Boundary observability for the finite-difference space semi-discretizations of the $2-d$ wave equation in the square

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Abstract

We extend our previous results on the boundary observability of the finite-difference space semidiscretizations of the $1-d$ wave equation to $2-d$ in the case of the square. As in the $1-d$ case, we prove that the constants on the boundary observability inequality blow-up as the mesh-size tends to zero. However, we prove a uniform observability inequality in a subspace of solutions generated by the low frequencies. The dimension of these subspaces grows as the mesh size tends to zero and eventually, in the limit, covers the whole energy space. Our result is sharp in the sense that the uniformity of the observability inequality is lost when the dimension of the subspaces grows faster. Our method of proof combines discrete multiplier techniques and Fourier series developments.

1 Introduction

Let Ω be the square $\Omega=(0,\pi)\times(0,\pi)$ of $I\!R^2$ and consider the wave equation with Dirichlet boundary conditions

$$\left\{\begin{array}{lll} u''-\Delta u=0 & \text{in} & Q=\Omega\times(0,T)\\ u=0 & \text{on} & \Sigma=\partial\Omega\times(0,T)\\ u(x,0)=u^0(x),\ u'(x,0)=u^1(x) & \text{in} & \Omega. \end{array}\right. \tag{1.1}$$

In (1.1) $'=\partial/\partial t$ denotes partial derivation with respect to time and Δ is the Laplacian in the space variable $x=(x_1,x_2)\in\Omega$.

Given $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ system (1.1) admits a unique solution $u\in C\left([0,T];H_0^1(\Omega)\right)\cap C^1\left([0,T];L^2(\Omega)\right)$. Moreover, the energy

$$E(t)=\frac{1}{2}\int_\Omega\left[\mid u_t(x,t)\mid^2+\mid\nabla u(x,t)\mid^2\right]dx \tag{1.2}$$

*Supported by grant PB96-0663 of the DGES (Spain) and ERB-FMRX-CT960033 of the EU.

remains constant, i.e.

$$E(t) = E(0), \quad \forall 0 < t < T. \tag{1.3}$$

Let Γ_0 denote a subset of the boundary of Ω constituted by two consecutive sides, for instance,

$$\Gamma_0 = \{(x_1, \pi) : x_1 \in (0, \pi)\} \cup \{(\pi, x_2) : x_2 \in (0, \pi)\}. \tag{1.4}$$

It is by now well-known (see [L]) that for $T > 2\sqrt{2}\pi$ there exists $C(T) > 0$ such that

$$E(0) \leq C(T) \int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \tag{1.5}$$

holds for every finite-energy solution of (1.1).

In (1.5), n denotes the outward unit normal to Ω, $\partial \cdot / \partial n$ the normal derivative and $d\sigma$ the surface measure.

Remark 1.1 (a) The lower bound $2\sqrt{2}\pi$ on the minimal observability time is sharp. On the other hand inequality (1.5) fails if in the right hand side of (1.5) insted of Γ_0 we only consider the energy concentrated on a strict subset of Γ_0. These two facts can be proved with the aid of the gaussian beams solutions by J. Ralston [R] as in [BLR].

(b) We refer to [Bu], [BLR] and [BuG] for sharp sufficient conditions in terms of geometric optics for the boundary observability of the wave equation in smooth domains.

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The goal of this paper is to analyze the boundary observability of some semi-discrete approximations in space of the wave equation (1.1).

Let us consider first finite-difference semi-discretizations. Given $J, K \in \mathbb{N}$ we set

$$h_1 = \frac{\pi}{J+1}, \; h_2 = \frac{\pi}{K+1}. \tag{1.6}$$

We denote by $u_{j,k}(t)$ the approximation of the solution u of (1.1) at the point $x_{j,k} = (jh_1, kh_2)$. The finite-difference semi-discretization of (1.1) is as follows:

$$\begin{cases} u''_{jk} - \frac{u_{j+1,k}+u_{j-1,k}-2u_{j,k}}{h_1^2} - \frac{u_{j,k+1}+u_{j,k-1}-2u_{j,k}}{h_2^2} = 0 \\ \qquad 0 < t < T, j = 1, \cdots, J; k = 1, \cdots, K \\ u_{j,k} = 0, \; 0 < t < T, j = 0, J+1; k = 0, K+1 \\ u_{j,k}(0) = u^0_{j,k}, u'_{j,k}(0) = u^1_{j,k}, \; j = 1, \cdots, J; k = 1, \cdots, K. \end{cases} \tag{1.7}$$

In (1.7), the first equation provides a 5−point approximation of the wave equation. The second equation takes account of the homogeneous Dirichlet boundary conditions. The last one provides the initial conditions guaranteeing the uniqueness of the solution. System (1.7) is a systen of JK linear differential equations of second order.

Let us now introduce the *discrete energy* associated with system (1.7):

$$E_{h_1,h_2}(t) = \frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \left[| \, u'_{jk}(t) \, |^2 + \left| \frac{u_{j+1,k}(t) - u_{j,k}(t)}{h_1} \right|^2 + \left| \frac{u_{j,k+1}(t) - u_{j,k}(t)}{h^2} \right|_2 \right]. \tag{1.8}$$

It is easy to see that the energy remains constant in time, i.e.,

$$E_{h_1,h_2}(t) = E_{h_1,h_2}(0), \quad \forall 0 < t < T \tag{1.9}$$

for every solution of (1.7).

We now observe that the discrete version of the energy observed on the boundary (i.e., of $\int_0^T \int_{\Gamma_0} | \, \partial u / \partial_n \, |^2 \, d\sigma dt$) is given by

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \sim \int_0^T \left[h_1 \sum_{j=1}^{J} \left| \frac{u_{j,K}(t)}{h_2} \right|^2 + h_2 \sum_{k=1}^{K} \left| \frac{u_{J,k}(t)}{h_1} \right|^2 \right] dt. \tag{1.10}$$

The discrete version of (1.5) is then an inequality of the form

$$E_{h_1,h_2}(0) \leq C_{h_1,h_2}(T) \int_0^T \left[h_1 \sum_{j=1}^{J} \left| \frac{u_{j,K}(t)}{h_2} \right|^2 + h_2 \sum_{k=1}^{K} \left| \frac{u_{J,K}(t)}{h_1} \right|^2 \right] dt. \tag{1.11}$$

As we shall see, (1.11) holds for any $T > 0$ and any $h_1, h_2 > 0$ as in (1.6), for a suitable constant $C_{h_1,h_2}(T) > 0$.

The problem we discuss here can be formulated as follows: *Assuming* $T > 2\sqrt{2\pi}$, *is the constant* $C_{h_1,h_2}(T)$ *in* (1.11) *uniformly bounded as* $h_1, h_2 \to 0$*? Or, in othe words, can we recover the observability inequality* (1.5) *as the limit as* $h_1, h_2 \to 0$ *of the inequalities* (1.11) *for the semi-discrete systems* (1.7)?

This problem is motivated by the numerical implementation of the boundary controllability property of the wave equation (see [G], [GL], [GLL], and [AL]).

As it was already observed in [G], the constants $C_{h_1,h_2}(T)$ in (1.11) necesarily blow-up as $h_1, h_2 \to 0$. This is due to the fact that spurious high frequency oscillations are present in the semi-discrete system (1.7). This result may be rigorously stated as follows:

Theorem 1.1 *For any* $T > 0$ *we have*

$$\sup_{u \text{ solution of } (1.17)} \left[\frac{E_{h_1,h_2}(0)}{\int_0^T \left[h_1 \sum_{j=1}^{J} \left| \frac{u_{j,K}(t)}{h_2} \right|^2 + h_2 \sum_{k=1}^{K} \left| \frac{u_{J,k}}{h_1} \right|^2 \right] dt} \right] \to \infty \text{ as } h_1, h_2 \to 0. \tag{1.12}$$

This result will be proved in section 2 through the spectral analysis of system (1.7).

In order to prove the positive counterpart of Theorem 1.1 we have to filter the high frequencies. To do that we consider the eigenvalue problem associated with (1.7):

$$\begin{cases} -\frac{\varphi_{j+1,k}+\varphi_{j-1,k}-2\varphi_{j,k}}{h_1^2}-\frac{\varphi_{j,k+1}-\varphi_{j,k-1}-2\varphi_{j,k}}{h_2^2}=\lambda\varphi_{j,k} \\ \qquad\qquad j=1,\cdots,J; k=1,\cdots,K \\ \varphi_{j,k}=0,\ j=0,J+1;\ k=0,\ K+1. \end{cases} \tag{1.13}$$

System (1.13) admits JK eigenvalues. The following is a sharp upper bound for the eigenvalues of (1.13):

$$\lambda \leq 4\left[\frac{1}{h_1^2}+\frac{1}{h_2^2}\right]. \tag{1.14}$$

As we shall see in section 2, (1.12) is due to solutions of (1.7) of the form $u=e^{\sqrt{\lambda}t}\varphi, \lambda$ being the largest eigenvalue of (1.13) and φ the corresponding eigenfunction. Indeed, as we shall see, the high frequency eigenfunctions of system (1.13) are such that the energy concentrated on the observed subset of the boundary is asymptotically smaller than the total energy.

