# Exact boundary controllability of two Euler-Bernoulli beams connected by a point mass

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December 20, 2002

**Abstract**

We consider a hybrid system consisting of two flexible beams connected by a point mass. In a previous work we showed that when the constant of rotational inertia $\gamma$ is positive, due to the presence of the mass, the system is well-posed in asymmetric spaces, i.e. spaces with different regularity to both sides of the mass. As a consequence of this, the space of controllable data when we act on the free extreme of the system is also an asymmetric space when $\gamma > 0$.

In this paper we study the case $\gamma = 0$ in which we recover the classical Euler-Bernoulli model for the beams. We prove in this case that the system is not well-posed in asymmetric spaces and then the presence of the point mass does not affect the controllability of the system. The proofs are based in the development of solutions in Fourier series and the use of non-harmonic Fourier series.

**Key words and phrases:** Flexible beams, point mass, Fourier series, controllability.

**AMS Subject Classification:** 35L30, 35P15, 42A55.

# 1 Introduction

In this paper we study the boundary controllability of a linear system modelling the vibrations of two Euler-Bernoulli beams connected by a point mass.

We assume that the beams occupy the intervals $(-1,0)$ and $(0,1)$ and that the point mass is located at $x=0$. By means of the scalar function $u=u(x,t)$ defined for $x\in(-1,1)$ and $t>0$ we describe the vertical displacements of the beams and the point mass. The linear equations describing the small vibrations of this system can be written as follows

$$\begin{cases} u_{tt}+\partial^4 u=0, & \text{for } x\in(-1,0), t>0 \\ u_{tt}+\partial^4 u=0, & \text{for } x\in(0,1), t>0 \\ [u]\,(0,t)=[\partial u](0,t)=[\partial^2 u](0,t)=0, & \text{for } t>0 \\ u_{tt}(0,t)+[\partial^3 u](0,t)=0, & \text{for } t>0. \end{cases} \tag{1.1}$$

Here $\partial$ denotes partial derivation with respect to $x$ and the index $t$ derivation with respect to time. $[u](0)=u(0^+)-u(0^-)$ denotes the jump of the function $u$ at the point $x=0$ where the mass is located.

The first two equations in (1.1) describe the vibrations of the beams. The third equation guarantees that $u$, $\partial u$ and $\partial^2 u$ are continuous across $x=0$ while the last equation describes the vibrations of the point mass at $x=0$. To simplify the exposition we have assumed that the mass concentrated at $x=0$ is one.

Assuming that the beams are posed at their extremes, system (1.1) has to be completed with the following boundary conditions:

$$u(\pm 1,t)=\partial^2 u(\pm 1,t)=0, \quad \text{for } t>0. \tag{1.2}$$

System (1.1) can be viewed as the singular limit as $\epsilon\to 0$ of a system consisting of three flexible beams occupying the intervals $(-1,-\epsilon)$, $(-\epsilon,\epsilon)$ and $(\epsilon,1)$ respectively, the middle one having density $1/2\epsilon$. In fact, the solutions of this approximate system with boundary conditions (1.2) converge in suitable weak and strong topologies to the solutions of the limit system (1.1). We refer to [1] for a detailed analysis in the case of vibrating strings instead of flexible beams.

System (1.1) is a particular case of the following one:

$$\begin{cases} u_{tt}-\gamma\partial^2 u_{tt}+\partial^4 u=0, & \text{for } x\in(-1,0), t>0 \\ u_{tt}-\gamma\partial^2 u_{tt}+\partial^4 u=0, & \text{for } x\in(0,1), t>0 \\ [u]\,(0,t)=[\partial u](0,t)=0, & \text{for } t>0, \\ [\partial^2 u](0,t)+\gamma\partial u_{tt}(0,t)=0, & \text{for } t>0, \\ [\partial^3 u](0,t)+u_{tt}(0,t)=0, & \text{for } t>0. \end{cases} \tag{1.3}$$

Here the equations for the vibrations of the beams have the extra term $\gamma\partial^2 u_{tt}$ of rotational inertia. This term of rotational inertia is usually small compared with the other terms of the equations, i.e. constant $\gamma$ is small. Observe that system (1.1) corresponds to the limit case $\gamma = 0$.

