol{f}(\varepsilon)\}_{\varepsilon>0}$ satisfy all the hypotheses of theorem 4.2. At the limit we obtain (cf. (4.25))

$$\left\{ \begin{aligned} 0 \; &=< \psi_3, -\partial_3 f > + \int_{\omega \times (0,T)} \left[2\rho \dot{\theta}_3 \dot{\phi}_3 + 3 m_{\alpha\beta}(\theta_3) \partial_{\alpha\beta} \phi_3 \right] d\omega dt \\ &- \int_{\partial\omega \times (0,T)} (q_\zeta \nu_\zeta) m_{\alpha\beta}(\theta_3) \partial_{\alpha\beta} \phi_3 d\gamma dt, \end{aligned} \right. \tag{4.33}$$

where θ_3 is the solution of the 2D problem (cf. (4.11))

$$\left\{ \begin{aligned} 2\rho \ddot{\theta}_3 - \partial_{\alpha\beta} m_{\alpha\beta}(\theta_3) = \int_{-1}^{+1} (-\partial_3 f) dx_3, &\qquad \text{in} \quad \omega \times (0,T), \\ \theta_3 = \frac{\partial \theta_3}{\partial \nu} = 0, &\qquad \text{on} \quad \partial\omega \times (0,T), \\ \theta_3(0) = 0, \quad \dot{\theta}_3(0) = 0, &\qquad \text{in} \quad \omega. \end{aligned} \right.$$

But $\int_{-1}^{+1} (-\partial_3 f) dx_3 = 0$, since $f_3 \in \mathcal{D}(\Omega \times (0,T))$, and so $\theta_3 = 0$. Therefore we conclude that $< \psi_3, -\partial_3 f >= 0$.

ii) To prove that $\psi_\alpha = \hat{\psi}_\alpha - x_3\partial_\alpha\psi_3$, with $\hat{\psi}_\alpha$ independent x_3, we must prove that $\partial_3\psi_\alpha + \partial_\alpha\psi_3 = 0$ in $\mathcal{D}'(\Omega \times (0,T))$, that is

$$< \partial_3\psi_\alpha + \partial_\alpha\psi_3, f_\alpha >= 0, \qquad \forall f_\alpha \in \mathcal{D}(\Omega \times (0,T)), \quad \text{for} \quad \alpha = 1,2, \tag{4.34}$$

or equivalentely

$$< \psi_\alpha, -\partial_3 f_\alpha > + < \psi_3, -\partial_\alpha f_\alpha >= 0, \qquad \forall f_\alpha \in \mathcal{D}(\Omega \times (0,T)), \quad \text{for} \quad \alpha = 1,2.$$

As in i) we can choose suitable sequences $\{\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon)\}_{\varepsilon>0}$, $\{\boldsymbol{f}(\varepsilon)\}_{\varepsilon>0}$, such that, as $\varepsilon \to 0$

$$\begin{array}{rll} \boldsymbol{\theta}_0(\varepsilon) \longrightarrow \boldsymbol{\theta}_0 = (\theta_{0\alpha}, 0), & \text{strongly in} & \boldsymbol{V}(\Omega), \\ \boldsymbol{\theta}_1(\varepsilon) \longrightarrow \boldsymbol{\theta}_1 = (\theta_{1\alpha}, 0), & \text{strongly in} & [L^2(\Omega)]^3, \\ f_\alpha(\varepsilon) \longrightarrow -\partial_3 f_\alpha, & \text{strongly in} & X, \\ f_\beta(\varepsilon) \longrightarrow 0, & \text{strongly in} & X, \qquad \beta \neq \alpha, \\ f_3(\varepsilon) \longrightarrow -\partial_\alpha f_\alpha, & \text{strongly in} & L^1(0,T;L^2(\Omega)) \quad (\text{no sum on } \alpha). \end{array} \tag{4.35}$$

Again we remark that, with this choice, all the hypotheses of theorem 4.2 are verified. By (4.25) we obtain

$$\left\{ \begin{array}{l} 0 \;\; =< \psi_\alpha, -\partial_3 f_\alpha > + < \psi_3, -\partial_\alpha f_\alpha > + \int_{\omega\times(0,T)} \left[2\rho\dot{\theta}_3\dot{\phi}_3 + 3m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3\right] d\omega dt \\ \quad - \int_{\partial\omega\times(0,T)} (q_\zeta\nu_\zeta) m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\gamma dt, \end{array} \right. \tag{4.36}$$

where θ_3 is the solution of the 2D equation (cf. (4.11))

$$\left\{ \begin{array}{rll} 2\rho\ddot{\theta}_3 - \partial_{\alpha\beta} m_{\alpha\beta}(\theta_3) = \int_{-1}^{+1} \left[(-\partial_\alpha f_\alpha) - \partial_\alpha[x_3(\partial_3 f_\alpha)]\right] dx_3, & \text{in} & \omega \times (0,T), \\ \theta_3 = \frac{\partial\theta_3}{\partial\nu} = 0, & \text{on} & \partial\omega \times (0,T), \\ \theta_3(0) = 0, \quad \dot{\theta}_3(0) = 0, & \text{in} & \omega. \end{array} \right. \tag{4.37}$$