In order to get uniform observability estimates we first observe that solutions of (1.7) can be developed in Fourier series of the form

$$u=\sum_{\lambda \text{ e.v. of } (13)}\left[a_\lambda^+ e^{i\sqrt{\lambda}t}+a_\lambda^- e^{-i\sqrt{\lambda}t}\right]\varphi_\lambda \tag{1.15}$$

where the sum runs over al eigenvalues of (1.13), $a_\lambda^\pm$ are complex coefficients and φ_λ are the eigenvectors of (1.13).

We then introduce the following classes of solutions of (1.7) in which the high frequencies have been truncated or filtered.

For any $0<\gamma\leq 4$ we set

$$\mathcal{C}_\gamma(h_1,h_2)=\left\{u \text{ solution of (1.7) of the form } u=\sum_{\lambda\leq\gamma\left[h_1^{-2}+h_2^{-2}\right]}\left[a_\lambda^+ e^{i\sqrt{\lambda}t}+a_\lambda^- e^{-i\sqrt{\lambda_k t}}\right]\varphi_\lambda\right\}. \tag{1.16}$$

Note that, according to the upper bound (1.14), when $\gamma=4, \mathcal{C}_\gamma\left(h_1,h_2\right)=\mathcal{C}_4\left(h_1,h_2\right)$ coincides with the space of all solutions of (1.16). However, when $0<\gamma<4$, solutions in the class $\mathcal{C}_\gamma\left(h_1,h_2\right)$ do not contain the contribution of the high frequencies $\lambda>\gamma\left(h_1^{-2}+h_2^{-2}\right)$ that have been truncated or filtered.

The following result asserts that, whatever $0<\gamma<4$ is, the uniform observability does not hold.

Theorem 1.2 *For any $T > 0$ and $0 < \gamma \leq 4$, there exist sequences $h_1, h_2 \to 0$ such that*

$$\sup_{u \in \mathcal{C}_\gamma(h_1,h_2)} \left[\frac{E_{h_1,h_2}(0)}{\int_0^T \left[h_1 \sum_{j=1}^{J} \left| \frac{u_{j,K}(t)}{h_2} \right|^2 + h_2 \sum_{k=1}^{k} \left| \frac{u_{J,k}}{h_1} \right|^2 \right] dt} \right] \to \infty. \tag{1.17}$$

Remark 1.2 Let us compare Theorem 1.2 with the $1-d$ results in [IZ1,2]. In $1-d$ there is one single parameter for the mesh size. Let us denote it by $h > 0$. The $1-d$ upper bound for the spectrum is then $\lambda \leq 4h^{-2}$. The analogue fo Theorem 1.1 was proved in [IZ1,2]. In other words, due to spurious high frequency vibrations the observability constant blows up as $h \to 0$ in $1-d$ too. However, in [IZ1,2] it was shown that if $0 < \gamma < 4$, in the class $\mathcal{C}_\gamma(h)$ of solutions of the semi-discrete wave equation in which the Fourier components vanish for $\lambda \geq \gamma^{-2}$, then, for $T > 0$ large enough, the observability constant remains bounded as $h \to 0$.

Theorem 1.2 shows that the $2-d$ analogue is not true. This is due to the fact that, even when $\lambda \leq \gamma \left(h_1^{-2} + h_2^{-2} \right)$ with $0 < \gamma < 4$, the eigenfunctions may present spurious oscillations in some space direction for high frequencies. As we shall see in Section 3 the result is sharp since when $h_1 = h_2 = h$, the uniform observability holds in the class $\mathcal{C}_\gamma(h_1, h_2)$ as soon as $\gamma > 2$.

Note however that Theorem 1.2 does not exclude the existence of other sequences $h_1, h_2 \to 0$ for which the supremum in (1.17) remains bounded.

Our proof of Theorem 1.2 requires $h_1, h_2 \to 0$ so that

$$\sup |h_2/h_1| < \sqrt{\gamma/(4-\gamma)},$$

or, by symmetry,

$$\sup |h_1/h_2| < \sqrt{\gamma/(4-\gamma)}.$$

■

The positive counterpart of Theorem 1.1 and 1.2 will be stated and proved in Section 3 since the description of the appropriate filtering of high frequencies requires a precise analysis of the spectrum of the system.

2 Spectral analysis: Non-uniform observability

The eigenvalues and eigenvectors of system (1.13) may be computed explicitly (see [IK], p. 459).

The eigenvalues of system (1.13) are as follows

$$\lambda^{p,q}(h_1, h_2) = 4 \left[\frac{1}{h_1^2} \sin^2 \left(\frac{ph_1}{2} \right) + \frac{1}{h_2^2} \sin^2 \left(\frac{qh_2}{2} \right) \right], p = 1, \cdots, J; q = 1, \cdots, K, \tag{2.1}$$

and the corresponding eigenvectors:

$$\varphi^{p,q} = \left(\varphi^{p,q}_{j,k}\right)_{\substack{1\le j\le J\\ 1\le k\le K}}, \varphi^{p,q}_{j,k} = \sin(jph_1)\sin(kqh_2). \tag{2.2}$$

Let us also recall what the spectrum of the continuous system is. The eigenvalue problem associated with (1.1) is

$$-\Delta\varphi = \lambda\varphi \text{ in } \Omega;\ \varphi = 0 \text{ on } \partial\Omega. \tag{2.3}$$

The eigenvalues of the continuous problem are

$$\lambda^{p,q} = p^2 + q^2 \tag{2.4}$$

and the corresponding eigenfunctions

$$\varphi^{p,q}(x_1, x_2) = \sin(px_1)\sin(qx_2). \tag{2.5}$$

The following properties are easy to check:

Proposition 2.1 *The following properties hold:*

(a) $\lambda^{p,q}(h_1, h_2) \to \lambda^{p,q}$ *as* $h_1, h_2 \to 0$ *for all* $p, q \in I\!N$.

(b) *The eigenvectors of the discrete system coincide with the eigenfunctions of the continuous one evaluated at the mesh points* $x_{j,k} = (jh_1, kh_2)$.

(c) $\lambda^{p,q}(h_1, h_2) \le \lambda^{p,q}$, $\forall(p, q)$, $\forall h_1, h_2 > 0$.

(d) $\lambda^{p,q}(h_1, h_2) \le 4\left[\frac{1}{h_1^2} + \frac{1}{h_2^2}\right]$, $\forall(p, q)$, $\forall h_1, h_2 > 0$.

(e) $\lambda^{p,q}(h_1, h_2)/\left(h_1^{-2} + h_2^{-2}\right) \to 4$ *for* $p = J, q = K$ *as* $h_1, h_2 \to \infty$.

(f) *For* (p, q) *fixed,*

$$\begin{cases} \lambda^{p,q}(h_1, h_2) \to \lambda^{p,q}(0, h_2) = 4\left[\frac{p^2}{4} + \frac{1}{h_2^2}\sin^2\left(\frac{qh_2}{2}\right)\right] & \text{as } h_1 \to 0, \\ \lambda^{p,q}(h_1, h_2) \to \lambda^{p,q}(h_1, 0) = 4\left[\frac{1}{h_1^2}\sin^2\left(\frac{qh_1}{2}\right) + \frac{q^2}{4}\right] & \text{as } h_2 \to 0. \end{cases} \tag{2.6}$$

Remark 2.1 The statement (a) guarantees the pointwise convergence of the spectrum of the discrete system towards the spectrum of the continuous one. Convergence (e) guarantees that the upper bound (d) (see also (1.14)) on the spectrum is sharp.

The statement (f) of the Proposition provides the pointwise limit of the spectrum when one of the mesh parameters tends to zero the other one being fixed. Obviously, the eigenvalues $\lambda^{p,q}(h_1, 0)$ correspond to the discretization of the continuous eigenvalue problem with respect to the variable x_1, i.e.,

$$\begin{cases} \varphi = (\varphi_1(x_2), \cdots, \varphi_J(x_2)) : \\ -\left[\frac{\varphi_{j+1}(x_2)+\varphi_{j-1}(x_2)-2\varphi_j(x_2)}{h_1^2}\right] - \varphi_j''(x_2) = \lambda\varphi_j(x_2), 0 < x_2 < \pi,\ j = 1, \cdots, J \\ \varphi_j \equiv 0,\ j = 0, J \\ \varphi_j(x_2) = 0,\ x_2 = 0, \pi,\ j = 0, \cdots, J. \end{cases} \tag{2.7}$$

In (2.7) we denote by $'$ derivation with respect to x_2. The eigenvalues $\lambda^{p,q}(0, h_2)$ correspond to the semi-discrete problem in which the Laplacian is discretized in the variable x_2 but not with respect to x_1.

■

When proving Theorems 1.1 and 1.2 the following identity from [IZ1,2] will be useful.

