Perfectly Matched Layers in 1-d : Energy decay for continuous and semi-discrete waves. *

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Abstract

In this paper we investigate the efficiency of the method of Perfectly Matched Layers (PML) for the 1-d wave equation. The PML method furnishes a way to compute solutions of the wave equation for exterior problems in a finite computational domain by adding a damping term on the matched layer. In view of the properties of solutions in the whole free space, one expects the energy of solutions obtained by the PML method to tend to zero as $t \to \infty$, and the rate of decay can be understood as a measure of the efficiency of the method. We prove, indeed, that the exponential decay holds and characterize the exponential decay rate in terms of the parameters and damping potentials entering in the implementation of the PML method. We also consider a space semi-discrete numerical approximation scheme and we prove that, due to the high frequency spurious numerical solutions, the decay rate fails to be uniform as the mesh size parameter h tends to zero. We show however that adding a numerical viscosity term allows us to recover the property of exponential decay of the energy uniformly on

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h. Although our analysis is restricted to finite differences in 1-d, most of the methods and results apply to finite elements on regular meshes and to multi-dimensional problems.

1 Introduction.

When numerically solving wave propagation problems in unbounded domains, because of the finite computational possibilities, one has to truncate the computational domain. This makes it necessary to choose boundary conditions at the newly formed exterior boundary. These boundary conditions are relevant, for example in problems arising in acoustics and electrodynamics, since they may have a significant impact on the whole solution due to reflections.
In order to avoid those spurious reflections a natural method, introduced by Engquist and Majda in [16], is based on the use of the so-called transparent boundary conditions. This method is, however, hard to implement. The transparent boundary conditions are often of non-local nature, depend on the geometry of the domain, etc. For the state of the art, we refer to the survey article [29].
An alternate approach, proposed by Bérenger in [10] in 1994, is the so-called method of the Perfectly Matched Layers (PML). The idea consists in surrounding the computational domain by a layer and extending the equation to it adding damping terms designed to dissipate the energy entering in it, such that no spurious reflection waves are created. This method, first introduced to deal with Maxwell's equations, has been successfully adapted to many other wave models, see the survey article [26].

This article is aimed to develop a complete rigorous analysis in 1-d for the PML model associated to the scalar wave equation. Our work is inspired on the existing literature on the control and stabilization of waves.
More precisely, the object of this paper is twofold. First, we analyze the continuous 1-d wave equation to accurately describe the efficiency of the PML method in terms of the various parameters entering in it and second, we consider semi-discrete numerical approximation schemes. To the best of our knowledge, the study of this system has only been developed by a plane wave analysis ([9]), where explicit formulas are given for the amplitudes of the reflected and transmitted waves around the interface.
To be more precise, we consider the wave equation in an unbounded domain

of the form $(0,\infty)$ with homogeneous Neuman boundary conditions at $x=0$ and initial data in $L^2(0,\infty)$ with compact support:

$$\begin{cases} \partial_{tt}^2 u - \partial_{xx}^2 u = 0, \ x>0, \ t>0, \\ \partial_x u(0,t) = 0. \end{cases} \tag{1.1}$$

In the hyperbolic form, considering the physical variables $P = -\partial_x u$ and $V = \partial_t u$, the system under consideration can be written as follows

$$\begin{cases} \partial_t P + \partial_x V = 0 & \text{in } (0,\infty)\times(0,T) \\ \partial_t V + \partial_x P = 0 & \text{in } (0,\infty)\times(0,T) \\ P(0,t) = 0 \\ P(x,0) = P_0(x), \quad V(x,0) = V_0(x). \end{cases} \tag{1.2}$$

Its solution can be computed explicitly by the method of characteristics (which gives D'Alembert's formula). Since we assume the initial data to be compactly supported, for instance in $(0,a)$ for some $a>0$, it follows that the solutions (P,V) will vanish in $(0,a)$ for $t \geq 2a$, which is the time needed for waves to go from $x=a$ to $x=0$ and back to $x=a$ after reflection. The fact that P and V reach the zero state in time $t=2a$ in $(0,a)$ is also reflected on u, that stabilizes to the constant $\int_0^a V_0(x)\ dx$ for $t\geq 2a$ on the interval $(0,a)$.

The goal of the PML method, when applied to this 1-d model, is to reproduce this very property of P,V but by solving a problem in a bounded domain. For convenience, we translate the domain $(0,\infty)$ where waves propagate to $(-1,\infty)$ and focus on the restriction of solutions on the compact domain $(-1,0)$. This can be done, by scaling, without loss of generality. Then, solutions (P,V) with initial data compactly supported in $(-1,0)$ vanish on $(-1,0)$ for $t\geq 2$ and we expect that the approximate solutions, obtained by the PML method in a bounded domain, will reproduce this property. A way of measuring how small is the restriction of the approximate solutions to $(-1,0)$ is analyzing the time decay properties of its energy as $t\to\infty$.
The PML method is designed to give an accurate approximation of the solutions of (1.2) in $(-1,0)$, by solving the following system on the domain $(-1,1)$, in which the space-layer $(0,1)$ has been added:

$$\begin{cases} \partial_t P + \partial_x V + \chi_{(0,1)}\sigma P = 0 & \text{in } (-1,1)\times(0,T) \\ \partial_t V + \partial_x P + \chi_{(0,1)}\sigma V = 0 & \text{in } (-1,1)\times(0,T) \\ P(-1,t) = P(1,t) = 0 \\ P(x,0) = P_0(x), \quad V(x,0) = V_0(x). \end{cases} \tag{1.3}$$

Here σ is a positive function defined on $(0,1)$, which is assumed to be in $L^1(0,1)$. Note that within the added layer $(0,1)$ the equations in (1.3) have been modified by adding the terms involving the dissipative potential σ. Throughout the paper the function σ is extended on $(-1,1)$ by zero in $(-1,0)$. Actually, one can recover most of the results presented here in the case where the added space-layer is $(0,r)$ by a scaling argument, which maps $(-1,r)$ to $(-1,1)$ and by considering functions σ in (1.3) vanishing in $(-1,2/(1+r))$. We analyze (1.3) for all initial data though, as we have said, the relevant ones in the context of the PML method, are those with compact support in $(-1,0)$. Recall that the true solution (P,V) of (1.2) vanishes in $(-1,0)$ for $t>2$ when the initial data have support in $(-1,0)$. So we expect the energy of the PML solutions localized in $(-1,0)$ to be small when $t>2$. Then the exponential decay rate of the restriction of solutions of (1.3) to $(-1,0)$ is a way of measuring the efficiency of the PML method and the chosen damping potential σ. Actually, as we shall see, it coincides with the decay rate of the total energy of solutions. Thus, most of the paper will be devoted to analyze the later.
System (1.3) is well-posed, and the total energy of solutions

$$E(t)=E(P(t),V(t))=\frac{1}{2}\int_{-1}^{1}\left(|P(t,x)|^2+|V(t,x)|^2\right)\,dx \tag{1.4}$$

is dissipated according to the following law

$$\frac{dE}{dt}(t)=-\int_{0}^{1}\sigma(x)\big(|P(t,x)|^2+|V(t,x)|^2\big)\,dx. \tag{1.5}$$

This last equation shows the well-posedness of the 1-d PML equations in the space $(P,V)\in C([0,\infty);L^2(-1,1)^2)$.

As far as we know, the problem of the exponential decay of the energy for the PML method has not been addressed in detail so far. In [6] it was stated that a first order energy of solutions for Maxwell's PML model with a constant σ decays, but no decay rate was given.
In our analysis we will follow the techniques of [14], which, actually, in the present setting, can be applied more simply. Note that system (1.3) and its dissipative properties are similar to those of the classical damped wave equation:

$$\begin{cases}\partial_{tt}^2 w-\partial_{xx}^2 w+2a(x)\partial_t w=0 & \text{in } (-1,1)\times(0,T)\\ w(0,t)=w(1,t)=0.\end{cases} \tag{1.6}$$

In this case, the energy dissipation law reads :

$$\frac{d}{dt}\Big(\frac{1}{2}\int_0^1 \big(|\partial_t w|^2 + |\partial_x w|^2\big)\, dx\Big) = -2\int_0^1 a(x)|\partial_t w|^2 \, dx. \tag{1.7}$$

For system (1.6), it is well-known that the energy decays exponentially as $t \to \infty$ provided $a \geq 0$ is strictly positive on some subinterval. Moreover, in [14] the exponential decay rate was characterized as the spectral abscissa, for $a \in BV(-1,1)$.
Actually, in the special case where σ is constant, the PML equations (1.3) in $(0,1)$ read as follows:

$$\partial^2_{tt} u - \partial^2_{xx} u + 2\sigma\partial_t u + \sigma^2 u = 0 \quad \text{in } (0,1)\times(0,T), \tag{1.8}$$

which is a dispersive variant of system (1.6), since (1.8) contains the extra term $\sigma^2 u$. As we shall see, the presence of this added dispersive term simplifies the spectral analysis of the system.

We define the exponential decay rate of solutions of (1.3) as a function of σ, by

$$\begin{aligned}\omega(\sigma) = \sup\{\omega :\ \exists C,\ \forall (P_0, V_0) \in (L^2(-1,1))^2, \\ \forall t,\ E(t) \leq C E(P_0, V_0)\exp(-\omega t)\}.\end{aligned} \tag{1.9}$$

For each $\omega \leq \omega(\sigma)$, we define $C(\omega)$ as the best constant such that

$$\forall (P_0, V_0) \in (L^2(-1,1))^2,\ \forall t,\ E(t) \leq C(\omega) E(P_0, V_0)\exp(-\omega t). \tag{1.10}$$

Note that this actually measures the decay rate of the energy of solutions of (1.3) in the whole domain, not only in $(-1,0)$. However, we will prove that the decay rates of the energy of the restriction of solutions of (1.3) to $(-1,0)$ and in the whole domain coincide.
Let us also define the space operator L by

$$\begin{aligned} &L(P,V) = (\partial_x V + \chi_{(0,1)}\sigma P, \partial_x P + \chi_{(0,1)}\sigma V) \\ &D(L) = H^1_0(-1,1) \times H^1(-1,1). \end{aligned} \tag{1.11}$$

This unbounded operator on $L^2(-1,1)$ is the generator of a semi-group of contractions solving the equations (1.3). We prove that the decay rate $\omega(\sigma)$

satisfies $\omega(\sigma) = 2S(\sigma)$, where $S(\sigma)$ is the spectral abscissa, defined in terms of $\Lambda(L)$, the spectrum of the operator L, as follows:

$$S(\sigma) = \sup\{Re(\lambda) \; | \lambda \in \Lambda(L)\}. \tag{1.12}$$

This is done by means of a complete description of the spectrum of L, that also shows that $\omega(\sigma)$ coincides with

$$I = \int_0^1 \sigma(x)\; dx, \tag{1.13}$$

which is a measure of the total damping entering in the system.
This result confirms the ones in [7]-[9]-[8] about the efficiency of taking a singular damping $\sigma \notin L^1$ for the PML method for the Helmholtz equation. Our characterization (1.13) of the decay rate as the integral of σ confirms that, when taking σ singular, the decay rate may be made arbitrarily large.

In the second part of this article, we investigate the decay of the energy for the following semi-discrete finite-difference approximation scheme for PML:

$$\begin{cases} \partial_t P_j + \frac{V_{j+1}-V_j}{h} + \sigma_j P_j = 0, & j \in \{-N+1, \dots, N-1\} \\ \partial_t V_j + \frac{P_j - P_{j-1}}{h} + \sigma_{j-1/2} V_j = 0, & j \in \{-N+1, \dots, N\} \\ P_{-N} = P_N = 0. \end{cases} \tag{1.14}$$

The notations we employ are the classical ones for finite differences: $h = 1/N$, for some $N \in \mathbb{N}$, is the mesh size, $x_j = jh$, $j = -N, \cdots, N$ constitute the mesh points and P_j and V_j are approximations of P on x_j and of V on $(x_{j-1} + x_j)/2$. We approximate the function σ by a piecewise constant function taking the value $\sigma_{j+1/2}$ on each (x_j, x_{j+1}) and denote by σ_j the mean value of $\sigma_{j-1/2}$ and $\sigma_{j+1/2}$.