But as $f_\alpha \in \mathcal{D}(\Omega \times (0,T))$ we have

$$\int_{-1}^{+1} \left[(-\partial_\alpha f_\alpha) - \partial_\alpha[x_3(\partial_3 f_\alpha)]\right] dx_3 = -\int_{-1}^{+1} \partial_\alpha f_\alpha dx_3 + \partial_\alpha\Big(\int_{-1}^{+1} f_\alpha dx_3\Big) = 0,$$

so $\theta_3 = 0$ is the unique solution of (4.37) and the result follows from (4.36).

iii) To prove that $\hat{\psi}_\alpha = 0$, let us consider in (4.25), $\theta_{03} = \theta_{13} = 0$, $f_3 = 0$ and $f_\alpha \neq 0$, independent of x_3 and $f_\alpha \in \mathcal{D}(\omega \times (0,T))$, considered as a test function in $\omega \times (0,T)$. Then, by (4.25), $\theta_3 = 0$ and we have

$$0 =< \psi_\alpha, f_\alpha >=< \hat{\psi}_\alpha - x_3\partial_\alpha\psi_3, f_\alpha >=< \hat{\psi}_\alpha, f_\alpha > + < -x_3\partial_\alpha\psi_3, f_\alpha > . \tag{4.38}$$

But as ψ_3 and f_α are independent of x_3

$$< -x_3\partial_\alpha\psi_3, f_\alpha >= -\int_0^T \int_\Omega \partial_\alpha\psi_3 x_3 f_\alpha d\Omega dt = -\int_0^T \int_\omega \partial_\alpha\psi_3 \Big(\int_{-1}^{+1} x_3 f_\alpha dx_3\Big) d\omega dt = 0,$$

and we have $< \hat{\psi}_\alpha, f_\alpha >= 0$, that is $\hat{\psi}_\alpha = 0$ in $\omega \times (0,T)$. ∎

4.3 Proof of theorems 1.2 and 1.3

Finally we are able to identify the limit problem (4.25) and consequently to justify the results of theorems 1.2 and 1.3. We consider the 2D plate problem

$$\left\{\begin{aligned} 2\rho\ddot{y}_3+\tfrac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2 y_3=2\rho\ddot{\phi}_3+\tfrac{8\mu(\lambda+\mu)}{\lambda+2\mu}\Delta^2\phi_3, &\quad \text{in} \quad \omega\times(0,T),\\ y_3=0, &\quad \text{on} \quad \partial\omega\times(0,T),\\ \tfrac{\partial y_3}{\partial\nu}=(q_\alpha\nu_\alpha)\Delta\phi_3, &\quad \text{on} \quad \partial\omega\times(0,T),\\ y_3(T)=\dot{y}_3(T)=0, &\quad \text{in} \quad \omega, \end{aligned}\right. \tag{4.39}$$

where the function ϕ_3 is the unique solution of the homogeneous 2D plate problem

$$\left\{\begin{aligned} 2\rho\ddot{\phi}_3+\tfrac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\phi_3=0, &\quad \text{in} \quad \omega\times(0,T),\\ \phi_3=\tfrac{\partial\phi_3}{\partial\nu}=0, &\quad \text{on} \quad \partial\omega\times(0,T),\\ \phi_3(0)=\phi_{03},\quad \dot{\phi}_3(0)=\phi_{13}, &\quad \text{in} \quad \omega. \end{aligned}\right. \tag{4.40}$$

Remark 4.3 Clearly, system (4.39) presents two types of controls. First we have a boundary control on $\partial\omega$. But, on the other hand there is an internal control, which is the right–hand side of the equation of the motion. The internal control is defined as indicated in (1.18)-(1.20). ∎

The solution y_3 of problem (4.39) is defined by transposition in the following way :