Let us denote by ψ^ℓ the vector

$$\psi^\ell = \left(\psi_1^\ell, \cdots, \psi_N^\ell\right); \psi_j^\ell = \sin(j\ell h) \tag{2.8}$$

with $N + 1 = 1/h$, for $\ell = 1, \cdots, N$.

The following identity holds:

Lemma 2.1 ([IZ1,2])

For any $h > 0$ such that $N = 1/h - 1 \in \mathbb{N}$ it follows that

$$\frac{4}{h}\sin^2\left(\frac{h\ell}{2}\right)\sum_{j=1}^{N}\left|\psi_j^\ell\right|^2 = \sum_{j=0}^{N}\left|\frac{\psi_{j+1}^\ell - \psi_j^\ell}{h}\right|^2 = \frac{\pi}{2\left(1-\sin^2(h\ell/2)\right)}\left|\frac{\psi_N^\ell}{h}\right|^2 \tag{2.9}$$

for all $\ell = 1, \cdots, N$.

Remark 2.2 Identity (2.10) provides the ratio between the total energy of the eigenvectors of the $1 - d$ semi-discrete wave equation and the energy concentrated on the extreme $x = 1$, since the eigenvectors are of the form (2.8)-(2.9) and the corresponding eigenvalue is

$$\mu^\ell(h) = \frac{4}{h^2}\sin^2\left(\frac{h\ell}{2}\right).$$

■

Remark 2.3 Note that the following holds as a consequence of (2.9):

$$h\sum_{j=1}^{N}\left|\psi_j^\ell\right|^2 = \frac{\pi\sin^2(\ell N h)}{2\sin^2(h\ell)} = \frac{\pi\sin^2(\ell(\pi-h))}{2\sin^2(h\ell)} = \frac{\pi}{2}. \tag{2.10}$$

■

Observe that the $2 - d$ eigenvectors in (2.2) are products of vectors of the form (2.8). Thus identity (2.10) allows us to stablish the corresponding $2 - d$ observability identity.

Proposition 2.2 *Let $\varphi^{p,q}(h_1, h_2)$ be the eigenvector of* (1.13) *with indexes $(p, q) \in \{1, \cdots, J\} \times \{1, \cdots, K\}$ and $h_1, h_2 > 0$ as in* (2.2)*. Then*

$$h_1 h_2 \sum_{j=0}^{J}\sum_{k=0}^{K}\left[\left|\frac{\varphi_{j+1,k} - \varphi_{j,k}}{h_1}\right|^2 + \left|\frac{\varphi_{j,k+1} - \varphi_{j,k}}{h_2}\right|^2\right] \tag{2.11}$$

$$= \frac{\pi}{2\left(1-\sin^2(ph_1/2)\right)} h_2 \sum_{k=0}^{K}\left|\frac{\varphi_{J,k}}{h_1}\right|^2 + \frac{\pi}{2\left(1-\sin^2(qh_2/2)\right)} h_1 \sum_{j=0}^{J}\left|\frac{\varphi_{j,K}}{h_2}\right|^2.$$

Remark 2.4 In identity (2.11) we have avoided the superscripts (p,q) of φ to simplify the notation.

■

Proof of Proposition 2.2. According to (2.2) we have $\varphi_{j,k} = \sin(jph_1)\sin(kqh_2)$. Then, in view of (2.9)

$$h_1 h_2 \sum_{j=0}^{J} \left| \frac{\varphi_{j+1,k} - \varphi_{j,k}}{h_1} \right|^2 = h_2 \frac{\pi}{2\left(1-\sin^2\left(ph_1/2\right)\right)} \left| \frac{\varphi_{J,k}}{h_1} \right|^2$$

and

$$h_1 h_2 \sum_{k=0}^{K} \left| \frac{\varphi_{j,k+1} - \varphi_{j,k}}{h_2} \right|^2 = h_1 \frac{\pi}{2\left(1-\sin^2\left(qh_2/2\right)\right)} \left| \frac{\varphi_{j,K}}{h_2} \right|^2 .$$

Therefore

$$\begin{aligned} h_1 h_2 \sum_{j=0}^{J} \sum_{k=0}^{K} \left[\left| \frac{\varphi_{j+1,k} - \varphi_{j,k}}{h_1} \right|^2 + \left| \frac{\varphi_{j,k+1} - \varphi_{j,k}}{h_2} \right|^2 \right] &= \frac{\pi}{2\left(1-\sin^2\left(ph_1/2\right)\right)} h_2 \sum_{k=0}^{K} \left| \frac{\varphi_{J,k}}{h_1} \right|^2 \\ &+ \frac{\pi}{2\left(1-\sin^2\left(qh_2/2\right)\right)} h_1 \sum_{j=0}^{J} \left| \frac{\varphi_{j,K}}{h_2} \right|^2 . \end{aligned}$$

■

In view of identity (2.11), Theorems 1.1 and 1.2 are easy to prove.

Proof of Theorem 1.1. For, given (p,q) and $h_1, h_2 > 0$ we consider the solution of (1.7) in separated variables associated to the eigenfunction $\varphi^{p,q}\left(h_1,h_2\right)$. We have

$$u = \cos\left(\sqrt{\lambda^{p,q}\left(h_1,h_2\right)t}\right) \varphi^{p,q}\left(h_1,h_2\right). \tag{2.12}$$

The initial energy $E_{h_1 h_2}(0)$ can be computed easily with the aid of identity (2.11):

$$\begin{aligned} E_{h_1,h_2}(0) &= \frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \left[\left| \frac{\varphi_{j+1,k} - \varphi_{j,k}}{h_1} \right|^2 + \left| \frac{\varphi_{j,k+1} - \varphi_{j,k}}{h_2} \right|^2 \right] \\ &= \frac{\pi}{4\left(1-\sin^2\left(ph_1/2\right)\right)} h_2 \sum_{k=0}^{K} \left| \frac{\varphi_{J,k}}{h_1} \right|^2 + \frac{\pi}{4\left(1-\sin^2\left(qh_2/2\right)\right)} h_1 \sum_{j=0}^{J} \left| \frac{\varphi_{j,K}}{h_2} \right|^2 . \end{aligned} \tag{2.13}$$

On the other hand, the energy concentrated on the boundary is given by

$$\int_0^T \cos^2\left(\sqrt{\lambda} t\right) dt \left[h_1 \sum_{j=1}^{J} \left| \frac{\varphi_{j,K}}{h_2} \right|^2 + h_2 \sum_{k=1}^{K} \left| \frac{\varphi_{J,k}}{h_1} \right|^2 \right]. \tag{2.14}$$

Therefore

$$
\begin{aligned}
Q(h_1,h_2) &= \frac{E_{h_1,h_2}(0)}{\displaystyle\int_0^T \left[h_1 \sum_{j=1}^{J} \left| \frac{u_{j,K}(t)}{h_2} \right|^2 + h_2 \sum_{k=1}^{K} \left| \frac{u_{J,k}(t)}{h_1} \right|^2 \right] dt} \\
&= \frac{\displaystyle\frac{\pi}{4\left(1-\sin^2(qh_2/2)\right)} h_1 \sum_{j=0}^{J} \left| \frac{\varphi_{j,K}}{h_2} \right|^2 + \frac{\pi}{4\left(1-\sin^2(ph_1/2)\right)} h_2 \sum_{k=0}^{K} \left| \frac{\varphi_{J,k}}{h_1} \right|^2}{\displaystyle\int_0^T \cos^2\left(\sqrt{\lambda} t\right) dt \left[h_1 \sum_{j=1}^{J} \left| \frac{\varphi_{j,k}}{h_2} \right|^2 + h_2 \sum_{k=0}^{K} \left| \frac{\varphi_{J,k}}{h_1} \right|^2 \right]}
\end{aligned}
\tag{2.15}
$$

To prove Theorem 1.1 we take $p = J$, $q = K$ in the quotient (2.15), i.e. we consider the solution u associated to the largest eigenvalue. Let us now analyze the limit of the quotient $Q(h_1,h_2)$ as $h_1, h_2 \to 0$, i.e. $p = J, q = K \to \infty$.

Taking into account $\lambda^{p,q}(h_1,h_2) \to \infty$ it is easy to see that

$$
\int_0^T \cos^2\left(\sqrt{\lambda} t\right) dt \to \frac{T}{2}. \tag{2.16}
$$

On the other hand

$$
\lim_{h_2 \to 0} \frac{\pi}{4\left(1-\sin^2(qh_2/2)\right)} = \frac{\pi}{4} \lim_{h_2 \to 0} \left[\frac{1}{1-\sin^2(\pi/2 - h_2/2)} \right] = \infty.
$$

In a similar way, we deduce that $\pi/4\left(1-\sin^2(ph_1/2)\right) \to \infty$ as $h_1 \to 0$.

In view of this, it is immediate to see that $Q(h_1,h_2) \to \infty$ as $h_1, h_2 \to 0$. This concludes the proof of Theorem 1.1.

