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A HYBRID SYSTEM CONSISTING OF TWO FLEXIBLE BEAMS CONNECTED BY A POINT MASS: SPECTRAL ANALYSIS AND WELL-POSEDNESS IN ASYMMETRIC SPACES

CARLOS CASTRO AND ENRIQUE ZUAZUA

Abstract. We consider a hybrid system consisting on two flexible beams connected by a point mass. We prove that the presence of the point mass affects the spectral gap when the inertia term does not vanish. This allows us to show that the system is well-posed in asymmetric spaces in which solutions have one more degree of regularity to one side of the mass. The proofs combine classical techniques from asymptotic analysis and the theory of non-harmonic Fourier series.

Key words: Flexible beams, point mass, asymmetric spaces, Fourier series.
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1. Introduction

In this paper we study a linear system modeling the vibrations of two flexible beams connected by a point mass.

We assume that the beams occupy the intervals (-1,0) and (0,1) and that the point mass is located at $x = 0$. By means of the scalar function $u = u(x,t)$ defined for $x \in (-1,1)$ and $t > 0$ we describe the vertical displacements of the beams and the point mass. The linear equations describing the small vibrations of this system can be written as follows

$$\begin{cases} \gamma\partial^2 u_{tt} - u_{tt} - \partial^4 u = 0, & \text{for } x \in (-1,0), t > 0 \\ \gamma\partial^2 u_{tt} - u_{tt} - \partial^4 u = 0, & \text{for } x \in (0,1), t > 0 \\ [u]\,(0,t) = [\partial u](0,t) = 0, & \text{for } t > 0 \\ u_{tt}(0,t) + [\partial^3 u](0,t) = 0, & \text{for } t > 0 \\ \gamma\partial u_{tt}(0,t) - [\partial^2 u](0,t) = 0 & \text{for } t > 0, \end{cases} \tag{1.1}$$

where ∂ denotes partial derivation with respect to x and the index t derivation with respect to time. $[u](0) = u(0^+) - u(0^-)$ denotes the jump of the function u at the point $x = 0$ where the mass is located. Assuming that the beams are posed at their extremes, system (1.1) has to be completed with the following boundary conditions:

$$u(\pm 1,t) = \partial^2 u(\pm 1,t) = 0, \quad \text{for } t > 0. \tag{1.2}$$

In this system $\gamma \geq 0$ is the constant of rotational inertia. The third equation guarantees that u and ∂u are continuous across $x = 0$ while the last two equations describe the vibrations of the point mass at $x = 0$. To simplify the expos ition we have assumed that the mass concentrated at $x = 0$ is one.

This system can be viewed as the singular limit as $\varepsilon \to 0$ of a system consisting of three flexible beams occupying the intervals $(-1,-\varepsilon), (-\varepsilon,\varepsilon)$ and $(\varepsilon,1)$ respectively, the middle one having density $1/2\varepsilon$. In this case, if $\rho^{\varepsilon}(x) = 1 + \chi_{(-\varepsilon,\varepsilon)}/2\varepsilon$, $\chi_{(-\varepsilon,\varepsilon)}$ being the characteristic function of the interval $(-\varepsilon, \varepsilon)$, the equations of mot ion read as follows:

$$\gamma\partial\left(\rho^{\varepsilon}(x)\partial u_{tt}\right) - \rho^{\varepsilon}(x)u_{tt} - \partial^4 u = 0, \quad \text{for } -1 < x < 1,\, t > 0. \tag{1.3}$$

It is easy to see, formally, that, as $\varepsilon \to 0$, solutions of (1.3) with appropriate boundary and initial conditions converge to the solutions of (1.1). This can be proved rigorously in suitable weak and strong topologies. We refer to [C] for a detailed analysis in the case of flexible strings instead of flexible beams. System (1.1) can also be written as

$$\gamma\partial\left(\rho\partial u_{tt}\right) - \rho u_{tt} - \partial^4 u = 0 \quad \text{in } (-1,1)\times I\!R$$

where $\rho = 1+\delta_0$, δ_0 being the Dirac delta at the origin. In this way it becomes clear why solutions of (1.3) approach the solutions of (1.1) as $\varepsilon \to 0$.

It is worth noting that, in the particular case in which the constant γ of rotational inertia vanishes ($\gamma = 0$) $\partial^2 u$ is continuous across $x = 0$ too. This implies that the effect of the mass point is weaker on the behavior of the syste m when $\gamma = 0$ than when $\gamma > 0$. Thus the properties of system (1.1)-(1.2) when $\gamma = 0$ are much closer to the case in which the point mass is not present and we will not address it here.

All along this paper we assume that $\gamma > 0$.

System (1.1)-(1.2) has to be completed with suitable initial conditions for $u(x,t), u(0,t)$ and $\partial u(0,t)$. The last two quantities will be denoted by y and z respectively, i.e.

$$u(0,t) = y(t); \qquad \partial u(0,t) = z(t). \tag{1.4}$$

The initial conditions are then:

$$\begin{cases} u(x,0) = u^0 & \text{in } (-1,0)\cup(0,1)\,; y(0) = y^0,\ z(0) = z^0 \\ u_t(x,0) = u^1(x) & \text{in } (-1,0)\cup(0,1);\ y_t(0) = y^1,\ z_t(0) = z^1. \end{cases} \tag{1.5}$$

Under appropriate regularity and compatibility conditions on the initial data it is easy to see that system (1.1)-(1.2) with the initial conditions above admits a unique solution in a suitable class. On the other hand, its energy

$$\begin{aligned} E(t) &= \int_{-1}^{1}\left[\left|\partial^2 u(x,t)\right|^2 + \gamma\left|\partial u_t(x,t)\right|^2 + \left|u_t(x,t)\right|^2\right]dx \\ &\qquad + \left|u_t(0,t)\right|^2 + \gamma\left|\partial u_t(0,t)\right|^2 \end{aligned} \tag{1.6}$$

is constant along trajectories. We deduce then that system (1.1) is well posed in the energy space

$$\begin{aligned} \mathcal{H} &= \{((u^0,y^0,z^0),(u^1,y^1,z^1)) \in (H^2(-1,1)\times I\!R\times I\!R) \\ &\quad \times(H_0^1(-1,1)\times I\!R\times I\!R) \text{ such that} \\ &\quad u^0(0) = y^0,\ \partial u^0(0) = z^0,\ u^1(0) = y^1,\ \partial u^1(0) = z^1\} \end{aligned}$$

in the sense that if $U^0 = ((u^0,y^0,z^0),(u^1,y^1,z^1)) \in \mathcal{H}$ then the solution

$$U(t) = ((u(t),y(t),z(t)),(u_t(t),y_t(t),z_t(t)))$$

of (1.1)-(1.2) with initial data U^0 belongs to $\mathcal{H}$ for every $t > 0$.

In this work we prove that there exist spaces of solutions with different regularity to both sides of $x = 0$ where system (1.1) is also well posed without having an associated natural energy.

The main result we prove is the following:

Theorem 1. *Consider Y the subspace of elements*

$$U^0 = ((u^0, y^0, z^0), (u^1, y^1, z^1)) \in \mathcal{H}$$

such that the restriction of (u^0, u^1) to $(0,1)$ belongs to $H^3(0,1) \times H^2(0,1)$ and verify the following conditions:

$$\partial u^1(0^+) = z^1, \quad \partial^2 u^0(1) = 0.$$

Then, the solution $U(t) = ((u(t), y(t), z(t)), (u_t(t), y_t(t), z_t(t)))$ of (1.1)-(1.2) with initial data $U^0 \in Y$ belongs to Y for every $t > 0$.

The same phenomena was observed in [HZ] in the case of two flexible strings connected by a point mass. In [HZ] this was proved by using the explicit formula for solutions of the one-dimensional wave equation in terms of its initial data and it was seen th at this is a consequence of the fact that solutions gain one derivative when crossing the mass. In [HZ] it was also observed that the spectral gap of the wave equation vanishes in the presence of a point mass and it was conjectured these two facts (i.e. the existence of an asymmetric space where the system is well posed and the lack of the spectral gap) to be closely related. Later on, in [C], it was proved that these two properties are equivalent.

When analyzing the fourth order system (1.1) we do not have explicit formulas of solutions. Therefore, we adopt the point of view of [C] based on a careful analysis of the spectrum of the system and on the theory of non-harmonic Fourier series.

We prove that the eigenvalues of (1.1) are simple and that the presence of the point mass affects the spectral gap. We also do a detailed asymptotic analysis of the eigenfunctions. This analysis shows that system (1.1)-(1.2) is also well-posed in an asymmetric space defined in terms of Fourier series. The most technical part of the paper is devoted to prove that this space is also asymmetric in the sense that its elements are more regular to one side of the mass, the difference in the number of L^2-derivatives being exactly one. This result applies only when $\gamma > 0$ since, as we said above, when $\gamma = 0$ the presence of the mass has a much weaker effect on the behavior of the system. In this case system (1.1)-(1.2) is not well-posed in asymmetric spaces of this kind.

As it was shown in [HZ] for the strings connected by a point mass, the existence of asymmetric spaces where system (1.1) is well posed has some consequences concerning the controllability of system (1.1) when we act in one extreme $x = \pm 1$. Using the results of the present paper one can prove that, if we change the condition $\partial^2(1,t) = 0$ by $\partial^2(1,t) = q(t)$ where $q(t) \in L^2(0,T)$ is the control and we take T large enough, the space of initial data is not an usual energy space (as it happens if the point mass is not present) but an asymmetric space. The detailed proof of this result will be given elsewhere.

The rest of the paper is organized as follows. In section 2 we give some basic results on the spectral decomposition of the energy spaces and the development of solutions in Fourier series. In section 3 we perform a careful analysis of the spectrum of the system. In section 4 we introduce and identify the asymmetric space mentioned above. In section 5 we derive some final comments.

2. Preliminary spectral results

When decomposing solutions of (1.1)-(1.2) in Fourier series one is led to consider solutions in separated variables $u = e^{i\lambda t}\varphi(x)$. In this class of solutions system (1.1)-(1.2) becomes:

$$\begin{cases} \partial^4\varphi = \lambda^2\varphi - \gamma\lambda^2\partial^2\varphi\,, & \text{for } x \in (-1,0) \\ \partial^4\varphi = \lambda^2\varphi - \gamma\lambda^2\partial^2\varphi\,, & \text{for } x \in (0,1) \\ [\varphi](0) = [\partial\varphi](0) = 0 \\ [\partial^2\varphi](0) = -\gamma\lambda^2\partial\varphi(0) \\ [\partial^3\varphi](0) = \lambda^2\varphi(0) \\ \varphi(\pm 1) = \partial^2\varphi(\pm 1) = 0. \end{cases} \tag{2.1}$$

In order to solve the eigenvalue problem (2.1) it is convenient to introduce its variational formulation: *Find real positive numbers λ^2 such that there exists a non-trivial function $\varphi \in H^2 \cap H_0^1(-1,1)$ satisfying*

$$\begin{aligned} \int_{-1}^{1} \partial^2\varphi\partial^2 v dx = \int_{-1}^{1} \left[\lambda^2\varphi v + \gamma\lambda^2\partial\varphi\partial v\right] dx + \gamma\lambda^2\partial\varphi(0)\partial v(0) \\ +\lambda^2\varphi(0)v(0), \qquad \forall v \in H^2 \cap H_0^1(-1,1). \end{aligned} \tag{2.2}$$

In what follows the space $H^2\cap H_0^1(-1,1)$ will be considered as endowed with the norm

$$\| \varphi \|_{H^2\cap H_0^1} = \left[\int_{-1}^{1} | \partial^2\varphi |^2 \, dx\right]^{1/2} \tag{2.3}$$

which, in this space, is equivalent to the norm induced by $H^2(-1,1)$.

The classical theory on the decomposition of self-adjoint compact operators allows us to prove the following result:

Proposition 2. *The eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ of system (2.1) constitute a sequence of positive real numbers of multiplicity less or equal than two:*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$$

such that $\lambda_k \longrightarrow \infty$ as $k \longrightarrow \infty$.

Moreover, the corresponding eigenfunction $\{\phi_k\}_{k\in\mathbb{N}}$ may be normalized to form an orthonormal basis of $H^2 \cap H_0^1(-1,1)$.

Remark 3. In the next section we will prove that all the eigenvalues are simple.

Proof. Let us consider the linear bounded operator $K : H^2 \cap H_0^1(-1,1) \longrightarrow H^2 \cap H_0^1(-1,1)$ such that for each $F \in H^2 \cap H_0^1(-1,1)$, $KF = u \in H^2 \cap H_0^1(-1,1)$ is the unique solution of

$$\begin{aligned} \int_{-1}^{1} \partial^2 u\partial^2 v dx = \int_{-1}^{1} \left[\gamma\partial F\partial v + Fv\right] dx + \gamma\partial F(0)\partial v(0) + F(0)v(0), \\ \forall v \in H^2 \cap H_0^1(-1,1). \end{aligned} \tag{2.4}$$

By Lax-Milgram's Lemma, the solution of (2.4) exists and it is unique.

On the other hand, it is easy to see that there exists $C > 0$ such that

$$\| u \|_{H^2\cap H_0^1} \leq C \left[\| F \|_{H^2\cap H_0^1} + | \partial F(0) | + | F(0) |\right] \leq C \| F \|_{H^2\cap H_0^1}, \tag{2.5}$$

for all $F \in H^2 \cap H_0^1(-1,1)$.

Let us see that K is compact and self-adjoint.

To see that K is compact, let us consider a sequence $\{F_k\}$ in $H^2 \cap H_0^1(-1,1)$ that weakly converges to F in $H^2 \cap H_0^1(-1,1)$. Then $u_k = KF_k$ is bounded in $H^2 \cap H_0^1(-1,1)$. By extracting subsequences (that we denote by the index k) we can deduce that u_k weakly converges to some $u \in H^2 \cap H_0^1(-1,1)$ in $H^2 \cap H_0^1(-1,1)$. Passing to the limit in the weak formulation (2.4) of the identity $u_k = KF_k$ it is easy to see that $u = KF$. Therefore it suffices to show that the norms converge. Taking the test function $v = u_k$ in the weak equation (2.4) that u_k satisfies we deduce that

$$\| u_k \|^2_{H^2 \cap H_0^1(-1,1)} = \int_{-1}^{1} | \partial^2 u_k |^2 \, dx = \tag{2.6}$$
$$\int_{-1}^{1} [\gamma \partial F_k \partial u_k + F_k u_k] \, dx + \gamma \partial F_k(0) \partial u_k(0) + F_k(0) u_k(0).$$

Taking into account that the embedding $H^2(-1,1) \subset C^1([-1,1])$ is compact it is easy to pass to the limit on the right hand side of (2.6). Then, using the weak equation (2.4) that the limit $u = KF$ satisfies we deduce that $\| u_k \|_{H^2 \cap H_0^1}$ converges to $\| u \|_{H^2 \cap H_0^1}$. This concludes the proof of the compactness.

Since K is bounded, in order to see that it is self-adjoint it is sufficient to check that K is symmetric. This is straightforward from its definition.

From the theory of compact, self-adjoint operators we deduce that K has a non-increasing sequence of positive eigenvalues $\{\mu_k\}$ such that $\mu_k \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, it is sufficient to observe that $\lambda > 0$ solves (2.1) if and only if $\lambda = 1/\mu_k$ for some k. We also deduce that the corresponding eigenfunctions form an orthonormal basis of $H^2 \cap H_0^1(-1,1)$.

Let us prove that the multiplicity is at most two. If it were greater than two for some index k, then it would exist a non-trivial solution u of (2.1) with $\lambda = \lambda_k$ such that, in addition to $u(-1) = \partial^2 u(-1) = 0$, it would satisfy $\partial u(-1) = \partial^3 u(-1) = 0$. The equations in (2.1) being of order four, this would imply that $u \equiv 0$, leading us to a contradiction. □

In the sequel we will often identify the elements of $H^2 \cap H_0^1(-1,1)$ with those of $\left(H^2 \cap H_0^1(-1,1)\right) \times \mathbb{R} \times \mathbb{R}$ by means of the linear mapping

$$\begin{array}{rcl} \tau : H^2 \cap H_0^1(-1,1) & \longrightarrow & \left(H^2 \cap H_0^1(-1,1)\right) \times \mathbb{R} \times \mathbb{R} \\ u & \longrightarrow & \tau u = (u, u(0), \partial u(0)) . \end{array} \tag{2.7}$$

From Proposition 2 the space $H^2 \cap H_0^1(-1,1)$ can be also written as follows

$$H^2 \cap H_0^1(-1,1) = \left\{ u : u = \sum_{k \in \mathbb{N}} a_k \phi_k, \| u \|^2_{H^2 \cap H_0^1} = \sum_{k \in \mathbb{N}} | a_k |^2 < \infty \right\}. \tag{2.8}$$

We can also define the following fractional Hilbert spaces $(H_\alpha, \| \cdot \|_\alpha)_{\alpha \in \mathbb{R}}$:

$$H_\alpha = \left\{ u = \sum_{k \in \mathbb{N}} a_k \phi_k : \| u \|^2_\alpha = \sum_{k \in \mathbb{N}} | a_k |^2 \lambda_k^{4\alpha} < \infty \right\}. \tag{2.9}$$

We will denote by $< \cdot, \cdot >_\alpha$ the scalar product in H_α

Clearly $H_0 = H^2 \cap H_0^1(-1,1)$ and $\| \cdot \|_{H^2 \cap H_0^1(-1,1)} = \| \cdot \|_0$.