Definition 4.1 *The function y_3 is a solution of the 2D plate problem (4.39) in the transposition sense if $y_3\in L^\infty(0,T;L^2(\omega))$, the traces $\{y_3(0),\dot{y}_3(0)\}$ make sense in $L^2(\omega)\times H^{-2}(\omega)$, and y_3 verifies*

$$\left\{\begin{aligned} &<\{\rho\dot{y}_3(0),-\rho y_3(0)\},\{\theta_{03},\theta_{13}\}>=<y_3,g_3>+\int_{\omega\times(0,T)}\Big[2\rho\dot{\theta}_3\dot{\phi}_3+3m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3\Big]d\omega dt\\ &-\int_{\partial\omega\times(0,T)}(q_\zeta\nu_\zeta)m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\gamma dt, \end{aligned}\right. \tag{4.41}$$

for any $g_3\in L^1(0,T;L^2(\omega))$ and any $\{\theta_{03},\theta_{13}\}\in H^2_0(\omega)\times L^2(\omega)$, with θ_3 the solution of the 2D plate problem with initial data $\{\theta_{03},\theta_{13}\}$ and applied body force g_3, that is,

$$\left\{\begin{aligned} 2\rho\ddot{\theta}_3+\tfrac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\theta_3=g_3, &\quad \textit{in} \quad \omega\times(0,T),\\ \theta_3=\tfrac{\partial\theta_3}{\partial\nu}=0, &\quad \textit{on} \quad \partial\omega\times(0,T),\\ \theta_3(0)=\theta_{03},\quad \dot{\theta}_3(0)=\theta_{13}, &\quad \textit{in} \quad \omega.\ \blacksquare \end{aligned}\right. \tag{4.42}$$

Theorem 4.3 *There exists a unique solution of problem (4.39) in the sense of transposition.*

Sketch of the Proof : The justification of the existence of a unique solution of problem (4.39), in the sense of definition 4.1, is similar to that of theorem 3.3. We multiply the first equation of (4.39) by θ_3, solution of (4.42), and integrate by parts on $\omega \times (0,T)$, assuming that y_3 is smooth enough. By density arguments, the existence of a unique solution $y_3 \in L^\infty(0,T;L^2(\omega))$ is proved. Then, the fact that $y_3(0)$ and $\dot{y_3}(0)$ make sense in $L^2(\omega)$ and $H^{-2}(\omega)$ respectively, can be proved as in Lions [7], Remarque 3.16, p.267. ∎

We can now prove theorems 1.2 and 1.3.

i) Proof of theorem 1.2

To prove theorem 1.2 we must show that $\psi_3 = y_3$, where y_3 satisfies (4.41) and ψ_3 verifies (4.25). We remark that formula (4.25) will coincide with formula (4.41), if we are able to prove that (4.25) is valid for $f_\alpha = 0$, any pair $\{\theta_{03}, \theta_{13}\} \in H_0^2(\omega) \times L^2(\omega)$ and any $f_3 \in L^1(0,T;L^2(\omega))$. In order to do that we first establish the following lemma :

Lemma 4.4 *For any $g_3 \in L^1(0,T;L^2(\omega))$ and any $\{\theta_{03}, \theta_{13}\} \in H_0^2(\omega) \times L^2(\omega)$, there exist sequences $\{\boldsymbol{f}(\varepsilon)\}_{\varepsilon>0} \in X^2 \times L^1(0,T;L^2(\Omega))$, and $\{\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon)\}_{\varepsilon>0} \in \boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$, such that*

$$
\begin{aligned}
&\boldsymbol{f}(\varepsilon) \longrightarrow (0,0,g_3), && \textit{strongly in} \quad X^2 \times L^1(0,T;L^2(\Omega)),\\
&\boldsymbol{\theta}_0(\varepsilon) \longrightarrow (-x_3\partial_1\theta_{03}, -x_3\partial_2\theta_{03}, \theta_{03}), && \textit{strongly in} \quad \boldsymbol{V}(\Omega),\\
&\tfrac{1}{\varepsilon} e_{\alpha 3}(\boldsymbol{\theta}_0(\varepsilon)) \longrightarrow 0, && \textit{strongly in} \quad L^2(\Omega),\\
&\tfrac{1}{\varepsilon^2} e_{33}(\boldsymbol{\theta}_0(\varepsilon)) \longrightarrow \tfrac{\lambda}{\lambda+2\mu} x_3 \Delta(\theta_{03}), && \textit{strongly in} \quad L^2(\Omega).\\
&\varepsilon\theta_{1\alpha}(\varepsilon) \longrightarrow 0, && \textit{strongly in} \quad L^2(\Omega),\\
&\theta_{13}(\varepsilon) \longrightarrow \theta_{13}, && \textit{strongly in} \quad L^2(\Omega).
\end{aligned}
\tag{4.43}
$$