Remark 2.5 It is clear that the method of proof of Theorem 1.1 fails when p, q are restricted to satisfy

$$
p \leq \delta J,\ q \leq \delta K
$$

with $0 < \delta < 1$.

Indeed, in that case, the quotient in (2.15) can be easily bounded above by

$$
\frac{\max\left[\frac{\pi}{4\cos^2(\delta(\pi-h_2)/2)}, \frac{\pi}{4\cos^2(\delta(\pi-h_1)/2)}\right]}{\displaystyle\int_0^T \cos^2\left(\sqrt{\lambda} t\right) dt} \sim \frac{\pi}{4\cos^2(\delta\pi/2)\left(T/2 - 1/4\sqrt{\lambda}\right)}
$$

as $h_1, h_2 \to 0$.

Note that the factor $\cos^2(\delta\pi/2)$ in the denominator tends to zero as $\delta \to 1$, which explains the fact that Theorem 1.1 holds.

■

Proof of Theorem 1.2. We choose $p = J$. The value of q is chosen so that the eigenvalue $\lambda^{J,q}(h_1, h_2)$ is such that the solution u as in (2.12) is in the class $\mathcal{C}_\gamma(h_1, h_2)$. For that we need

$$\begin{aligned}
\lambda^{J,q}(h_1,h_2) &= 4\left[\frac{1}{h_1^2}\sin^2\left(\frac{Jh_1}{2}\right)+\frac{1}{h_2^2}\sin^2\left(\frac{qh_2}{2}\right)\right] \\
&= 4\left[\frac{1}{h_1^2}\sin^2\left(\frac{\pi}{2}-\frac{h_1}{2}\right)+\frac{1}{h_2^2}\sin^2\left(\frac{qh_2}{2}\right)\right] \\
&= 4\left[\frac{\cos^2(h_1/2)}{h_1^2}+\frac{1}{h_2^2}\sin^2(qh_2/2)\right] \le \gamma\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right),
\end{aligned} \tag{2.17}$$

or, equivalently,

$$4\cos^2(h_1/2)-\gamma \le \left[\gamma - 4\sin^2(qh_2/2)\right]|h_1/h_2|^2. \tag{2.18}$$

Let us choose h_1, h_2 such that

$$\sup\left|\frac{h_2}{h_1}\right| < \sqrt{\frac{\gamma}{4-\gamma}}. \tag{2.19}$$

Of course this can be done by taking $h_1 \to 0$ and then $h_2 = ah_1$ with $a < \sqrt{4/(4-\gamma)}$. Under assumption (2.19) it is clear that taking

$$q \le \delta/h_2 \tag{2.20}$$

with $0 < \delta < \pi$ small enough, (2.18) holds.

Let us now pass to the limit in the ratio $Q(h_1, h_2)$. It is easy to see that (2.16) holds. On the other hand, in view of (2.20),

$$\lim_{h_2\to 0}\frac{\pi}{4\left(1-\sin^2(qh_2/2)\right)} = \lim_{h_2\to 0}\frac{\pi}{4\cos^2(qh_2/2)} < \infty, \tag{2.21}$$

while

$$\begin{aligned}
\frac{\pi}{4\left(1-\sin^2(Jh_1/2)\right)} &= \frac{\pi}{4\cos^2(Jh_1/2)} = \frac{\pi}{4\cos^2(\pi/2-h_1/2)} = \frac{\pi}{4\sin^2(h_1/2)} \\
&\sim \pi/h_1^2 \to \infty \text{ as } h_1 \to 0.
\end{aligned} \tag{2.22}$$

In view of (2.21) and (2.22), to conclude that

$$Q(h_1,h_2) \to \infty \text{ as } h_1, h_2 \to 0, \tag{2.23}$$

it is sufficient to show that

$$h_1^2\left[\frac{h_1\sum_{j=0}^{J}\left|\frac{\varphi_{j,k}}{h_2}\right|^2}{h_2\sum_{k=0}^{K}\left|\frac{\varphi_{J,k}}{h_1}\right|^2}\right] \to 0 \text{ as } h_1, h_2 \to 0. \tag{2.24}$$

In view of the form of the eigenvectors (2.2) and identity (2.9) it follows that

$$h_1 \sum_{j=0}^{J} \left| \frac{\varphi_j, K}{h_2} \right|^2 = \frac{\sin^2(Kqh_2)}{h_2^2} h_1 \sum_{j=0}^{J} \sin^2(jJh_1) \tag{2.25}$$

$$= \frac{\pi \sin^2(Kqh_2)}{2h_2^2} = \frac{\pi \sin^2\left((\pi - h_2)q\right)}{2h_2^2}.$$

On the other hand,

$$h_2 \sum_{k=0}^{K} \left| \frac{\varphi_{J,k}}{h_1} \right|^2 = \frac{\pi \sin^2\left(J^2 h_1\right)}{2h_1^2} = \frac{\pi \sin^2(h_1)}{2h_1^2}. \tag{2.26}$$

Combining (2.25) and (2.26) we get

$$h_1^2 \left[\frac{h_1 \sum_{j=0}^{J} |\varphi_{j,K}/h_2|^2}{h_2 \sum_{k=0}^{K} |\varphi_{J,k}/h_1|^2} \right] = \frac{h_1^4 \sin^2\left((\pi - h_2)q\right)}{h_2^2 \sin^2(h_1)} \tag{2.27}$$

$$\sim \frac{h_1^2}{h_2^2} \sin^2\left((\pi - h_2)q\right) = \left|\frac{h_1}{h_2}\right|^2 \sin^2(qh_2) \to 0 \text{ as } h_1, h_2 \to 0$$

provided

$$\sup \left| \frac{h_1}{h_2} \right| < \infty \tag{2.28}$$

and q is fixed independent of h_2.

Note that (2.19) and (2.28) are perfectly compatible. As we said above, it is sufficient to take $h_1 \to 0$ and $h_2 = ah_1$ with $a < \sqrt{\gamma/(4-\gamma)}$.

■

Remark 2.6 Our proof works when $q = o\,(1/h_2)$.

■

Remark 2.7 Our proof of Theorem 1.2 works under the condition

$$\sup \left| \frac{h_2}{h_1} \right| < \sqrt{\frac{\gamma}{4-\gamma}} \tag{2.29}$$

or, the symmetric one,

$$\sup \left| \frac{h_1}{h_2} \right| < \sqrt{\frac{\gamma}{4-\gamma}}. \tag{2.30}$$

Condition (2.29) coincides with (2.19). By, symmetry, taking $q = K$ and $p = 0(1/h_1)$, the proof of Theorem 1.2 works under assumption (2.30) as well.

Note that conditions (2.29) and (2.30) are sharp. Indeed, as indicated in Remark 2.4, to prove Theorem 1.2 for solutions generated by a single eigenvector we need to take $p = J$ (resp. $q = K$). Then (2.29) (resp. (2.30)) is a necessary condition for the existence of eigenvalues in the range

$$\lambda \leq \gamma \left(h_1^{-2} + h_2^{-2} \right).$$

Observe that, if we take the same net spacing in x_1 and x_2, i.e., $h_1 = h_2 = h$, Theorem 1.2 only applies when $\gamma > 2$.

■

3 Uniform observability estimates

This section is devoted to prove uniform observability estimates in classes of solutions in which the high frequencies have been filtered or truncated. Instead of applying directly $2-d$ discrete multiplier techniques we employ discrete Fourier series developments and $1-d$ discrete multipliers. First we prove some basic identities that are valid for all solutions of (1.7). Then we derive the uniform observability estimates by a suitable filtering of the high frequencies.

We develop solutions of (1.7) in Fourier series

$$u = \sum_{p=1}^{J} \sum_{q=1}^{K} \left(a_{p,q} e^{i\mu^{p,q}t} + b_{p,q} e^{-i\mu^{p,q}t} \right) \varphi^{p,q} \tag{3.1}$$

where

$$\mu^{p,q} = \sqrt{\lambda^{p,q}}. \tag{3.2}$$

In (3.1) we omit the dependence on h_1, h_2 to simplify the notation. When this becomes important we shall also use the subscript $\overrightarrow{h} = (h_1, h_2) : \varphi^{p,q} = \varphi^{p,q}_{\overrightarrow{h}}, \mu^{p,q} = \mu^{p,q}_{\overrightarrow{h}}, \ldots$

In view of the form of the eigenvectors (2.2) the solution u may be decomposed as

$$u = \sum_{p=1}^{J} \psi^p v^p \tag{3.3}$$

with

$$v^p = \sum_{q=1}^{K} \left(a_{p,q} e^{i\mu^{p,q}t} + b_{p,q} e^{-i\mu^{p,q}t} \right) \xi^q \tag{3.4}$$

and

$$\psi^p = (\psi_1^p, \cdots, \psi_J^p) \, ; \, \psi_j^p = \sin(pjh_1), \tag{3.5}$$

$$\xi^q = \left(\xi_1^1, \cdots, \xi_K^q \right) ; \, \xi_k^q = \sin(qkh_2). \tag{3.6}$$