In a previous work (see [4]) we showed that system (1.2)-(1.3), when $\gamma > 0$, is well-posed in asymmetric spaces, i.e. spaces with one more degree of regularity to one side of $x = 0$. This is due to the presence of the point mass because when the mass is not present, system (1.3) is reduced to the hyperbolic equation $u_{tt} - \gamma \partial^2 u_{tt} + \partial^4 u = 0$ and the singularities propagate along the characteristics reaching any part of the domain in finite time.

In [4] it is proved that the well-posedness of system (1.2)-(1.3) in asymmetric spaces is a consequence of the lack of asymptotic spectral gap, i.e.

$$\liminf_{k \to \infty} \lambda_k - \lambda_{k+1} = 0$$

where $\lambda_k$ are the square roots of the eigenvalues. Observe that, in absence of mass, the asymptotic spectral gap is strictly positive.

The same phenomena was observed in [6] in the case of two flexible strings connected by a point mass. In [6] this property was proved by using the explicit formula for solutions of the one-dimensional wave equation in terms of its initial data. In [6] it was also observed that the spectral gap of the wave equation vanishes in the presence of a point mass and it was conjectured these two facts (i.e. the asymmetry of the controllable space and the lack of the spectral gap) to be closely related. Later on, in [3], it was proved that these two properties are equivalent (see also [1]).

In this paper we show that system (1.1)-(1.2) is not well posed in asymmetric spaces. To prove it we analyze the spectral gap. We will see that it converges to infinity and that there is no asymmetric spaces where the system is well-posed. So, the effect of the point mass is less singular in system (1.1).

We also study the consequences of the above property in the controllability of system (1.1)-(1.2).

System (1.1)-(1.2) has to be completed with suitable initial conditions for $u(x,t)$ and $u(0,t)$. The last quantity will be denoted by $z$, i.e.

$$u(0,t) = z(t). \tag{1.4}$$

The initial conditions are then:

$$\begin{cases} u(x,0) = u^0(x) & \text{in } (-1,0) \cup (0,1)\,; \quad z(0) = z^0, \\ u_t(x,0) = u^1(x) & \text{in } (-1,0) \cup (0,1); \quad z_t(0) = z^1. \end{cases} \tag{1.5}$$

The energy of the system is given by

$$E(t) = \int_{-1}^{1} \left[ \left| \partial^2 u(x,t) \right|^2 + |u_t(x,t)|^2 \right] dx + |u_t(0,t)|^2 \tag{1.6}$$

which is constant along trajectories.

We study the control problem in which a control function $q = q(t)$ acts on the system through the extreme $x = 1$ on the quantity $\partial^2 u(1,t)$. Then the boundary conditions in (1.2) have to be replaced by

$$u(\pm 1,t) = \partial^2 u(-1,t) = 0; \quad \partial^2 u(1,t) = q(t) \text{ for } t > 0. \tag{1.7}$$

The problem of exact controllability can be formulated as follows: *Given* $T > 0$, *find the class* $H$ *of initial conditions for which there exists a control* $q$, *say in* $L^2(0,T)$, *such that the solution of* (1.1), (1.4) *with boundary conditions* (1.7) *is at rest at time* $t = T$, i.e. *it satisfies*

$$\begin{cases} u(x,T) = 0 \quad \textit{for } x \in (-1,0) \cup (0,1), \quad z(T) = 0 \\ u_t(x,T) = 0 \quad \textit{for } x \in (-1,0) \cup (0,1), \quad z_t(T) = 0. \end{cases} \tag{1.8}$$

In this formulation of the control problem we have chosen the control to belong to $L^2(0,T)$. This is not, of course, the unique choice but it is the one that comes more naturally when studying the problem of controllability by means of J.-L. Lions' HUM method (see [9]).