Observe that, if $u = \sum_{k\in I\!\!N} a_k\phi_k$, then $Ku = \sum_{k\in I\!\!N} \frac{a_k}{\lambda_k^2}\phi_k$. Clearly K is an isomorphism from H_α into $H_{\alpha+1}$. We can also write explicitely K^{-1}:

$$K^{-1}u = \sum_{k\in I\!\!N} \lambda_k^2 a_k\phi_k$$

which is continuous from $H_{\alpha+1}$ into H_α.

We need to identify the spaces H_α for some values of the parameter $\alpha \in I\!\!R$. To do that we denote by $H^s\left((-1,1)\backslash\{0\}\right)\cap H^2\cap H_0^1(-1,1)$ the subspace of $H^2\cap H_0^1(-1,1)$ constituted by the elements such that its restrictions to $(-1,0)$ and $(0,1)$ belong to H^s.

We also introduce the operator $(I-\gamma\partial^2)^{-1} : H^{-1}(-1,1) \to H_0^1(-1,1)$ such that $u = (I-\gamma\partial^2)^{-1}F$ if and only if $u \in H_0^1(-1,1)$ satisfies

$$\int_{-1}^{1} [\gamma\partial u\partial v + uv]\,dx = \langle F, v\rangle,\ \forall v \in H_0^1(-1,1). \tag{2.10}$$

We have the following characterizations of the fractional spaces H_α:

Proposition 4. *Assume that* $\gamma > 0$. *Then:*
(a) H_1 *coincides algebraically and topologically with the subspace of*

$$H^4\left((-1,1)\backslash\{0\}\right)\cap H^2\cap H_0^1(-1,1)\times I\!\!R\times I\!\!R$$

constituted by the elements (u,y,z) *such that*

$$\begin{cases} \partial^2u(\pm 1) = 0, \quad u(0) = y, \quad \partial u(0) = z \\ \left[\partial^3 u\right](0) = F(0); \quad [\partial^2 u](0) = -\gamma\partial F(0), \end{cases} \tag{2.11}$$

where $F = (I-\gamma\partial^2)^{-1}\bar{F}, \bar{F}$ *being the* $L^2(-1,1)$*-function that coincides with* $\partial^4 u$ *to both sides of* $x=0$ *that the will denote by*

$$\bar{F} = \left[\partial^4\mid_{(-1,0)} + \partial^4\mid_{(0,1)}\right]u.$$

(b) $H_{1/2}$ *coincides algebraically and topologically with the subspace of*

$$H^3\left((-1,1)\backslash\{0\}\right)\cap H^2\cap H_0^1(-1,1)\times I\!\!R\times I\!\!R$$

constituted by the elements (u,y,z) *such that*

$$\partial^2u(\pm 1) = 0, \quad u(0) = y, \quad \partial u(0) = z. \tag{2.12}$$

(c) $H_{-1/2}$ *coincides with the subspace of* $H_0^1(-1,1)\times I\!\!R\times I\!\!R$ *constituted by the elements* (u,y,z) *such that* $u(0) = y$.
Moreover

$$\|\,(u,y,z)\,\|_{-1/2}^2 = \int_{-1}^{1}\left[\gamma\mid\partial u\mid^2 + \mid u\mid^2\right]dx + \mid y\mid^2 + \gamma\mid z\mid^2. \tag{2.13}$$

(d) H_{-1} *coincides algebraically and topologically with the quotient space of* $L^2(-1,1)\times I\!\!R\times I\!\!R$ *constituted by the classes* (u,y,z) *characterized in the following way: Two elements* (u^1,y^1,z^1) *and* (u^2,y^2,z^2) *belong to the same class if and only if*

$$(u^1-u^2, y^1-y^2, z^1-z^2) = \alpha(m,-1,0) + \beta(n,0,\gamma^{-1})$$

where $\alpha, \beta \in I\!R$ and m and n are the functions

$$m(x) = \begin{cases} \dfrac{\sinh(\frac{1+x}{\sqrt{\gamma}})}{2\sqrt{\gamma}\cosh(\frac{1}{\sqrt{\gamma}})} & \text{if } x \in [-1,0] \\ \dfrac{\sinh(\frac{1-x}{\sqrt{\gamma}})}{2\sqrt{\gamma}\cosh(\frac{1}{\sqrt{\gamma}})} & \text{if } x \in [0,1] \end{cases},$$

$$n(x) = \begin{cases} \dfrac{\sinh(\frac{1+x}{\sqrt{\gamma}})}{2\gamma\sinh(\frac{1}{\sqrt{\gamma}})} & \text{if } x \in [-1,0] \\ \dfrac{-\sinh(\frac{1-x}{\sqrt{\gamma}})}{2\gamma\sinh(\frac{1}{\sqrt{\gamma}})} & \text{if } x \in [0,1] \end{cases}. \tag{2.14}$$

(e) *$H_{-3/2}$ coincides algebraically and topologically with the quotient space of $H^{-1}(-1,1) \times I\!R \times I\!R$ constituted by the classes (u, y, z) characterized in the following way: Two elements (u^1, y^1, z^1) and (u^2, y^2, z^2) belong to the same class if and only if*

$$(u^1 - u^2, y^1 - y^2, z^1 - z^2) = \alpha(m, -1, 0) + \beta(n, 0, \gamma^{-1})$$

where $\alpha, \beta \in I\!R$, m and n are the functions given in (2.14).

Proof. **(a)** Let (u, y, z) be a element of $H^4\left((-1,1)\backslash\{0\}\right) \cap H^2 \cap H_0^1(-1,1) \times I\!R \times I\!R$ satisfying (2.12). We set $F = \left[(I - \gamma\partial^2)^{-1}\left[\partial^4\left|_{(-1,0)} + \partial^4\right|_{(0,1)}\right]\right] u$. Observe that $\left[\partial^4\left|_{(-1,0)} + \partial^4\right|_{(0,1)}\right] u = \partial^4 u - \left[\partial^3 u\right](0)\delta_0 - \left[\partial^2 u\right](0)\partial\delta_0$ in the sense of distributions.

We have $F \in H_0$ and $u = KF$ since the compatibility conditions (2.12) are satisfied. Therefore $u \in H_1$.

To prove the other embedding, let u be any element of H_1. Then, there exists some $F \in H^2 \cap H_0^1(-1,1)$ such that $KF = u$, i.e.

$$\int_{-1}^{1} \partial^2 u \partial^2 v dx = \gamma \int_{-1}^{1} \partial F \partial v dx + \int_{-1}^{1} F v dx + \gamma \partial F(0) \partial v(0) + F(0) v(0),$$
$$\forall v \in H^2 \cap H_0^1(-1,1).$$

By choosing test functions such that $v(0) = \partial v(0) = 0$, it is easy to see that $\partial^4 u = -\gamma\partial^2 F + F$ in $(-1,0) \cup (0,1)$. Therefore $u \in H^4\left((-1,1)\backslash\{0\}\right) \cap H^2 \cap H_0^1(-1,1)$. On the other hand, integrating by parts in the left hand side of this identity we obtain

$$\begin{aligned} &\int_{-1}^{0} \partial^4 u v dx + \int_{0}^{1} \partial^4 u v dx - [\partial^2 u](0)\partial v(0) + [\partial^3 u](0) v(0) \\ &= \int_{-1}^{1} (-\gamma\partial^2 F + F) v dx - [\partial^2 u](0)\partial v(0) + [\partial^3 u](0) v(0) \\ &= \gamma \int_{-1}^{1} \partial F \partial v dx + \int_{-1}^{1} F v dx - [\partial^2 u](0)\partial v(0) + [\partial^3 u](0) v(0), \\ &\qquad \forall v \in H^2 \cap H_0^1(-1,1). \end{aligned}$$

Therefore, the compatibility conditions (2.12) hold.

We have proved the continuity and bijectivity of the map $u \to F$. The equivalence of the norms is then a consequence of the open map theorem.

(b) Let us take $u \in H_1$ and compute its norm in $H_{1/2}$. We have

$$\| u \|_{1/2}^2 = \langle u, u\rangle_{1/2} = \langle u, K^{-1}u\rangle_0 = \langle KK^{-1}u, K^{-1}u\rangle_0 =$$

$$= \int_{-1}^{1} \partial^2(KK^{-1}u)\partial^2(K^{-1}u) = \int_{-1}^{1} \left[\gamma \left|\partial K^{-1}u\right|^2 + \left|K^{-1}u\right|^2\right] dx$$

$$+\gamma \left|\partial K^{-1}u(0)\right|^2 + \left|K^{-1}u(0)\right|^2 .$$

Since $u \in H_1$, in view of (2.12), the right hand side of this identity coincides with

$$\int_{-1}^{1} \left|\partial(I-\gamma\partial^2)^{-1}\partial\left[\left(\partial^3 \left.\right|_{(-1,0)} + \partial^3\left.\right|_{(0,1)}\right)\right] u - H(x)\left[\partial^3 u\right](0)\right|^2$$

$$+\left|\left[\partial^3 u\right](0)\right|^2 + \frac{1}{\gamma^2}\left|\left[\partial^2 u\right](0)\right|^2$$

where H denotes the Heaviside function ($H(x) = 0$ for $x \leq 0$ and $H(x) = 1$ for $x \geq 0$).

On H_1 this norm is equivalent to the H^3 one to both sides of the point mass. Since $H_{1/2}$ is the completion of H_1 with respect to this norm we deduce that $H_{1/2}$ coincides with the space given in the statement of the proposition.

(c) Let $u \in H_0 = H^2 \cap H_0^1(-1,1)$ be given. Then

$$\| u \|_{-1/2}^2 = \langle u, u\rangle_{-1/2} = \langle u, Ku\rangle = \int_{-1}^{1} \partial^2 u \partial^2 (Ku) dx = \tag{2.15}$$

$$= \int_{-1}^{1} \left[\gamma \mid \partial u \mid^2 + \mid u \mid^2\right] dx + \gamma \mid \partial u(0) \mid^2 + \mid u(0) \mid^2 .$$

The completion of H_0 with respect to the norm $\| \cdot \|_{-1/2}$ (that coincides with $H_{-1/2}$) is precisely the space of point (a) of the statement of the proposition. On the other hand, (2.11) is a consequence of (2.14).

(d) First of all let us obtain a suitable identity for $K^{-1}u$. Suppose that $u \in H_1$. It is easy to check that, in the sense of distributions,

$$K^{-1}u = \left[(1-\gamma\partial^2)^{-1}\left[\partial^4|_{(-1,0)} + \partial^4|_{(0,1)}\right]\right] u = -\gamma^{-1}G + \gamma^{-1}(1-\gamma\partial^2)^{-1}G$$

where

$$G = \left[\partial^2|_{(-1,0)} + \partial^2|_{(0,1)}\right] u - H(x)[\partial^2 u](0) - Y(x)[\partial^3 u](0)$$
$$+ \left([\partial^2 u](0) + [\partial^3 u](0)\right)\frac{x+1}{2}.$$

Here $H(x)$ denotes the Heaviside function and $Y(x)$ its primitive given by $Y(x) = 0$ if $x \in [-1,0]$ and $Y(x) = x$ if $x \in [0,1]$. Now, we observe that $K^{-1}u \in H^2 \cap H_0^1(-1,1)$ and

$$K^{-1}u = -\gamma^{-1}\left[\partial^2|_{(-1,0)} + \partial^2|_{(0,1)}\right] u - \gamma^{-2}u + \gamma^{-2}(1-\gamma\partial^2)^{-1}u$$
$$+\gamma^{-1}\left(H(x) - \frac{x+1}{2} - (1-\gamma\partial^2)^{-1}H(x) + (1-\gamma\partial^2)^{-1}\frac{x+1}{2}\right)[\partial^2 u](0)$$
$$+\gamma^{-1}\left(Y(x) - \frac{x+1}{2} - (1-\gamma\partial^2)^{-1}Y(x) + (1-\gamma\partial^2)^{-1}\frac{x+1}{2}\right)[\partial^3 u](0).$$

On the other hand, $K^{-1}u(0) = [\partial^3 u](0)$ and $(\partial K^{-1}u)(0) = -\gamma^{-1}[\partial^2 u](0)$.

We define

$$\begin{aligned} n(x) &= -\gamma^{-1}\left(H(x) - \frac{x+1}{2} - (1-\gamma\partial^2)^{-1}H(x) + (1-\gamma\partial^2)^{-1}\frac{x+1}{2}\right) \\ m(x) &= -\gamma^{-1}\left(Y(x) - \frac{x+1}{2} - (1-\gamma\partial^2)^{-1}Y(x) + (1-\gamma\partial^2)^{-1}\frac{x+1}{2}\right). \end{aligned}$$

A straightforward computation shows that m and n defined as above are those of (2.14). The image by K^{-1} of a element $u \in H_1$ is then

$$\begin{aligned} K^{-1}u = \big(&-\gamma^{-1}\partial^2 u - \gamma^{-2}u + \gamma^{-2}(1-\gamma\partial^2)^{-1}u - n[\partial^2 u](0) - m[\partial^3 u](0), \\ &[\partial^3 u](0), -\gamma^{-1}[\partial^2 u](0)\big). \end{aligned} \tag{2.16}$$

Now we observe that $H_{-1} = K^{-1}H_0$. Thus, to identify H_{-1} we characterize the image by K^{-1} of $H_0 = H^2 \cap H_0^1(-1,1)$.

Consider an element u of H_0. For any α,$\beta \in I\!R$ we can find a sequence $u_k \in H_1$ such that $u_k \to u$ in H_0 with $[\partial^3 u_k](0) = \alpha$ and $[\partial^2 u_k](0) = \beta$. Then, by (2.16) we see that $K^{-1}u_k$ converges in $L^2(-1,1)\times I\!R \times I\!R$ to

$$\begin{aligned} &\big(-\gamma^{-1}\partial^2 u - \gamma^{-2}u + \gamma^{-2}(1-\gamma\partial^2)^{-1}u, 0, 0\big) - \alpha(m,-1,0) - \beta(n,0,\gamma^{-1}) \\ &\quad = L(u,\alpha,\beta). \end{aligned} \tag{2.17}$$

The image by K^{-1} of u is then the two dimensional subspace of $L^2(-1,1)\times I\!R\times I\!R$ composed by the elements $L(u,\alpha,\beta)$ with $\alpha,\beta \in I\!R$. This subspace can be identified with a class of the quotient space of $L^2(-1,1)\times I\!R\times I\!R$ given in the statement of the Proposition.

As a consequence of the open map Theorem, K^{-1} is in fact an isomorphism from H_0 to the quotient space defined above.

(e) Following the ideas of the proof of (d) we can characterize $H_{-3/2}$ as $K^{-1}(H_{-1/2})$. Now we observe that if $u \in H_1$ then

$$\begin{aligned} K^{-1}u &= -\gamma^{-1}\left[\partial^2|_{(-1,0)} + \partial^2|_{(0,1)}\right]u - \gamma^{-1}\delta_0\partial u(0) - \gamma^{-2}u \\ &\quad +\gamma^{-2}(1-\gamma\partial^2)^{-1}u - n(x)[\partial^2 u](0) + m(x)[\partial^3 u](0). \end{aligned}$$

Let be now

$$(u,y,z) \in H_{-1/2} = \left\{(u,y,z) \in H_0^1(-1,1)\times I\!R\times I\!R \,:\, y = u(0)\right\}.$$

Given α,$\beta \in I\!R$ we can find a sequence $u_k \in H_1$ such that $(u_k, u_k(0), \partial u_k(0)) \to (u,y,z)$ in $H_{-1/2}$ with $[\partial^2 u_k](0) = \beta$, $[\partial^3 u_k](0) = \alpha$ and $[\partial u_k](0) \to z$. Then, $K^{-1}u_k$ converges in $H^{-1}(-1,1)\times I\!R\times I\!R$ to

$$\begin{aligned} &\big(-\gamma^{-1}\partial^2 u - \gamma^{-2}u + \gamma^{-2}(1-\gamma\partial^2)^{-1}u, 0, 0)\big) + \gamma^{-1}z(\delta_0, 0, 0) \\ &-\alpha(m,-1,0) - \beta(n,0,\gamma^{-1}). \end{aligned} \tag{2.18}$$

The image by K^{-1} of u is then the two dimensional subspace of $H^{-1}(-1,1)\times I\!R\times I\!R$ composed by the elements (2.18) with $\alpha,\beta \in I\!R$ and we can conclude as in (d). □

Let us now recall briefly how solutions of (1.1)-(1.2) can be developed in Fourier series.