The solution u of (1.7) can also be decomposed as

$$u = \sum_{q=1}^{K} \xi^q w^q \tag{3.7}$$

with

$$w^q = \sum_{p=1}^{J} \left(a_{p,q} e^{i\mu^{p,q}t} + b_{p,q} e^{-i\mu^{p,q}t} \right) \psi p. \tag{3.8}$$

Observe that for any $p = 1, \cdots, J$, $v^p = v$ solves the $1-d$ semi-discrete wave equation

$$\begin{cases} v_k'' - \left[\frac{v_{k+1}+v_{k-1}-2v_k}{h_2^2}\right] + \alpha^p v_k = 0,\ 0 < t < T,\ k = 0, \cdots, K \\ v_0 = v_{K+1} = 0,\ 0 < t < T, \end{cases} \tag{3.9}$$

with

$$\alpha^p = \frac{4}{h_1^2} \sin^2 \left(\frac{ph_1}{2} \right). \tag{3.10}$$

On the other hand, $w = w^q$ satisfies

$$\begin{cases} w_j'' - \left[\frac{w_{j+1}+w_{j-1}-2w_j}{h_1^2}\right] + \beta^q w_j = 0,\ 0 < t < T,\ j = 1, \cdots, J \\ w_0 = w_{J+1} = 0,\ 0 < t < T, \end{cases} \tag{3.11}$$

with

$$\beta^q = \frac{4}{h_2^2} \sin^2 \left(\frac{qh_2}{2} \right). \tag{3.12}$$

The energy

$$F(t) = \frac{1}{2} \sum_{k=0}^{K} \left[\left|v_k'\right|^2 + \left| \frac{v_{k+1} - v_k}{h_2} \right|^2 + \alpha^p \left|v_k\right|^2 \right] \tag{3.13}$$

is conserved for solutions of (3.9). More precisely,

$$F(t) = F(0),\ \forall 0 < t < T. \tag{3.14}$$

The conserved energy for solutions of (3.11) is given by

$$G(t) = \frac{1}{2} \sum_{j=0}^{J} \left[\left|w_j'\right|^2 + \left| \frac{w_{j+1} - w_j}{h_1} \right|^2 + \beta^q \left|w_j\right|^2 \right], \tag{3.15}$$

i.e.

$$G(t) = G(0),\ \forall 0 < t < T. \tag{3.16}$$

On the other hand, the energy conservation properties (3.14) and (3.16) for the $1-d$ systems (3.9) and (3.11) and the orthogonality properties

$$\sum_{j=1}^{J} \psi_j^p \psi_j' = \sum_{j=0}^{J} \left(\psi_{j+1}^p - \psi_j^p \right) \left(\psi_{j+1}^{p'} - \psi_j^{p'} \right) = 0 \tag{3.17}$$

for $p \neq p'$ and

$$\sum_{k=1}^{K} \xi_k^q \xi_k^{q'} = \sum_{k=0}^{K} \left(\xi_{k+1}^q - \xi_k^q\right)\left(\xi_{k+1}^{q'} - \xi_k^{q'}\right) = 0 \tag{3.18}$$

for $q \neq q'$ imply the conservation property (1.9) for the energy E of solutions of the $2-d$ system (1.7).

The following identities hold:

Lemma 3.1 *For any solution v of* (3.9) *the following identity holds:*

$$\frac{h_2}{2}\sum_{k=0}^{K}\int_0^T \left[\left|_k'\right|^2 + \left|\frac{v_{k+1}-v_k}{h_2}\right|^2 - \alpha^p v_k v_{k+1}\right] dt + \tag{3.19}$$

$$-\frac{h_2}{4}\sum\int_0^T \left|v_k' - v_{k+1}'\right|^2 dt + X_1(t)\Big|_0^T = \frac{\pi}{2}\int_0^T \left|\frac{v_K}{h_2}\right|^2 dt$$

with

$$X_1(t) = h_2 \sum_{k=0}^{K} k\left(\frac{v_{k+1}-v_{k-1}}{2}\right) v_k'. \tag{3.20}$$

In a similar way any solution w of (3.11) *satisfies*

$$\frac{h_1}{2}\sum_{j=0}^{J}\int_0^T \left[\left|w_j'\right|^2 + \left|\frac{w_{j+1}-w_j}{h_1}\right|^2 - \beta^q w_j w_{j+1}\right] dt + \tag{3.21}$$

$$-\frac{h_1}{4}\sum_{j=0}^{J}\int_0^T \left|w_j' - w_{j+1}'\right|^2 dt + X_2(t)\Big|_0^T = \frac{\pi}{2}\int_0^T \left|\frac{w_J}{h_1}\right|^2 dt$$

with

$$X_2(t) = h_1 \sum_{j=0}^{J} j\left(\frac{w_{j+1}-w_{j-1}}{2}\right) w_j'. \tag{3.22}$$

Proof. We briefly sketch the proof of (3.19), since that of (3.21) is the same.

We proceed as in [IZ1,2] using the discrete multiplier $k\frac{(v_{k+1}-v_{k-1})}{2}$ (which is the discrete version of the classical multiplier $y\partial_y v$ for solutions of the continuous wave equation). Arguing as in [IZ1,2] we obtain

$$\frac{h_2}{2}\sum_{k=0}^{K}\int_0^T \left[v_k' v_{k+1}' + \left|\frac{v_{k+1}-v_k}{h_2}\right|^2 + \alpha^p v_k k\left(\frac{v_{k+1}-v_{k-1}}{2}\right)\right] dt + X(t)\Big|_0^T = 0. \tag{3.23}$$

We then observe that

$$\sum_{k=0}^{K} v_k k\left(\frac{v_{k+1}-v_{k-1}}{2}\right) = -\frac{1}{2}\sum_{k=0}^{K} v_k v_{k+1}. \tag{3.24}$$

On the other hand,

$$\sum_{k=0}^{K} v_k' v_{k+1}' = \sum_{k=0}^{K} |v_k'|^2 - \frac{1}{2}\sum_{k=0}^{K} |v_k' - v_{k+1}'|^2 . \tag{3.25}$$

Combining (3.23)-(3.25), identity (3.19) follows immediately.

■

We may now stablish the following identity for solutions of the $2-d$ system (1.7):

Lemma 3.2 *Every solution u of* (1.7) *satisfies*

$$\begin{aligned}
&\frac{h_1 h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}\left[2\left|u_{j,k}'\right|^2 + \left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2 + \left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2\right] dt \qquad (3.26)\\
&-\frac{h_1 h_2}{4}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\left|u_{j+1,k}'-u_{j,k}'\right|^2 + \left|u_{j,k+1}'-u_{j,k}'\right|^2\right] dt\\
&-\frac{h_1 h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\frac{(u_{j+1,k}-u_{j,k})}{h_1}\frac{(u_{j+1,k+1}-u_{j,k+1})}{h_1} + \frac{(u_{j,k+1}-u_{j,k})}{h_2}\frac{(u_{j+1,k+1}-u_{j+1,k})}{h_2}\right] dt\\
&+X(t)\Big|_0^T = \frac{\pi}{2}\left[h_2\sum_{k=0}^{K}\int_0^T\left|\frac{u_{J,k}}{h_1}\right|^2 dt + h_1\sum_{j=0}^{J}\int_0^T\left|\frac{u_{j,K}}{h_2}\right|^2 dt\right]
\end{aligned}$$

with

$$X(t) = h_1 h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\left[k\left(\frac{u_{j,k+1}-u_{j,k-1}}{2}\right)u_{j,k}' + j\left(\frac{u_{j+1,k}-u_{j-1,k}}{2}\right)u_{j,k}'\right]. \tag{3.27}$$

Proof. Combining identity (3.19), the decomposition (3.3) and the orthogonality properties (3.17) we deduce that

$$\begin{aligned}
&\frac{h_1 h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\left|u_{j,k}'\right|^2 + \left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2 - \frac{(u_{j+1,k}-u_{j,k})}{h_1}\frac{(u_{j+1,k+1}-u_{j,k+1})}{h_1}\right] dt\\
&-\frac{h_1 h_2}{4}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|u_{j,k}'-u_{j,k+1}'\right|^2 dt + \frac{h_1 h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K} k\left(u_{j,k+1}-u_{j,k-1}\right)u_{j,k}'\Big|_0^T\\
&= \frac{\pi h_1}{2}\sum_{j=0}^{J}\int_0^T\left|\frac{u_{j,k}}{h_2}\right|^2 dt. \qquad (3.28)
\end{aligned}$$

In a similar way, one can show that

$$\frac{h_1 h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\left|u_{j,k}'\right|^2 + \left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2 - \frac{(u_{j,k+1}-u_{j,k})}{h_2}\frac{(u_{j+1,k+1}-u_{j+1,k})}{h_2}\right] dt$$

$$
-\frac{h_1h_2}{4}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|u'_{j+1,k}-u'_{j,k}\right|^2dt+\frac{h_1h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}j\left(u_{j+1,k}-u_{j-1,k}\right)u'_{j,k}\Big|_0^T
$$

$$
=\frac{\pi h_2}{2}\sum_{k=0}^{K}\int_0^T\left|\frac{u_{J,k}}{h_1}\right|^2dt. \tag{3.29}
$$

Combining (3.28) and (3.29) we obtain (3.26).