Proof : It is enough to choose

$$
\begin{aligned}
&\boldsymbol{f}(\varepsilon) = (0,0,g_3), \qquad \boldsymbol{\theta}_1(\varepsilon) = (0,0,\theta_{13}),\\
&\boldsymbol{\theta}_0(\varepsilon) = (-x_3\partial_1\theta_{03}, -x_3\partial_2\theta_{03}, \theta_{03} + \varepsilon^2 v(\varepsilon)),
\end{aligned}
\tag{4.44}
$$

such that

$$
\begin{aligned}
&\varepsilon^2 v(\varepsilon) \longrightarrow 0, && \text{strongly in} \quad H^1_{|\Gamma_0}(\Omega),\\
&\partial_3 v(\varepsilon) \longrightarrow \tfrac{\lambda}{\lambda+2\mu} x_3 \Delta(\theta_{03}), && \text{strongly in} \quad L^2(\Omega).
\end{aligned}
\tag{4.45}
$$

We remark that (4.45) is possible, because $H^1_{|\Gamma_0}(\Omega)$ is dense in $L^2(\Omega)$.∎

Due to the last lemma and lemmas 4.2 and 4.3 we can finish the proof of theorem 1.2. In fact, the last lemma enables us to conclude the (4.25) is valid for $f_\alpha = 0$, for any $\{\theta_{03}, \theta_{13}\} \in H_0^2(\omega) \times L^2(\omega)$ and any $f_3 \in L^1(0,T;L^2(\omega))$. Therefore $\psi_3 = y_3$, that is ψ_3 is the solution in the transposition sense of equations (1.16). ∎

ii) Proof of theorem 1.3

In order to prove theorem 1.3, let us now define the functions ξ_3^ε and η_3^ε and the initial data $\{\xi_{03}^\varepsilon, \xi_{13}^\varepsilon\}$, $\{\eta_{03}^\varepsilon, \eta_{13}^\varepsilon\}$ as indicated in (1.21), and let us rewrite formula (4.25), with $f_\alpha = 0$, in the original plate $\overline{\Omega}^\varepsilon = \overline{\omega} \times [-\varepsilon, \varepsilon]$ instead of the plate $\overline{\Omega} = \omega \times [-1, +1]$. We must also introduce the functions $\hat{\theta}_3^\varepsilon$, $\hat{\theta}_{03}^\varepsilon$, $\hat{\theta}_{13}^\varepsilon$ and F_3^ε defined by

$$\begin{aligned} \hat{\theta}_3^\varepsilon(x_1, x_2, t) &= \varepsilon\theta_3(x_1, x_2, t), \\ \hat{\theta}_{03}^\varepsilon(x_1, x_2, t) &= \varepsilon\theta_{03}(x_1, x_2, t), \\ \hat{\theta}_{13}^\varepsilon(x_1, x_2, t) &= \varepsilon\theta_{13}(x_1, x_2, t), \\ F_3^\varepsilon(x_1, x_2, t) &= \int_{-\varepsilon}^{+\varepsilon} f_3\varepsilon^3 dx_3^\varepsilon, \end{aligned} \tag{4.46}$$

for all $(x_1, x_2) \in \overline{\omega}$ and all $t \in [0, T]$. It is immediately seen that if θ_3 is the solution of problem (4.42), with $g_3 = \int_{-1}^{+1} f_3 dx_3$, then $\hat{\theta}_3^\varepsilon$ is the solution of the following 2D plate problem (cf. Ciarlet [2] and Raoult [14], [15])

$$\left\{ \begin{aligned} 2\rho^\varepsilon\varepsilon\ddot{\hat{\theta}}_3^\varepsilon - \varepsilon^3\partial_{\alpha\beta}m_{\alpha\beta}(\hat{\theta}_3) = F_3^\varepsilon, &\qquad \text{in} \quad \omega \times (0, T), \\ \hat{\theta}_3^\varepsilon = \frac{\partial\hat{\theta}_3^\varepsilon}{\partial\nu} = 0, &\qquad \text{on} \quad \partial\omega \times (0, T), \\ \hat{\theta}_3^\varepsilon(0) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{+\varepsilon} \hat{\theta}_{03}^\varepsilon dx_3^\varepsilon, \quad \dot{\hat{\theta}}_3^\varepsilon(0) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{+\varepsilon} \hat{\theta}_{13}^\varepsilon dx_3^\varepsilon, &\qquad \text{in} \quad \omega. \end{aligned} \right. \tag{4.47}$$