Given initial data $(u^0, y^0, z^0) \in H_0, (u^1, y^1, z^1) \in H_{-1/2}$ we compute its Fourier coefficients

$$\widetilde{a}_k = \langle u^0, \phi_k \rangle_0 = \int_{-1}^{1} \partial^2 u^0 \partial^2 \phi_k dx \tag{2.19}$$

$$\begin{aligned} \widetilde{b}_k &= \left\langle \left(u^1, y^1, z^1\right), \left(\phi_k, \phi_k(0), \partial\phi_k(0)\right) \right\rangle_{-1/2} \\ &= \int_{-1}^{1} \left(\gamma \partial u^1 \partial \phi_k + u^1 \phi_k\right) dx + y^1 \phi_k(0) + \gamma z^1 \partial \phi_k(0). \end{aligned} \tag{2.20}$$

Then

$$\begin{cases} u^0 = \sum_{k \in I\!N} \widetilde{a}_k \phi_k, \ u^0(0) = y^0 = \sum_{k \in I\!N} \widetilde{a}_k \phi_k(0), \\ \partial u^0(0) = z^1 = \sum_{k \in I\!N} \widetilde{a}_k \partial \phi_k(0), \\ \\ u^1 = \sum_{k \in I\!N} \widetilde{b}_k \lambda_k \phi_k, \ u^1(0) = y^1 = \sum_{k \in I\!N} \widetilde{b}_k \lambda_k \phi_k(0), \ z^1 = \sum_{k \in I\!N} \widetilde{b}_k \lambda_k \partial \phi_k(0) \end{cases} \tag{2.21}$$

and $\{\widetilde{a}_k\}, \left\{\widetilde{b}_k\right\} \in \ell^2$.

Then the unique solution of (1.1), (1.2), (1.4) and (1.5),

$$(u, y, z) \in C\left([0, \infty); H_0\right) \cap C^1\left([0, \infty); H_{-1/2}\right) \tag{2.22}$$

can be written as follows

$$u(x,t) = \sum_{k \in I\!N} \left(\widetilde{a}_k \cos(\lambda_k t) \phi_k(x) + \widetilde{b}_k \sin(\lambda_k t) \phi_k(x)\right). \tag{2.23}$$

To simplify these expresions we set

$$\lambda_{-k} = -\lambda_k, \quad \phi_{-k} = \phi_k \tag{2.24}$$

and we introduce the complex Fourier coefficients

$$a_k = \frac{\widetilde{a}_k - i\widetilde{b}_k}{2}, \quad a_{-k} = \frac{\widetilde{a}_k + i\widetilde{b}_k}{2}. \tag{2.25}$$

Then, u can be written as follows

$$u(x,t) = \sum_{k \in Z\!\!\!Z} a_k e^{i\lambda_k t} \phi_k(x). \tag{2.26}$$

It is clear that $\left\{\bar{\phi}_k\right\}_{k \in Z\!\!\!Z}$ with $\bar{\phi}_k = (\phi_k, i\lambda_k \phi_k)$ constitutes an orthonormal basis of the energy space $\mathcal{H} = H_0 \times H_{-1/2}$ (at this level we are identify $i\lambda_k \phi_k$ with $(i\lambda_k \phi_k, i\lambda_k \phi_k(0), i\lambda_k \partial \phi_k(0))$ by means of the mapping τ in (2.7)).

Then, the vector-valued unknown

$$U = ((u, y, z), (u_t, y_t, z_t)) \tag{2.27}$$

can be written as

$$U(t) = \sum_{k \in Z\!\!\!Z} a_k e^{i\lambda_k t} \bar{\phi}_k. \tag{2.28}$$

The conservation of the energy E in (1.6) is equivalent the fact that system (1.1)-(1.2) generates a group of isometries in $\mathcal{H}$. More precisely

$$E(t) = \| U(t) \|_{\mathcal{H}}^2 = \sum_{k \in Z\!\!\!Z} \left| a_k e^{i\lambda_k t} \bar{\phi}_k \right|^2 = \sum_{k \in Z\!\!\!Z} | a_k |^2 = \| U(0) \|_{\mathcal{H}}^2 = E(0). \tag{2.29}$$

Obviously, one can also obtain developments in Fourier series of the form (2.24) for solution of (1.1)-(1.2) in other classes.

3. Spectral analysis.

In this section we obtain precise estimates on the eigenvalues of (2.1).

First of all we observe that it suffices to consider eigenvalues associated to even or odd eigenfunctions. Indeed, if $u = u(x)$ is an eigenfunction corresponding to the eigenvalue λ, then $v(x) = u(-x)$ is an eigenfunction too. Thus, $w_1(x) = u(x) - u(-x)$ and $w_2(x) = u(x) + u(-x)$ are also eigenfunctions with the same eigenvalue. When λ is simple this implies that w_1 and w_2 are necessarily proportional. Since w_1 is odd and w_2 even, this means that one of them has to vanish and then, u to be either odd or even. When the multiplicity is two (recall that, by Proposition 2 it can not be greater than two), if one of the eigenfunctions is not even or odd it generates two eigenfunctions w_1 and w_2 as above and then necessarily, there are two even eigenfunctions, two odd eigenfunctions or one even and one odd. In any case, we can reduce our study to the analysis of even and odd eigenfunctions.

3.1. Even eigenfunctions.

Proposition 5. *The eigenvalues corresponding to even eigenfunctions are simple and we will denote then by* $\{\lambda_{2k-1}\}_{k\in\mathbb{N}}$. *They satisfy*

$$\lambda_{2k-1} = \mu^+_{2k-1}\sqrt{\frac{\left(\mu^+_{2k-1}\right)^2}{\gamma\left(\mu^+_{2k-1}\right)^2+1}} \tag{3.1}$$

where μ^+_{2k-1} *are the positive roots of*

$$\frac{4}{x} + 2\gamma x + \sqrt{\gamma x^2+1}\tanh\sqrt{\frac{x^2}{\gamma x^2+1}} = \tan x. \tag{3.2}$$

The corresponding eigenfunctions are

$$\varphi_{2k-1} = C_k \begin{cases} \sin\left(\mu^+_{2k-1}(1+x)\right) - \dfrac{\mu^+_{2k-1}\cos\mu^+_{2k-1}\sinh\left(\mu^-_{2k-1}(1+x)\right)}{\mu^-_{2k-1}\cosh\mu^-_{2k-1}} \\ \qquad \text{if } x\in(-1,0), \\ \\ \sin\left(\mu^+_{2k-1}(1-x)\right) - \dfrac{\mu^+_{2k-1}\cos\mu^+_{2k-1}\sinh\left(\mu^-_{2k-1}(1-x)\right)}{\mu^-_{2k-1}\cosh\mu^-_{2k-1}} \\ \qquad \text{if } x\in(0,1) \end{cases} \tag{3.3}$$

where $\mu^-_{2k-1} = \sqrt{\dfrac{\left(\mu^+_{2k-1}\right)^2}{\gamma\left(\mu^+_{2k-1}\right)^2+1}}$. *Moreover,* $\partial\varphi_{2k-1}(1) \neq 0$.

Proof. Eigenvalues λ with even eigenfunctions are characterized by the following reduced system:

$$\begin{cases} \partial^4\varphi = \lambda^2\varphi - \gamma\lambda^2\partial^2\varphi & \text{in } (0,1) \\ \varphi(1) = \partial^2\varphi(1) = \partial\varphi(0) = 0 \\ \partial^3\varphi(0) - \frac{\lambda^2}{2}\varphi(0) = 0. \end{cases} \tag{3.4}$$

Taking into account that $\varphi(1) = \partial^2\varphi(1) = 0$, solutions of this system are of the form

$$\varphi(x) = A\sin\left(\mu^+(1-x)\right) + B\sinh\left(\mu^-(1-x)\right) \tag{3.5}$$

where A and B are constants and μ^- and $i\mu^+$ are the real and imaginary roots of the polynomial $x^4 + \gamma\lambda^2x^2 - \lambda^2 = 0$ given by

$$\mu^- = \sqrt{-\frac{\gamma\lambda^2}{2} + \sqrt{\frac{\gamma^2\lambda^4}{4} + \lambda^2}}; \quad \mu^+ = \sqrt{\frac{\gamma\lambda^2}{2} + \sqrt{\frac{\gamma^2\lambda^4}{4} + \lambda^2}}. \tag{3.6}$$

Imposing the other boundary conditions we obtain

$$\partial\varphi(0) = -A\mu^+\cos(\mu^+) - B\mu^-\cosh(\mu^-) = 0; \tag{3.7}$$

$$\partial^3\varphi(0) - \frac{\lambda^2}{2}\varphi(0) = A\left[(\mu^+)^3\cos(\mu^+) - \frac{\lambda^2}{2}\sin(\mu^+)\right]$$
$$-B\left[(\mu^-)^3\cosh(\mu^-) + \frac{\lambda^2}{2}\sinh(\mu^-)\right] = 0. \tag{3.8}$$

This system admits a non-trivial solution (A, B) if and only if

$$\mu^+\cos(\mu^+)\left((\mu^-)^3\cosh(\mu^-) + \frac{\lambda^2}{2}\sinh(\mu^-)\right)$$
$$= \mu^-\cosh(\mu^-)\left(\frac{\lambda^2}{2}\sin(\mu^+) - (\mu^+)^3\cos(\mu^+)\right). \tag{3.9}$$

Observe that $\lambda \neq 0$ and therefore both μ^- and μ^+ are non-zero. On the other hand $\cos(\mu^+) \neq 0$, too. Indeed, otherwise, from (3.7) we would get $B = 0$ and then, by (3.8), $A\lambda^2\sin(\mu^+) = 0$ and this implies $A = 0$. Thus in (3.9) we may divide by $\mu^+\mu^-\cos\mu^+\cosh\mu^-$ to get

$$(\mu^-)^2 + \frac{\lambda^2}{2\mu^-}\tanh\mu^- = \frac{\lambda^2}{2\mu^+}\tan(\mu^+) - (\mu^+)^2. \tag{3.10}$$

From the definition of μ^+ and μ^- (notice that $(\mu^-)^2$ and $-(\mu^+)^2$ are the roots of $z^2 + \gamma\lambda^2z - \lambda^2 = 0$) we deduce that

$$(\mu^-)^2 - (\mu^+)^2 = -\gamma\lambda^2; \quad -(\mu^-)^2(\mu^+)^2 = -\lambda^2.$$

This allows us to write μ^- and λ in terms of μ^+:

$$\lambda = \mu^+\sqrt{\frac{(\mu^+)^2}{\gamma(\mu^+)^+ + 1}}; \quad \mu^- = \sqrt{\frac{(\mu^+)^2}{\gamma(\mu^+)^2 + 1}}. \tag{3.11}$$

Replacing the values of λ and μ^- in (3.9) we deduce that

$$(\mu^+)^2 + \frac{(\mu^+)^3}{2}\sqrt{\gamma(\mu^+)^2 + 1}\tanh\sqrt{\frac{(\mu^+)^2}{\gamma(\mu^+)^2 + 1}} + (\mu^+)^2\left(1 + \gamma(\mu^+)^2\right)$$
$$-\frac{(\mu^+)^3}{2}\tan(\mu^+) = 0.$$

Multiplying by $2/(\mu^+)^3$ we deduce that μ^+ is a positive root of (3.2). From (3.7) we can write B in terms of A and we obtain the identity (3.3) for the eigenfunctions.

Let us see that these eigenvalues are simple. Assume that there exist two eigenfunctions corresponding to the same eigenvalue, both being even functions. Then we can find a combination φ of them such that $\partial\varphi(1) = 0$, i.e. $A\mu^+ + B\mu^- = 0$. This identity and (3.7) imply

$$B\mu^- \left(\cos(\mu^+) - \cosh(\mu^-)\right) = 0.$$

Since $B\mu^- \neq 0$ we deduce that $\cos(\mu^+) = \cosh(\mu^-)$. But then $\mu^- = 0$ and this implies that $\varphi \equiv 0$.

Notice that the same argument shows that $\partial\varphi(1) \neq 0$. □

3.2. Odd eigenfunctions.

Proposition 6. *The eigenvalues corresponding to odd eigenfunctions are simple and we will denote then by* $\{\lambda_{2k}\}_{k\in\mathbb{N}}$. *They satisfy*

$$\lambda_{2k} = \mu_{2k}^+ \sqrt{\frac{(\mu_{2k}^+)^2}{\gamma(\mu_{2k}^+)^2 + 1}} \tag{3.12}$$

where μ_{2k}^+ *are the positive roots of*

$$\left(\frac{2+\gamma x^2}{x^2}\right)\frac{2}{\gamma x} - \sqrt{\frac{1}{\gamma x^2 + x}}\coth\sqrt{\frac{x^2}{\gamma x^2+1}} = -\cot x. \tag{3.13}$$

The corresponding eigenfunction is given by

$$\varphi_{2k} = C_k \begin{cases} -\sin\left(\mu_{2k}^+(1+x)\right) + \dfrac{\sin\mu_{2k}^+}{\sinh\mu_{2k}^-}\sinh\left(\mu_{2k}^-(1+x)\right) & \text{in } (-1,0) \\ \\ \sin\left(\mu_{2k}^+(1-x)\right) - \dfrac{\sin\mu_{2k}^+}{\sinh\mu_{2k}^-}\sinh\left(\mu_{2k}^-(1-x)\right) & \text{in } (0,1), \end{cases} \tag{3.14}$$

where $\mu_{2k}^- = \sqrt{\frac{(\mu_{2k}^+)^2}{\gamma(\mu_{2k}^+)^2+1}}$. *Moreover* $\partial\varphi_{2k}(1) \neq 0$.

Proof. Eigenvalues corresponding to odd eigenfunctions are characterized by the following system:

$$\begin{cases} \partial^4\varphi + \gamma\lambda^2\partial^2\varphi - \lambda^2\varphi = 0 & \text{in } (0,1) \\ \varphi(0) = \varphi(1) = \partial^2\varphi(1) = 0 \\ \partial^2\varphi(0) + \frac{\gamma\lambda^2}{2}\partial\varphi(0) = 0. \end{cases} \tag{3.15}$$

Notice that, although the eigenfunctions under consideration are odd we do not impose the condition $\partial^2\varphi(0) = 0$ since φ is not smooth at $x = 0$ due to presence of the mass.

Proceeding as in the proof of Proposition 5 we see that the general form of the solution of (3.15) is given by (3.5) where A and B are constants and μ^- and $i\mu^+$ are as in (3.7).

From the boundary conditions that φ satisfies at $x = 0$ we obtain the equations

$$\varphi(0) = A\sin(\mu^+) + B\sinh(\mu^-) = 0, \tag{3.16}$$

$$\partial^2\varphi(0) + \frac{\gamma\lambda^2}{2}\partial\varphi(0) = A\left[-(\mu^+)^2\sin(\mu^+) - \frac{\gamma\lambda^2}{2}\mu^+\cos(\mu^+)\right] + B\left[(\mu^-)^2\sinh(\mu^-) - \frac{\gamma\lambda^2}{2}\mu^-\cosh(\mu^-)\right] = 0. \tag{3.17}$$

This system has a non-trivial solution (A, B) if and only if

$$\sin(\mu^+)\left[(\mu^-)^2\sinh(\mu^-)-\frac{\gamma\lambda^2}{2}\mu^-\cosh(\mu^-)\right]=$$
$$-\sinh(\mu^-)\left[(\mu^+)^2\sin(\mu^+)+\frac{\gamma\lambda^2}{2}\mu^+\cos(\mu^+)\right].$$

As in Proposition 3.1, it is easy to see that $\gamma\lambda^2\mu^+\sin\mu^+\sinh\mu^-$ may not vanish and, dividing in the last identity, we obtain:

$$\frac{2\left[(\mu^+)^2+(\mu^-)^2\right]}{\gamma\lambda^2\mu^+}-\frac{\mu^-}{\mu^+}\coth(\mu^-)=-\cot\mu^+.$$

Replacing the values of λ and μ^- from (3.11) in this identity we obtain

$$\frac{2\left[(\mu^+)^2\gamma+2\right]}{\gamma(\mu^+)^3}-\frac{1}{\sqrt{\gamma(\mu^+)^2+1}}\coth\mu^-=-\cot\mu^+.$$

Thus, μ^+ is necessarily a positive root of

$$\frac{2}{\gamma x}\left(\frac{\gamma x^2+2}{\gamma x^2}\right)-\sqrt{\frac{1}{\gamma x^2+1}}\coth\sqrt{\frac{x^2}{\gamma x^2+1}}=-\cot x. \tag{3.18}$$

Concerning the eigenfunctions, if we compute B from (3.16) and replace this value in (3.5) we obtain the expression (3.14) in (0,1). The value of the eigenfunction in (-1,0) is obtained taking into account that it is odd.

Let us check that these eigenvalues are simple too. Otherwise it would exist a non-trivial linear combination φ of eigenfunctions such that $\partial\varphi(1)=-A\mu^+-B\mu^-=0$. Then $B=-A\mu^+/\mu^-$ and identities (3.16)-(3.17) would yield

$$\sinh\mu^-=\frac{\mu^-}{\mu^+}\sin\mu^+ \tag{3.19}$$

$$\mu^+\sin\mu^++\mu^-\sinh\mu^-=\frac{\gamma\lambda^2}{2}\left(\cos\mu^+-\cosh\mu^-\right). \tag{3.20}$$

Combining these two identities we get

$$\mu^+\left(1+\frac{(\mu^-)^2}{(\mu^+)^2}\right)\sin\mu^+=\frac{\gamma\lambda^2}{2}\left(\cos\mu^+-\cosh\mu^-\right)\le 0$$

and this contradicts the fact that, in view of (3.19), $\sin\mu^+\ge 0$, except if $\sin\mu^+=0$. But then, from (3.19), $\mu^-=0$ and this implies $\varphi\equiv 0$.