∎

We now need the equipartition of energy identity for the $2-d$ system (1.7):

Lemma 3.3 *Every solution u of* (1.7) *satisfies*

$$
h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|u'_{j,k}\right|^2dt=Y(t)\Big|_0^T+h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2+\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2\right]dt \tag{3.30}
$$

with

$$
Y(t)=h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}u_{j,k}u'_{j,k}. \tag{3.31}
$$

Proof. We multiply in (1.7) by $u_{j,k}$, add for $j=1,\cdots,J$, $k=1,\cdots,K$ and integrate with respect to $t\in(0,T)$. Identity (3.30) follows immediately taking into account that

$$
h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T u''_{j,k}u_{j,k}dt=h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}u'_{j,k}u_{j,k}\Big|_0^T-h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|u'_{j,k}\right|^2dt,
$$

$$
h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left(\frac{u_{j+1,k}+u_{j-1,k}-2u_{j,k}}{h_1^2}\right)u_{j,k}dt=-h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|\frac{u_{j,k}-u_{j+1,k}}{h_1}\right|^2dt
$$

and

$$
h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\frac{(u_{j,k+1}+u_{j,k-1}-2u_{j,k})}{h_2^2}u_{j,k}dt=-h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2dt.
$$

∎

Combining the identities in Lemmas 3.2 and 3.3 and the conservation of energy E it follows that:

Lemma 3.4 *Every solution u of (1.7) satisfies*

$$
\begin{aligned}
&TE(0) - \frac{h_1 h_2}{4} \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left[\left| u'_{j,k+1} - u'_{j,k} \right|^2 + \left| u'_{j+1,k} - u'_{j,k} \right|^2 \right] dt \\
&+ \frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left[\left| \frac{u_{j+1,k} - u_{j,k}}{h_1} \right|^2 - \frac{(u_{j+1,k} - u_{j,k})(u_{j+1,k+1} - u_{j,k+1})}{h_1^2} \right. \\
&\left. + \left| \frac{u_{j,k+1} - u_{j,k}}{h_2} \right|^2 - \frac{(u_{j,k+1} - u_{j,k})(u_{j+1,k+1} - u_{j+1,k})}{h_2^2} \right] dt \\
&+ Z(t)\big|_0^T = \frac{\pi}{2} \left[h_1 \sum_{j=0}^{J} \int_0^T \left| \frac{u_{j,K}}{h_2} \right|^2 dt + h_2 \sum_{k=0}^{K} \int_0^T \left| \frac{u_{J,k}}{h_1} \right|^2 dt \right], \qquad (3.32)
\end{aligned}
$$

with

$$
Z(t) = X(t) + \frac{Y(t)}{2} = h_1 h_2 \sum_{j=0}^{J} \sum_{k=0}^{K} \left[k \frac{(u_{j,k+1} - u_{j,k})}{2} u'_{j,k} + j \frac{(u_{j+1,k} - u_{j,k})}{2} u'_{j,k} + \frac{1}{2} u_{j,k} u'_{j,k} \right]. \qquad (3.33)
$$

Remark 3.1 Identity (3.32) is the semi-discrete version of the identity

$$
TE(0) + \int_\Omega u_t \left(x \cdot \nabla u + \frac{u}{2} \right) \Big|_0^T = \frac{\pi}{2} \int_0^T \int_0^\pi \left[\left| \frac{\partial u}{\partial x_1}(\pi, x_2) \right|^2 + \left| \frac{\partial u}{\partial x_2}(x_1, \pi) \right|^2 \right] \qquad (3.34)
$$

that solutions of the continuous wave equation (1.1) satisfy. This identity may be proved using the multipliers $x \cdot \nabla u$ and u (see [K] and [L] for instance). In the case of the square $\Omega = (0, \pi) \times (0, \pi)$ it can also be obtained by means of Fourier decomposition and using $1 - d$ multipliers. This is the method we have employed in the semi-discrete case.

Note however that (3.32) contains two error terms:

$$
\frac{h_1 h_2}{4} \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left[\left| u'_{j,k+1} - u'_{j,k} \right|^2 + \left| u'_{j+1,k} - u'_{j,k} \right|^2 \right] dt
$$

and

$$
\begin{aligned}
&\frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left[\left| \frac{u_{j+1,k} - u_{j,k}}{h_1} \right|^2 - \frac{(u_{j+1,k} - u_{j,k})}{h_1} \frac{(u_{j+1,k+1} - u_{j,k+1})}{h_1} \right. \\
&\left. + \left| \frac{u_{j,k+1} - u_{j,k}}{h_2} \right|^2 - \frac{(u_{j,k+1} - u_{j,k})}{h_2} \frac{(u_{j+1,k+1} - u_{j+1,k})}{h_2} \right] dt.
\end{aligned}
$$

Following the developments in [IZ1,2] we shall get bounds on these error terms imposing upper bounds on the eigenvalues. Note however that upper bounds of the form $\lambda \leq \gamma \left(h_1^{-2} + h_2^{-2} \right)$ will not be sufficient. We shall rather impose upper bounds of the form $\lambda \leq 2\gamma \min \left(h_1^{-2}, h_2^{-2} \right)$.

As a consequence of Lemma 3.4 the following inequality holds:

Lemma 3.5 *Every solution u of* (1.7) *satisfies*

$$\begin{aligned} TEh_1, h_2(0) \quad &- \quad \frac{\Lambda}{4} \max\left(h_1^2, h_2^2\right) h_1 h_2 \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left|u'_{j,k}\right|^2 dt \\ &+ \quad Z(t)\Big|_0^T \leq \frac{\pi}{2} \left[h_1 \sum_{j=0}^{J} \int_0^T \left|\frac{u_{j,k}}{h_2}\right|^2 dt + h_2 \sum_{k=0}^{K} \int_0^T \left|\frac{u_{J,K}}{h_1}\right|^2 \right] \end{aligned} \tag{3.35}$$

where Λ is the largest eigenvalue involved in the Fourier development of u.

Proof. In view of Lemma 3.4, it is sufficiet to estimate the remainders mentioned in Remark 3.1.

Let us consider first

$$R_1 = \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left[\left|u'_{j,k+1} - u'_{j,k}\right|^2 + \left|u'_{j+1,k} - u'_{j,k}\right|^2 \right] dt. \tag{3.36}$$

We claim that

$$R_1 \leq \Lambda \max\left(h_1^2, h_2^2\right) \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left|u'_{jk}\right|^2. \tag{3.37}$$

Indeed, in fact, the following more general fact is true:

Lemma 3.6 *Let I be a family of indexes (p, q). Let*

$$\Lambda = \max_{(p,q)\in I} \lambda^{p,q}. \tag{3.38}$$

Then

$$\sum_{j=0}^{J} \sum_{k=0}^{K} \left[|\phi_{j,k+1} - \phi_{j,k}|^2 + |\phi_{j+1,k} - \phi_{j,k}|^2 \right] \leq \Lambda \max\left(h_1^2, h_2^2\right) \sum_{j=0}^{J} \sum_{k=0}^{K} |\phi_{j,k}|^2, \forall \phi \in \operatorname*{span}_{(p,q)\in I} \{\varphi^{p,q}\}. \tag{3.39}$$

Proof of Lemma 3.6. We first observe that

$$\sum_{j=0}^{J} \sum_{k=0}^{K} \left[\frac{|\varphi_{j,k+1} - \varphi_{j,k}|^2}{h_2^2} + \frac{|\varphi_{j+1,k} - \varphi_{j,k}|^2}{h_1^2} \right] = \lambda \sum_{j=0}^{J} \sum_{k=0}^{K} |\varphi_{j,k}|^2 \tag{3.40}$$

when φ is an eigenvector of (1.13) with eigenvalue λ

On the other hand, if φ and ψ are eigenvectors with non-equal indexes $(p, q) \neq (p', q')$ the following orthogonality properties hold:

$$\sum_{j=0}^{J} \sum_{k=0}^{K} \left[\frac{(\varphi_{j,k+1} - \varphi_{j,k})}{h_2} \frac{(\psi_{j,k+1} - \psi_{j,k})}{h_2} + \frac{(\varphi_{j+1,k} - \varphi_{j,k})}{h_1} \frac{(\psi_{j+1,k} - \psi_{j,k})}{h_1} \right] = 0 \tag{3.41}$$

and

$$\sum_{j=0}^{J}\sum_{k=0}^{K}\varphi_{j,k}\psi_{j,k}=0. \tag{3.42}$$

Combining (3.40)-(3.42) we deduce that

$$\sum_{j=0}^{J}\sum_{k=0}^{K}\left[\left|\frac{\phi_{j,k+1}-\phi_{j,k}}{h_2}\right|^2+\left|\frac{\phi_{j+1,k}-\phi_{j,k}}{h_1}\right|^2\right]\leq\Lambda\sum_{j=0}^{J}\sum_{k=0}^{K}\left|\phi_{j,k}\right|^2, \forall\phi\in\underset{(p,q)\in I}{\text{span}}\left\{\varphi^{p,q}\right\}. \tag{3.43}$$