With the de-scalings (1.21), (4.46), the assumptions (1.3) on the Lamé constants and density of mass and the formula of change of variables, we obtain that formula (4.25) is equivalent to

$$\left\{ \begin{aligned} & < \left\{\tfrac{1}{\varepsilon^5}\rho^\varepsilon \int_{-\varepsilon}^{+\varepsilon} \xi_{13}^\varepsilon dx_3^\varepsilon, -\tfrac{1}{\varepsilon^5}\rho^\varepsilon \int_{-\varepsilon}^{+\varepsilon} \xi_{03}^\varepsilon dx_3^\varepsilon\right\}, \left\{\hat{\theta}_{03}^\varepsilon, \hat{\theta}_{13}^\varepsilon\right\} > \\ & =< \tfrac{1}{\varepsilon^5}\xi_3^\varepsilon, F_3^\varepsilon > + \int_{\omega\times(0,T)} \left[\tfrac{1}{\varepsilon^4}2\rho^\varepsilon\dot{\hat{\theta}}_3^\varepsilon\dot{\eta}_3^\varepsilon + \tfrac{1}{\varepsilon^2}3m_{\alpha\beta}(\hat{\theta}_3^\varepsilon)\partial_{\alpha\beta}\eta_3^\varepsilon\right] d\omega dt \\ & - \int_{\partial\omega\times(0,T)} \tfrac{1}{\varepsilon^2}(q_\zeta\nu_\zeta)m_{\alpha\beta}(\hat{\theta}_3^\varepsilon)\partial_{\alpha\beta}\eta_3^\varepsilon d\gamma dt, \end{aligned} \right.$$

or equivalentely, multiplying by ε^5

$$\left\{ \begin{aligned} & < \left\{\rho^\varepsilon \int_{-\varepsilon}^{+\varepsilon} \xi_{13}^\varepsilon dx_3^\varepsilon, -\rho^\varepsilon \int_{-\varepsilon}^{+\varepsilon} \xi_{03}^\varepsilon dx_3^\varepsilon\right\}, \left\{\hat{\theta}_{03}^\varepsilon, \hat{\theta}_{13}^\varepsilon\right\} > \\ & =< \xi_3^\varepsilon, F_3^\varepsilon > + \int_{\omega\times(0,T)} \left[\varepsilon 2\rho^\varepsilon\dot{\hat{\theta}}_3^\varepsilon\dot{\eta}_3^\varepsilon + 3\varepsilon^3 m_{\alpha\beta}(\hat{\theta}_3^\varepsilon)\partial_{\alpha\beta}\eta_3^\varepsilon\right] d\omega dt \\ & - \int_{\partial\omega\times(0,T)} \varepsilon^3(q_\gamma\nu_\gamma)m_{\alpha\beta}(\hat{\theta}_3^\varepsilon)\partial_{\alpha\beta}\eta_3^\varepsilon d\gamma dt. \end{aligned} \right. \tag{4.48}$$

Formula (4.48) means exactly (cf. definition 4.1) that the de-scaled function ξ_3^ε is the solution, in the transposition sense, of the 2D plate problem (1.22), and so theorem 1.3 is proved. ∎

4.4 Controllabilty of the 2D plate problem

It is well known that the 2D plate problem (4.39) is exactly controllable in $L^2(\omega) \times H^{-2}(\omega)$ in an arbitrary small time with only one boundary control, that is, without any control distributed in $\omega \times (0,T)$ (acting in the equations of the motion) and with boundary conditions

$$\begin{aligned} y_3 &= 0, \quad \text{on} \quad \partial\omega \times (0,T), \\ \tfrac{\partial y_3}{\partial \nu} &= v, \quad \text{on} \quad \partial\omega \times (0,T), \end{aligned}$$

with $v \in L^2(\partial\omega \times (0,T))$.

However, in the limit process we have obtained an internal control which is roughly, the contribution of the controls imposed at the top and bottom of the 3D plate.

Let us briefly explain how the type of controls we have obtained for the 2D plate problem by the taking the limit, as $\varepsilon \to 0$, can also be constructed applying directly HUM, to the 2D plate equation.