The energy $E_h(t)$ of the semi-discrete system (1.14) is given by

$$E_h(t) = \frac{h}{2} \sum_{j=-N+1}^{N} \big(|P_j(t)|^2 + |V_j(t)|^2\big), \tag{1.15}$$

and can be interpreted as a discretization of the continuous energy E in (1.4). It decays exponentially as $t \to \infty$. But, as we shall see, the decay rate is not uniform on h. This is due to the spurious high frequency numerical oscillations whose group velocity is close to zero. The effect of these spurious oscillations has already been noticed in a number of articles in connection with the qualitative properties of numerical waves since [28] and further developed in the survey article [32]. We give a precise analysis of the spectrum in terms of h and σ, when σ is a constant on $(0,1)$, that will further clarify this lack of uniform (on h) exponential decay.

Inspired by [27], in order to remedy this lack of uniform decay, we consider the following viscous scheme, which is again convergent of order 2:

$$\begin{cases} \partial_t P_j + \frac{V_{j+1}-V_j}{h} + \sigma_j P_j - h^2(\Delta_h P)_j = 0, \; j = -N+1, \dots, N-1, \\ \partial_t V_j + \frac{P_j - P_{j-1}}{h} + \sigma_{j-1/2} V_j - h^2 (\Delta_h V)_j = 0, \; j = -N+1, \dots, N, \\ P_{-N} = P_N = 0, \quad V_{-N} = V_{-N+1}, \quad V_{N+1} = V_N. \end{cases} \tag{1.16}$$

Here and in the sequel Δ_h denotes the classical discretization of the Laplace operator:

$$(\Delta_h A)_j = \frac{1}{h^2}(A_{j+1} + A_{j-1} - 2A_j).$$

The energy of this modified system is further dissipated by the added numerical viscosity terms:

$$\frac{dE_h}{dt}(t) = -h \sum_{j=-N+1}^{N} \sigma_j |P_j|^2 - h \sum_{j=-N+1}^{N} \sigma_{j-1/2} |V_j|^2 \\ - h^3 \sum_{j=-N}^{N-1} \Big(\Big(\frac{P_{j+1} - P_j}{h} \Big)^2 + \Big(\frac{V_{j+1} - V_j}{h} \Big)^2 \Big). \tag{1.17}$$

In particular, the viscosity terms provide an efficient dissipation on the high frequency waves and, accordingly, as we shall see in Theorem 5.1, the decay rate is uniform on h.

Furthermore, we prove in Theorem 5.2 that the decay rate of the energy of the semi-discrete approximation schemes (1.16) coincides with the continuous one, that is I, under an appropriate choice of the viscosity parameter. In other words, we can recover the dynamical properties of the continuous PML at the semi-discrete level.

This numerical technique of adding numerical viscosity provides a way to keep the PML method accurate at the semi-discrete level. Inspired on previous work on the control of waves ([32]), we may expect that other remedies

will also allow preserving the uniform (on h) decay properties of the energy, for instance a mixed-finite element method as in [4] and [12] or a multi-grid scheme as in [17] and [19].
Actually, most of the results presented here at the semi-discrete level have a very wide range of validity, and can be extended to different approximation schemes, for instance using finite elements, and even in higher dimension. In particular, the construction in subsection 4.1 works and proves that in general the discrete energy cannot be uniformly exponentially decaying, if a numerical viscosity is not added.

Here is a brief overview on the PML method and its possible applications. The mathematical analysis of the continuous model was done in [21] and [13], where it was proved that the solution of the continuous PML for the Helmholtz equation with an infinite layer corresponds exactly to the unbounded solution in the computational domain. Moreover, it was also stated that, if the layer is bounded but large enough, solutions provide a good approximation in the computational domain. Moreover, it was proved in [8] that when the layer is bounded, the PML method for the Helmholtz equation recovers the exact solution in the computational domain if we choose a radial damping potential $\sigma \notin L^1$.
Unfortunately, it was proved in [1] that the PML method is only weakly well-posed for Maxwell's equations in the sense that the functions involved in the splitting induced by the PML method do not stay in the same functional space as the initial data, thus requiring smoother initial data. This also implies that instabilities may arise under small perturbations. A number of articles has been devoted to gain a better comprehension of these problems on well-posedness and instabilities in the continuous case ([6], [5], [23] and [30]). New absorbing layers were also proposed in the continuous case for Maxwell's equations and advective acoustics, in particular, in [2], [3], [26].

The structure of the present paper is the following. In section 2, we carefully analyze the spectral properties of the space operator L, by using a shooting method. This will allow us to give an explicit formula for its spectrum in Theorem 2.1. In section 3, we prove that the quantities I, S, and $\omega(\sigma)$ above coincide. We will also prove that the inequality (1.10) holds for $\omega = \omega(\sigma)$ and give some estimates on the best constant $C(\omega(\sigma))$ in this inequality. We also give an explicit representation formula for the solutions of the continuous PML equations and deduce the optimality of our estimates. In section 4, we address the same issues for the space semi-discrete system. We show that the high frequency spurious numerical solutions are responsible for a lack of uniform exponential decay of the energy and, in the special case where σ is constant, we give an asymptotic description of the spectrum of the discretized operator. Finally, in section 5, we consider the viscous scheme (1.16) and prove the exponential decay of the energy, uniformly in h.

2 Analysis of the space operator L.

The aim of this section is to give a complete description of the spectral properties of L defined as in (1.11).

Theorem 2.1. *Let $\sigma \in L^1(0,1)$ be a non-trivial and non-negative function. Then:*

1. *The operator L has a compact inverse.*
2. *The spectrum of the operator L coincides with the set of the eigenvalues*

$$\lambda_k = \frac{1}{2}\Big(\int_0^1 \sigma(x)\ dx + ik\pi\Big),\ \ k \in \mathbb{Z}. \tag{2.1}$$

3. *The eigenvectors (P_k, V_k) form a Riesz basis of $L^2(-1,1)^2$.*

Let us first remark that the first statement implies that the spectrum is discrete. The interest of the second statement is that it provides an explicit description of the eigenvalues. The last claim allows characterizing the decay rate in terms of the spectral abscissa. The following subsections will be devoted to the proof of each of these three statements.

2.1 Inverse of the operator L.

Consider the system

$$(P,V) \in D(L) \quad ; \quad L(P,V) = (f,g).$$

where f and g are two given functions in $L^2(-1,1)$.

To solve this problem, we consider $Q = P + V$ and $R = V - P$ that satisfy

$$\partial_x Q + \sigma(x)Q(x) = f(x) + g(x), \quad \partial_x R - \sigma(x)R(x) = f(x) - g(x). \tag{2.2}$$

Introducing the boundary conditions $P = 0$ at $x = \pm 1$, this yields

$$Q = R, \quad x = \pm 1. \tag{2.3}$$

Then straightforward computations show that equations (2.2)-(2.3) have a unique solution if and only if $I \neq 0$, which is true since σ is a non-trivial non-negative function.
By (2.2) and (2.3) we deduce that L^{-1} defines a bounded operator

$$L^{-1} : L^2(-1,1)^2 \to H_0^1(-1,1) \times H^1(-1,1),$$

which turns out to be compact as an operator from $L^2(-1,1)^2$ into itself.

2.2 Analysis of the spectrum : Eigenvalues of L.

The system characterizing the spectrum is as follows:

$$\begin{cases} \partial_x V + \sigma P = \lambda P, \ \partial_x P + \sigma V = \lambda V, \ x \in (-1,1) \\ P(-1) = P(1) = 0. \end{cases}$$

Using the functions Q and R as in the previous section gives

$$Q(x) = Q(-1)e^{-\int_{-1}^{x}(\sigma(z)-\lambda)\ dz}, \quad R(x) = R(-1)e^{\int_{-1}^{x}(\sigma(z)-\lambda)\ dz}.$$

The boundary conditions yield (2.3). Then λ is an eigenvalue if and only if

$$\exp(-\int_{-1}^{1}(\sigma(z)-\lambda)\ dz) = \exp(\int_{-1}^{1}(\sigma(z)-\lambda)\ dz). \tag{2.4}$$

Hence the result (2.1).

Remark: Note that the eigenvalues are totally explicit for all damping potentials σ. This is not the case for the damped wave equation (1.6), which, when written as a first order system, corresponds to adding the damping potential only in one of the equations of the system. In that case, (2.1) only holds asymptotically for high frequencies and this under the assumption that $\sigma \in BV(-1,1)$ (see [14]).

2.3 Analysis of the spectrum : Eigenvectors.

Define the function θ by

$$\theta(x) = \int_{-1}^{x} \Big(\sigma(z) - \frac{I}{2}\Big)\, dz. \tag{2.5}$$

This function can be seen as a measure of the difference between the dissipative term σ and the average dissipation $I/2$. Note also that $\theta(-1) = \theta(1) = 0$. We remark that for all eigenvectors P_k, V_k, the functions Q_k, R_k as in the previous section satisfy (taking $Q(-1) = R(-1) = 1$) :

$$Q_k(x)\exp(\theta(x)) = e^{\frac{ik\pi}{2}(x+1)}, \quad R_k(x)\exp(-\theta(x)) = e^{-\frac{ik\pi}{2}(x+1)}.$$

Our purpose now is to check that the family (P_k, V_k) constitutes a Riesz basis in $L^2(-1,1)^2$ (see [31] for an introduction to that subject). This means in particular that any pair of functions $(f,g) \in L^2(-1,1)^2$ can be written in an unique way as follows:

$$(f,g) = \sum a_k (P_k, V_k), \tag{2.6}$$

with

$$\sum |a_k|^2 \simeq \|(f,g)\|^2. \tag{2.7}$$

To prove this, we observe that (2.6) is equivalent to:

$$\begin{cases} (f+g)(x)e^{\theta(x)} = \sum a_k Q_k(x) e^{\theta(x)} = \sum a_k e^{\frac{ik\pi}{2}(x+1)} \\ (g-f)(x)e^{-\theta(x)} = \sum a_k R_k(x) e^{-\theta(x)} = \sum a_k e^{-\frac{ik\pi}{2}(x+1)}. \end{cases}$$

Then, the coefficients $\{a_k\}$ of the decomposition (2.6) of (f,g) on the basis $\{(P_k, V_k)\}$ can be identified as the Fourier coefficients of the function W defined in $(-3,1)$ by

$$W(x) = \begin{cases} (f+g)(x)\exp(\theta(x)), & -1 < x < 1 \\ (g-f)(-2-x)\exp(-\theta(-2-x)), & -3 < x < -1. \end{cases} \tag{2.8}$$

In other words (2.6) holds if and only if

$$W(x) = \sum_k a_k \exp\Big(\frac{ik\pi}{2}(x+1)\Big), \ x \in (-3,1). \tag{2.9}$$

Obviously W is in $L^2(-3,1)$ if and only if (f,g) is in $L^2(-1,1)^2$, and therefore (2.7) holds.
This construction defines an isomorphism $\mathcal{I}$, which maps the eigenvectors $\psi_k = (P_k, V_k)$ to the classical Fourier basis of $L^2(-3,1)$:

$$\mathcal{I}(f,g) = W, \tag{2.10}$$

where W is the function given in (2.8). Note that this implies that any function $\psi \in (L^2(-1,1))^2$ can be expanded as $\sum a_k \psi_k$, where the coefficients a_k satisfies:

$$\|\mathcal{I}\psi\|^2_{L^2(-3,1)} = 4 \sum |a_k|^2.$$

Remark: In [14], it was proved (see Theorem 5.5) that the solution $y_2(x,\lambda)$ of the Cauchy-Lipschitz system

$$\begin{cases} -\partial^2_{xx} u + \lambda^2 u + 2a(x)\lambda u = 0, \ x \in (-1,1) \\ u(-1,\lambda) = 0, \quad \partial_x u(-1,\lambda) = 1, \end{cases}$$

which naturally arises when dealing with the spectral problem associated to a damped string, satisfies the following properties:

$$\begin{cases} y_2(x,\lambda_n) = 2\frac{\sinh(\xi(x) + in\pi(x+1)/2)}{in\pi - \int_{-1}^{1} a(x)\ dx} + O(1/n^2) \\ \partial_x y_2(x,\lambda_n) = \cosh(\xi(x) + in\pi(x+1)/2) + O(1/|n|), \end{cases}$$

where λ_n is the n-th root of $\lambda \to y_2(1,\lambda)$ and ξ is

$$\xi(x) = \int_{-1}^{x} a(s)\ ds - (x+1)\frac{1}{2}\int_{-1}^{1} a(x')\ dx'.$$

As indicated in the introduction, the dissipative potential $\sigma(x)$ of the PML method plays the same role as $a(x)$ in the dissipative wave equation (1.6). Obviously, the function $\xi(x)$ plays the same role as $\theta(x)$ in (2.5). We conclude that the eigenvectors of the damped wave equation are asymptotically close to the ones of the PML system.