From (3.43), inequality (3.39) follows immediately taking into account that

$$\begin{aligned}&\sum_{j=0}^{J}\sum_{k=0}^{K}\left[\left|\phi_{j,k+1}-\phi_{j,k}\right|^2+\left|\phi_{j+1,k}-\phi_{j,k}\right|^2\right]\\&\leq\max\left(h_1^2,h_2^2\right)\sum_{j=0}^{J}\sum_{k=0}^{K}\left[\left|\frac{\phi_{j,k+1}-\phi_{j,k}}{h_2}\right|^2+\left|\frac{\phi_{j+1,k}-\phi_{j,k}}{h_1}\right|^2\right].\end{aligned}$$

■

In view of Lemma 3.5, estimate (3.37) is immediate. It is sufficient to apply (3.39) to $\phi=u(t)$ for any $t\in(0,T)$ and to integrate the resulting inequality for $t\in(0,T)$.

We now proceed to estimate

$$\begin{aligned}R_2 &= \sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2-\frac{\left(u_{j+1,k}-u_{j,k}\right)\left(u_{j+1,k+1}-u_{j,k+1}\right)}{h_1^2}\right]dt \qquad (3.44)\\&+\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2-\frac{\left(u_{j,k+1}-u_{j,k}\right)\left(u_{j+1,k+1}-u_{j+1,k}\right)}{h_2^2}\right]dt=R_2^1+R_2^2.\end{aligned}$$

Both terms have a similar structure. Let us analyze the first one R_2^1. We have

$$\begin{aligned}&\left|\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\frac{\left(u_{j+1,k}-u_{j,k}\right)\left(u_{j+1,k+1}-u_{j,k+1}\right)}{h_1^2}dt\right| \qquad (3.45)\\&\leq\left(\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2dt\right)^{1/2}\left(\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|\frac{u_{j+1,k+1}-u_{j,k+1}}{h_1}\right|^2dt\right)^{1/2}\\&=\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2dt.\end{aligned}$$

In view of (3.45) we observe that $R_2^1\geq 0$. In a similar way we get $R_2^2\geq 0$. Therefore $R_2\geq 0$.

Combining identity (3.32) with (3.37) and the fact $R_2\geq 0$, inequality (3.35) follows immediately. ■

Combining Lemma 3.5 with the equipartition of energy identity the following holds:

Lemma 3.7 *Every solution of* (1.7) *satisfies*

$$T\left(1-\frac{\Lambda}{4}\max\left(h_1^2h_2^2\right)\right)E(0)+\widehat{Z}(t)\Big|_0^T \le \frac{\pi}{2}\left[\frac{h_1}{2}\sum_{j=0}^{J}\int_0^T\left|\frac{u_{j,K}}{h_2}\right|^2dt+\frac{h_2}{2}\sum_{k=0}^{K}\int_0^T\left|\frac{u_{J,k}}{h_1}\right|^2dt\right] \tag{3.46}$$

Λ *being the largest eigenvalue entering in the Fourier development of* u, *and*

$$\widehat{Z}(t)=Z(t)-\frac{\Lambda}{8}\max\left(h_1^2,h_2^2\right)Y(t). \tag{3.47}$$

Proof. Combining the equipartition of energy identity and the conservation of energy property it follows that

$$h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|u'_{j,k}\right|^2dt=TE(0)+\frac{1}{2}Y(t)\Big|_0^T. \tag{3.48}$$

Combining (3.35) and (3.48) inequality (3.46) follows immediately.

■

We have to estimate now the quantity $\widehat{Z}$ in (3.46)-(3.47). The following holds:

Lemma 3.8 *Every solution* u *of* (1.7) *satisfies*

$$\begin{aligned}&\left(T\left(1-\frac{\Lambda}{4}\max\left(h_1^2,h_2^2\right)\right)-2\sqrt{2\pi^2+\left(\eta^2+8\mid\eta\mid\right)/\lambda_1}\right)E(0) \qquad (3.49)\\ &\le\frac{\pi}{2}\left[h_1\sum_{j=0}^{J}\int_0^T\left|\frac{u_{j,K}}{h_2}\right|^2dt+h_2\sum_{k=0}^{K}\int_0^T\left|\frac{u_{J,k}}{h_1}\right|^2dt\right],\end{aligned}$$

with λ_1 *the least eigenvalue of* (1.13), Λ *the largest eigenvalue entering in the Fourier expansion of* u *and*

$$\eta=\frac{1}{2}-\frac{\Lambda}{8}\max\left(h_1^2,h_2^2\right). \tag{3.50}$$

Proof. Note that

$$\widehat{Z}=X+\eta Y \tag{3.51}$$

with X as in (3.27), Y as in (3.31) and η as in (3.50). In (3.51) and in the sequel we do not make explicit in the notation the dependence with respect to time.

We have

$$\widehat{Z}=h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\left[u'_{j,k}\left[\frac{k\left(u_{j,k+1}-u_{j,k-1}\right)}{2}+\frac{j\left(u_{j+1,k}-u_{j-1,k}\right)}{2}+\eta u_{j,k}\right]\right]. \tag{3.52}$$

Thus

$$| \widehat{Z} |\leq h_1h_2\left(\sum_{j=0}^{J}\sum_{k=0}^{K}\left|u'_{j,k}\right|^2\right)^{1/2}\left(\sum_{j=0}^{J}\sum_{k=0}^{K}\left|\frac{k\left(u_{j,k+1}-u_{j,k-1}\right)}{2}+\frac{j\left(u_{j+1,k}-u_{j-1,k}\right)}{2}+\eta u_{j,k}\right|^2\right)^{1/2}. \tag{3.53}$$

On the other hand

$$\sum_{j=0}^{J}\sum_{k=0}^{K}\left|\frac{k\left(u_{j,k+1}-u_{j,k-1}\right)}{2}+\frac{j\left(u_{j+1,k}-u_{j-1,k}\right)}{2}+\eta u_{j,k}\right|^2 \tag{3.54}$$

$$\begin{aligned}
\leq\ & \sum_{j=0}^{J}\sum_{k=0}^{K}\left[\left|\frac{k\left(u_{j,k+1}-u_{j,k-1}\right)}{2}+\frac{j\left(u_{j+1,k}-u_{j-1,k}\right)}{2}\right|^2+\eta^2\left|u_{j,k}\right|^2+\right.\\
& \left.+\eta k\left(u_{j,k+1}-u_{j,k-1}\right)u_{j,k}+\eta j\left(u_{j+1,k}-u_{j-1,k}\right)u_{j,k}\right]\\
\leq\ & \sum_{j=0}^{J}\sum_{k=0}^{K}\left[2\pi^2\left[\left|\frac{u_{j,k+1}-u_{j,k-1}}{2h_2}\right|^2+\left|\frac{u_{j+1,k}-u_{j-1,k}}{2h_1}\right|^2\right]+\eta^2\left|u_{j,k}\right|^2-\eta\left(u_{j,k+1}u_{j,k}+u_{j+1,k}u_{j,k}\right)\right]\\
\leq\ & \sum_{j=0}^{J}\sum_{k=0}^{K}\left[2\pi^2\left(\frac{1}{2}\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2+\frac{1}{2}\left|\frac{u_{j,k}-u_{j,k-1}}{h_2}\right|^2+\frac{1}{2}\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2+\frac{1}{2}\left|\frac{u_{j,k}-u_{j-1,k}}{h_1}\right|^2\right)\right.\\
& \left.+\eta^2\left|u_{j,k}\right|^2-\eta\left(u_{j,k+1}u_{j,k}+u_{j+1,k}u_{j,k}\right)\right]\\
=\ & 2\pi^2\sum_{j=0}^{J}\sum_{k=0}^{K}\left[\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2+\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2\right]+\left(\eta^2+8\mid\eta\mid\right)\sum_{j=0}^{J}\sum_{k=0}^{K}\left|u_{j,k}\right|^2\\
\leq\ & \left[2\pi^2+\frac{\left(\eta^2+8\mid\eta\mid\right)}{\lambda_1}\right]\sum_{j=0}^{J}\sum_{k=0}^{K}\left[\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2+\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2\right]
\end{aligned}$$

where λ_1 is the least eigenvalue of (1.13).