For the system (1.3), when $\gamma > 0$, it turns that the space of controllable initial data can not be found among the family of energy spaces but it is asymmetric in the sense that its elements have one more degree of regularity to the left of $x = 0$ (see [5]).

The same phenomena was observed in [6] in the case of two flexible strings connected by a point mass.

For the system (1.1) that we are considering here the space of controllable initial data is completely symmetric. In fact it coincides with the space of controllable data in the absence of mass. This is due to the fact that system (1.1) is not well-posed in asymmetric spaces.

After this work was completed the authors were informed about the work of S. W. Taylor [11]. In that paper using different techniques based on the method by W. Littman and S. W. Taylor [10] for exact controllability similar results to those presented here are proved.

The rest of the paper is organized as follows. In section 2 we study the eigenvalue problem associated to system (1.1) and (1.2). In section 3 we state the existence and uniqueness of solutions using Fourier series. In section 4 we prove the asymptotic properties of the spectrum that are needed to state, in section 5, observability inequalities. In particular we will see that system (1.1) is not well-posed in asymmetric spaces. In section 6 we solve system (1.1) with non-homogeneous boundary conditions to give sense to the solutions of the controlled problem. Finally, in section 7 we obtain the main controllability results.

## 2 The eigenvalue problem

In this section we solve the eigenvalue problem associated to (1.1). Our analysis provides implicit formulas for the eigenvalues and the characterization of some fractional powers of the associated linear operator which will allow us to give a description of the solutions of (1.1) in terms of Fourier series.

The problem is to find the values $\lambda \in I\!R$ for which there exists a non-trivial solution $u = u(x)$ of

$$\begin{cases} u'''' = \lambda^2 u & \text{in } (-1,0) \\ u'''' = \lambda^2 u & \text{in } (0,1) \\ [u](0) = [\partial u](0) = [\partial^2 u](0) = 0 \\ [\partial^3 u](0) = \lambda^2 u(0) \\ u(-1) = u(1) = \partial^2 u(-1) = \partial^2 u(1) = 0. \end{cases} \tag{2.9}$$

The variational formulation of (2.9) is: *Find the values $\lambda^2$ for which there exists a non-trivial solution $u \in H^2 \cap H_0^1(-1,1)$ of*

$$\int_{-1}^{1} u''v'' = \lambda^2 \int_{-1}^{1} uv + \lambda^2 u(0)v(0), \qquad \text{for all } v \in H^2 \cap H_0^1(-1,1).$$

Here and in the sequel we consider the Sobolev space $H^2 \cap H_0^1(-1,1)$ with the norm

$$\|u\|^2_{H^2 \cap H_0^1(-1,1)} = \int_{-1}^{1} |u''(x)|^2 dx$$

which is equivalent to the norm of $H^2(0,1)$ over $H^2 \cap H_0^1(-1,1)$.

The following is a first classical result:

**Theorem 1** *There exists a sequence of positive eigenvalues $\{(\lambda_k)^2\}_{k \in I\!N}$ of the problem (2.9) such that*

$$0 < (\lambda_1)^2 \leq (\lambda_2)^2 \leq (\lambda_3)^2 \leq \cdots \to \infty.$$

*The associated eigenfunctions $(\phi_k)_{k \in I\!N}$ can be chosen to constitute an orthonormal system in $H^2 \cap H_0^1(-1,1)$.*

**Proof.-** Let us consider the unbounded linear operator $A^{-1}$ defined over $H^2 \cap H_0^1(-1,1)$ as $A^{-1}F = u$ where $u$ is the unique solution

$$\int_{-1}^{1} u''v'' = \int_{-1}^{1} Fv + F(0)v(0), \qquad \forall v \in H^2 \cap H_0^1(-1,1).$$

It is easy to see that $A^{-1}$ is a compact and selfadjoint operator. The theorem establishes the classical results for this type of linear operators (see [8]).

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Explicit computations give us the following more precise result.