3 On the decay of the energy.

3.1 On the decay rate.

Theorem 3.1. *The energy of the continuous PML system* (1.3) *is exponentially decaying. More precisely,*

$$\exists C > 0, \ s.t. \ \forall t > 0, \ E(t) \leq C\exp(-\omega(\sigma)t), \tag{3.1}$$

for all solution of (1.3) *with* $\omega(\sigma)$ *as in* (1.9). *Moreover,* $\omega(\sigma) = I = 2S(\sigma)$, *with* I *and* $S(\omega)$ *as in* (1.13) *and* (1.12), *and the best constant* $C(\omega(\sigma))$ *in* (3.1) *as defined in* (1.10) *satisfies:*

$$C(\omega(\sigma)) \leq \exp(4\|\theta\|_\infty), \tag{3.2}$$

where $\theta = \theta(x)$ *is as in* (2.5).

Proof. Equality $I = 2S(\omega)$ was actually proved in the last section. From the previous section, we also know that the family of eigenvectors $\psi_k = (P_k, V_k)$ constitutes a Riesz basis of $L^2(-1,1)^2$ and this is sufficient to characterize the exponential decay rate as the spectral abscissa, i.e. $\omega(\sigma) = 2S(\sigma)$.
We now give further estimates on the decay rate in order to obtain (3.2), using the explicit isomorphism $\mathcal{I}$ given in (2.8).
Given $U_0 = (P_0, V_0) \in L^2(-1,1)^2$, we expand U_0 in the basis ψ_k : $U_0 = \sum a_k \psi_k$. We have :

$$2E_0 = \|U_0\|^2_{L^2(-1,1)^2} \geq \|\mathcal{I}\|^{-2} \|\mathcal{I}U_0\|^2_{L^2(-3,1)} \geq 4\|\mathcal{I}\|^{-2} \sum |a_k|^2.$$

It is easy to check that

$$U(t) = \sum a_k \exp(-\lambda_k t)\psi_k,$$

and then

$$\|\mathcal{I}U(t)\|^2_{L^2(-3,1)^2} = \exp(-tI) \sum |a_k|^2.$$

But

$$2E(t) = \|U(t)\|^2_{L^2(-1,1)^2} \leq \|\mathcal{I}^{-1}\|^2 \|\mathcal{I}U(t)\|^2_{L^2(-3,1)^2}.$$

Combining these inequalities, we get

$$E(t) \leq \|\mathcal{I}\|^2 \|\mathcal{I}^{-1}\|^2 \exp(-tI) E_0. \tag{3.3}$$

On the other hand, obviously, the exponential decay rate I is optimal as one can see by analyzing the solutions in separated variables.

According to (3.3) we have $C(\omega(\sigma)) \leq \kappa(\mathcal{I})^2$, where $\kappa(\mathcal{I})$ is the conditioning number $\kappa(\mathcal{I}) = \|\mathcal{I}\| \cdot \|\mathcal{I}^{-1}\|$, but we would like to derive a more explicit expression in terms of the damping potential σ. By Parseval's identity applied to (2.9), for f and g in $L^2(-1,1)$ we get:

$$\|\mathcal{I}((f,g))\|^2_{L^2(-3,1)^2} = 4\sum |a_k|^2 = \int_{-1}^{1} |f(x)+g(x)|^2 \mathrm{exp}(2\theta(x))\ dx \\ + \int_{-1}^{1} |f(x)-g(x)|^2 \mathrm{exp}(-2\theta(x))\ dx. \quad (3.4)$$

As a consequence,

$$2\mathrm{exp}(-2\|\theta\|_\infty)\|(f,g)\|^2_{L^2(-1,1)^2} = 2\mathrm{exp}(-2\|\theta\|_\infty)\int_{-1}^{1}\left(|f(x)|^2+|g(x)|^2\right)\ dx \\ \leq \|\mathcal{I}((f,g))\|^2_{L^2(-3,1)^2} \leq 2\mathrm{exp}(2\|\theta\|_\infty)\|(f,g)\|^2_{L^2(-1,1)^2}.$$

Accordingly,

$$\|\mathcal{I}\|^2 \leq 2\mathrm{exp}(2\|\theta\|_\infty), \quad \|\mathcal{I}^{-1}\|^2 \leq \frac{1}{2}\mathrm{exp}(2\|\theta\|_\infty),$$

and (3.2) holds. □

In order to discuss the efficiency of the PML method and, more precisely, that of system (1.3), we recall that it has been designed to provide an approximation of the solution of (1.2) in $(-1,0)$ for initial data with support in $(-1,0)$. Accordingly, we define E_l and E_r as the energy on the left and right subdomains respectively:

$$\begin{aligned} E_l(P,V) &= \tfrac{1}{2}\int_{-1}^{0}\left(|P(x)|^2+|V(x)|^2\right)\ dx, \\ E_r(P,V) &= \tfrac{1}{2}\int_{0}^{1}\left(|P(x)|^2+|V(x)|^2\right)\ dx. \end{aligned} \quad (3.5)$$

Theorem 3.2. *Let P_0 and V_0 be the initial data for the PML equations* (1.3) *with support in* $(-1,0)$*. Then,*

$$\begin{aligned} E_l(P(t),V(t)) &\leq \mathrm{exp}(I(2-t))E_0 \\ E_r(P(t),V(t)) &\leq \mathrm{exp}(I+2\|\theta\|_\infty - It)E_0. \end{aligned} \quad (3.6)$$

Proof. The result follows from careful upper bounds in the previous proof, using (3.4), the conditions on the support of initial data, and the fact that the $L^\infty(-1,0)$ norm of θ is precisely $I/2$. This leads us to

$$E_0\mathrm{exp}(I) \geq \sum |a_k|^2 \geq E_l(P(t),V(t))\mathrm{exp}((t-1)I).$$

This establishes the first inequality. The second one is left to the reader. □

3.2 Comments

As a consequence of (3.6), if we fix a shape σ for the damping potential, and if we define the sequence of amplified potentials $\sigma_n(x) = n\sigma(x)$, then the corresponding solutions (P_n, V_n) to the PML system with initial data (P_0, V_0) supported in $(-1, 0)$ damped by σ_n tend to zero in $L^2((-1, 0))^2$ for $t > 2$ as $n \to \infty$.

Theorem 3.1 also confirms the results in [7] and [9], where it was proved by a plane wave analysis that the reflection coefficient on $x = 0$ is of order $\exp(-I)$ and that, taking a function $\sigma \notin L^1(0, 1)$, makes the PML method very efficient. In [7]-[9] numerical computations were done for different choices of σ : $\sigma_1(x) = (1 - x)^{-1} - 1$, $\sigma_2(x) = (1 - x)^{-2} - 1$ and $\sigma_3(x) = (1 - x)^2$. Numerical evidences in [7] show that the Helmholtz PML system is clearly more accurate for σ_1 and σ_2 than for σ_3. A precise proof was also given in [8] through the analysis of the Dirichlet-to-Neuman operator associated to the PML. Unfortunately, this kind of proof does not seem to hold anymore at the discrete level. Our result (3.1) on the decay rate of the energy also justifies these numerical evidences, since σ_1 and σ_2 do not belong to L^1 and have infinite average. As we shall see in the sequel, the methods we present here are more robust and will allow us to study the semi-discrete equations as well.

Let us now analyze the function θ entering in (3.2), which is obviously continuous on $(-1, 1)$. It is easy to see that the L^∞ norm of θ is exactly $I/2$ on $(-1, 0)$. On $(0, 1)$, the situation is more complex : θ is differentiable on $(0, 1)$, its derivative is $\theta'(x) = \sigma(x) - I/2$, and $\theta(0) = -I/2$, and $\theta(1) = 0$. We can also remark that $\|\theta\|_\infty = -\inf \theta \leq I$.
A natural question is trying to minimize the quantity $\|\theta\|_\infty$ on the positive potentials σ which have a given integral I_0. Easy considerations indicate that there are many different σ which satisfy $\|\theta\|_\infty = I_0/2$, the most natural one being the choice $\sigma = I_0$. However, in view of (3.6), this discussion is irrelevant if we are only considering the energy E_l concentrated in $(-1, 0)$.

3.3 Optimality of the decay rate.

We complete this section with some results on the optimality of the decay rates we observed.

Theorem 3.3. *The estimates given in* (3.2) *and in Theorem 3.2 are sharp.*

Proof. We rewrite the system (1.3) in the following way :

$$\left\{\begin{array}{ll} \partial_t(P+V)+\partial_x(P+V)+\sigma(P+V)=0 & \text{in } (-1,1)\times(0,T) \\ \partial_t(P-V)-\partial_x(P-V)+\sigma(P-V)=0 & \text{in } (-1,1)\times(0,T) \\ P(-1,t)=P(1,t)=0. & \end{array}\right.$$

Using characteristics leads to :

$$\begin{array}{l} (P-V)(x,t)=(P_0-V_0)(x+t)\exp(-\int_x^{x+t}\sigma(y)\ dy),\ x\le 1-t \\ (P-V)(x,t)=(P-V)(1,1-x)\exp(-\int_x^1\sigma(y)\ dy),\ x>1-t \\ (P+V)(x,t)=(P+V)(-1,x+1)\exp(-\int_{-1}^x\sigma(y)\ dy), x<t-1 \\ (P+V)(x,t)=(P_0+V_0)(x-t)\exp(-\int_{x-t}^x\sigma(y)\ dy),\ x\ge t-1. \end{array} \tag{3.7}$$

We deduce that :

$$\begin{aligned} E(t) &= \frac{1}{4}\int_{-1}^1\left(|(P+V)(x,t)|^2+|(P-V)(x,t)|^2\right)\ dx \\ &\le \frac{1}{4}\exp\Big(-2\inf_{\gamma\in\mathcal{R}_t}\int_\gamma\sigma(y)\ dy\Big)\int_{-1}^1\left(|(P_0+V_0)(x)|^2+|(P_0-V_0)(x)|^2\right)\ dx \\ &= \exp(-2\inf_{\gamma\in\mathcal{R}_t}\int_\gamma\sigma(y)\ dy)E_0, \end{aligned}$$

where $\mathcal{R}_t$ is the set of characteristic rays of length t. Besides, by these formulas it is easy to see that this estimate is sharp since we can concentrate waves around these rays (see subsection 4.1 where this analysis is carried out on the semi-discrete model).
Then, the best constant $C(\omega(\sigma))$ in (3.2) is precisely

$$C(\omega(\sigma))=\sup_{t>0}\left\{\frac{E(t)}{E_0}\exp(It)\right\}=\sup_{t>0}\exp(It-2\inf_{\gamma\in\mathcal{R}_t}\int_\gamma\sigma(y)\ dy).$$

It is then enough to compute

$$M=\sup_{t>0}\sup_{\gamma\in\mathcal{R}_t}\int_\gamma\Big(\frac{I}{2}-\sigma(y)\Big)\ dy.$$

Then, looking at rays γ_a^t starting at $a \in [-1,1]$ and traveling toward the left we get

$$
\begin{aligned}
M &\geq \sup_{t>0} \sup_{a} \int_{\gamma_a^t} \Big(\frac{I}{2} - \sigma(y) \Big)\, dy \\
&\geq \sup_{a} \sup_{t\in[1+a,3+a]} \Big(\int_{-1}^{a} \Big(\frac{I}{2} - \sigma(y) \Big)\, dy + \int_{-1}^{t-2-a} \Big(\frac{I}{2} - \sigma(y) \Big)\, dy \Big) \\
&\geq \sup_{a} \{ \int_{-1}^{a} \Big(\frac{I}{2} - \sigma(y) \Big)\, dy \} + \sup_{b} \{ \int_{-1}^{b} \Big(\frac{I}{2} - \sigma(y) \Big)\, dy \} \\
&\geq -2 \inf_{a} \theta(a) = 2\|\theta\|_\infty .
\end{aligned}
$$

This implies that $C(\omega(\sigma)) \geq \exp(4\|\theta\|_\infty)$. The optimality of (3.2) follows. The method of proof carries over to the other two estimates given in Theorem 3.2. The details are left to the reader. □

Note that all the results on the continuous model could have been obtained using this explicit representation formula along characteristics without using spectral analysis.