The same argument shows that these eigenfunctions satisfy $\partial\varphi(1)\neq 0$. □

3.3. Simplicity of the eigenvalues.

The goal of this section is to prove the following:

Proposition 7. *All the eigenvalues of system (2.1) are simple.*

Proof. In the section above we have shown that the number of odd or even eigenfunctions associated to each eigenvalue is at most one. Thus, it is sufficient to show that there is no eigenvalue solving simultaneously (3.1)-(3.2) and (3.12)-(3.13). Taking into account that the mapping that associates λ to μ^+ as in (3.1) and (3.12) is strictly increasing, it is sufficient to show that (3.2) and (3.13) do not have roots in common.

We set

$$f(x)=\left[\sqrt{\frac{1}{\gamma x^2+1}}\coth\sqrt{\frac{x^2}{\gamma x^2+1}}-\left(\frac{2+\gamma x^2}{x^2}\right)\frac{2}{\gamma x}\right]^{-1}$$

$$g(x)=\frac{4}{x}+2\gamma x+\sqrt{\gamma x^2+1}\tanh\sqrt{\frac{x^2}{\gamma x^2+1}}.$$

Recall that μ_{2k-1}^+ and μ_{2k}^+ are respectively the positive roots of $g(x)=\tan x$ and $f(x)=\tan x$. Clearly it is sufficient to show that $f(x)\neq g(x)$ for all $x>0$.

We argue by contradiction. Assume that there exists $x>0$ such that $g(x)/f(x)=1$. In other words,

$$-\frac{4}{\gamma}\left(\frac{2}{x^2}+\gamma\right)\tanh\sqrt{\frac{x^2}{\gamma x^2+1}}-\frac{2}{\gamma}\sqrt{\frac{\gamma x^2+1}{x^2}}\tanh^2\sqrt{\frac{x^2}{\gamma x^2+1}}$$
$$+2\sqrt{\frac{x^2}{\gamma x^2+1}}=0.$$

By the change of variables $y=\sqrt{x^2/(\gamma x^2+1)}$ this equation reduces to

$$-\frac{2}{\gamma}\left(\frac{2}{y^2}-\gamma\right)\tanh y-\frac{1}{\gamma y}\tanh^2 y+y=0$$

where $y\in(0,\gamma^{-1/2})$. From this identity we get

$$\tanh y=\gamma y-\frac{2}{y}\pm\sqrt{\left(\frac{2}{y}-\gamma y\right)^2+\gamma y^2}. \tag{3.21}$$

We have to show that the last equation does not posses any root in the interval $(0,\gamma^{-1/2})$.

First of all we observe that

$$\gamma y-\frac{2}{y}-\sqrt{\left(\frac{2}{y}-\gamma y\right)^2+\gamma y^2}\leq 0$$

in $(0,\gamma^{-1/2})$ and then, since $\tanh y\geq 0$ in this interval, we see that there are no roots for the minus sign on the right hand side of (3.21).

Thus, it is sufficient to show that $F(y,\gamma)<\tanh y$ for $0<y<\frac{1}{\sqrt{\gamma}}$ with

$$F(y,\gamma)=\gamma y-\frac{2}{\gamma}+\sqrt{\left(\frac{2}{y}-\gamma y\right)^2+\gamma y^2} \tag{3.22}$$

To see this we apply the following lemma:

Lemma 8. *Let $F(y,\gamma)$ be a positive function defined in $(0,\infty)\times(0,\infty)$. Assume that the following two conditions hold:*

$$\frac{\partial F}{\partial\gamma}(y,\gamma)\geq 0,\quad \forall y\in(0,\gamma^{-1/2}),\quad \forall\gamma\geq 0; \tag{3.23}$$

$$F(\gamma^{-1/2},\gamma)<\tanh(\gamma^{-1/2}),\quad \forall\gamma>0. \tag{3.24}$$

Then, $F(y,\gamma)<\tanh y$ for all $\gamma>0$ and $0<y<\gamma^{-1/2}$.

Proof. Assume that for some $\gamma > 0$ there exists $y_0 \in (0, \gamma^{-1/2})$ such that $F(y_0, \gamma) \geq \tanh(y_0)$. Let $\gamma_0 > \gamma$ be such that $y_0 = 1/\sqrt{\gamma_0}$. Then by (3.23), we have

$$F(y_0, \gamma_0) > F(y_0, \gamma) \geq \tanh(y_0)$$

and this contradicts (3.24). □

It is sufficient to show that the function F as in (3.22) satisfies (3.23) and (3.24). We have

$$\begin{aligned} \frac{\partial F}{\partial \gamma} &= y + \frac{1}{2\sqrt{\left(\frac{2}{y} - \gamma y\right)^2 + \gamma y^2}} \left(y^2 - 2\left(\frac{2}{y} - \gamma y\right) y \right) \\ &\geq y - y\sqrt{\frac{\left(\frac{2}{y} - \gamma y\right)^2}{\left(\frac{2}{y} - \gamma y\right)^2 + \gamma y^2}} \geq 0. \end{aligned}$$

On the other hand,

$$F\left(\gamma^{-1/2}, \gamma\right) = -\sqrt{\gamma} + \sqrt{\gamma + 1} < \tanh\left(\gamma^{-1/2}\right), \forall \gamma > 0. \tag{3.25}$$

Indeed, to see that (3.25) holds we use the following elementary lemma:

Lemma 9. *Assume that $f \in C^1(0, \infty)$ satisfies*

$$\lim_{\gamma \to \infty} f(\gamma) \geq \lim_{\gamma \to 0^+} f(\gamma) = 0 \tag{3.26}$$

$$f > 0 \textit{ in the set where } f' \geq 0. \tag{3.27}$$

Then, necessarily $f(\gamma) > 0$ for every $\gamma \in (0, \infty)$.

In view of this lemma, to conclude the proof of (3.25) and therefore, that of Proposition 7, it is sufficient to check that $f(\gamma) = \tanh\left(\frac{1}{\sqrt{\gamma}}\right) + \sqrt{\gamma} - \sqrt{\gamma + 1}$ satisfies (3.26) and (3.27). We have $\lim_{\gamma \to \infty} f(\gamma) = 0$; $\lim_{\gamma \to 0} f(\gamma) = 0$. On the other hand,

$$f'(\gamma) = \frac{-1}{2}\gamma^{-3/2}\left(1 - \tanh^2\left(\gamma^{-1/2}\right)\right) + \frac{1}{2\sqrt{\gamma}} - \frac{1}{2\sqrt{\gamma + 1}},$$

and $f'(\gamma) \geq 0$ if and only if

$$\frac{\sqrt{\gamma + 1} - \sqrt{\gamma}}{\sqrt{\gamma}\sqrt{\gamma + 1}} \geq \gamma^{-3/2}\left(1 - \tanh^2\left(\gamma^{-1/2}\right)\right)$$

or, equivalently,

$$\tanh^2\left(\gamma^{-1/2}\right) \geq 1 - \gamma\frac{\sqrt{\gamma + 1} - \sqrt{\gamma}}{\sqrt{\gamma + 1}}. \tag{3.28}$$

We have to distinguish two cases. First, if the right hand side of (3.28) is negative at some point, since it is an increasing function, we deduce that it is negative in an interval of the form $(0, \gamma_0)$. But then clearly $f'(\gamma) > 0$ in $(0, \gamma_0)$ and since $f(\gamma) \to 0$ as $\gamma \to 0$ we deduce that $f > 0$ in $(0, \gamma_0)$. Suppose now that the right hand side of (3.28) has positive sign.

In this case

$$f(\gamma) \geq \left[1 - \gamma\left(\frac{\sqrt{\gamma + 1} - \sqrt{\gamma}}{\sqrt{\gamma + 1}}\right)\right]^{1/2} + \sqrt{\gamma} - \sqrt{\gamma + 1} > 0$$

since

$$\left[1-\gamma\left(\frac{\sqrt{\gamma+1}-\sqrt{\gamma}}{\sqrt{\gamma+1}}\right)\right]^{1/2} > \sqrt{\gamma+1}-\sqrt{\gamma}. \tag{3.29}$$

This can be seen easily taking squares. Indeed, (3.29) is equivalent to

$$\gamma\left(\frac{\sqrt{\gamma+1}-\sqrt{\gamma}}{\sqrt{\gamma+1}}\right) < 2\sqrt{\gamma(\gamma+1)}-2\gamma$$

or

$$\sqrt{\gamma}\left(\sqrt{\gamma+1}-\sqrt{\gamma}\right) < 2(\gamma+1)-2\sqrt{\gamma(\gamma+1)}. \tag{3.30}$$

Taking squares it is easy to see that (3.30) holds for any $\gamma > 0$.

This completes the proof of Proposition 7. □

3.4. Asymptotic behavior of the eigenvalues.

In this section we obtain precise estimates on the asymptotic behavior of the eigenvalues as $k \to \infty$.

Concerning the eigenvalues associated to even eigenfunctions we have the following:

Proposition 10. *We have*

$$\lambda_{2k-1} = \frac{k\pi-\pi/2}{\sqrt{\gamma}} - \frac{c_1(\gamma)}{\left(k\pi-\frac{\pi}{2}\right)\sqrt{\gamma}} + O(k^{-2}), \ \text{as } k \to \infty \tag{3.31}$$

where $c_1(\gamma) = \left(2\gamma+\sqrt{\gamma}\tanh\left(\gamma^{-1/2}\right)\right)^{-1} + (2\gamma)^{-1}$.

Remark 11. In the absence of mass the asymptotic behavior of eigenvalues associated to even eigenfunctions is as follows:

$$\lambda_{2k-1} = \frac{k\pi-\frac{\pi}{2}}{\sqrt{\gamma}} + O(k^{-2}), \ \text{as } k \to \infty.$$

This shows that only the second term of the asymptotic expansion is affected by the presence of the mass.

Proof. Recall that λ_{2k-1} is given by (3.1) where μ^+_{2k-1} are the positive roots of (3.2). The function on the left hand side of (3.1) is positive, continuous and increasing for large x. Thus, for large k, equation (3.2) has an unique root in each interval $\left(k\pi-\pi,\, k\pi-\frac{\pi}{2}\right)$ in which the tangent is positive.

Keeping the leading terms in (3.2), for large k we obtain

$$\tan\mu^+_{2k-1} = \mu^+_{2k-1}\left(2\gamma+\gamma^{1/2}\tanh(\gamma^{-1/2})\right) + O(k^{-1})$$

and taking into account that $\mu^+_{2k-1} \in \left(k\pi-\pi,\, k\pi-\frac{\pi}{2}\right)$ this yields

$$\cot\mu^+_{2k-1} = \left(\mu^+_{2k-1}(2\gamma+\gamma^{1/2}\tanh(\gamma^{-1/2})\right)^{-1} + O(k^{-3})$$

and using Taylor's expansions we deduce that

$$\left(k\pi-\frac{\pi}{2}\right) - \mu^+_{2k-1} = \frac{1}{\left(k\pi-\frac{\pi}{2}\right)\left(2\gamma+\gamma^{1/2}\tanh(\gamma^{-1/2})\right)} + O(k^{-2}). \tag{3.32}$$

On the other hand, from (3.5) we have

$$\left(\mu^+_{2k-1}\right)^2 = \frac{\gamma(\lambda_{2k-1})^2}{2}\left(1+\sqrt{1+\frac{4}{\gamma^2\lambda^2_{2k-1}}}\right) = \gamma\lambda^2_{2k-1} + \frac{1}{\gamma} + O(k^{-2})$$

and therefore

$$\lambda_{2k-1} = \frac{\mu^+_{2k-1}}{\sqrt{\gamma}} - \frac{1}{2\gamma^{3/2}\mu^+_{2k-1}} + O(k^{-2}). \tag{3.33}$$

Combining (3.32) and (3.33) we get (3.31). □

Let us consider now the eigenvalues associated to odd eigenfunctions:

Proposition 12. *We have*

$$\lambda_{2k} = \frac{\left(k\pi - \frac{\pi}{2}\right)}{\sqrt{\gamma}} - \frac{c_2(\gamma)}{\sqrt{\gamma}\left(k\pi - \frac{\pi}{2}\right)} + O(k^{-2}), \ \textit{as } k \to \infty \tag{3.34}$$

where $c_2(\gamma) = \gamma^{-1/2}\coth\gamma^{-1/2} - 2 + (2\gamma)^{-1}$.

Remark 13. In the absence of mass these eigenvalues behave as follows: $\lambda_{2k} = \dfrac{k\pi}{\sqrt{\gamma}} + O(k^{-2})$.

Proof. We proceed as in the proof of Proposition 10. We observe that, in view of (3.13), μ_{2k} are the roots of an equation of the form

$$\left(2 - \gamma^{-1/2}\coth\gamma^{-1/2}\right)\frac{1}{x} + O(x^{-2}) = -\cot x, \ \text{as} \ \mid x \mid \to \infty.$$

Applying Taylor's development of the cotangent function at $k\pi - \dfrac{\pi}{2}$ we deduce that

$$\left(k\pi - \frac{\pi}{2}\right) - \mu^+_{2k} = \frac{\gamma^{-1/2}\coth\gamma^{-1/2} - 2}{\left(k\pi - \frac{\pi}{2}\right)} + O(k^{-2}). \tag{3.35}$$

From (3.33) and (3.35) the identity (3.34) holds. □

Concerning the spectral gap we have:

Proposition 14. *We have*

$$\lambda_{2k} - \lambda_{2k-1} = \frac{C(\gamma)}{\left(k\pi - \frac{\pi}{2}\right)\gamma^{1/2}} + O(k^{-2}), \ \textit{as } k \to \infty \tag{3.36}$$

where

$$C(\gamma) = c_1(\gamma) - c_2(\gamma) > 0, \forall\gamma > 0. \tag{3.37}$$

Remark 15. In the absence of mass we have $\lambda_{2k} - \lambda_{2k-1} = \frac{\pi}{2\sqrt{\gamma}} + O(k^{-1})$. Therefore the asymptotic gap is reduced by a multiplicative factor of the order of $1/k$ by the presence of the mass.

Proof. The fact that (3.36) holds is an immediate consequence of (3.31) and (3.34). Thus, it is sufficient to check that (3.37) holds.

We have

$$C(\gamma) = \left(2\gamma + \sqrt{\gamma}\tanh(\gamma^{-1/2})\right)^{-1} - \left(\gamma^{-1/2}\coth\gamma^{-1/2} - 2\right).$$

Therefore $C(\gamma) > 0$ if and only if

$$1 - \left(\frac{1}{\sqrt{\gamma}}\coth\frac{1}{\sqrt{\gamma}} - 2\right)\left(2\gamma + \sqrt{\gamma}\tanh\left(\frac{1}{\sqrt{\gamma}}\right)\right) > 0,$$

i.e.

$$4\gamma - 2\sqrt{\gamma}\coth\frac{1}{\sqrt{\gamma}} + 2\sqrt{\gamma}\tanh\frac{1}{\sqrt{\gamma}} > 0$$

or equivalently

$$y^2 + 2\sqrt{\gamma}y - 1 > 0 \tag{3.38}$$

with $y = \tanh\left(\frac{1}{\sqrt{\gamma}}\right)$.

The roots of the function on the left hand side of (3.38) are $y = -\sqrt{\gamma} \pm \sqrt{\gamma+1}$. In particular, (3.38) holds if $y > \sqrt{\gamma+1} - \sqrt{\gamma}$. Thus, it is sufficient that $\tanh \frac{1}{\sqrt{\gamma}} > \sqrt{\gamma+1} - \sqrt{\gamma}$ and this was obtained in (3.25). □

4. The asymmetric space.

In this section we are going to introduce and characterize an asymmetric subspace of the energy space $\mathcal{H} = H_0 \times H_{-1/2}$. As we will see this subspace is stable under the flow generated by system (1.1)-(1.2) and it is a natural space to solve the boundary control problem.

4.1. Construction and basic properties of the asymmetric space.

With the notations of section 2; when $\gamma > 0$ we set

$$\begin{aligned} Y &= \left\{ U = \sum_{k \in \mathbb{Z}\setminus\{0\}} a_k \bar{\phi}_k \in \mathcal{H} \;:\; \| U \|_Y^2 = \sum_{k \in \mathbb{Z}\setminus\{0\}} | a_{2k-\sigma_k} |^2 \right. \\ &\qquad \left. + \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{| a_{2k} - a_{2k-\sigma_k} |^2}{\delta_k^2} < \infty \right\} \end{aligned} \tag{4.1}$$

where $\delta_k = \lambda_{2k} - \lambda_{2k-\sigma_k}$ where $\sigma_k = \operatorname{sgn} k$, i.e. $\sigma_k = 1$ if $k > 0$ and $\sigma_k = -1$ if $k < 0$.