Given ϕ_3 solution of the homogeneous plate problem (4.40) we introduce the energy

$$E(\phi_3)(t) = \int_\omega \frac{1}{2}\Big[2\rho|\dot{\phi}_3|^2 - m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3\Big]d\omega. \tag{4.49}$$

By using multiplier techniques, the inverse inequality established in theorem 3.2, and taking the limit, as $\varepsilon \to 0$ we prove the following estimate :

Lemma 4.5 *Let $T > 0$ be fixed. Then, there exists constants C_1 and C_2, such that*

$$\left\{ \begin{array}{l} \Big| \int_{\omega\times(0,T)} \Big[2\rho|\dot{\phi}_3|^2 + 3m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3\Big]d\omega dt - \int_{\partial\omega\times(0,T)}(q_\zeta\nu_\zeta)m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3 d\gamma dt\Big| \\ \le C_1 E(\phi_3)(0), \end{array} \right. \tag{4.50}$$

and

$$\left\{ \begin{array}{l} E(\phi_3)(0) \\ \le C_1\Big| \int_{\omega\times(0,T)} \Big[2\rho|\dot{\phi}_3|^2 + 3m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3\Big]d\omega dt - \int_{\partial\omega\times(0,T)}(q_\zeta\nu_\zeta)m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3 d\gamma dt\Big|, \end{array} \right. \tag{4.51}$$

for every solution ϕ_3 of (4.40) with initial data $\{\phi_{03}, \phi_{13}\} \in D_{\Delta^2} \times H_0^2(\omega)$.

Proof : Applying (2.37) with $\theta_3 = \phi_3$ and $f_3 = 0$ we obtain

$$\left\{ \begin{array}{l} -\int_{\partial\omega\times(0,T)}(q_\zeta\nu_\zeta)m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3 d\gamma dt = 2\Big[\int_\omega 2\rho\dot{\phi}_3(q_\alpha\partial_\alpha\phi_3)d\omega\Big]_0^T \\ +2\int_{\omega\times(0,T)} 2\rho|\dot{\phi}_3|^2 d\omega dt - 2\int_{\omega\times(0,T)} m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3 d\omega dt. \end{array} \right. \tag{4.52}$$

Note that this identity holds for smooth solutions thanks to the regularity results of Niane [9]. This identity shows in particular, that

$$\Big| \int_{\partial\omega\times(0,T)} (q_\zeta\nu_\zeta)m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3 d\gamma dt\Big| \le C E(\phi_3)(0), \tag{4.53}$$

for some constant $C > 0$, that does not depend on ϕ_3 (but that does depend on T). This implies (4.50).

In order to prove estimate (4.51) we remark that problem (4.40) can be interpreted as the asymptotic limit of the 3D plate problem (2.6), whose solution is $\boldsymbol{\theta}(\varepsilon)$, with applied body forces

$f_i(\varepsilon) = 0$ and initial data $\{\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon)\}$, verifying the assumptions (4.5)–(4.9) of theorem 4.2. That is (cf. lemma 4.4), if $\{\phi_{03}, \phi_{13}\} \in H_0^2(\omega) \times L^2(\omega)$ there exists a sequence $\{\boldsymbol{\theta}_0(\varepsilon), \boldsymbol{\theta}_1(\varepsilon)\}_{\varepsilon>0} \in \boldsymbol{V}(\Omega) \times [L^2(\Omega)]^3$ such that as $\varepsilon \to 0$ the strong limit of $\boldsymbol{\theta}_0(\varepsilon)$ in $\boldsymbol{V}(\Omega)$ is equal to

$$(-x_3\partial_1\phi_{03}, -x_3\partial_2\phi_{03}, \phi_{03}),$$

and the strong limit of $\theta_{13}(\varepsilon)$ in $L^2(\Omega)$ is equal to ϕ_{13}. Moreover the conditions (4.10)–(4.12) are satisfied.

So, if we take the limit, as $\varepsilon \to 0$ in (3.2), we obtain the inequality (4.51), due to the strong convergences (4.10), the lemma 4.1 and formula (2.37), that shows

$$\left\{ \begin{array}{l} \int_{\omega\times(0,T)} \Big[2\rho|\dot{\phi}_3|^2 + 3m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3\Big] d\omega dt - \int_{\partial\omega\times(0,T)} (q_\zeta\nu_\zeta) m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3 d\gamma dt = \\ 2\Big[\int_\omega 2\rho\dot{\phi}_3(q_\alpha\partial_\alpha\phi_3)d\omega\Big]_0^T + 3\int_{\omega\times(0,T)} 2\rho|\dot{\phi}_3|^2 d\omega dt + \int_{\omega\times(0,T)} m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3 d\omega dt. \end{array} \right. \blacksquare$$

Remark 4.4 We emphasize that the inverse inequality (4.51) is not obtained by multiplier techniques, as usual, but as the limit as $\varepsilon \to 0$ of the inverse inequality (3.2) for the 3D plate problem. ∎