4 On the semi-discrete PML equations.

In this section, we analyze the space semi-discrete PML system (1.14). For this purpose, we need to define a discrete space operator L_h, the discretization of L, defined in (1.11).
System (1.14) can be written as

$$\partial_t(P,V) + L_h(P,V) = 0,$$

where L_h is the discretization of L derived from (1.14). If we use a matrix representation, writing (P,V) as the vector $(V_{-N+1}, P_{-N+1}, V_{-N+2}, \cdots, P_{N-1}, V_N)$, L_h is the matrix defined by

$$
\begin{cases}
L_h(j,j) = \sigma_{j/2-N}, \quad \forall j \in \{1,\ldots,4N-1\} \\
L_h(j,j+1) = \frac{1}{h}, \quad \forall j \in \{1,\cdots,4N-2\} \\
L_h(j+1,j) = -\frac{1}{h}, \quad \forall j \in \{1,\cdots,4N-2\} \\
L_h(i,j) = 0, \quad \text{if } |i-j| > 1.
\end{cases}
\tag{4.1}
$$

If $\sigma_{j-1/2} = \sigma_j = \sigma_{j+1/2}$, then both P_j and V_j satisfy

$$\partial_{tt}^2 U_j - \frac{1}{h^2}(U_{j+1} + U_{j-1} - 2U_j) + 2\sigma_j \partial_t U_j + \sigma_j^2 U_j = 0,$$

which is a discretization of (1.8).
The energy E_h in (1.15) of the semi-discrete PML satisfies the dissipation law:

$$\frac{dE_h}{dt}(t) = -h \sum_{j=-N+1}^{N} \Big(\sigma_j |P_j|^2 + \sigma_{j-1/2} |V_j|^2 \Big). \tag{4.2}$$

It is then natural to investigate the decay rate of this discrete energy E_h when $h \to 0$. Our first result is of negative nature and states the lack of uniform exponential decay due to high frequency spurious oscillations:

Theorem 4.1. *There are no positive constants C and μ such that for all h small enough*

$$E_h(t) \le C \ E_h(0) \exp(-\mu t), \tag{4.3}$$

for all solutions of (1.14).

One could have expected this behavior : Indeed, it is well known since [28] that the group velocity for numerical schemes differs from the continuous case, because of the numerical dispersion relations. This often produces wave packets captured in the undamped subinterval $(-1, 0)$ and it is natural to expect them to have a very low exponential decay.
We will propose two proofs in the sequel. The first one is based on a very general construction of waves concentrated along the rays of Geometric Optics for system (1.14). More precisely, we construct non propagating waves concentrated in $(-1, 0)$, whose exponential decay rate tends to zero as $h \to 0$. In the second approach, we do a precise description of the spectrum of the operator L_h in (4.1) in the particular case where σ is constant. In particular, we prove that the real part of the high frequency eigenvalues can be small of order $o(1)$, which provides another proof of Theorem 4.1.

4.1 Construction of non propagating waves.

We only sketch this construction, whose details can be done similarly as in [24]. To make it easier, we do not consider the PML system (1.14), but rather the semi-discrete 1-d wave equation in an infinite lattice $h\mathbb{Z}$, where h is the mesh size:

$$\begin{cases} \partial_{tt}^2 u_j - \Delta_h u_j = 0, \ (t,j) \in (0,\infty) \times \mathbb{Z}, \\ u_j(0) = u_j^0, \quad \partial_t u_j(0) = u^1(0). \end{cases} \tag{4.4}$$

We claim that this is sufficient to exhibit non propagating waves for system (4.4) to prove Theorem 4.1. Indeed, the uniform exponential decay is equivalent to an observability inequality

$$E_h(0) \leq C \int_0^T \sigma(x)(|P(t,x)|^2 + |V(t,x)|^2) \ dx$$

for the conservative system (1.14) (i.e with $\sigma = 0$), with two uniform constants C and T (see the proof of Theorem 5.1). But the conservative system (1.14) coincides with system (4.4), up to the boundary conditions, which can be easily handled.

To properly define the rays of Geometric Optics, we need to use the space discrete Fourier transform defined for $\xi h \in (-\pi, \pi]$ by:

$$\begin{array}{l} \hat{\phi}(\xi) = h \sum_j \phi_j \mathrm{exp}(-i\xi jh), \ \xi h \in (-\pi,\pi], \\ \phi^h(x) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \hat{\phi}(\xi) \mathrm{exp}(i\xi x) \ d\xi, \ x \in \mathbb{R} \end{array} \tag{4.5}$$

Note that the inverse Fourier transform provides a natural extension of ϕ_j as a continuous function, denoted ϕ^h in the sequel.
The symbol of the operator (4.4) is given by

$$\tau^2 - \omega_h(\xi)^2, \quad \omega_h(\xi) = \frac{2}{h} \sin\Big(\frac{\xi h}{2}\Big). \tag{4.6}$$

Thus, the rays of Geometric Optics for frequencies ζ_0/h are the trajectories ([32]):

$$X_{\pm}^{\zeta_0} : (x_0, t) \to x_0 \pm t\cos(\zeta_0/2). \tag{4.7}$$

Taking $\zeta_0 \in (-\pi, \pi]$, we look for solutions concentrated along the trajectory $t \to X_+^{\zeta_0}(0,t)$. Note that we can take $x_0 = 0$ without loss of generality because of the translation invariance of system (4.4).

For we consider initial data of the form

$$u_j^{0,h} = \phi(jh)\mathrm{exp}(i\zeta_0 j), \quad u_j^{1,h} = i\omega_h(\xi_0^h)\phi(jh)\mathrm{exp}(i\zeta_0 j), \tag{4.8}$$

where ϕ is a smooth positive function of compact support in $(-a, a)$. Then, from the smoothness assumption on ϕ, one can prove that $\hat{u}^0$ and $\hat{u}(t)$ are concentrated in the region $\xi h \in [\zeta_0 - \epsilon_0, \zeta_0 + \epsilon_0]$, where ϵ_0 is a small parameter:

$$\begin{array}{l} \left| u^{0,h}(x) - \frac{h}{2\pi} \int_{|\xi - \xi_0^h| < \epsilon_0/h} \hat{u}^0(\xi) \exp(i\xi x)\ d\xi \right| \leq \frac{C}{\omega_h(\epsilon_0/h)^2} \\ \left| u^h(t,x) - \frac{h}{2\pi} \int_{|\xi - \xi_0^h| < \epsilon_0/h} \hat{u}(t,\xi) \exp(i\xi x)\ d\xi \right| \leq C \frac{(1 + T\omega_h(\xi_0^h))}{\omega_h(\epsilon_0/h)^2}. \end{array} \tag{4.9}$$

On the other hand,

$$\hat{u}(t,\xi) = \hat{u}^0(\xi) \Big(\cos(t\omega_h(\xi)) + it\omega_h(\xi_0^h) \mathrm{sinc}(t\omega_h(\xi)) \Big), \tag{4.10}$$

where $\mathrm{sinc}(y) = \sin(y)/y$. But, for ξ such that $|\xi - \xi_0| < \epsilon_0/h$, it is easy to see that this behaves as $\hat{u}^0(\xi)\exp(it\omega_h(\xi))$, and then the analysis of the oscillating integral in (4.9) gives that, when $h \to 0$,

$$\Big| |u(t, x + t\cos(\zeta_0/2))| - |u^0(x)| \Big| \leq C\epsilon_0. \tag{4.11}$$

This gives a sequence of solutions of (1.14) of unit energy such that the energy outside $\{(t,x) \in (0,T) \times \mathbb{R}, x \in X_+(t, [-a,a])\}$ tends to zero.
Note that the construction given above proves that the lack of uniform exponential decay of the energy actually takes its origin from the discretization scheme employed rather than from the PML method in itself.

4.2 Spectral analysis for constant σ.

From now, we make the assumption that the continuous damping function σ is a piecewise constant function vanishing in $(-1, 0)$ and taking the value σ in $(0,1)$. This leads to set $\sigma_j = \sigma_{j-1/2} = \sigma$ if $j \geq 1$, $\sigma_j = \sigma_{j+1/2} = 0$ for $j \leq -1$ and $\sigma_0 = \sigma/2$.

In the sequel, as we did for the operator L, we perform a spectral analysis of the operator L_h. As we shall see some numerical pathologies appear at high frequencies. More precisely, for frequencies of the order $2/h$ there appear eigenvalues whose real part is close to zero. This makes the exponential decay rate of the corresponding semigroups not to be uniform on h.

Accordingly, we analyze the asymptotic properties of the spectrum. We fix σ, and analyze the behavior of the eigenvalues of L_h when h goes to zero.

Proposition 4.1. *For $\sigma > 0$, we consider the spectral problem :*

$$\left\{\begin{array}{l} \frac{V_{j+1}-V_j}{h} + \sigma\chi_{j\geq 1}P_j = \lambda P_j,\ j\in\{-N+1,\cdots,N-1\}\backslash\{0\} \\ \frac{P_j-P_{j-1}}{h} + \sigma\chi_{j\geq 1}V_j = \lambda V_j,\ j\in\{-N+1,\cdots,N\} \\ \frac{V_1-V_0}{h} + \frac{\sigma}{2}P_0 = \lambda P_0 \\ P_{-N} = P_N = 0. \end{array}\right. \tag{4.12}$$

The following properties hold :

- *For any eigenvalue λ, its conjugate $\bar{\lambda}$ is also an eigenvalue.*
- *All the eigenvalues are simple.*
- *All the eigenvalues satisfy $0 < \mathcal{R}e(\lambda) < \sigma$ and $|\mathcal{I}m(\lambda)| \leq 2/h$.*
- *If λ is an eigenvalue, $\sigma - \lambda$ is also an eigenvalue.*

Proof. The first statement is obvious since the coefficients of system (4.12) are real. The second one is classical and follows from easy algebraic considerations. The third one is a consequence of the energy dissipation law (4.2):

$$0 \geq \frac{dE_h}{dt}(t) \geq -2\sigma E_h(t).$$

To analyze the imaginary part of the eigenvalues, we use the matrix representation of L_h given in (4.1): if $|\mathcal{I}m(\lambda)| > 2/h$, then the matrix $L_h - \lambda I$ is invertible, since it is diagonally dominant.
The last statement follows from this remark: If (P, V) is an eigenvector corresponding to λ, then $(\tilde{P}, \tilde{V})$ defined by $\tilde{P}_j = P_{-j}$ and $\tilde{V}_j = V_{-j+1}$ is an eigenvector for the eigenvalue $\sigma - \lambda$. □

From the previous proposition, we can assume that λ has a positive imaginary part, since the other eigenvalues can be obtained by reflection. Setting $\mu = \lambda - \sigma$, P satisfies

$$\begin{array}{l} \frac{P_{j+1}+P_{j-1}-2P_j}{h^2} = \lambda^2 P_j,\ j\leq -1 \\ \frac{P_{j+1}+P_{j-1}-2P_j}{h^2} = \mu^2 P_j,\ j\geq 1 \\ P_{-N} = P_N = 0. \end{array}$$