Clearly Y endowed with the norm $\| \cdot \|_Y$ is a Hilbert space. On the hand, it is clear that if all the δ_k were uniformly positive and bounded above, then Y would coincide algebraically and topologically with $\mathcal{H}$.

Notice that $\| U \|_Y < \infty$ if and only if

$$\sum_{k \in \mathbb{Z}\setminus\{0\}} | a_{2k} + a_{2k-\sigma_k} |^2 + \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{| a_{2k} - a_{2k-\sigma_k} |^2}{\delta_k} < \infty \tag{4.2}$$

or, in other words, $\{a_{2k} + a_{2k-\sigma_k}\} \in \ell^2$ and $\left\{\delta_k^{-1}\left(a_{2k} - 2_{2k-\sigma_k}\right)\right\} \in \ell^2$. Since $\delta_k \to 0$ as $k \to \infty$ it is clear that Y is a strict subspace of $\mathcal{H}$.

Let us see that system (1.1)-(1.2) is well-posed in Y:

Proposition 16. *Let $U^0 = \left((u^0, y^0, z^0), (u^1, y^1, z^1)\right)$ be an element of Y. Then, the solution $U(t) = \left((u(t), y(t), z(t)), (u_t(t), y_t(t), z_t(t))\right)$ of (1.1)-(1.2) with initial data U^0 belongs to Y for every $t > 0$. Furthermore, for any $T > 0$ there exists a constant $C(T) > 0$ such that*

$$\| U(t) \|_Y \leq C(T) \| U^0 \|_Y, \quad \forall 0 \leq t \leq T, \forall U^0 \in Y. \tag{4.3}$$

Proof. We have

$$\begin{aligned}
\| U(t) \|_Y^2 = & \sum_{k\in\mathbb{Z}\setminus\{0\}} \left| a_{2k-\sigma_k} e^{i\lambda_{2k-\sigma_k} t} \right|^2 \\
& + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{| a_{2k} e^{i\lambda_{2k} t} - a_{2k-\sigma_k} e^{i\lambda_{2k-\sigma_k} t} |^2}{\delta_k^2} \le \sum_{k\in\mathbb{Z}\setminus\{0\}} | a_{2k-\sigma_k} |^2 \\
& +2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \left[\left| e^{i\lambda_{2k} t} \right|^2 \frac{| a_{2k} - a_{2k-\sigma_k} |^2}{\delta_k^2} + |a_{2k-\sigma_k}|^2 \frac{\left| e^{i\lambda_{2k} t} - e^{i\lambda_{2k-\sigma_k} t} \right|^2}{\delta_k^2} \right] \\
& \le \sum_{k\in\mathbb{Z}\setminus\{0\}} \left[| a_{2k-\sigma_k} |^2 + 2 \frac{| a_{2k} - a_{2k-\sigma_k} |^2}{\delta_k^2} + 2t^2 \, | a_{2k-\sigma_k} |^2 \right] \\
& \le \left(2 + 2t^2\right) \| U^0 \|_Y^2
\end{aligned}$$

since

$$\left| e^{i\lambda_{2k} t} - e^{i\lambda_{2k-\sigma_k} t} \right| / \, \delta_k \le t, \forall k \in \mathbb{Z}, \forall t > 0.$$

Therefore, the result holds with $C(T) = 2 + 2T^2$. □

We set

$$p_k = \frac{\bar{\phi}_{2k} + \bar{\phi}_{2k-\sigma_k}}{2}; \quad q_k = \delta_k \frac{\bar{\phi}_{2k} - \bar{\phi}_{2k-\sigma_k}}{2}, \forall k \in \mathbb{Z}\setminus\{0\}. \tag{4.4}$$

We have:

Proposition 17. *The set* $\{p_k\}_{k\in\mathbb{Z}\setminus\{0\}} \cup \{q_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ *forms a Riesz basis of* Y*.*

Proof. Observe that

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} a_k \bar{\phi}_k = \sum_{k\in\mathbb{Z}\setminus\{0\}} \left[\left(a_{2k} + a_{2k-\sigma_k}\right) p_k + \left(\frac{a_{2k} - a_{2k-\sigma_k}}{\delta_k} \right) q_k \right]. \tag{4.5}$$

With the scalar product

$$\begin{aligned}
\langle \sum_{k\in\mathbb{Z}\setminus\{0\}} a_k \bar{\phi}_k, \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k \bar{\psi}_k \rangle = & \sum_{k\in\mathbb{Z}\setminus\{0\}} \left[\left(a_{2k} + a_{2k-\sigma_k}\right)\left(b_{2k} + b_{2k-\sigma_k}\right) \right. \\
& \left. + \frac{\left(a_{2k} - a_{2k-\sigma_k}\right)\left(b_{2k} - b_{2k-\sigma_k}\right)}{\delta_k^2} \right]
\end{aligned} \tag{4.6}$$

the set $\{p_k\}_{k\in\mathbb{Z}\setminus\{0\}} \cup \{q_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ is orthonormal and the norm associated to this scalar product is equivalent to $\| \cdot \|_Y$ as observed in (4.2). □

The presence of the point mass makes the amplitude of the oscillation at $x = 0$ to be much smaller than elsewhere. Thus, the even eigenfunctions $\phi_{2k-\sigma_k}$, in addition to $\partial\phi_{2k-\sigma_k}(0) = 0$, are such that $\phi_{2k-\sigma_k}(0)$ is small while the odd ones, in addition to $\phi_{2k}(0) = 0$, are such that $\partial\phi_{2k}(0)$ is small. Taking this into account and since the gap between consecutive eigenvalues vanishes asymptotically even and odd eigenfunctions are expected to be very close one to each other to one side of the point mass. Then $\left(\bar{\phi}_{2k} + \bar{\phi}_{2k-\sigma_k}\right)/2$ and $\left(\bar{\phi}_{2k} - \bar{\phi}_{2k-\sigma_k}\right)/2$ will be, roughly, one the even reflection of the other one with respect to $x = 0$. Since we have weighted differently p_k and q_k when introducing the factor δ_k in the definition of the later and taking into account that $\delta_k \sim k^{-1}$ as $k \to \infty$, it is natural to expect Y to be constituted by functions whose degree of regularity differs by an order to one side of the point mass and another.

In figures 1 and 2 bellow we give an approximate graph of p_k and q_k/δ_k exhibiting this fact:

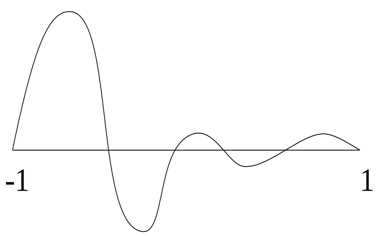

Figure 1: p_k

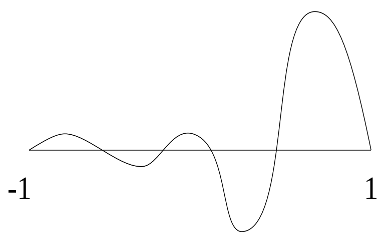

Figure 2: q_k/δ_k

4.2. Characterization of the asymmetric space.

The following theorem provides a precise characterization of Y:

Theorem 18. *Y is the subspace of elements $U^0 = \big((u^0, y^0, z^0), (u^1, y^1, z^1)\big)$ of $\mathcal{H}$ such that the restriction of (u^0, u^1) to (0,1) belongs to $H^3(0,1) \times H^2(0,1)$ and in addition to the compatibility conditions of $\mathcal{H}$ ($u^0(0) = y^0$, $\partial u^0(0) = z^0$, $u^1(0) = y^1$) the following hold:*

$$\partial u^1(0^+) = z^1, \partial^2 u^0(1) = 0. \tag{4.7}$$

Furthermore, the norm $\| \cdot \|_Y$ is equivalent to

$$\left[\| U \|_{\mathcal{H}}^2 + \left\| \left(u^0 \mid_{(0,1)} u^1 \mid_{(0,1)}\right) \right\|_{H^3 \times H^2(0,1)}^2 \right]^{1/2}.$$

To prove Theorem 18 we need some preliminary results:

Lemma 19. *We set*

$$\begin{aligned} K &= \big\{(u,v) \in \left(H^3(0,1) \times H^2(0,1)\right) : \\ &\qquad u(0) = \partial^2 u(0) = \partial u(1) = v(0) = \partial v(1) = 0\big\}. \end{aligned}$$

The vectors

$$\psi_j = -\left(1, \frac{i}{\sqrt{\gamma}}(j\pi - \sigma_j \frac{\pi}{2})\right) \frac{1}{\left(j\pi - \sigma_j \frac{\pi}{2}\right)^3} \cos\left(\left(j\pi - \sigma_j \frac{\pi}{2}\right)(1-x)\right),$$

with $j \in \mathbb{Z} \setminus \{0\}$ form an orthonormal basis of K for the norm

$$\|(u,v)\|_K = \left[\int_0^1 \mid \partial^3 u \mid^2 dx + \gamma \int_0^1 \mid \partial^2 v \mid^2 dx\right]^{1/2}.$$

Moreover,

$$\|\widetilde{p_j} - \psi_j\|_K = O(j^{-1}), \ \text{as } j \to \infty \tag{4.8}$$

where

$$\widetilde{p_j} = 2C \left(\frac{p_j \mid_{(0,1)} - \left(\frac{h_j}{\mu_{2j-\sigma j}^+}, \frac{h_j}{2\sqrt{\gamma}}\right) \phi_{2j-\sigma_j}}{1-x} + S_j \right)$$

where $h_j = \mu^+_{2j} - \mu^+_{2j-\sigma_k}$ *and* $S_j = a_j + b_j x + c_j x(x-2)/2$ *with coefficients*

$$a_j = -p_j(0) + \left(\frac{h_j}{\mu^+_{2j-\sigma_j}}, \frac{h_j}{2\sqrt{\gamma}}\right)\phi_{2j-\sigma_j}(0),$$

$$b_j = -\partial\left(\frac{p_j\mid_{(0,1)} - \left(\frac{h_j}{\mu^+_{2j-\sigma_j}}, \frac{h_j}{2\sqrt{\gamma}}\right)\phi_{2j-\sigma_j}}{1-x}\right)\Bigg|_{x=1},$$

$$c_j = -\left(\partial^2\left(\frac{p_j^{(1)}\mid_{(0,1)} - \frac{h_j}{\mu^+_{2j-\sigma_j}}\phi_{2j-\sigma_j}}{1-x}\right)\Bigg|_{x=0}, 0\right),$$

and $C = 1/C(\gamma)$ *is the constant in* (3.37).

Remark 20. In the definition of $\widetilde{p_j}$ the functions p_j appear. Here there is some ambiguity in the notation since p_j has six components. For the definition of $\widetilde{p_j}$ we only use the first and fourth components of p_j. When defining S_j its coefficients are two-dimensional vectors. In the definition of the coefficient c_j of S_j, $p_j^{(1)}$ denotes the first component of the vector p_j.

Proof. We first observe that the components of ψ_j are the eigenfunctions solution of

$$-\partial^2 u = \lambda^2 u, 0 < x < 1;\ u(0) = \partial u(1) = 0.$$

It is easy to see that ψ_j have been normalized to be orthonormal in K.

Let us prove the second part of the lemma. For simplicity, we assume that $j > 0$.

In view of Proposition 5 and 6 and the asymptotic formulas we have obtained for the eigenvalues in section 3.4 it follows that

$$\begin{aligned}\bar{\phi}_{2j-1} = (1, i\lambda_{2j-1})\frac{\rho_{2j-1}}{\left(\mu^+_{2j-1}\right)^2}\Bigg[\sin\left(\mu^+_{2j-1}(1-x)\right) \\ -\frac{\mu^+_{2j-1}\cos\mu^+_{2j-1}}{\mu^-_{2j-1}\cosh\mu^-_{2j-1}}\sinh\left(\mu^-_{2j-1}(1-x)\right)\Bigg]\end{aligned}$$

$$\bar{\phi}_{2j} = -(1, i\lambda_{2j})\frac{\rho_{2j}}{\left(\mu^+_{2j}\right)^2}\left[\sin\left(\mu^+_{2j}(1-x)\right) - \frac{\sin\left(\mu^+_{2j}\right)\sinh\left(\mu^-_{2j}(1-x)\right)}{\sinh\left(\mu^-_{2j}\right)}\right]$$

in the interval (0,1), where ρ_j are of the order of $1 + O(j^{-1})$ as we will see below. Recall that $\bar{\phi}_j$ have been normalized to constitute an orthonormal basis of the energy space $\mathcal{H} = H_0 \times H_{-1/2}$ and that their sign has been chosen so that the first component of $(-1)^j\partial\bar{\phi}_j(1)$ has positive sign.

On the other hand, by (3.32) and (3.35) we have

$$\mu_{2j}^{+}=j\pi-\frac{\pi}{2}+O(j^{-1});\ \mu_{2j}^{-}=\sqrt{\frac{\left(\mu_{2j}^{+}\right)^{2}}{\gamma\left(\mu_{2j}^{+}\right)^{2}+1}}=\gamma^{-1/2}+O(j^{-1})$$

$$\mu_{2j-1}^{+}=j\pi-\frac{\pi}{2}+O(j^{-1});\ \mu_{2j-1}^{-}=\sqrt{\frac{\left(\mu_{2j-1}^{+}\right)^{2}}{\gamma\left(\mu_{2j-1}^{+}\right)^{2}+1}}=\gamma^{-1/2}+O(j^{-1})$$

and therefore $\cos\left(\mu_{j}^{+}\right)=O(j^{-1})$; $\sin\left(\mu_{j}^{+}\right)=O(1)$.

Thus, the terms in $\bar{\phi}_j$ or in some of its derivatives in which hyperbolic functions appear are of the order of $1/j$ in $L^{\infty}(0,1)$.

Before computing the norm $\| \widetilde{p}_j-\psi_j \|_K$ some remarks are in order.

When computing the norm $\| \widetilde{p}_j-\psi_j \|_K$ one is led to estimate the norm of the second derivative of $\phi_{2j-1}/(1-x)$. This is actually the quantity in which hyperbolic functions appear and that produces the largest contribution. The term one obtains is as follows:

$$i\lambda_{2j-1}\frac{\rho_{2j-1}\cos\left(\mu_{2j-1}^{+}\right)}{\mu_{2j-1}^{+}\mu_{2j-1}^{-}\cosh\left(\mu_{2j-1}^{-}\right)}\partial^{2}\left[\frac{\sinh\left(\mu_{2j-1}^{-}(1-x)\right)}{1-x}\right]$$

which is of the order of j^{-1}, since the function $\sinh\left(\mu_{2j-1}^{-}(1-x)\right)/(1-x)$ and its derivatives are bounded in $L^{\infty}(0,1)$.

If we denote by $S_j^{(1)}$ and $S_j^{(2)}$ the first and second components of the polynomial S_j, we have $\partial^3 S_j^{(1)}=\partial^2 S_j^{(2)}=0$. Therefore, when computing the norm of $\widetilde{p}_j-\psi_j$ in K, the polynomial S_j does not affect the computations.

We now proceed to the proof of (4.8) in two steps. In the first one we estimate the H^3-norm of the first component of $\widetilde{p}_j-\psi_j$ while in the second step we compute the H^2-norm of the second component.

Step 1. We have to prove that

$$\int_0^1\left|\partial^3\left[\frac{\left(\frac{-\rho_{2j}}{(\mu_{2j}^{+})^2}\sin\left(\mu_{2j}^{+}(1-x)\right)+\frac{\rho_{2j-1}}{(\mu_{2j-1}^{+})^2}\sin\left(\mu_{2j-1}^{+}(1-x)\right)\right)}{C(\gamma)(1-x)}\right.\right.$$

$$\left.\left.-\frac{2\rho_{2j-1}h_j\sin\left(\mu_{2j-1}^{+}(1-x)\right)}{(\mu_{2j-1}^{+})^3C(\gamma)(1-x)}\right]-\sin\left((j\pi-\pi/2)(1-x)\right)+O(j^{-1})^2\right|^2$$

$$\leq\frac{(\rho_{2j}-\rho_{2j-1})^2}{C^2(\gamma)}\int_0^1\left|\partial^3\left(\frac{\sin\left(\mu_{2j}^{+}(1-x)\right)}{\left(\mu_{2j}^{+}\right)^2(1-x)}\right)\right|^2dx$$

$$+2\rho_{2j-1}^2\int_0^1\left|\frac{1}{C(\gamma)}\partial^3\left[\frac{\left(\frac{-\sin\left(\mu_{2j}^{+}(1-x)\right)}{\left(\mu_{2j}^{+}\right)^2}+\frac{\sin\left(\mu_{2j-1}^{+}(1-x)\right)}{\left(\mu_{2j-1}^{+}\right)^2}\right)}{1-x}\right.\right. \tag{4.9}$$

$$-\frac{2h_j\sin\left(\mu^+_{2j-1}(1-x)\right)}{\left(\mu^+_{2j-1}\right)^3(1-x)}\Bigg] - \frac{\sin\left((j\pi-\pi/2)(1-x)\right)+O(j^{-1})}{\rho_{2j-1}}\Bigg|^2 dx \quad (4.10)$$

is of the order of $O(j^{-1})$. Observe that in the integral above all the terms in which hyperbolic functions appear have been bounded by $O(j^{-1})$.