**Proposition 1** *The eigenvalues of the problem (2.9) are given by the values $\{\lambda_k^2\}_{k \in I\!N}$ such that $\lambda_{2k} = (k\pi)^2$ and $\lambda_{2k-1} = \mu_{2k-1}^2$ where $\mu_{2k-1}$ are the positive roots of*

$$\frac{4}{x} + \tanh x = \tan x.$$

*The corresponding eigenfunctions are given by*

$$\phi_{2k}(x) = \sin(k\pi x), \tag{2.10}$$

$$\phi_{2k-1}(x) = \begin{cases} \sin(\mu_{2k-1}(1+x)) - \frac{\cos\mu_{2k-1}}{\cosh\mu_{2k-1}}\sinh(\mu_{2k-1}(1+x)) & \textit{if } x \in (-1,0) \\ \sin(\mu_{2k-1}(1-x)) - \frac{\cos\mu_{2k-1}}{\cosh\mu_{2k-1}}\sinh(\mu_{2k-1}(1-x)) & \textit{if } x \in (0,1). \end{cases} \tag{2.11}$$

*Furthermore,* $\phi_k'(1) \neq 0$.

**Remark 1** *The roots* $\lambda^{1/2}$ *of the eigenvalues corresponding to odd eigenfunctions are the zeros of* $\tan x = 0$ *while those corresponding to even eigenfunctions are the roots of* $\tan x = 4/x + \tanh x > 0$. *Then, all the eigenvalues are simple.*

**Proof of Proposition 1.-** Due to the symmetry of the problem we can suppose that the eigenfunctions are either odd or even.

Odd eigenfunctions must solve the following system:

$$\begin{cases} u'''' = \lambda^2 u & \text{in } (0,1) \\ u(0) = \partial^2 u(0) = 0 \\ u(1) = \partial^2 u(1) = 0. \end{cases} \tag{2.12}$$

The values of $\lambda$ for which we have non-trivial solutions are $\lambda_k = k\pi$ and the eigenfunctions are clearly given by (2.10).

Even eigenfunctions must solve the following system:

$$\begin{cases} u'''' = \lambda^2 u & \text{in } (0,1) \\ \partial^3 u(0) - \frac{\lambda^2}{2} u(0) = 0 \\ \partial u(0) = u(1) = \partial^2 u(1) = 0. \end{cases} \tag{2.13}$$

Taking into account the conditions $u(1) = \partial^2 u(1) = 0$, the solution of (2.13) is of the form

$$u(x) = A\sin(\lambda(1-x)) + B\sinh(\lambda(1-x)). \tag{2.14}$$

Imposing the other boundary conditions we get

$$\begin{aligned} \partial u(0) &= -A\lambda\cos(\lambda) - B\lambda\cosh(\lambda) = 0 \\ \partial^3 u(0) - \frac{\lambda^2}{2}u(0) &= A\left[\lambda^3\cos(\lambda) - \frac{\lambda^2}{2}\sin(\lambda)\right] \\ &+ B\left[-\lambda^3\cosh(\lambda) - \frac{\lambda^2}{2}\sinh(\lambda)\right] = 0 \end{aligned} \tag{2.15}$$

which has non-trivial solutions if and only if

$$\cos(\lambda)\left[\lambda\cosh(\lambda) + \frac{1}{2}\sinh(\lambda)\right] = \cosh(\lambda)\left[-\lambda\cos(\lambda) + \frac{1}{2}\sin(\lambda)\right].$$

This is equivalent to

$$\tan(\lambda) = \frac{4}{\lambda} + \tanh(\lambda).$$

The equation for the eigenfunctions can be obtained combining (2.14) and (2.15).

■

In the sequel we will identify the elements $u \in H^2 \cap H_0^1(-1,1)$ with the vectors $(u, u(0))$ of elements of $H^2 \cap H_0^1(-1,1) \times I\!R$ by means of the map

$$\begin{array}{rcl} H^2 \cap H_0^1(-1,1) & \longrightarrow & H^2 \cap H_0^1(-1,1) \times I\!R \\ u & \longrightarrow & (u, u(0)). \end{array}$$

Observe that $(u, u(0))$ has the two unknowns of the system: $u$, the displacement of the beams and $u(0)$, el displacement of the mass located at $x = 0$.