As for the classical discrete Laplace operator, we define α and β, two complex numbers with imaginary parts in $(-\pi/h, \pi/h]$ and satisfying the numerical dispersion relations :

$$\sinh\left(\frac{\alpha h}{2}\right) = \frac{\lambda h}{2} \quad ; \quad \sinh\left(\frac{\beta h}{2}\right) = \frac{\mu h}{2}. \tag{4.13}$$

Then, we can express P for $j \leq -1$ and for $j \geq 1$ as

$$P_j = A\sinh(\alpha(jh+1)),\ j \leq -1, \quad P_j = B\sinh(\beta(jh-1)),\ j \geq 1.$$

These two quantities have to coincide at $j = 0$ and therefore:

$$A\sinh(\alpha) = -B\sinh(\beta).$$

We can then compute the corresponding value for V:

$$\begin{array}{l} V_j = A\cosh(\alpha(jh+1-h/2)),\ j \leq 0 \\ V_j = B\cosh(\beta(jh-1-h/2)),\ j \geq 1. \end{array}$$

The transmission conditions are given by the equation on P_0:

$$V_1 - \sinh\left(\frac{\beta h}{2}\right) P_0 = V_0 + \sinh\left(\frac{\alpha h}{2}\right) P_0.$$

Then if λ is an eigenvalue, there exists a non trivial solution (A, B) to the system:

$$\left\{ \begin{array}{l} 0 = A\sinh(\alpha) + B\sinh(\beta) \\ 0 = A\cosh(\alpha)\cosh\left(\frac{\alpha h}{2}\right) - B\cosh(\beta)\cosh\left(\frac{\beta h}{2}\right), \end{array} \right.$$

where (α, β) are given by (4.13), μ being $\sigma - \lambda$. It is well-known that this system has non trivial solutions if and only if its determinant vanishes, that is to say:

$$\sinh(\alpha)\cosh(\beta)\cosh\left(\frac{\beta h}{2}\right) + \cosh(\alpha)\sinh(\beta)\cosh\left(\frac{\alpha h}{2}\right) = 0. \tag{4.14}$$

This equation actually is a polynomial in λ. Indeed, using Tchebychev polynomials P_{2k} and Q_{2k} defined by

$$\forall a \in \mathbb{C},\ \sinh(2ka) = \cosh(a)P_{2k}(\sinh(a)), \quad \cosh(2ka) = Q_{2k}(\sinh(a)),$$

the condition (4.14) is equivalent to

$$\cosh\Big(\frac{\alpha h}{2}\Big)\cosh\Big(\frac{\beta h}{2}\Big)\Big(P_{2N}\Big(\sinh\Big(\frac{\alpha h}{2}\Big)\Big)Q_{2N}\Big(\sinh\Big(\frac{\beta h}{2}\Big)\Big) \\ + P_{2N}\Big(\sinh\Big(\frac{\beta h}{2}\Big)\Big)Q_{2N}\Big(\sinh\Big(\frac{\alpha h}{2}\Big)\Big)\Big) = 0. \quad (4.15)$$

This equation has two particular solutions corresponding to $\alpha h = i\pi$ and $\beta h = i\pi$. Nevertheless, although these two solutions allow a non-trivial choice (A, B), the corresponding solutions are identically zero, and therefore they do not correspond to eigenvalues. Since the degree of this polynomial in (4.15) is exactly $4N-1$ and since all the eigenvalues are simple, the roots of (4.14) are exactly the eigenvalues of the problem, except the special solutions $\lambda = 2i/h$ and $\lambda = \sigma + 2i/h$.

Our interest now is to compute the eigenvalues, or at least to give their asymptotic form. We present in Figure 1 the distribution of eigenvalues for different values of σ.
Three different cases occur. When σ is very small (of order h or less), then the real parts of the eigenvalues are very close to $\sigma/2$ at all frequencies. When σ is such that $h << \sigma << 1/h$, two branches appear at the high frequencies, their abscissa having two accumulation points, namely 0 and σ. Finally, Figure 1 illustrates the well-known fact ([13]) that, on the numerical approximation of PML equations, taking σ too large deteriorates the decay rate, in opposition to the continuous case.

To study the asymptotic behavior of the spectrum, we will need a number of notations.
We rewrite (4.14) as $f(\alpha, \beta, h) = 0$, where f is defined by

$$f(\alpha,\beta,h) := \sinh(\alpha+\beta)\Big(\cosh\Big(\frac{\alpha h}{2}\Big) + \cosh\Big(\frac{\beta h}{2}\Big)\Big) \\ + \sinh(\alpha-\beta)\Big(\cosh\Big(\frac{\beta h}{2}\Big) - \cosh\Big(\frac{\alpha h}{2}\Big)\Big). \quad (4.16)$$

In the sequel, we use the function Argsh defined as the inverse function of sinh, which coincides with $\log(z+\sqrt{1+z^2})$, which is holomorphic on the set $\Omega = \mathbb{C}\backslash\{z:\ \mathcal{R}e(z) = 0,\ |\mathcal{I}m(z)| \geq 1\}$ and continuous at the points $z = \pm i$:

$$\forall z \in \Omega,\ \sinh(\mathrm{Argsh}(z)) = z$$
$$\forall z \in \mathbb{C},\ s.t.\ \mathcal{I}m(z) \in (-\pi/2, \pi/2),\ \mathrm{Argsh}(\sinh(z)) = z.$$

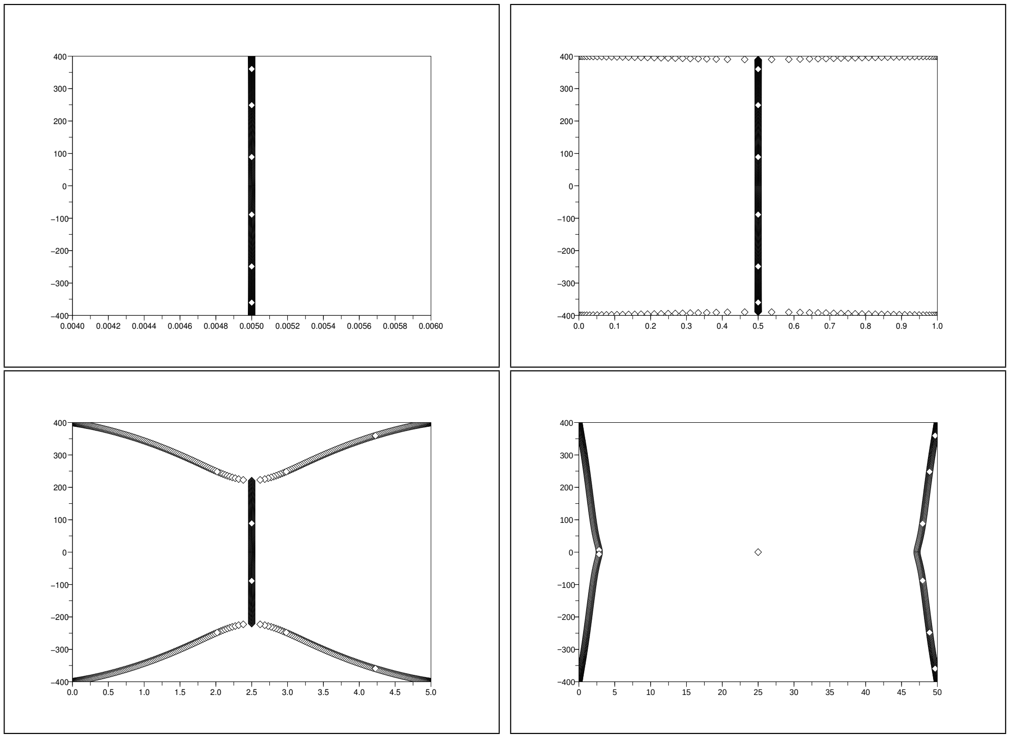

Figure 1: Eigenvalues for $N = 200$ and various values of σ : $\sigma = 0.01$ on the upper left, $\sigma = 1$ on the upper right, $\sigma = 5$ on the bottom left, $\sigma = 50$ on the bottom right.

Then, β given by the relation (4.13) is an holomorphic function of α:

$$\beta(\alpha, h) = \frac{2}{h}\text{Argsh}\Big(\sinh\left(\frac{\alpha h}{2}\right) - \frac{\sigma h}{2}\Big). \tag{4.17}$$

Hence the solutions of (4.14) correspond precisely to the roots α of the holomorphic function g

$$g(\alpha, h) = \cosh\left(\frac{\alpha h}{2}\right)\sinh(\alpha + \beta) \\ + \Big(\cosh\left(\frac{\beta h}{2}\right) - \cosh\left(\frac{\alpha h}{2}\right)\Big)\sinh(\alpha)\cosh(\beta), \tag{4.18}$$

where $\beta = \beta(\alpha)$ as in (4.17). Of course, α given by (4.13) is a holomorphic function of λ and we can also define $\tilde{g}$ as a holomorphic function of λ by

$$\tilde{g}(\lambda, h) := g(\alpha(\lambda), h).$$

The analysis of the roots of (4.14) can be carried out using tools from complex analysis, as for instance Rouché's theorem.

The low frequencies. We choose a number $\delta < 1$ and study the eigenvalues λ of the operator L_h such that $|\mathcal{I}m(\lambda)h| \leq 2\delta$ when $h \to 0$.

Theorem 4.2. *Assume $\delta < 1$. There exists C_δ such that for h small enough, the set of the eigenvalues λ_k^h of the operator L_h such that $|\mathcal{I}m(\lambda)h| \leq 2\delta$ is composed by one point in each disk D_k^h*

$$|\lambda - \hat{\lambda}_k^h| \leq C_\delta h, \quad \hat{\lambda}_k^h = \frac{2i}{h}\sin\Big(\frac{k\pi h}{4}\Big) + \frac{\sigma}{2}, \tag{4.19}$$

k being an integer satisfying $|\sin\Big(\frac{k\pi h}{4}\Big)| \leq \delta$.

Let us first remark that these disks D_k^h are disconnected for h small enough since the distance between two consecutive eigenvalues λ_k^h and λ_j^h is bounded from below by $\cos(\arcsin(\delta)) = \sqrt{1-\delta^2} > 0$. This implies that for h small enough, the number of eigenvalues in the range $|\mathcal{I}m(\lambda)h| \leq 2\delta$ is exactly $\lfloor \frac{8}{\pi h} \arcsin(\delta) \rfloor$ ($\lfloor \cdot \rfloor$ denotes the integer part).
Moreover, their real part being essentially $\sigma/2$, the energy of the solutions $\exp(-\lambda_k t)(P^{k,h}, V^{k,h})$, where $(P^{k,h}, V^{k,h})$ is an eigenvector associated to λ_k, is decreasing exponentially, the decay rate being $\sigma + o(h)$.

Proof. The proof is divided into two steps. First we derive some basic estimates on the parameters entering in (4.18). Second we approximate the function g by another holomorphic function $\hat{g}$ in order to apply Rouché's theorem.