To do that we need the following Lemma:

Lemma 21. *We have $\rho_j = 1+O(j^{-1})$ as $j\to\infty$. Moreover, $\rho_{2j}-\rho_{2j-1} = O(j^{-3})$.*

Proof. Let us consider first the even eigenfunctions. According to the normalization in $H^2\cap H^1_0(-1,1)$ we have

$$1 = \parallel \phi_{2j-1}\parallel^2_{H^2\cap H^1_0(-1,1)} = \int_{-1}^1 \mid \partial^2\phi_{2j-1}\mid^2 dx = 2\int_0^1 \mid \partial^2\phi_{2j-1}\mid^2 dx$$

$$= 2\rho^2_{2j-1}\int_0^1 \left|\sin\left(\mu^+_{2j-1}(1-x)\right) + \frac{\mu^-_{2j-1}\cos\mu^+_{2j-1}\sinh\left(\mu^-_{2j-1}(1-x)\right)}{\mu^+_{2j-1}\cosh\mu^-_{2j-1}}\right|^2 dx$$

$$= 2\rho^2_{2j-1}\int_0^1 \Big[\sin^2\left(\mu^+_{2j-1}(1-x)\right)$$
$$+2\frac{\mu^-_{2j-1}\cos\mu^+_{2j-1}}{\mu^+_{2j-1}\cosh\mu^-_{2j-1}}\sin\left(\mu^+_{2j-1}(1-x)\right)\sinh\left(\mu^-_{2j-1}(1-x)\right)$$
$$+\left(\frac{\mu^-_{2j-1}\cos\mu^+_{2j-1}}{\mu^+_{2j-1}\cosh\mu^-_{2j-1}}\right)\sinh^2\left(\mu^-_{2j-1}(1-x)\right)\Big] dx = 2\rho^2_{2j-1}\left(I_1+I_2+I_3\right).$$

Let us analyze now each of these integrals:

$$I_1 = \int_0^1 \sin^2\left(\mu^+_{2j-1}(1-x)\right)dx = \frac{1}{2} - \frac{\cos(2\mu^+_{2j-1})}{4\mu^+_{2j-1}};$$

$$I_2 = 2\frac{\mu^-_{2j-1}\cos\mu^+_{2j-1}}{\mu^+_{2j-1}\cosh\mu^-_{2j-1}}\int_0^1 \sin\left(\mu^+_{2j-1}(1-x)\right)\sinh\left(\mu^-_{2j-1}(1-x)\right)dx$$

and, by integrating by parts,

$$\int_0^1 \sin\left(\mu^+_{2j-1}(1-x)\right)\sinh\left(\mu^-_{2j-1}(1-x)\right)dx =$$
$$-\left(\frac{\mu^-_{2j-1}}{\mu^+_{2j-1}}\right)^2\int_0^1 \sin\left(\mu^+_{2j-1}(1-x)\right)\sinh\left(\mu^+_{2j-1}(1-x)\right)dx$$
$$-\frac{\cos\mu^+_{2j-1}\sinh\mu^-_{2j-1}}{\mu^+_{2j-1}} + \frac{\mu^-_{2j-1}}{\left(\mu^+_{2j-1}\right)^2}\sin\left(\mu^+_{2j-1}\right)\cosh\left(\mu^-_{2j-1}\right)$$

and therefore

$$\int_0^1 \sin\left(\mu^+_{2j-1}(1-x)\right)\sinh\left(\mu^-_{2j-1}(1-x)\right)dx$$

$$= \frac{\left(\frac{\mu^-_{2j-1}}{(\mu^+_{2j-1})^2}\sin\left(\mu^+_{2j-1}\right)\cosh\left(\mu^-_{2j-1}\right) - \frac{\cos\mu^+_{2j-1}\sinh\mu^-_{2j-1}}{\mu^+_{2j-1}}\right)}{1+\left(\frac{\mu^-_{2j-1}}{\mu^+_{2j-1}}\right)^2}$$

$$= \frac{O(j^{-2})}{1+O(j^{-2})} = O(j^{-2}).$$

Thus $I_2 = O(j^{-4})$.

Finally, it is easy to see that $I_3 = O(j^{-4})$. Therefore

$$\begin{aligned} \rho_{2j-1} &= \left(\frac{1-\cos\left(2\mu^+_{2j-1}\right)}{2\mu^+_{2j-1}} + O(j^{-4})\right)^{-1/2} \\ &= \frac{1+\cos\left(2\mu^+_{2j-1}\right)}{4\mu^+_{2j-1}} + \frac{3\sin^2\left(2\mu^+_{2j-1}\right)}{32\left(\mu^+_{2j-1}\right)^2} + O(j^{-4}). \end{aligned}$$

On the other hand,

$$\begin{aligned} \cos\left(2\mu^+_{2j-1}\right) &= 1 - \frac{\left(2j\pi-\pi-2\mu^+_{2j-1}\right)^2}{2} + O\left(2j\pi-\pi-2\mu^+_{2j-1}\right)^4 \\ &= 1+O(j^{-2}) \end{aligned}$$

and

$$\frac{1}{\mu^+_{2j-1}} = \frac{1}{j\pi-\pi/2+O(j^{-1})} = \frac{1}{j\pi-\pi/2} + O(j^{-3}).$$

Therefore $\rho_{2j-1} = 1 + O\left(j^{-1}\right)$.

In a similar way we can compute ρ_{2j}:

$$\begin{aligned} 1 &= \| \phi_{2j} \|^2_{H^2\cap H^1_0(-1,1)} \\ &= 2\rho^2_{2j}\int_0^1 \left|\sin\left(\mu^+_{2j}(1-x)\right) + \frac{\left(\mu^-_{2j}\right)^2\sin\mu^+_{2j}}{\left(\mu^+_{2j}\right)^2\sinh\mu^-_{2j}}\sinh\left(\mu^-_{2j}(1-x)\right)\right|^2 dx \\ &= 2\rho^2_{2j}\int_0^1 \Big[\sin^2\left(\mu^+_{2j}(1-x)\right) \\ &\quad + 2\frac{\left(\mu^-_{2j}\right)^2\sin\mu^+_{2j}}{\left(\mu^+_{2j}\right)^2\sinh\mu^-_{2j}}\sin\left(\mu^+_{2j}(1-x)\right)\sinh\left(\mu^-_{2j}(1-x)\right) \\ &\quad + \frac{\left(\mu^-_{2j}\right)^4\sin^2\mu^+_{2j}}{\left(\mu^+_{2j}\right)^4\sinh^2\mu^-_{2j}}\sinh^2\left(\mu^-_{2j-1}(1-x)\right)\Big] dx \\ &= 2\rho^2_{2j}\int_0^1 \sin^2\left(\mu^+_{2j}(1-x)\right)dx + O(j^{-4}). \end{aligned}$$

Notice that the last integral in this identity coincides with I_1 above. Therefore, proceeding as in the estimate of ρ_{2j-1} we deduce that

$$\rho_{2j} = 1 + \frac{\cos(2\mu^+_{2j})}{4\mu^2_{2j}} + \frac{3\cos^2(2\mu^+_{2j}}{32(\mu^+_{2j})^+} + O(j^{-4})$$

but, again,

$$\cos\left(2\mu_{2j}^+\right) = 1 - \frac{\left(2j\pi - \pi - 2\mu_{2j}^+\right)^2}{2} + O\left(2j\pi - \pi - 2\mu_{2j}^+\right)^4 = O(j^{-1})$$

and

$$\frac{1}{\mu_{2j}^+} = \frac{1}{j\pi - \pi/2 + O(j^{-1})} = \frac{1}{j\pi - \pi/2} + O(j^{-3}).$$

Therefore, $\rho_{2j} = 1 + O(j^{-1})$.

Observe however that $\rho_{2j} - \rho_{2j-1} = O(j^{-3})$. □

Let us go back to the proof of Lemma 19 and more precisely to the inequality (4.9).

Observe that

$$\begin{aligned} \left|\partial^3\left(\frac{\sin(\mu_{2j}^+(1-x))}{(\mu_{2j}^+)^2(1-x)}\right)\right| &= \left(\mu_{2j}^+\right)^2\left|\partial_y^3\left(\frac{\sin y}{y}\right)\right|_{y=\mu_{2j}^+(1-x)} \\ &\leq C(\mu_{2j-1}^+)^2 = O(j^2). \end{aligned}$$

Thus, in view of Lemma 21, the first term on the right hand side of (4.9) is of the order of $O(j^{-2})$.

Let us consider now the second term on the right hand side of (4.9). To simplify the notation we set

$$\alpha_j = \sin\left(\mu_j^+(1-x)\right)/(1-x); \quad \beta_j = \cos\left(\mu_j^+(1-x)\right)/(1-x).$$

Then, the second term on the right hand side of (4.9) can be bounded above by:

$$\begin{aligned} &\rho_{2j-1}^2 h_j^2\left|\frac{1}{C(\gamma)}\partial^3\left(\frac{\alpha_{2j}}{h_j(\mu_{2j}^+)^2} - \frac{\alpha_{2j-1}}{h_j(\mu_{2j-1}^+)^2} + \frac{2\alpha_{2j-1}}{(\mu_{2j-1}^+)^3} - \frac{\beta_{2j-1}(1-x)}{(\mu_{2j-1}^+)^2}\right)\right|^2 \\ &+4\rho_{2j-1}^2 h_j^2\int_0^1\left|\frac{\partial^3\left(\frac{\beta_{2j-1}(1-x)}{(\mu_{2j-1}^+)^2}\right)}{C(\gamma)} + \frac{\sin\left((j\pi-\pi/2)(1-x)\right)}{2h_j} + O(1)\right|^2 \\ &= 4\rho_{2j-1}^2 h_j^2\int_0^1\left(\frac{|A_1|^2}{C^2(\gamma)} + 4\,|A_2|^2\right)dx. \end{aligned} \tag{4.11}$$

Since $h_j = O(j^{-1})$ and $\rho_{2j-1} = O(1)$, it is sufficient to prove that $|A_1|$ and $|A_2|$ are uniformly bounded with respect to j.

Let us consider first $|A_1|$. Writing $\mu_{2j}^+ = \mu_{2j-1}^+ + h_j$ we have

$$\begin{aligned} |A_1| &= \left|\partial^3\left[\frac{\alpha_{2j}-\alpha_{2j-1}}{(\mu_{2j-1}^+ + h_j)^2 h_j} + \frac{\alpha_{2j-1}}{h_j}\left(\frac{1}{(\mu_{2j-1}^+ + h_j)^2} - \frac{1}{(\mu_{2j-1}^+)^2}\right) + \right.\right. \\ &\qquad \left.\left. + \frac{2\alpha_{2j-1}}{(\mu_{2j-1}^+)^3} - \frac{\beta_{2j-1}}{(\mu_{2j-1}^+)^2}(1-x)\right]\right| \\ &\leq \left|\frac{\partial^3\left(\frac{\alpha_{2j}-\alpha_{2j-1}}{h_j}\right)}{(\mu_{2j-1}^+ + h_j)^2} - \frac{\partial^3\left(\beta_{2j-1}(1-x)\right)}{(\mu_{2j-1}^+)^2}\right| \\ &\quad + \left|\left(\frac{1}{h_j(\mu_{2j-1}^+ + h_j)^2} - \frac{1}{h_j(\mu_{2j-1}^+)^2} + \frac{2}{(\mu_{2j-1}^+)^3}\right)\partial^3\alpha_{2j-1}\right| \end{aligned}$$

$$\begin{aligned}
&\leq \left|\left(\frac{1}{(\mu_{2j-1}^{+}+h_j)^2}-\frac{1}{(\mu_{2j-1}^{+})^2}\right)\partial^3\left(\frac{\alpha_{2j}-\alpha_{2j-1}}{h_j}\right)\right|+\\
&\quad+\frac{1}{(\mu_{2j-1}^{+})^2}\left|\partial^3\left(\left(\frac{\alpha_{2j}-\alpha_{2j-1}}{h_j}\right)-\beta_{2j-1}(1-x)\right)\right|\\
&+ \left|\left(\frac{1}{h_j(\mu_{2j-1}^{+}+h_j)^2}-\frac{1}{h_j(\mu_{2j-1}^{+})^2}+\frac{2}{(\mu_{2j-1}^{+})^3}\right)\partial^3\alpha_{2j-1}\right|\\
&= B_1+B_2+B_3. \qquad (4.12)
\end{aligned}$$

Let us see that the terms $B_i, i=1,2,3$ are uniformly bounded with respect to j. We first observe that

$$\begin{aligned}
\alpha_{2j}(1-x) &= \sin\left(\mu_{2j}^{+}(1-x)\right)=\sin\left(\left(\mu_{2j-1}^{+}+h_j\right)(1-x)\right)\\
&= \sin\left(\mu_{2j-1}^{+}(1-x)\right)\cos\left(h_j(1-x)\right)+\cos\left(\mu_{2j-1}^{+}(1-x)\right)\sin\left(h_j(1-x)\right)
\end{aligned}$$

and therefore

$$\begin{aligned}
B_2 &= \frac{1}{(\mu_{2j-1}^{+})^2}\left|\partial^3\left(\frac{\alpha_{2j-1}\cos\left(h_j(1-x)\right)+\beta_{2j-1}\sin\left(h_j(1-x)\right)}{h_j}\right.\right.\\
&\qquad\left.\left.-\frac{\alpha_{2j-1}}{h_j}-\beta_{2j-1}(1-x)\right)\right|\\
&\leq \frac{1}{(\mu_{2j-1}^{+})^2}\left|\partial^3\left(\beta_{2j-1}(1-x)\frac{\sin(h_j(1-x))-h_j(1-x)}{h_j(1-x)}\right)\right|+\\
&\quad+\frac{1}{(\mu_{2j-1}^{+})^2}\left|\partial^3\left(\alpha_{2j-1}(1-x)\frac{\cos(h_j(1-x))-1}{h_j(1-x)}\right)\right|. \qquad (4.13)
\end{aligned}$$

In order to estimate these two terms we need the following Lemma:

Lemma 22. *For $n=0,1,2,3$ we have*

$$\partial^n\left(\frac{\cos\left(h_j(1-x)\right)-1}{h_j(1-x)}\right)=O(h_j)=O(j^{-1});$$

$$\partial^n\left(\frac{\sin\left(h_j(1-x)\right)-h_j(1-x)}{h_j(1-x)}\right)=O(h_j).$$

Proof. We only prove the first identity since the second one can be proved in a similar way.

We have

$$\partial^n\left(\frac{\cos(h_j(1-x))-1}{h_j(1-x)}\right)=\partial^n\left(\frac{-\frac{h_j^2(1-x)^2}{2}+\frac{h_j^4(1-x)^4}{4!}\cdots}{h_j(1-x)}\right)$$

$$=\partial^n\left(-\frac{h_j(1-x)}{2!}+\frac{h_j^3(1-x)^3}{4!}\cdots\right).$$

The power series inside the derivation operator ∂^n in the last term converges uniformly for $x\in[0,1]$. Therefore its derivate can be computed term by term and the result follows immediately. □

Going back to (4.12) and taking into account that when taking derivatives of $\alpha_{2j-1}(1-x)$ we get μ_{2j-1}^{+} as multiplicative factor it follows that

$$\frac{1}{(\mu_{2j-1}^{+})^2}\left|\partial^3\left(\alpha_{2j-1}(1-x)\frac{\cos(h_j(1-x))-1}{h_j(1-x)}\right)\right|\leq C\frac{(\mu_{2j-1}^{+})^3}{(\mu_{2j-1}^{+})^2}h_j=O(1).$$

In a similar way we get that the first term on the right hand side of (4.12) is uniformly bounded. This shows that B_2 is uniformly bounded.

Let us consider now the term B_1. Let us first observe that

$$\frac{1}{(\mu^+_{2j-1}+h_j)^2}-\frac{1}{(\mu^+_{2j-1})^2}=\frac{-h_j^2-2\mu^+_{2j-1}h_j}{(\mu^+_{2j-1}+h_j)^2(\mu^+_{2j-1})^2}=O(j^{-4}).$$

On the other hand,

$$\begin{aligned}\left|\partial^3\left(\frac{\alpha_{2j}-\alpha_{2j-1}}{h_j}\right)\right| \quad &\le \quad \left|\partial^3\left(\frac{\alpha_{2j}-\alpha_{2j-1}}{h_j}-\beta_{2j-1}(1-x)\right)\right| \\ &\quad +\left|\partial^3\left(\beta_{2j-1}(1-x)\right)\right|. \end{aligned} \tag{4.14}$$

As we have seen when we estimated above the term B_2, the first term on the right hand side of (4.13) is of order $O(j^2)$. The second one is of order $O(j^3)$. Thus $B_1=O(j^{-1})$.