By Theorem 1, if we normalize the eigenfunctions so that $\|\phi_k\|^2_{H^2\cap H_0^1} = \lambda_k^2$ then the space $H^2 \cap H_0^1(-1,1)$ can be characterized as:

$$H^2 \cap H_0^1(-1,1) = \left\{ u \ : \ u = \sum_{k\in I\!N} a_k \phi_k \text{ with } \sum_{k\in I\!N} |a_k|^2 \lambda_k^2 = \|u\|^2_{H^2\cap H_0^1(-1,1)} < \infty \right\}.$$

The domain of the operator $A$ is then

$$D(A) = \left\{ u \ : \ u = \sum_{k\in I\!N} a_k \phi_k \text{ with } \sum_{k\in I\!N} |a_k|^2 (\lambda_k)^4 < \infty \right\}.$$

Let us introduce now for each $\alpha \in I\!R$ the following Hilbert spaces $(X_\alpha, \|\cdot\|_{X_\alpha})$:

$$X_\alpha = \left\{ u \ : \ u = \sum_{k\in I\!N} a_k \phi_k \quad \text{with} \quad \|u\|^2_{X_\alpha} = \sum_{k\in I\!N} |a_k|^2 (\lambda_k)^{4\alpha} < \infty \right\}.$$

Observe that $X_{1/2} = H^2 \cap H_0^1(-1,1)$.

**Proposition 2** *We have the following characterizations of the spaces $X_\alpha$:*

1. *$X_0$ coincides algebraically and topologically with the space $L^2(-1,1) \times I\!R$. Furthermore*

$$\|(u,z)\|^2_{X_0} = \int_{-1}^{1} |u|^2 + |z|^2.$$

2. *$X_{1/4}$ coincides with the subspace of $H_0^1(-1,1) \times I\!R$ of the elements $(u(x), z)$ verifying $u(0) = z$.*

3. *$X_{-1/4}$ coincides with the dual space of $X_{1/4}$, i.e. the quotient subspace of $H^{-1}(-1,1)\times I\!R$ constituted by the classes $(\varphi,\eta)$ characterized in the following way: two elements $(\varphi^1,\eta^1)$ and $(\varphi^2,\eta^2)$ belong to the same class if and only if*

$$(\varphi^1-\varphi^2,\eta^1-\eta^2)=\alpha(\delta_0,-1)$$

*where $\alpha\in I\!R$ and $\delta_0$ denotes the Dirac distribution at $x=0$.*

**Proof.- 1.-** Consider $u\in X_{1/2}=H^2\cap H_0^1(-1,1)$. If we denote by $<\cdot,\cdot>_{X_\alpha}$ the scalar product in $X_\alpha$, we have:

$$\|u\|^2_{X_0}=<u,u>_{X_0}=<u,A^{-1}u>_{X_{1/2}}=\int_{-1}^1 u''(A^{-1}u)''=\int_{-1}^1|u|^2+|u(0)|^2.$$

The completion of $X_{1/2}$ with this norm is $L^2(-1,1)\times I\!R$. On the other hand $X_0$ is also the completion of $X_{1/2}$ with respect to the norm $X_0$. So, we deduce that $X_0$ must coincide topologically and algebraically with $L^2(-1,1)\times I\!R$.

**2.-** To characterize $X_{1/4}$ we are going to use the interpolation theory (see [14]). Following the notation introduced in [14] (section 2.1) we have $X_{1/4}=\left[X_{1/2},X_0\right]_{1/2}$. Then we have to prove that $\left[X_{1/2},X_0\right]_{1/2}$ is the subspace of $H_0^1(-1,1)\times I\!R$ constituted by the elements $(u,z)$ verifying $u(0)=z$.