We first need to derive some basic estimates on $\alpha(\lambda)$ given in (4.13), mainly by using the previous theorem. In the strip $|\mathcal{I}m(z)| \leq \delta$ and $|\mathcal{R}e(z)| \leq \sigma h$, if $z = a + ib$, we have that

$$z + \sqrt{1+z^2} = a + \sqrt{1-b^2} + ib\Big(1 + \frac{a}{\sqrt{1-b^2}}\Big) + O(h).$$

Then, we can check that the (complex) logarithm of that quantity satisfies:

$$|\mathcal{R}e(\mathrm{Argsh}(z))| \leq Ch \quad ; \quad |\tan(\mathcal{I}m(\mathrm{Argsh}(z)))| \leq \frac{\delta}{\sqrt{1-\delta^2}} + o(1),$$

where the constant C depends on δ. Then, using (4.13), we obtain the following estimates :

$$|\mathcal{R}e(\alpha)| \leq C \quad ; \quad |\mathcal{I}m(\alpha)| \leq \gamma = \arctan\Big(\frac{\delta}{\sqrt{1-\delta^2}}\Big). \tag{4.20}$$

Using (4.17) and the Taylor's formula applied to the function Argsh in $\sinh(\alpha h/2)$, we get that

$$\Big|\beta - \Big(\alpha - \frac{\sigma}{\cosh(\frac{\alpha h}{2})}\Big)\Big| \leq Ch. \tag{4.21}$$

Again using the estimates (4.20), we get

$$\left|\cosh\Big(\frac{\alpha h}{2}\Big)\sinh(\alpha+\beta) - \cosh\Big(\frac{\alpha h}{2}\Big)\sinh\left(2\alpha - \frac{\sigma}{\cosh\left(\frac{\alpha h}{2}\right)}\right)\right| \leq Ch.$$

The well-known formula $\cosh^2(x) = 1 + \sinh^2(x)$ and the estimates (4.20), (4.21) give

$$\Big|\cosh\Big(\frac{\beta h}{2}\Big) - \cosh\Big(\frac{\alpha h}{2}\Big)\Big| \leq Ch. \tag{4.22}$$

Combining all these inequalities and (4.18), we get that

$$|g(\alpha,h) - \hat{g}(\alpha,h)| \leq C_1 h, \tag{4.23}$$

where $\hat{g}$ is the function defined by :

$$\hat{g}(\alpha,h) = \cosh\Big(\frac{\alpha h}{2}\Big)\sinh\left(2\alpha h - \frac{\sigma}{\cosh\left(\frac{\alpha h}{2}\right)}\right). \tag{4.24}$$

The roots of $\hat{g}$ satisfy

$$\hat{\alpha}_k^h = \frac{1}{2}\left(ik\pi + \frac{\sigma}{\cosh\left(\frac{\hat{\alpha}_k^h h}{2}\right)}\right).$$

From the estimate (4.20) on α, we can give the following approximation

$$\left| \hat{\alpha}_k^h - \frac{1}{2}\left(ik\pi + \frac{\sigma}{\cos\left(\frac{k\pi h}{4}\right)} \right) \right| \leq Ch.$$

For each h, we define $K_h = \lfloor \frac{4}{h\pi} \arcsin(\delta) \rfloor$. We consider the rectangle R_h delimited by the lines $|\mathcal{R}e(\alpha)| = M$ and $|2\mathcal{I}m(\alpha)| = \pi((K_h - 1) + \epsilon)$, where $\epsilon < 1$ is a positive number. On its boundary, we can check that

$$|\hat{g}(\alpha, h)| \geq |\sin(\pi\epsilon)| - Ch.$$

Using (4.23), there exists h_0 such that for all $h < h_0$, on the boundary of R_h,

$$|g(\alpha, h) - \hat{g}(\alpha, h)| < |\hat{g}(\alpha, h)|.$$

Then for all $h < h_0$, the number of roots in R_h is precisely $2K_h - 1$.
We can go further in the description of the zeros of $g(., h)$. We define

$$\tilde{\alpha_k^h} = \frac{1}{2}\left(ik\pi + \frac{\sigma}{\cos\left(\frac{k\pi h}{4}\right)} \right).$$

Now we fix the rectangle R_k^h by $|2\mathcal{I}m(\alpha - \tilde{\alpha}_k^h)| = \pi\epsilon_1$ and $|\mathcal{R}e(\alpha - \tilde{\alpha}_k^h)| = \epsilon_2$. On the boundary of R_k^h, again we can check that

$$|\hat{g}(\alpha, h)| \geq \inf\{|\sin(\pi\epsilon_1)|, |\sinh(\epsilon_2)|\} - Ch.$$

Then it exists a constant C independent of k such that the conditions $|\epsilon_1| \geq Ch$ and $|\epsilon_2| \geq Ch$ are enough to prove that the following inequality holds on the boundary R_k^h :

$$|g(\alpha, h) - \hat{g}(\alpha, h)| < |\hat{g}(\alpha, h)|.$$

By Rouché's theorem, this establishes that $g(., h)$ has only one root α_k^h in R_k^h satisfying

$$|\alpha_k^h - \tilde{\alpha}_k^h| \leq Ch. \tag{4.25}$$

Back in the variable λ, it gives that for h small enough, each eigenvalue λ such that $|\mathcal{I}m(\lambda)h| \leq 2\delta$ is in one of the disks defined by

$$|\lambda - \hat{\lambda}_k^h| \leq Ch, \quad \hat{\lambda}_k^h = \frac{2i}{h}\sin(\frac{k\pi h}{4}) + \frac{\sigma}{2}.$$

□

The high frequencies. Here we will deal with the limit case $\delta = 1$.

Theorem 4.3. *For any $\epsilon > 0$, there exists h_ϵ such that for all $h < h_\epsilon$, the set of eigenvalues satisfying $|h\mathcal{I}m(\lambda_h) - 2| \leq \epsilon$ is non empty. The set of accumulation points of the abscissa $\mathcal{R}e(\lambda_h)$ for sequences λ_h satisfying $\lambda_h h \to 2i$ when $h \to 0$ is exactly $\{0, \sigma\}$.*

Proof. The first point comes from the fact that a set of accumulation points is closed. Indeed, from the previous theorem, taking $\epsilon > 0$ and setting $\delta = 1 - \epsilon/4$, there exists a sequence of eigenvalues λ_h such that $\mathcal{I}m(\lambda_h)h \to 2\delta > 2 - \epsilon$.

Now we assume we have a sequence of eigenvalues λ_h for the operator L_h, such that $\lambda_h h \to 2i$, and we analyze the behavior of their real parts a_h. For that purpose, we need to know precisely how $\lambda_h h$ is converging to $2i$. We assume that

$$\frac{\mathcal{I}m(\lambda_h)h}{2} = 1 - \epsilon(h) \tag{4.26}$$

with $\epsilon(h)$ a positive function of h continuous at zero, such that $\epsilon(0) = 0$. To simplify notations, we will skip the index h in the sequel.
Remark that the difficulty comes from the fact that $\lambda h/2 \to i$, which is precisely a point where Argsh is not holomorphic anymore. However, from the explicit form of Argsh, we may derive some estimates on α and β. Indeed, recall that:

$$\text{Argsh}(z) = \log(z + \sqrt{1 + z^2}) \quad ; \quad \cosh(z) = \sqrt{1 + \sinh(z)^2}.$$

Actually, it is sufficient to estimate these functions. Since

$$1 + \left(\frac{\lambda h}{2}\right)^2 = 2\epsilon(h) - \epsilon(h)^2 + \left(\frac{ah}{2}\right)^2 + i(1 - \epsilon(h))\frac{ah}{2},$$

we will need to distinguish several cases depending which is the dominant term.

The case $h = o(\epsilon(h))$: In that case, we get that

$$\cosh\left(\frac{\alpha h}{2}\right) = \sqrt{2\epsilon(h)} + o(\sqrt{\epsilon(h)}).$$

This also implies that

$$\mathcal{R}e\left(\frac{\alpha h}{2}\right) = \frac{1}{2}\log|z + \sqrt{1 + z^2}|^2 = \epsilon(h) + o(\epsilon(h)).$$

And the same estimates hold true for β.
It follows that $f(\alpha,\beta,h)$ defined in (4.16) cannot vanish. Indeed, our estimates imply that the real parts of both α and β blow up, which implies that

$$\begin{aligned}
&|\sinh(\alpha+\beta)| \simeq \exp(4\tfrac{\epsilon(h)}{h} + o(\tfrac{\epsilon(h)}{h})),\\
&|\sinh(\alpha-\beta)| \leq \exp(o(\tfrac{\epsilon(h)}{h})),\\
&|\cosh\left(\tfrac{\alpha h}{2}\right) + \cosh\left(\tfrac{\beta h}{2}\right)| \simeq \sqrt{2\epsilon(h)} + o(\sqrt{\epsilon(h)}),\\
&|\cosh\left(\tfrac{\alpha h}{2}\right) - \cosh\left(\tfrac{\beta h}{2}\right)| \leq o(\sqrt{\epsilon(h)}).
\end{aligned}$$

The case $\epsilon(h) = o(h)$: Under this assumption, we get

$$\cosh\left(\frac{\alpha h}{2}\right) = \sqrt{i\frac{ah}{2}} + o(\sqrt{h}), \quad \cosh\left(\frac{\beta h}{2}\right) = \sqrt{-i\frac{(\sigma-a)h}{2}} + o(\sqrt{h}).$$

Besides, using the explicit formula of the function Argsh, we obtain :

$$\mathcal{R}e\Big(\frac{\alpha h}{2}\Big) = \frac{\sqrt{ah}}{2} + o(\sqrt{h}), \quad \mathcal{R}e\Big(\frac{\beta h}{2}\Big) = -\frac{\sqrt{(\sigma-a)h}}{2} + o(\sqrt{h}).$$

But these estimates lead to

$$\begin{aligned}
&|\cosh\left(\tfrac{\alpha h}{2}\right) + \cosh\left(\tfrac{\beta h}{2}\right)| = \sqrt{\sigma h} + o(\sqrt{h}),\\
&|\cosh\left(\tfrac{\alpha h}{2}\right) - \cosh\left(\tfrac{\beta h}{2}\right)| = \sqrt{\sigma h} + o(\sqrt{h})
\end{aligned}$$

and

$$\begin{aligned}
&|\sinh(\alpha+\beta)| \simeq \exp(\tfrac{1}{2}|\sqrt{ah} - \sqrt{(\sigma-a)h}|),\\
&|\sinh(\alpha-\beta)| \simeq \exp(\tfrac{1}{2}\sqrt{ah} + \sqrt{(\sigma-a)h}).
\end{aligned}$$

Thus, if $f(\alpha,\beta,h) = 0$, f being as in (4.16), we need that $|\,|\sqrt{\sigma-a} - \sqrt{a}| - (\sqrt{\sigma-a} + \sqrt{a})| \to 0$, which implies $a \to 0$ or $a \to \sigma$.

The case where $\epsilon(h) = Kh$ follows from similar considerations and is left to the reader.

Summarizing, we deduce the existence of a sequence of eigenvalues such that $\lambda h \to 2i$, and hence whose real part is converging to zero or σ. To finish the analysis, we only have to prove that both 0 and σ are accumulation points. This assertion is obvious since the spectrum is symmetric around $\sigma/2$. $\square$

Theorems 4.2 and 4.3 fully explain Figure 1 for $h << \sigma << 1/h$, since they state, roughly speaking, that the eigenvalues λ are close to the line $\mathcal{R}e(\lambda) = \sigma/2$ except when their imaginary part is close to $\pm 2/h$, in which case, their real parts tend to 0 or σ.

To describe the behavior of the eigenvectors, we define the energies in the left and right intervals $(-1, 0)$ and $(0, 1)$, respectively :

$$\begin{cases} E_h^l = \frac{h}{4}|P_0|^2 + \frac{h}{2}\sum_{j=1}^{N}(|P_j|^2 + |V_j|^2), \\ E_h^r = \frac{h}{4}|P_0|^2 + \frac{h}{2}\sum_{j=-N+1}^{0}(|V_j|^2 + |P_{j-1}|^2). \end{cases} \tag{4.27}$$

Proposition 4.2 (Distribution of the energy). *Let $(\lambda_k^h)_h$ be a sequence of eigenvectors of L_h such that $h\mathcal{I}m(\lambda_k^h) \to 2$, and that $a_k^h = \mathcal{R}e(\lambda_k^h)$ converges to a. Then*

$$\frac{E_h^r(P_k^h, V_k^h)}{E_h^l(P_k^h, V_k^h)} \to_{h\to 0} \frac{a}{\sigma - a}. \tag{4.28}$$

In particular, there exists a sequence of high frequency eigenvectors whose energy is concentrated on the left interval $(-1, 0)$.