Let us analyze B_3. We have $\mid \partial^3\alpha_{2j-1}\mid = O(j^4)$. On the other hand, the coefficient in B_3 is such that

$$\begin{aligned}&\left|\frac{(\mu^+_{2j-1})^3-(\mu^+_{2j-1}-2h_j)(\mu^+_{2j-1}+h_j)^2}{h_j(\mu^+_{2j-1})^2(\mu^+_{2j-1}+h_j)^2}\right| \\ &\qquad =\left|\frac{2h_j^3+4\mu^+_{2j-1}h_j-(\mu^+_{2j-1})^2h_j^2}{h_j(\mu^+_{2j-1})^3(\mu^+_{2j-1}+h_j)^2}\right|=O(j^{-4}).\end{aligned}$$

Thus, B_3 is uniformly bounded too.

Going back to (4.11) we see that $\mid A_1\mid$ is uniformly bounded.

In order to complete the Step 1 we have to show that $\mid A_2 \mid$ is also uniformly bounded. We have

$$\begin{aligned}\mid A_2\mid \quad &= \quad \left|\frac{1}{2C(\gamma)}\partial^3\left(\frac{\beta_{2j-1}(1-x)}{(\mu^+_{2j-1})^2}\right)+\frac{\sin((j\pi-\pi/2)(1-x))}{2h_j}+O(1)\right| \\ &= \quad \left|-\frac{\mu^+_{2j-1}}{2C(\gamma)}\sin\left(\mu^+_{2j-1}(1-x)\right)+\frac{\sin((j\pi-\pi/2)(1-x))}{2h_j}\right|+O(1)\end{aligned}$$

On the other hand, in view of (3.32) and (3.35) we have

$$\frac{1}{\mu^+_{2j}-\mu^+_{2j-1}}=\frac{j\pi-\pi/2}{C(\gamma)}+O(1)=\frac{\mu^+_{2j-1}}{C(\gamma)}+O(1)$$

and therefore

$$\left|\frac{\mu^+_{2j-1}}{C(\gamma)}-\frac{1}{\mu^+_{2j}-\mu^+_{2j-1}}\right|=O(1). \tag{4.15}$$

Thus

$$\mid A_2\mid=\frac{1}{2h_j}\left|\sin\left(\mu^+_{2j-1}(1-x)\right)-\sin\left((j\pi-\pi/2)(1-x)\right)\right|+O(1)=O(1)$$

since $\mu^+_{2j-1}-(j\pi-\pi/2)$ and h_j are of the order of $O(j^{-1})$. This completes the first step of the proof of (4.8).

Step 2. Let us estimate now the L^2-norm of the second derivative of the second component of the vector $\widetilde{p}_j - \psi_j$:

$$\int_0^1 \left| \frac{1}{C(\gamma)} \partial^2 \left[\frac{\left(\frac{-\lambda_{2j}\rho_{2j}}{(\mu_{2j}^+)^2} \sin(\mu_{2j}^+(1-x)) + \frac{\lambda_{2j-1}\rho_{2j-1}}{(\mu_{2j-1}^+)^2} \sin(\mu_{2j-1}^+(1-x)) \right)}{1-x} \right.\right.$$

$$\left.\left. - \frac{\frac{\rho_{2j-1}h_j}{\sqrt{\gamma}(\mu_{2j-1}^+)^2} \sin(\mu_{2j-1}^+(1-x))}{1-x} \right] - \frac{1}{2\sqrt{\gamma}} \cos((j\pi - \pi/2)(1-x)) + O(j^{-1}) \right|^2$$

$$\leq \frac{1}{2} (\rho_{2j} - \rho_{2j-1})^2 \int_0^1 \left| \frac{1}{C(\gamma)} \partial^2 \left(\frac{\lambda_{2j} \sin\left(\mu_{2j}^+(1-x)\right)}{\left(\mu_{2j}^+\right)^2 (1-x)} \right) \right|^2 +$$

$$+2\rho_{2j-1}^2 \int_0^1 \left| \frac{1}{C(\gamma)} \partial^2 \left((1-x)^{-1} \left[\frac{-\lambda_{2j} \sin\left(\mu_{2j}^+(1-x)\right)}{2\left(\mu_{2j}^+\right)^2} \right.\right.\right.$$

$$\left.\left. + \frac{\lambda_{2j-1} \sin\left(\mu_{2j-1}^+(1-x)\right)}{2\left(\mu_{2j-1}^+\right)^2} - \frac{h_j \sin\left(\mu_{2j-1}^+(1-x)\right)}{\sqrt{\gamma}\left(\mu_{2j-1}^+\right)^2} \right] \right)$$

$$\left. - \frac{1}{2\sqrt{\gamma}} \cos\left((j\pi - \pi/2)\right)(1-x) + O(j-1) \right|^2 = A_1 + A_2. \tag{4.16}$$

To do that we have to show that both terms A_1 and A_2 are of the order of $O(j^{-1})$.

In view of Lemma 21, $\rho_{2j} - \rho_{2j-1} = O(j^{-3})$ while

$$\left| \partial^2 \left(\frac{\sin\left(\mu_{2j}^+(1-x)\right)}{\mu_{2j}^+(1-x)} \right) \right| = O\left(\left(\mu_{2j}^+\right)^2 \right) = O(j^2).$$

Therefore $A_1 = O(j^{-2})$, since $\lambda_{2j}/\mu_{2j}^+ = O(1)$.

In what concerns A_2 we first observe that it can be easily bounded as follows:

$$A_2 \leq C h_j^2 \int_0^1 \left| \partial^2 \left(\frac{\lambda_{2j}\alpha_{2j}}{2h_j\left(\mu_{2j}^+\right)^2} - \frac{\lambda_{2j-1}\alpha_{2j-1}}{2h_j\left(\mu_{2j-1}^+\right)^2} \right.\right.$$

$$\left.\left. + \frac{\alpha_{2j-1}}{\left(\mu_{2j-1}^+\right)^2\sqrt{\gamma}} - \frac{\beta_{2j-1}(1-x)}{2\mu_{2j-1}^+\sqrt{\gamma}} \right) \right|^2$$

$$+ \frac{C h_j^2}{\gamma} \int_0^1 \left| \frac{1}{C(\gamma)} \partial^2 \left(\frac{\beta_{2j-1}(1-x)}{\mu_{2j-1}^+} \right) + \frac{\cos\left((j\pi - \pi/2)(1-x)\right)}{h_j} + O(1) \right|^2$$

$$\leq C h_j^2 \left(\frac{1}{C^2(\gamma)} \mid A_2^1 \mid^2 + \mid A_2^2 \mid^2 \right) \tag{4.17}$$

for a suitable choice of positive constant C.

Since $h_j = O(j^{-1})$, it is sufficient to see that both A_2^1 and A_2^2 are uniformly bounded.

Let us consider first the term A_2^1. We recall that (see Propositions 10 and 12)

$$\lambda_j = \frac{\mu_j^+}{\sqrt{\gamma}} - \frac{1}{2\gamma^{3/2}\mu_j^+} + O(j^{-2}),$$

and therefore

$$\begin{aligned}
A_2^1 \;\le\; & \left|\partial^2\left(\frac{\alpha_{2j}}{2\sqrt{\gamma}h_j\mu_{2j}^+} - \frac{\alpha_{2j}}{4\gamma^{3/2}\left(\mu_{2j}^+\right)^3 h_j} + \frac{O\left(j^{-2}\right)\alpha_{2j}}{h_j\left(\mu_{2j}^+\right)^2}\right.\right.\\
& - \frac{\alpha_{2j-1}}{2\sqrt{\gamma}h_j\mu_{2j-1}^+} + \frac{\alpha_{2j-1}}{4\gamma^{3/2}\left(\mu_{2j-1}^+\right)^3 h_j} - \frac{O(j^{-2})\alpha_{2j-1}}{h_j\left(\mu_{2j-1}^+\right)^2}\\
& \left.\left. + \frac{\alpha_{2j-1}}{\sqrt{\gamma}\left(\mu_{2j-1}^+\right)^2} - \frac{\beta_{2j-1}(1-x)}{2\sqrt{\gamma}\mu_{2j-1}^+}\right)\right|\\
\le\; & \frac{1}{2\sqrt{\gamma}}\left|\partial^2\left(\frac{\alpha_{2j}}{h_j\mu_{2j}^+} - \frac{\alpha_{2j-1}}{h_j\mu_{2j-1}^+} + \frac{2\alpha_{2j-1}}{\left(\mu_{2j-1}^+\right)^2} - \frac{\beta_{2j-1}(1-x)}{\mu_{2j-1}^+}\right)\right|\\
& + \frac{1}{4\gamma^{3/2}}\left|\partial^2\left(\frac{\alpha_{2j}}{\left(\mu_{2j}^+\right)^3 h_j} - \frac{\alpha_{2j-1}}{\left(\mu_{2j-1}^+\right)^3 h_j}\right)\right| + \left|\partial^2\left(\frac{O(j^{-2})\alpha_{2j}}{h_j\left(\mu_{2j}^+\right)}\right)\right|\\
& + \left|\partial^2\left(\frac{O(j^{-2})\alpha_{2j-1}}{h_j\left(\mu_{2j-1}^+\right)^2}\right)\right| = B_1 + B_2 + B_3 + B_4. \qquad (4.18)
\end{aligned}$$

Let us see that $B_i, i = 1, \cdots, 4$ are uniformly bounded with respect to j.

The structure of the term B_1 is the same as term A_1 in step 1 (see (4.10)). By similar arguments it is easy to see that B_1 is uniformly bounded (notice that term A_1 is of the order of $\frac{1}{j}\partial B_1$ in (4.17) and that the multiplicative factor $\frac{1}{j}$ compensates the boundedness of an extra derivative).

Concerning B_2 we first observe that

$$\begin{aligned}
B_2 \le & \frac{1}{4\gamma^{3/2}}\left|\partial^2\left(\frac{\alpha_{2j}}{\left(\mu_{2j}^+\right)^3 h_j} - \frac{\alpha_{2j-1}}{\left(\mu_{2j-1}^+\right)^3 h_j} + \frac{2\alpha_{2j-1}}{\left(\mu_{2j-1}^+\right)^4} - \frac{\beta_{2j-1}(1-x)}{\left(\mu_{2j-1}^+\right)^3}\right)\right|\\
& + \frac{1}{2\gamma^{3/2}}\left|\partial^2\left(\frac{\alpha_{2j-1}}{\left(\mu_{2j-1}^+\right)^4}\right)\right| + \frac{1}{4\gamma^{3/2}}\left|\partial^2\left(\frac{\beta_{2j-1}(1-x)}{\left(\mu_{2j-1}^+\right)^3}\right)\right| = C_1 + C_2 + C_3.
\end{aligned}$$

The term C_1 can be estimated as B_1 in (4.17). Concerning C_2 we have:

$$C_2 \le \frac{C}{\left(\mu_{2j-1}^+\right)^3}\left|\partial^2\left(\frac{\alpha_{2j-1}}{\mu_{2j-1}^+}\right)\right| = O\left(1/\mu_{2j-1}^+\right) = O(j^{-1}).$$

We also have:

$$C_3 = \frac{C}{\left(\mu_{2j-1}^+\right)^3}\left|\partial^2\left(\cos\left(\mu_{2j-1}^+(1-x)\right)\right)\right| = O\left(1/\mu_{2j-1}^+\right) = O(j^{-1}).$$

Therefore B_2 is uniformly bounded too.

Finally, it is also easy to see that the term B_3 and B_4 in (4.16) are uniformly bounded. This completes the proof of the uniform boundedness of A_2^1.

To conclude the proof of this Lemma we have to show now that A_2^2 as in (4.16) is uniformly bounded too.

We have

$$\begin{aligned}
\mid A_2^2 \mid \quad &\leq \quad \left| \frac{1}{C(\gamma)} \partial^2 \left(\frac{\beta_{2j-1}(1-x)}{\mu_{2j-1}^+} \right) + \frac{\cos\left((j\pi - \pi/2)(1-x)\right)}{h_j} \right| + O(1) \\
&= \quad \left| \frac{\mu_{2j-1}^+}{C(\gamma)} \cos\left(\mu_{2j-1}^+(1-x)\right) - \frac{\cos\left((j\pi - \pi/2)(1-x)\right)}{h_j} \right| + O(1) \\
&\leq \quad \left| \frac{\mu_{2j-1}^+}{C(\gamma)} - \frac{1}{h_j} \right| \left| \cos\left(\mu_{2j-1}^+(1-x)\right) \right| \\
&\qquad + \frac{1}{h_j} \left| \cos\left(\mu_{2j-1}^+(1-x)\right) - \cos\left((j\pi - \pi/2)(1-x)\right) \right| + O(1)
\end{aligned}$$

and each of these can be shown to be uniformly bounded since, in view of (4.14), $\mu_{2j-1}^+/C(\gamma) - 1/h_j = O(1)$ and, on the other hand, $\mu_{2j-1}^+ - (j\pi - \pi/2) = O(j-1)$.

□

We also need the following result:

Lemma 23. *Let us set*

$$\Phi_{2j-\sigma_j} = \begin{cases} -\frac{\bar{\phi}_{2j}+\bar{\phi}_{2j-\sigma_j}}{2} & in \quad (-1,0) \\ \frac{\bar{\phi}_{2j}-\bar{\phi}_{2j-\sigma_j}}{2} & in \quad (0,1) \end{cases} \tag{4.19}$$

and define its regularization

$$\widetilde{\Phi}_{2j-\sigma_j} = \frac{1}{C(\gamma)} \left(\Phi_{2j-\sigma_j} + \delta_j^{-1} \partial q_j(0) r(x) \right) \tag{4.20}$$

where

$$r(x) = \begin{cases} (1+x)^3/3 & in \quad (-1,0) \\ (1-x)^3/3 & in \quad (0,1), \end{cases} \tag{4.21}$$

$\delta_j = \lambda_{2j} - \lambda_{2j-\sigma_j}$ *and* $C(\gamma)$ *is as in* (3.37).

Then

$$\| \delta_j \widetilde{\Phi}_{2j-\sigma_j} + \frac{1}{\gamma \lambda_{2j-\sigma_j}} \bar{\phi}_{2j-\sigma_j} \|_{1/2}^2 = O(j^{-2}). \tag{4.22}$$

Remark 24. $\delta_j \Phi_{2j-\sigma_j}$ is the even extension of the restriction of q_j to (0,1) and $\delta_j \widetilde{\Phi}_{2j-\sigma_j}$ is obtained from $\delta_j \Phi_{2j-\sigma_j}$ by adding a polynomial function so that the jump of the first derivative at $x = 0$ vanishes to guarantee that $\delta_j \widetilde{\Phi}_{2j-\sigma_j} \in H_{1/2}$. Recall that the first component of the elements of $H_{1/2}$ belongs to $H^2(-1,1)$ and therefore it is continuous with continuous derivative.

Proof. Without loss of generality we can assume that $j > 0$. Therefore $\sigma_j = 1$.

In order to prove (4.21) we will use the following norm, which is equivalent to $\| \cdot \|_{1/2}$,

$$|||(u,v)||| = \left(\int_{-1}^{0} \mid \partial^3 u \mid^2 + \int_{0}^{1} \mid \partial^3 u \mid^2 + \int_{-1}^{1} \mid \partial^2 u \mid^2 \right)^{1/2}.$$

Since $\widetilde{\Phi}_{2j-1}$ and $\bar{\Phi}_{2j-1}$ are even we can work on the interval (0,1) only. In this way, taking into account that $\bar{\phi}_j = (\phi_j, i\lambda_j\phi_j)$ we have to estimate

$$2\int_0^1 \left| \delta_j \frac{\partial^3\phi_{2j} - \partial^3\phi_{2j-1}}{2C(\gamma)} + \frac{\delta_j \partial\phi_{2j}(0)\partial^3 r_j}{2C(\gamma)} + \frac{\lambda_{2j-1}^{-1}}{\gamma}\partial^3\phi_{2j-1} \right|^2 \tag{4.23}$$
$$+2\int_0^1 \left| \delta_j \frac{i\lambda_{2j}\partial^2\phi_{2j} - i\lambda_{2j-1}\partial^2\phi_{2j-1}}{2C(\gamma)} + C(\gamma)\delta_j \frac{\partial\phi_{2j(0)}}{2C(\gamma)} i\lambda_{2j}\partial^2 r + \frac{i\partial^2\phi_{2j-1}}{\gamma} \right|^2$$

since $\partial q_j(0) = \delta_j \left(\partial\phi_{2j}(0) - \partial\phi_{2j-1}(0)\right)/2 = \delta_j\partial\phi_{2j}(0)/\,2 = O(j^{-2})$. Therefore, all the terms in which the polynomial r appears are of the order of $O(j^{-1})$.