Lemma 4.5 implies that the map

$$\begin{aligned} D_{\Delta^2} \times H_0^2(\omega) &\longrightarrow \mathbb{R} \\ \{\phi_{03}, \phi_{13}\} &\longrightarrow ||\{\phi_{03}, \phi_{13}\}||, \end{aligned}$$

where

$$\left\{ \begin{array}{rl} ||\{\phi_{03}, \phi_{13}\}|| = & \Big\{ \int_{\omega\times(0,T)} \Big[2\rho|\dot{\phi}_3|^2 + 3m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3\Big] d\omega dt \\ & \qquad - \int_{\partial\omega\times(0,T)} (q_\zeta\nu_\zeta) m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\phi_3 d\gamma dt \Big\}^{\frac{1}{2}} \end{array} \right.$$

is a norm in $D_{\Delta^2} \times H_0^2(\omega)$, equivalent to the usual norm in $H_0^2(\omega) \times L^2(\Omega)$. But, since $D_{\Delta^2} \times H_0^2(\omega)$ is dense in $H_0^2(\omega) \times L^2(\Omega)$, we infer that $||.||$ is a norm equivalent to the usual norm in $H_0^2(\omega) \times L^2(\Omega)$. Inspired on HUM, we define the operator

$$\begin{aligned} \Lambda\colon H_0^2(\omega) \times L^2(\Omega) &\longrightarrow H^{-2}(\omega) \times L^2(\omega), \\ \{\phi_{03}, \phi_{13}\} &\longrightarrow \{\dot{y}_3(0), -y_3(0)\}, \end{aligned} \tag{4.54}$$

where y_3 is the solution of (4.39), in the transposition sense (cf. definition 4.1) and $\{\phi_{03}, \phi_{13}\}$ are the initial data for problem (4.40).

As a consequence of the definition of Λ and the estimates (4.50)–(4.51) we deduce the following :

Theorem 4.4 *The operator* $\Lambda\colon H_0^2(\omega) \times L^2(\Omega) \to H^{-2}(\omega) \times L^2(\omega)$ *is an isomorphism.* ∎

So, theorem 4.4 confirms that, for any $\{y_3(0), \dot{y}_3(0)\}$ in the space $L^2(\omega) \times H^{-2}(\omega)$, the control obtained in the limit process, as in (4.39), can be obtained directly by HUM.

5 Further comments

i) In all this work we have assumed that the middle surface ω, of the plate $\overline{\Omega}^{\varepsilon}$, is a polygon in $\mathbb{R}^2$, and this imposes a restriction on the geometry of the plate. This assumption is made to guarantee the regularity $H^{3/2+\delta}(\Omega)$ (for some $\delta > 0$) for the solution of the 3D plate problem (2.1), which has mixed boundary conditions. This regularity $H^{3/2+\delta}(\Omega)$ is needed to establish the identity (2.21). The polygonal geometry of ω guarantees also that the solutions of the limit 2D plate problem lye in $H^{5/2+\delta}(\omega)$. These regularity results have been proved respectively by Nicaise [11] and Niane [9]. All the results presented in this work would hold to plates with more general middle surfaces ω provided these regularity results were proved.

ii) We have assumed that we control the 3D plate $\overline{\Omega} = \overline{\omega} \times [-1, 1]$ on the lateral boundary Γ_0 and on the top and bottom $\Gamma_{\pm}$ (cf. (1.10)). Instead of controlling both the top and bottom we could think of controlling only on the top (or only on the bottom), for example. If we control only on the top it suffices to take $q_3(\boldsymbol{x}) = x_3 + 1$, that is to take the vector $\boldsymbol{x}_0 = (x_{01}, x_{02}, -1)$. From the point of view of the exact controllability, this does not produce a considerable change. We remark that, the exact controllability results of theorem 1.1 also apply in that case, with the change that the controls $v_i(\varepsilon)$ defined on (1.10) would be equal to zero on $\Gamma_- \times (0, T)$ and equal to the expressions given in (1.10) on $\Gamma_+ \times (0, T)$.

But, to control only on the top (or only on the bottom) of the plate originates difficulties in the identification of the asymptotic limit. In fact, if $q_3(\boldsymbol{x}) = x_3 + 1$, then we would obtain (cf. (4.19)–(4.20))

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega f_i(\varepsilon)(\boldsymbol{q} \cdot \nabla \phi_i(\varepsilon)) d\Omega dt = < \int_{-1}^{+1} \Big(f_3 + x_3 \partial_\beta f_\beta \Big) dx_3, q_\alpha \partial_\alpha \phi_3 > - \int_0^T \int_\Omega f_\alpha \partial_\alpha \phi_3 d\Omega dt,$$

and the term $\int_0^T \int_\Omega f_\alpha \partial_\alpha \phi_3 d\Omega dt$ does not allow us to identify completely the limit of the transposition formula (3.10). We could suppose that $f_\alpha = 0$, the term $\int_0^T \int_\Omega f_\alpha \partial_\alpha \phi_3 d\Omega dt$ would disappear, but then we could not succeed in proving lemma 4.3 completely. It would be possible to prove that ψ_3 is independent of x_3 (as stated in lemma 4.3) and that it is the unique solution of (1.16), but to prove that $\psi_\alpha = \hat{\psi}_\alpha - x_3 \partial_\alpha \psi_3$ would be impossible.