Proof. In view of (4.2), the solution $\exp(-\lambda_k^h t)(P_k^h, V_k^h)$ corresponding to the eigenvector (P_k^h, V_k^h) satisfies

$$\frac{dE_h}{dt}(t) = -2\mathcal{R}e(\lambda_k^h)E_h(t) = -2\sigma E_h^r(t).$$

The result follows. □

Remark: According to this result we have a new evidence of the lack of uniform exponential decay, as stated in Theorem 4.1. There this was proved by means of a wave packet construction, whereas here we have built concentrated eigenvectors.

4.3 Connections with the theory of stabilization.

In this subsection, we discuss the links between our analysis and the existing controllability and stabilization theory and reread our results in this context.

Let us consider the 1-d damped wave equation (1.6) on $(0, 1)$. The decay rate of the solutions of this damped wave equation has been analyzed in several articles: see [14], [11], [15] and [22] for the multi-dimensional case. The exponential decay rate was characterized as the minimum of the spectral abscissa and the minimal value of the damping potential along the rays of geometric optics (In 1-d, these two quantities coincide as shown in [14]). One of the main features of system (1.6) is that an overdamping phenomenon occurs, in the sense that increasing the damping potential does not necessarily increase the decay rate. This is not the case for the PML system since, as observed in Theorem 2.1 and 3.1 the decay rate is $I = \int_0^1 \sigma(x)\ dx$, and this is precisely what makes PML so efficient.

We may now investigate the same questions in the semi-discrete 1-d case on a regular mesh of size $h = 1/N$. Then the finite difference approximation of (1.6) gives :

$$\begin{cases} \partial_{tt}^2 u_j - \Delta_h u_j + 2a_j \partial_t u_j = 0 \\ u_{-N} = u_N = 0. \end{cases} \tag{4.29}$$

It was proved in [20], [25] and [27] that the energy of solutions of (4.29) does not decay exponentially uniformly with respect to the mesh size h. Actually, this lack of uniform exponential decay can be deduced from the construction given in Subsection 4.1. As pointed out in [18], this has also interesting consequences when analyzing the optimal choice of dampers in which one observes also a different behavior from the continuous to the discrete case. We claim that this lack of uniform exponential decay can also be seen at the level of the spectrum. If we set $v_j = u_j'$, the system takes the form:

$$\frac{d}{dt}(u_{-N+1}, \cdots, u_{N-1}, v_{-N+1} \cdots, v_{N-1})^* + A(u_{-N+1}, \cdots, v_{N-1})^* = 0,$$

where A is the following matrix:

$$A = \begin{pmatrix} 0 & -I_{2N-1} \\ \Delta_h & 2\text{diag } (a_{-N+1}, \cdots, a_{N-1}) \end{pmatrix}.$$

We have performed the spectral computation of this matrix for piecewise constant damping potentials vanishing in $(-1, 0)$ and taking a constant value a on $(0, 1)$. The spectrum exhibits a behavior which is very close to the one we have observed for the PML system (see Figure 2), except at the low frequencies, where we observe the so-called overdamping phenomenon, which is reminiscent of the continuous system.

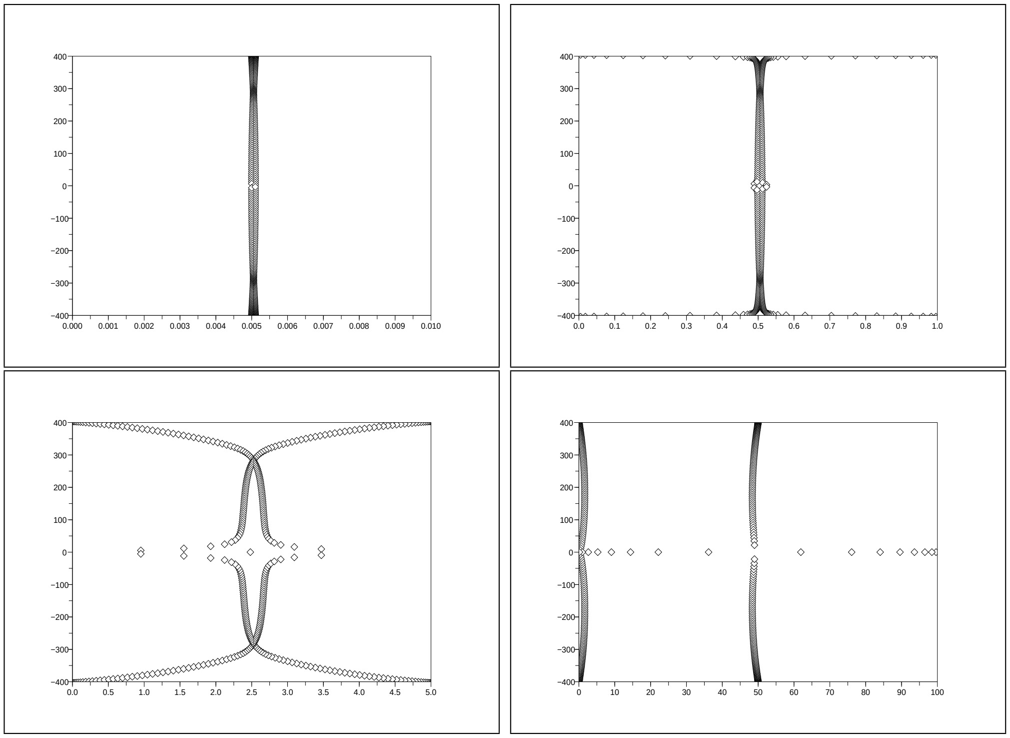

Figure 2: Eigenvalues of the semi-discrete damped wave equation (1.6) for $N = 200$ and various values of the damping potentials a : $a = 0.01$ in the upper left, $a = 1$ in the upper right, $a = 5$ on the bottom left, $a = 50$ on the bottom right.

5 A semi-discrete viscous PML.

The goal of this section is to propose a remedy to the defect of exponential decay proved in the previous section (see Theorem 4.1) for the semi-discrete approximation (1.14) of the PML system.

Along this section, we assume that $\sigma \in L^\infty((-1,1))$ is a positive function strictly positive on a subinterval (r_1, r_2) of $(0,1)$. To be more precise :

$$0 \le \sigma(x) \le M, \ x \ a.e. \in (-1,1), \quad \sigma(x) \ge m > 0, \ x \ a.e. \in (r_1, r_2). \tag{5.1}$$

For each h, we define σ_j^h as an approximation of σ in the points $x_j = jh$ satisfying

$$0 \le \sigma_j^h \le M, \ \forall j, \quad \sigma_j^h \ge m, \ \forall j \ s.t. \ jh \in (r_1, r_2). \tag{5.2}$$

To simplify the notations, we will write σ_j in the sequel, the dependence in h being clear within the context.
We propose to analyze system (1.16), which is a variant of the semi-discrete scheme (1.14), where a numerical viscosity term damping out the high frequencies has been added. Recall that, for system (1.16), the energy dissipation law (1.17) holds. In this way, the new semi-discrete problem satisfies the required property of uniform exponential decay:

Theorem 5.1. *Under the hypothesis* (5.2), *there exist two positive constants* C *and* μ *such that for all* $h > 0$, *for all initial data* (P_0^h, V_0^h), *the energy of the solution* (P, V) *of* (1.16) *satisfies*

$$E_h(t) \leq C\exp(-\mu t), \; t > 0. \tag{5.3}$$

Furthermore, we will see in Theorem 5.2 that one can choose the numerical viscosity such that this decay rate coincides with the continuous one I.

Proof. The method of proof we will use is classical in the theory of stabilization.
We claim that the energy of this viscous numerical approximation scheme (1.16) is exponentially decaying, uniformly in h, if and only if the following observability inequality holds for some time T and a constant C :

$$E_h(0) \leq C\Bigg(h\sum_j \int_0^T \sigma_j|P_j|^2 + \sigma_{j-1/2}|V_j|^2)\ dt \\ + h^3 \sum_j \int_0^T \Big[\Big(\frac{P_{j+1}-P_j}{h}\Big)^2 + \Big(\frac{V_{j+1}-V_j}{h}\Big)^2 \Big]\ dt \Bigg). \tag{5.4}$$

Indeed, combining the energy dissipation law (1.17) and (5.4), we easily deduce the existence of a constant $0 < \gamma < 1$ such that $E(T) \leq \gamma E(0)$. Iterating this inequality by means of the semi-group property, the exponential decay property (5.3) holds.
On the other hand, the proof of (5.4) for the solutions of (1.16) can be reduced to prove it for the corresponding conservative system, which turns out to be (1.14) with $\sigma = 0$. Indeed, since these two systems coincide up to a term which can be bounded by the right hand-side quantity in (5.4), it can be shown that the inequalities (5.4) for these systems are equivalent.

From now, we focus on the observability inequality (5.4) for the conservative system (1.14), that we prove using a multiplier method. Given $M > \sup\{1+r_1,\ 1-r_2\}$, where r_1 and r_2 are given by (5.1) and (5.2), we define a discrete function η^h satisfying the following properties:

$$\begin{array}{l} \eta^h_{-N} = \eta^h_N = 0, \quad |\eta^h_j| \leq M, \quad \forall j \\ \frac{\eta^h_{j+1}-\eta^h_j}{h} = 1 \quad \forall j \ s.t.\ jh \in (-1,1)\backslash(r_1,r_2) \\ |\frac{\eta^h_{j+1}-\eta^h_j}{h}| \leq \frac{3}{r_2-r_1} \quad \forall j. \end{array} \tag{5.5}$$

In the sequel we write η instead of η^h to simplify the notations. As before we use the multipliers $\eta_j(v_j+v_{j+1})$ and $\eta_j(p_j+p_{j-1})$ and we get :

$$\begin{aligned} & h\sum_{j=-N+1}^{N-1} \eta_j[p_j(T)v_j(T)-p_j(0)v_j(0)]+\Big(\frac{\eta_j+\eta_{j+1}}{2}\Big)[p_j(T)v_{j+1}(T)-p_j(0)v_{j+1}(0)] \\ & -h\sum_{j=-N+1}^{N}\int_0^T \Big(\frac{\eta_{j+1}-\eta_j}{h}\Big)|v_j|^2\,dt - h\sum_{j=-N+1}^{N-1}\int_0^T\Big(\frac{\eta_j-\eta_{j-1}}{h}\Big)|p_j|^2\,dt \\ & \qquad + h^2\sum_{j=-N+1}^{N-1}\int_0^T\Big(\frac{\eta_{j+1}-\eta_j}{2h}\Big)(p_j\partial_t v_{j+1}-\partial_t p_j v_{j+1})\,dt = 0. \end{aligned} \tag{5.6}$$

The classical inequality $2ab \leq a^2+b^2$ and the conservation of the energy allows us to bound the time boundary term by $4ME_h(0)$.
The only term in which numerical viscosity is needed is the last one:

$$A = \frac{h^2}{2}\sum_{j=-N+1}^{N-1} |\partial_t v_{j+1}p_j - \partial_t p_j v_{j+1}|$$

But, for any $\alpha > 0$, we have $2|ab| \leq \alpha a^2 + \frac{1}{\alpha}b^2$. Using the conservative system (1.14), we get:

$$\begin{aligned} |A| \ \leq\ & \alpha\frac{h}{4}\sum_{j=-N+1}^{N}|p_j|^2+|v_j|^2 \\ & +\frac{1}{4\alpha}h^3\sum_{j=-N+1}^{N-1}\Big[\Big(\frac{p_{j+1}-p_j}{h}\Big)^2+\Big(\frac{v_{j+1}-v_j}{h}\Big)^2\Big]. \end{aligned}$$

Combining these inequalities we get

$$h \sum_{j=-N+1}^{N} \int_0^T \Big(\frac{\eta_{j+1}-\eta_j}{h}\Big)|v_j|^2 \, dt + h \sum_{j=-N+1}^{N-1} \int_0^T \Big(\frac{\eta_j-\eta_{j-1}}{h}\Big)|p_j|^2 \, dt$$

$$\le (4M + \frac{1}{2}\alpha T)E_h(0) + C_\alpha \int_0^T h^3 \sum_j \int_0^T \Big[\Big(\frac{p_{j+1}-p_j}{h}\Big)^2 + \Big(\frac{v_{j+1}-v_j}{h}\Big)^2\Big] \, dt.$$