On the other hand, recall that, in view of (3.36):

$$\begin{aligned} \frac{\delta_j}{C(\gamma)} &= \frac{\lambda_{2j} - \lambda_{2j-1}}{C(\gamma)} = \frac{1}{(j\pi - \pi/2)\sqrt{\gamma}} + O(j^{-2}) = \frac{1}{\mu_{2j-1}^{+}\sqrt{\gamma}} + O(j^{-1}) \\ &= \frac{1}{\gamma\lambda_{2j-1}} + O(j^{-1}) = \frac{1}{\gamma\lambda_{2j}} + O(j^{-1}). \end{aligned}$$

Going back to (4.22) we obtain

$$\begin{aligned} &\frac{2}{\gamma}\int_0^1 \left| \frac{\partial^3\phi_{2j}}{2\mu_{2j}^{+}} - \frac{\partial^3\phi_{2j-1}}{2\mu_{2j-1}^{+}} + \frac{\partial^3\phi_{2j-1}}{\mu_{2j-1}^{+}} + O(j^{-1}) \right|^2 \\ &\quad + \frac{2}{\gamma^2}\int_0^1 \left| \frac{\partial\phi_{2j} - \partial^2\phi_{2j-1}}{2} + \partial\phi_{2j-1} + O(j^{-1}) \right|^2 \\ &\quad = \frac{1}{2\gamma}\int_0^1 \left| \frac{\partial^3\phi_{2j}}{\mu_{2j}^{+}} + \frac{\partial^3\phi_{2j-1}}{\mu_{2j-1}^{+}} + O(j^{-1}) \right|^2 \\ &\quad + \frac{1}{2\gamma^2}\int_0^1 \left| \partial^2\phi_{2j} + \partial^2\phi_{2j-1} + O(j^{-1}) \right|^2 \end{aligned} \tag{4.24}$$

since $\left|\partial^3\phi_j\right| = O(1)$ and $\left|\partial^2\phi_j\right| = O(j^{-1})$.

Rewriting (4.23) in view of the explicit form of ϕ_j we obtain

$$\begin{aligned} &\frac{1}{2\gamma}\int_0^1 \left| \rho_{2j} \left(\cos\left(\mu_{2j}^{+}(1-x)\right) + \frac{\left(\mu_{2j}^{-}\right)^3 \sin\mu_{2j}^{+}}{\left(\mu_{2j}^{+}\right)^3 \sinh\mu_{2j}^{+}} \cosh\left(\mu_{2j}^{-}(1-x)\right) \right) \right. \\ &-\rho_{2j-1} \left(\cos\left(\mu_{2j-1}^{+}(1-x)\right) + \frac{\left(\mu_{2j-1}^{-}\right)^2 \cos\mu_{2j-1}^{+}}{\left(\mu_{2j-1}^{+}\right)^2 \cosh\mu_{2j-1}^{-}} \cosh\left(\mu_{2j-1}^{-}(1-x)\right) \right) \\ &\left. +O(j^{-1}) \right|^2 dx \end{aligned}$$

$$+\frac{1}{2\gamma^2}\int_0^1 \left| \rho_{2j}\left(\sin\left(\mu_{2j}^+(1-x)\right)+\frac{\left(\mu_{2j}^-\right)^2 \sin\mu_{2j}^+}{\left(\mu_{2j}^+\right)\sinh\mu_{2j}^-}\sinh\left(\mu_{2j}^-(1-x)\right)\right)\right.$$
$$-\frac{\rho_{2j-1}}{2}\left(\sin\left(\mu_{2j-1}^+(1-x)\right)+\frac{\mu_{2j-1}^-\cos\mu_{2j-1}^+}{\mu_{2j-1}^+\cosh\mu_{2j-1}^-}\sinh\left(\mu_{2j-1}^-(1-x)\right)\right)$$
$$\left.+O(j^{-1})\right|^2 dx. \tag{4.25}$$

The terms in which the hyperbolic functions appear are of the order of $O(j^{-2})$. On the other hand, in view of Lemma 21 we know that ρ_{2j} and ρ_{2j-1} are of the order of $1+O(j^{-2})$. Therefore, the quantity in (4.24) can be written as

$$\frac{1}{2\gamma}\int_0^1 \left|\cos\left(\mu_{2j}^+(1-x)\right)-\cos\left(\mu_{2j-1}^+(1-x)\right)+O(j^{-1})\right|^2+$$
$$\frac{1}{2\gamma^2}\int_0^1 \left|\sin\left(\mu_{2j}^+(1-x)\right)-\sin\left(\mu_{2j-1}^+(1-x)\right)+O(j^{-1})\right|^2$$

which is of the order of $O(j^{-2})$ since $\mu_{2j}^+-\mu_{2j-1}^+=h_j=O(j^{-1})$. $\square$

We can now proceed to prove Theorem 18.

Proof. We proceed in two steps.

Step 1. First, let us show that any element $U\in Y$ is such that $U\mid_{(0,1)}\in H^3(0,1)\times H^2(0,1)$ and such that the compatibility conditions (4.7) holds.

In view of Proposition 17, U can be written as follows:

$$U=\sum_{j\in\mathbb{Z}\setminus\{0\}}\left(a_jp_j+b_jq_j\right),\ (a_j),\ (b_j)\in\ell^2.$$

We set $U=U^1+U^2$ with $U^1=\sum_{j\in\mathbb{Z}\setminus\{0\}}a_jp_j$ and $U^2=\sum_{j\in\mathbb{Z}\setminus\{0\}}b_jq_j$ and show that $U^1\mid_{(0,1)}\in H^3(0,1)\times H^2(0,1)$.

We set $U_r^1=U^1\mid_{(0,1)}$. Thus: $U_r^1=\sum_{j\in\mathbb{Z}\setminus\{0\}}a_jp_j\mid_{(0,1)}$.

Let us define now $\widetilde{U}_r^1=\sum_{j\in\mathbb{Z}\setminus\{0\}}a_j\widetilde{p}_j\mid_{(0,1)}$ where $\widetilde{p}_j$ is as in the statement of Lemma 19.

Let us prove that $\widetilde{U}_r^1\in K$, where K is as in Lemma 19 and that U_r^1 is as regular as $\widetilde{U}_r^1$. In view of Lemma 19 we have

$$\left\|\widetilde{U}_r^1\right\|_K^2=\left\|\sum_{j\in\mathbb{Z}\setminus\{0\}}a_j\widetilde{p}_j\right\|_K^2$$
$$\leq 2\left\|\sum_{j\in\mathbb{Z}\setminus\{0\}}a_j\left(\widetilde{p}_j-\psi_j\right)\right\|_K^2+2\left\|\sum_{j\in\mathbb{Z}\setminus\{0\}}a_j\psi_j\right\|_K^2$$
$$\leq 2\left(\sum_{j\in\mathbb{Z}\setminus\{0\}}\mid a_j\mid\|\widetilde{p}_j-\psi_j\|_K\right)^2+2\sum_{j\in\mathbb{Z}\setminus\{0\}}\mid a_j\mid^2$$

$$\leq \; 2 \sum_{j \in \mathbb{Z} \setminus \{0\}} | a_j |^2 \left(\sum_{j \in \mathbb{Z} \setminus \{0\}} \|p_j - \psi_j\|_K^2 + 1 \right)$$

$$\leq \; C \sum_{j \in \mathbb{Z} \setminus \{0\}} | a_j |^2 \left(1 + \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{1}{j^2} \right) \leq C \sum_{j \in \mathbb{Z} \setminus \{0\}} | a_j |^2 < \infty.$$

On the other hand,

$$\begin{aligned} \widetilde{U}_r^1 &= \sum_{j \in \mathbb{Z} \setminus \{0\}} a_j \widetilde{p}_j |_{(0,1)} \\ &= \sum_{a_j \in \mathbb{Z} \setminus \{0\}} a_j \left[\frac{p_j \, |_{(0,1)} - \left(\frac{h_j}{\mu^+_{2j-\sigma_j}}, \frac{h_j}{2\sqrt{\gamma}} \right) \phi_{2j-\sigma_j}}{1-x} + S_j(x) \right] \\ &= \frac{U_r^1}{1-x} - \sum_{j \in \mathbb{Z} \setminus \{0\}} a_j \left(\frac{h_j}{\mu^+_{2j-\sigma_j}}, \frac{h_j}{2\sqrt{\gamma}} \right) \frac{\phi_{2j-\sigma_j}}{1-x} + S(x) \end{aligned}$$

where $S(x) = a + bx + cx(x-2)/2$ is the polynomial with coefficients

$$a = -U^1(0) + \sum_{j \in \mathbb{Z} \setminus \{0\}} a_j \left(\frac{h_j}{\mu^+_{2j-\sigma_j}}, \frac{h_j}{2\sqrt{\gamma}} \right) \phi_{2j-\sigma_j}(0)$$

$$b = -\partial \left(\frac{U^1 \, |_{(0,1)} - \sum_{j \in \mathbb{Z} \setminus \{0\}} a_j \left(\frac{h_j}{\mu^+_{2j-\sigma_j}}, \frac{h_j}{2\sqrt{\gamma}} \right) \phi_{2j-\sigma_j}}{1-x} \right) \Bigg|_{x=1}$$

$$c = -\left(\partial \left(\frac{U^{1,(1)} \, |_{(0,1)} - \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{a_j h_j}{\sqrt{\gamma}} \phi_{2j-\sigma_j}}{1-x} \right) \Bigg|_{x=0} , 0 \right).$$

It is easy to see that none of these three coefficients is singular. On the other hand, since $a_j \in \ell^2, \; h_j = O(j^{-1})$ and $\mu^+_{2j-\sigma_j} = O(j)$,

$$\sum_{j \in \mathbb{Z} \setminus \{0\}} a_j \left(\frac{h_j}{\mu^+_{2j-\sigma_j}}, \frac{h_j}{\sqrt{\gamma}} \right) \phi_{2j-\sigma_j} \in H_{1/2}$$

and therefore, when restricted to the interval (0,1), belongs to $H^3(0,1) \times H^2(0,1)$ too. This implies that $U^1/(1-x)\,\big|_{(0,1)} \in H^3(0,1) \times H^2(0,1)$ and therefore $U^1\,\big|_{(0,1)} \in H^3(0,1) \times H^2(0,1)$ too.

The fact that the compatibility conditions (4.7) holds follows easily from the regularity we have proved for $U\,\big|_{(0,1)}$ and its Fourier series representation.

Step 2. Consider an element $U \in \mathcal{H}$ such that $U\big|_{(0,1)} \in H^3(0,1) \times H^2(0,1)$ and satisfying

$$\partial U(0^+) = \sum_{j\in \mathbb{Z}\setminus\{0\}} \left(a_k \partial p_k(0) + b_k \partial q_k(0)\right),\ \partial^2 u^0(1) = 0.$$

Since $U \in \mathcal{H}$ we have $U = \sum_{j\in\mathbb{Z}\setminus\{0\}} c_j \bar{\phi}_j$ for some coefficients $c_j \in \ell^2$. Clearly, U can also be written as follows

$$U = \sum_{j\in \mathbb{Z}\setminus\{0\}} \left(a_j p_j + \frac{b_j}{\delta_j} q_j\right),\ (a_j),\ (b_j) \in \ell^2.$$

It is sufficient to show that $(b_j/\delta_j) \in \ell^2$.

We set $U^1 = \sum_{j\in\mathbb{Z}\setminus\{0\}} a_j p_j$. As we have seen in Step 1, since $a_j \in \ell^2$ we deduce immediately that $U^1\big|_{(0,1)} \in H^3(0,1) \times H^2(0,1)$ and that U^1 also satisfies the compatibility conditions (4.7). Therefore since both U and U^1 verify the same properties, it follows that $U^2 = U - U^1$ is also such that, when restricted to (0,1), belongs to $H^3(0,1) \times H^2(0,1)$ and such that (4.7) holds. Let us denote by U_r^2 the even extension of $U^2\big|_{(0,1)}$ to the whole interval $(-1,1)$. We set

$$\widetilde{U}_r^2 = U_r^2 + (x) \sum_{j\in\mathbb{Z}\setminus\{0\}} b_j \delta_j^{-1} \partial q_j(0) \tag{4.26}$$

where r is the polynomial of (4.20). It is easy to see that the serie in (4.25) converges. On the other hand, by construction and in view of the characterization of $H_{1/2}$ given in Proposition 4 we deduce that $\widetilde{U}_r^2 \in H_{1/2}$. Moreover,

$$\begin{aligned}\widetilde{U}_r^2(x) &= \sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{b_j}{\delta_j}\delta_j \Phi_{2j-\sigma_j}(x) + \sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{b_j}{\delta_j}\partial q_j(0) r(x)\\ &= C(\gamma) \sum_{j\in\mathbb{Z}\setminus\{0\}} b_j \widetilde{\Phi}_{2j-\ sigma_j}(x).\end{aligned}$$

In view of Lemma 23 we know that, for N large enough,

$$\sum_{|j|>N} \left\| \delta_j \widetilde{\Phi}_{2j-\sigma_j} + \frac{1}{\gamma\lambda_{2j-\sigma_j}} \bar{\phi}_{2j-\sigma_j} \right\|_{1/2}^2 < 1. \tag{4.27}$$

We are in the conditions to apply the following result by Paley-Wiener on the stability of Riesz basis on Hilbert spaces (see [Y]):

Theorem 25. (Paley-Wiener) *Let $\{e_j\}$ be an orthonormal basis in a Hilbert space H. Let $\{f_j\}$ be a sequence of elements of H such that*

$$\sum_j \| e_j - f_j \|_H^2 < 1.$$

Then, $\{f_j\}$ forms a Riesz basis of H.

Taking into account that $\left\{ \left(\frac{\bar{\phi}_{2j-\sigma_j}}{\lambda_{2j-\sigma_j}} \right) \cup \left(\frac{\bar{\phi}_{2j}}{\lambda_{2j}} \right) \right\}$ forms an orthonormal basis of $H_{1/2}$ and in view of (4.26) we deduce that

$$\left\{ \left(\frac{\bar{\phi}_{2j-\sigma_j}}{\lambda_{2j-\sigma_j}} \right)_{|j|\le N} \cup \left(-\delta_j \widetilde{\Phi}_{2j-\sigma_j} \right)_{|j|\ge N} \cup \left(\frac{\bar{\phi}_{2j}}{\lambda_{2j}} \right)_{j\in\mathbb{Z}\setminus\{0\}} \right\}$$

forms a Riesz basis of $H_{1/2}$.

On the other hand,

$$\sum_{|j|>N} b_j \widetilde{\Phi}_{2j-\sigma_j} = \widetilde{U}_r^2 - \sum_{|j|\leq N} b_j \widetilde{\Phi}_{2j-\sigma_j} \in H_{1/2}$$

since both $\widetilde{U}_r^2$ and $\sum_{|j|<N} b_j \widetilde{\Phi}_{2j-\sigma_j}$ belong to $H_{1/2}$.

This implies that

$$\sum_{|j|>N} \left| \frac{bj}{\delta_j} \right|^2 < \infty$$

which is equivalent to $(b_j/\delta_j) \in \ell^2$. □

5. Comments.

The space Y is not the only asymmetric space in which system (1.1)-(1.2) is well-posed. In fact, we can construct in terms of Fourier series, in a similar way as we did for Y, the following Hilbert spaces depending on the parameter α:

$$Y_\alpha = \left\{ U = \sum_{k\in\mathbb{Z}\setminus\{0\}} a_k \bar{\phi}_k \ : \| U \|^2_{Y_\alpha} < \infty \right\} \tag{5.28}$$

where

$$\| U \|^2_{Y_\alpha} = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{| a_{2k-\sigma_k} |^2}{\delta_k^{2\alpha}} + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{| a_{2k} - a_{2k-\sigma_k} |^2}{\delta_k^{2\alpha+2}}.$$

It is easy to reproduce the proof of Proposition 16 to obtain that system (1.1)-(1.2) is well-posed in the spaces Y_α too.

These spaces may be characterized in classical terms as well. For instance, using the characterization of H_α for $\alpha = -3/2, -1, -1/2, 0$ given in Proposition 4 and following the proof of Theorem 18 one is able to conclude after some calculations that

$$\begin{aligned} Y_{-2} \;\; = \;\; & \left\{ ((u^0, y^0, z^0), (u^1, y^1, z^1)) \in H_{-1} \times H_{-3/2} \ : \right. \\ & \left. (u^0|_{(0,1)}, u^1|_{(0,1)}) \in H^1(0,1) \times L^2(0,1), \;\; u^0(1) = 0, \; u^0(0^+) = y \right\}. \end{aligned}$$

The same result can be deduced in a easier way by means of a change of variables. Consider u the solution of system (1.1)-(1.2) with initial condition $u(0) = u^0$ and $u_t(0) = u^1$ satisfying $U^0 = (u^0, u^1) \in Y$. Then $v = u_{tt}$ is also solution of (1.1)-(1.2) with initial conditions $v(0) = -K^{-1}u^0$ and $v_t(0) = -K^{-1}u^1$.

As system (1.1)-(1.2) is well-posed in Y the solution $(u(t), u_t(t)) \in Y$ for any $t > 0$. Then, $(v, v_t) = (-K^{-1}u(t), -K^{-1}u_t(t)) \in K^{-1}Y$ for any $t > 0$. In other words, system (1.1)-(1.2) is well-posed in $K^{-1}Y$.

Thanks to the characterization of the space Y and following the arguments given in the proof of Proposition 4 to characterize the spaces H_{-1} and $H_{-3/2}$, it is not difficult to conclude that $K^{-1}Y = Y_{-2}$.

The same arguments can be used to identify the asymmetric spaces Y_α for other values of α as well.

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Departamento de Matemática Aplicada, Universidad Complutense, 28040 Madrid, Spain