Combining (3.53) and (3.54) we deduce that

$$\left|\widehat{Z}\right|\leq\sqrt{2\pi^2+\frac{\left(\eta^2+8\mid\eta\mid\right)}{\lambda_1}}E(0). \tag{3.55}$$

In view of (3.55) we deduce that

$$\left|\widehat{Z}(t)\Big|_0^T\right|\leq\left|\widehat{Z}(0)\right|+\left|\widehat{Z}(T)\right|\leq 2\sqrt{2\pi^2+\left(\eta^2+8\mid\eta\mid\right)/\lambda_1}E(0). \tag{3.56}$$

Combining (3.46) and (3.56) we deduce (3.49).

■

3.1 Uniform boundary observability

In view of Lemma 3.7 it is easy to obtain uniform (as $h_1, h_2 \to 0$) observability inequalities.For, we introduce the following classes of solutions of (1.7) for any $0 < \beta < 1$:

$$\widehat{\mathcal{C}}_\beta\left(h_1, h_2\right) = \left\{ \begin{array}{l} u \text{ solution of (1.7) generated by the eigenvectors of (1.13)} \\ \text{such that } \lambda \max\left(h_1^2, h_2^2\right) \leq 4\beta \end{array} \right\}. \tag{3.57}$$

The following holds:

Theorem 3.1 *Let $0 < \beta < 1$. Assume that*

$$T > \frac{2\sqrt{2\pi^2 + c(\beta)}}{1-\beta} = T(\beta) \tag{3.58}$$

with

$$c(\beta) = \left[\frac{1}{4}(1-\beta)^2 + 4(1-\beta)\right] \Big/ \lambda_1. \tag{3.59}$$

Then, there exists $C = C(\beta, T) > 0$ such that

$$E_{h,h_2}(0) \leq C(\beta, T) \int_0^T \left[h_1 \sum_{j=0}^{J} \left| \frac{u_{j,K}(t)}{h_2} \right|^2 + h_2 \sum_{k=0}^{K} \left| \frac{u_{J,k}(t)}{h_1} \right|^2 \right] dt \tag{3.60}$$

holds for every solution u of (1.7) *in the class $\widehat{\mathcal{C}}_\beta\left(h_1, h_2\right) > 0$ and every $h_1, h_2 > 0$.*

Moreover, the constant $C(\beta, T)$ may be taken to be

$$C(\beta, T) = \frac{\pi}{2\left[T(1-\beta) - 2\sqrt{2\pi^2 + c(\beta)}\right]}. \tag{3.61}$$

Proof. According to inequality (3.49) and taking into account that

$$\frac{\Lambda}{4} \max\left(h_1^2, h_2^2\right) = \beta$$

in the class $\widehat{\mathcal{C}}_\beta\left(h_1, h_2\right)$, it follows that

$$\left(T(1-\beta) - 2\sqrt{2\pi^2 + (\eta^2 + 8 \mid \eta \mid)/\lambda_1}\right) E_{h_1,h_2}(0) \leq \frac{\pi}{2} \int_0^T \left[h_1 \sum_{j=0}^{J} \left| \frac{u_{j,K}}{h_2} \right|^2 + h_2 \sum_{k=0}^{K} \left| \frac{u_{J,k}}{h_1} \right|^2 \right] dt \tag{3.62}$$

with

$$\eta = \frac{1}{2} - \frac{\beta}{2} = \frac{1}{2}(1-\beta).$$

The statement of Theorem 3.1 follows immediately from (3.62).

■

Remark 3.2 In the definition (3.59) of $c(\beta)$, the least eigenvalue λ_1 depends on h_1, h_2. However, as $h_1, h_2 \to 0$, λ_1 converges to the least eigenvalue for $-\Delta$ in $H_0^1\left((0,\pi)\times(0,\pi)\right)$. Thus,

$$\lambda_1 \to \sqrt{2} \text{ as } h_1, h_2 \to 0.$$

Thus the minimal observability time remains bounded as $h_1, h_2 \to 0$.

■

Remark 3.3 The minimal observability time $T(\beta)$ satisfies

$$T(\beta) \to \infty \text{ as } \beta \to 1. \tag{3.63}$$

Moreover

$$T(\beta) \to 2\sqrt{2\pi^2 + c(0)} = 2\sqrt{2\pi^2 + 17/4\lambda_1}, \text{ as } \beta \to 0. \tag{3.64}$$

This indicates that:

(a) We loose the observability inequality as $\beta \to 1$. This is in agreement with the $1-d$ results of [IZ1,2].

(b) The estimate on the observability times is not sharp since we do not recover the observability time $2\sqrt{2}\pi$ needed for the continuous wave equation.

At this respect note that, in the $1-d$ case, the sharp observability time was recovered.

This lack of optimality is due to the estimates of the proof of Lemma 3.9 on $\widehat{Z}$ and more precisely to the terms

$$h_1, h_2 \sum_{j=0}^{J}\sum_{k=0}^{K}\left[\eta^2\left|u_{j,k}\right|^2 - \eta\left(u_{j,k+1}u_{j,k} + u_{j+1,k}u_{j,k}\right)\right]. \tag{3.65}$$

Note that in the context of the continuous wave equation the corresponding term is

$$\left(\eta^2 - \eta\right)\int_\Omega |\, u \,|^2\, dx_1, dx_2$$

which is non-positive (and therefore may be neglected) as soon as $\eta^2 - \eta \le 0$. In Theorem 3.1, $\eta = \frac{1}{2}(1-\beta)$. Thus

$$\eta^2 - \eta = \frac{1}{4}(1-\beta)^2 - \frac{1}{2}(1-\beta) = \frac{1}{2}(1-\beta)\left[\frac{1}{2}(1-\beta) - 1\right] = -\frac{1}{4}(1-\beta)^2 \le 0, \text{ for } 0 \le \beta \le 1.$$

This sign property does not seem to hold for the discrete quantity (3.65).

In Section 3.4 we shall see how a compactness-uniqueness argument may be used to improve the observability time.

3.2 Optimality of the uniform observability inequality

As we have seen in Remark 3.3, the estimate provided by Theorem 3.1 on the observability time is suboptimal.

Let us now analyze the optimality of Theorem 3.1 in what concerns the frequencies involved in the class $\widehat{\mathcal{C}}_\beta(h_1, h_2)$. For, we compare Theorem 3.1 to the counterexamples of the previous sections.

In to order to analyze the optimality of Theorem 3.1 we distinguish the following three cases:

Case 1: $h_1 = h_2 = h$;

Case 2: $h_2 = \ell h_1,$ with $\ell > 1$;

Case 3: $h_2 = \ell h_1,$ with $\ell < 1$.

Case 1: When $h_1 = h_2 = h$, it is easy to check that

$$\widehat{\mathcal{C}}_\beta(h_1, h_2) = \mathcal{C}_{2\beta}(h_1, h_2). \tag{3.66}$$

Therefore, Theorem 3.1 guarantees the observability in the classes $\mathcal{C}_\gamma(h_1, h_2)$ for any $\gamma < 2$.

According to Remark 1.2, the result is sharp since the uniform controllability fails for any $\gamma > 2$. The case $\gamma = 2$ corresponding to $\beta = 1$ remains open.

Case 2: When $h_2 = \ell h_1$ with $\ell > 1$, the condition

$$\lambda \max\left(h_1^2, h_2^2\right) \leq 4\beta \tag{3.67}$$

that characterizes the class $\widehat{\mathcal{C}}_\beta(h_1, h_2)$ can be rewritten as

$$\lambda \leq 4\beta h_2^{-2} = \frac{4\beta}{h_2^2}. \tag{3.68}$$

On the other hand, the condition characterizing the class $\mathcal{C}_\gamma(h_1, h_2)$ is

$$\lambda \leq \gamma\left[\frac{1}{h_1^2} + \frac{1}{h_2^2}\right] = \frac{\gamma}{h_2^2}(1 + \ell^2). \tag{3.69}$$

Whe have $\mathcal{C}_\gamma(h_1, h_2) \subset \widehat{\mathcal{C}}_\beta(h_1, h_2)$ as soon as

$$\gamma(1 + \ell^2) \leq 4\beta$$

or, in other words,

$$\gamma \leq \frac{4\beta}{\ell^2 + 1}. \tag{3.70}$$

In view of Theorem 3.1 we deduce that, under the condition $h_2 = \ell h_1$ with $\ell > 1$, the uniform observability holds in $\mathcal{C}_\gamma(h_1, h_2)$ as soon as

$$\gamma < \frac{4}{1 + \ell^2} \Leftrightarrow \ell < \sqrt{\frac{4 - \gamma}{\gamma}}. \tag{3.71}$$

On the other hand, as we have seen in Remark 1.2, the counterexample of Theorem 1.2 applies as soon as

$$\sup\left|\frac{h_1}{h_2}\right| = \frac{1}{\ell} < \sqrt{\frac{\gamma}{4-\gamma}}.$$

Thus, the result of Theorem 3.1 is sharp. The limit case $\beta = 1$ which correspond to $\gamma = 4/(1+\ell^2)$ remains open.

Case 3: By symmetry, the situation is the same as in Case 2 above. Thus Theorem 3.1 is sharp and the limit case $\beta = 1$ remains open.

■

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