But we remark that

$$2TE_h(0) - C\int_0^T \sum_j \sigma_j(|p_j|^2+|v_j|^2) \, dt$$

$$\le h \sum_{j=-N+1}^{N} \int_0^T \Big(\frac{\eta_{j+1}-\eta_j}{h}\Big)|v_j|^2 \, dt + h \sum_{j=-N+1}^{N-1} \int_0^T \Big(\frac{\eta_j-\eta_{j-1}}{h}\Big)|p_j|^2 \, dt,$$

and this establishes that

$$(T - 2M - \frac{1}{2}\alpha T)E_h(0) \le C\int_0^T \sum_j \sigma_j(|p_j|^2+|v_j|^2) \, dt$$

$$+ C_\alpha \int_0^T h^3 \sum_j \int_0^T \Big[\Big(\frac{p_{j+1}-p_j}{h}\Big)^2 + \Big(\frac{v_{j+1}-v_j}{h}\Big)^2\Big] \, dt.$$

This completes the proof of Theorem 5.1. Note that, by this method, we find that this observability inequality actually holds for any $T > 2\sup\{1 + r_1,\ 1 - r_2\}$ (r_1 and r_2 as in (5.1) and (5.2)), which corresponds precisely to the optimal characteristic time in the continuous setting. □

Unfortunately, the method of proof of Theorem 5.1 does not give a good estimate on the decay rate in terms of the parameters entering in the system. Since the system under consideration is finite dimensional, the decay rate of the energy is obviously given by the spectral abscissa. Therefore we have computed the eigenvalues of the system (1.16) in Figure 3 for damping potentials vanishing in $(-1,0)$ and taking the value σ in $(0,1)$. We observe that, first, at low frequencies, the numerical viscosity does not seem to change the spectrum, as one can check by comparing the figures with the ones obtained

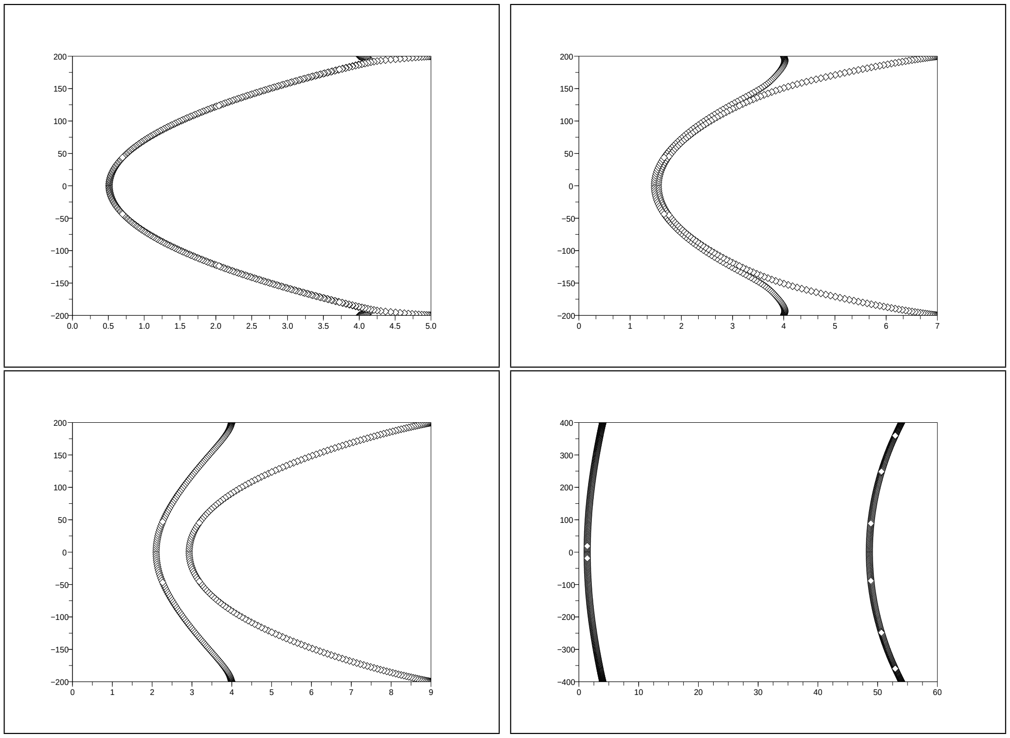

Figure 3: Eigenvalues of the viscous scheme (1.16) for $N = 100$ and various values of σ: $\sigma = 1$ on the upper left, $\sigma = 3$ on the upper right, $\sigma = 5$ on the bottom left and $\sigma = 50$ on the bottom right.

without the viscosity term (see Figure 1). This indicates that, as expected, the numerical viscosity does not modify the system at low frequencies. Second, at intermediate and high frequencies, one can see that the spectrum has a parabolic shape. Actually, one can easily check that, when $\sigma = 0$, the spectrum of (1.16) is exactly a parabolic curve $\mathcal{C}$. It is surprising to check that the spectrum given in Figure 3 fits quite well with the curve $\sigma/2 + \mathcal{C}$. Third, looking more closely at the high frequencies, the same phenomenon as before occurs, that is, two branches appear, corresponding to eigenvectors concentrated either in $(-1, 0)$, either in $(0, 1)$. But, thanks to the numerical viscosity, which efficiently damps them out, these two branches are away from zero. Moreover, it appears that the abscissa of the lowest branch is always 4. This precisely corresponds to the abscissa of the high frequency eigenvectors when $\sigma = 0$ in (1.16). In other words, this corresponds to waves concentrated in the undamped part $(-1,0)$, which are only dissipated by the additional viscosity.

In view of these spectral properties and with the purpose of recovering at the semi-discrete level the properties of the continuous PML system, it is natural to ask whether one can choose numerical viscosity coefficients α such that the decay rate μ_h of (1.16) as $h \to 0$ converges to I.
In the sequel, we address this issue. System (1.16) can be read as:

$$\partial_t(P,V) + (A_h + B_h)(P,V) = \alpha h^2 A_h^2(P,V), \tag{5.7}$$

where $A_h + B_h = L_h$, and

$$(A_h(P,V))_j = \Big(\frac{V_{j+1}-V_j}{h}, \frac{P_j - P_{j-1}}{h}\Big), \quad (B_h(P,V))_j = (\sigma_j^h P_j, \sigma_{j-1/2}^h V_j).$$

We need the following assumption:
There exists $\delta > 0$, such that for h small enough, the eigenvalues $\lambda_h = a_h + ib_h$ of $L_h = A_h + B_h$ with $|b_h| \leq \delta/h$ satisfy

$$a_h \geq I/2 + o_{h\to 0}(1). \tag{5.8}$$

Note that in the particular case where σ is constant, (5.8) holds for any $\delta < 2$ (see Theorem 4.2). We expect this property to hold for non constant σ as well, but this issue will be addressed elsewhere.

Theorem 5.2. *Fix $\alpha = \alpha_\delta = I/\delta$ in (5.7), with δ as in (5.8). Then, for all h small enough, there exists C_h such that the solutions (P,V) of (5.7) satisfy:*

$$E_h(t) \leq C_h E_h(0) \exp(-(I - o_{h\to 0}(1))t), \; t > 0. \tag{5.9}$$

Note that the constant C_h in (5.9) depends on h. In particular, we cannot guarantee C_h to be bounded.

Proof. Let us first consider the following modification of (5.7):

$$\partial_t(P,V) + (A_h + B_h)(P,V) = \alpha h^2 (A_h + B_h)^2(P,V), \tag{5.10}$$

It is straightforward to show that the eigenvalues $\mu(\alpha)$ of system (5.10) can be expressed in terms of $\mu(0)$, which coincide with the eigenvalues $\lambda = a + ib$ of system (4.12):

$$\mu(\alpha) = \lambda - \alpha h^2 \lambda^2, \quad \mathcal{R}e(\mu(\alpha)) = a + \alpha h^2(b^2 - a^2).$$

Under assumption (5.8), with the choice $\alpha = \alpha_\delta$, each eigenvalue $\mu(\alpha_\delta)$ satisfies

$$\mathcal{R}e(\mu(\alpha_\delta)) \geq I/2 - o_{h\to 0}(1). \tag{5.11}$$

Then, since the system is finite dimensional, there exists a constant C_h such that the solutions (P, V) of (5.10) satisfy

$$E_h(t) \leq C_h E_h(0)\exp(-(I - o_{h\to 0}(1))t), \ t > 0.$$

Now, we estimate the norm of the matrix $D_h = (A_h + B_h)^2 - A_h^2$:

$$D_h(P,V)_j = \left(2\sigma_j \frac{V_{j+1} - V_j}{h} + \sigma_j^2 P_j + \left(\frac{V_{j+1} + V_j}{2}\right)\left(\frac{\sigma_{j+1/2} - \sigma_{j-1/2}}{h}\right),\right.$$

$$\left.\left(\sigma_{j-1/2} + \frac{\sigma_j + \sigma_{j-1}}{2}\right)\frac{P_j - P_{j-1}}{h} + \sigma_{j-1/2}^2 V_j + \left(\frac{P_{j-1} + P_j}{2}\right)\frac{\sigma_{j+1/2} - \sigma_{j-3/2}}{2h}\right)$$

Note that systems (5.7) and (5.10) differ precisely by the term associated with $\alpha h^2 D_h$. Then, since

$$\|\alpha h^2 D_h\| \leq Ch, \tag{5.12}$$

a simple perturbation argument gives the result. Indeed, setting $L_h(\alpha) = L_h - \alpha h^2 L_h^2$, the solution $\psi = (P, V)$ of (5.7) is given by

$$\exp(tL_h(\alpha_0))\psi(t) = \psi(0) - \int_0^t \exp(sL_h(\alpha_0))\alpha_0 h^2 D_h \psi(s)\, ds.$$

Setting

$$f(t) = \exp(tI/2)\|\psi(t)\|,$$

this gives the equation

$$f(t) \leq f(0) + Ch\int_0^t f(s)\, ds,$$

and then Gronwall's lemma gives the result. □

6 Discussion and remarks.

In this paper we have presented a complete analysis of the decay of the energy of the 1-d PML system both at the continuous and semi-discrete level.

1. Analyzing the continuous system, we have shown that the two relevant parameters to describe the dissipation of the energy are $I = \int_0^1 \sigma(x)\, dx$ and $\|\theta\|_\infty$ as in (2.5). The exponential decay rate is exactly I while θ enters in the estimate of the multiplicative constant $C(\omega(\sigma))$ (see Theorem 3.1). This also confirms the interest in taking singular $\sigma \notin L^1$ as in [7]-[9]-[8].
2. An interesting question would be to investigate the decay of the energy in higher dimensions and to make precise which are the relevant parameters entering in it. According to [22], one could expect that the abscissa of the high frequency eigenvalues is related to the mean value of the damping along the rays of Geometric Optics. But the analysis of the low frequencies could be more complex, because of the possible overdamping phenomena, that could arise in the multi-dimensional case, although they have been excluded in 1-d.
3. At the semi-discrete level, we have studied in detail 1-d finite-difference approximation schemes. However, our analysis holds in a much more general setting. For instance, the same results holds for a finite element method. Besides, the construction we did in subsection 4.1 can also be done for semi-discrete multi-dimensional problems. Especially, the discrete energy will not decay uniformly on the mesh size, and a numerical viscosity will be needed to recover the property of exponential decay of the energy.
4. To the best of our knowledge, Theorem 5.2 is the first one where the uniform decay rate of the energy for an approximation scheme is proved to coincide with the decay rate of the energy of the continuous equation. This subject requires further investigation, for instance in the context of the damped wave equation. Moreover, this could be of significant importance in optimal design problems (see [18]), the goal being to design numerical schemes for which the optimal dampers converge to those of the continuous model. In view of Theorem 5.2 it is very likely that for a suitable viscous semi-discretization of the damped wave equation (1.6) this convergence property will hold.

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