In conclusion, we can say that taking a control on the lateral boundary and a control only on the top (or only on the bottom) of the 3D plate, does not allow us to conclude that at the limit, the displacements are of the Kirchhoff-Love type, but we can still conclude that the normal displacement is the solution of the 2D plate problem (1.16).

iii) Note that if in the 3D plate we only apply controls at the lateral boundary, then the exact controllability does not hold in any energy space (we refer to Bardos-Lebeau-Rauch [1], for a counter–example in that spirit). Thus it is rather natural to impose controls on the upper and lower surface of Ω.

Acknowledgments

The first author was supported by JNICT (Junta Nacional de Investigação Científica e Tecnológica – Portugal) and by the European Project *Junctions in Elastic Multi–Structures* of the program S.C.I.E.N.C.E. of the Commission of the European Comunities (contract SC1*043–C(EDB)). The second author was supported by DGICYT Project PB90–0245 and by EEC Grant SC1*–CT91–0732.

6 References

[1] – Bardos, C.; Lebeau, G. and Rauch, J : *Sharp sufficient conditions for the observation, control and stabilisation of the waves from the boundary* , SIAM J. Cont. Optim. 30, p.1024-1065, 1992.
[2] – Ciarlet, P.G. : *Plates and Junctions in Elastic–Multi–Structures : an Asymptotic Analysis*, Masson, Paris, 1990.
[3] – Cioranescu, D.; Donato, P. and Zuazua, E. : *Exact boundary controllability for the wave equation in domain with small holes* , J.Math. Pures Appl. 71, p.343-377, 1992.
[4] – Duvaut, G. ; Lions, J.L. : *Les inéquations en mécanique et en physique* , Dunod, Paris, 1972.
[5] – Grisvard, P. : *Contrôlabilité exacte des solutions de l'équation des ondes en présence de singularités* , J. Math. Pures et Appl., 68, p.215-259, 1989.
[6] – Lagnese, J. E. and Lions, J.L. : *Modelling and analysis of thin plates* , Masson, RMA6, Paris, 1988.
[7] – Lions, J.L. : *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1. Contrôlabilité exacte* , Masson, RMA8, Paris, 1988.
[8] – Lions, J.L. : *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 2. Perturbations* , Masson, RMA9, Paris, 1988.
[9] – Niane, M.T. : *Contrôlabilité exacte de l'équation des plaques vibrantes dans un polygone* , C.R. Acad. Sci. Paris, t. 307, Série 1, p. 517–521, 1988.
[10] – Nicaise, S.: *Exact controllability of a pluridimensional coupled problem* , Pub. Irma, Lille, Vol.20, n.4, 1990.
[11] – Nicaise, S. : *About the Lamé system in a polygonal or a poyhedral domain and a coupled problem between the Lamé system and the plate equation. 1: Regularity of the solutions* , Annali della Scuola Normale Superiori di Pisa, Classe de Scienze, Serie 4, Vol. 19, 3, p. 327-361, 1992.
[12] – Nicaise, S. : *About the Lamé system in a polygonal or a poyhedral domain and a coupled problem between the Lamé system and the plate equation. 2: Exact controllability* , Annali della Scuola Normale Superiori di Pisa, Classe de Scienze, Serie 4, Vol. 20, 2, p. 163-191, 1993.
[13] – Paulin, J.S.J. ; Vanninathan, M. : *Exact controllability of vibrations of thin bodies* , to appear in Portugalia Mathematica.
[14] – Raoult, A. : *Contributions à l'étude des modéles d'évolution de plaques et à l'approximation d'équations d'évolution linéaires du second ordre par des méthodes multipas* , Doctoral Dissertation (thèse de Troisième Cycle), Université Pierre et Marie Curie, Paris, 1980.
[15] – Raoult, A. : *Analyse mathématique de quelques modèles de plaques et de poutres élastiques ou elasto-plastique* , Thèse de Doctorat d'Etat, Université Pierre et Marie Curie, Paris, 1988.
[16] – Yan, L. : *Contrôlabilité exacte pour des domaines minces* , Asymptotic Analysis 5, p.461-471, 1992.