Observe that $X_{1/2}\subset(H^2\cap H_0^1(-1,1))\times I\!R$ and $X_0\subset L^2\times I\!R$. So, by the interpolation Theorem ([14], Section 5.1) we have

$$X_{1/4}=\left[X_{1/2},X_0\right]_{1/2}\subset\left[(H^2\cap H_0^1(-1,1))\times I\!R,L^2\times I\!R\right]_{1/2}=H_0^1(-1,1)\times I\!R$$

with continuous injection.

On the other hand, as $X_{1/2}$ is a dense subspace in $\left[X_{1/2},X_0\right]_{1/2}$ and the map $f(u,z)=u(0)-z$ is continuous in $H_0^1\times I\!R$, space which contains $X_{1/4}$, the elements of $X_{1/4}$ verify, as those of $X_{1/2}$, the relation $f(u,z)=u(0)-z=0$. We have then proved the first inclusion with continuity.

Consider now the reverse inclusion. Let us denote by $H^2\cap H_0^1(-1,0)\times\{0\}$ the subspace of $H^2\cap H_0^1(-1,1)\times I\!R$ constituted by the elements $(u,0)$ that verify $u|_{(0,1)}=0$ and by $L^2(-1,0)\times\{0\}$ the subspace of $L^2(-1,1)\times I\!R$ constituted by the elements $(u,0)$ which verify $u|_{(0,1)}=0$.

We have that $H^2\cap H_0^1(-1,0)\times\{0\}\subset X_{1/2}$ and $L^2(-1,0)\times\{0\}\subset L^2\times I\!R=X_0$. Using the interpolation Theorem mentioned above we deduce that

$$\left[H^2\cap H_0^1(-1,0)\times\{0\},L^2(-1,0)\times\{0\}\right]_{1/2}\subset\left[X_{1/2},X_0\right]_{1/2}=X_{1/4}$$

and then $H_0^1(-1,0)\times\{0\}\subset X_{1/4}$. In a similar way the imbedding $H_0^1(0,1)\times\{0\}\subset X_{1/4}$ is proved.

Observe that $(u,z)=((1-x^2),1)\in X_{1/2}$ and then $(u,z)\in X_{1/4}$. So, if we define $<(u,z)>$ the subspace generated by $(u,z)$ we have

$$[H_0^1(-1,0)\times\{0\}]\oplus[H_0^1(0,1)\times\{0\}]\oplus<(u,z)>\subset X_{1/4}$$

where $\oplus$ denotes the direct sum of subspaces. The space generated by these three subspaces is in fact the subspace of $H_0^1(-1,1)\times I\!R$ constituted by the elements $(u,z)$ such that $u(0)=z$. The proof of the reverse imbedding is now completed.

The equivalence of the norms is a consequence of the fact that the first inclusion was continuous and the open map Theorem.

**3.-** Observe that $X_{1/4}$ is a subspace of $H_0^1(-1,1)$ which has codimension 1. Its dual is then the quotient subspace of $H^{-1}(-1,1)\times I\!R$ where two elements $(\varphi^1,\eta^1)$ and $(\varphi^2,\eta^2)$ belong to the same class if and only if

$$<(u,z),(\varphi^1,\eta^1)>_{X_{1/4}\times X_{-1/4}}=<(u,z),(\varphi^2,\eta^2)>_{X_{1/4}\times X_{-1/4}},\quad\forall(u,z)\in X_{1/4}$$

where $<,>_{X_{1/4}\times X_{-1/4}}$ denotes the duality product.

Observe now that

$$<(u,z),(\delta_0,-1)>_{X_{1/4}\times X_{-1/4}}=<u,\delta_0>_{H^1\times H^{-1}}-z=u(0)-z=0,\quad\forall(u,z)\in X_{1/4}$$

and then $(\varphi^1,\eta^1)$ belongs to the same class as $(\varphi^2,\eta^2)$ if and only if $(\varphi^1-\varphi^2,\eta_1-\eta^2)=\alpha(\delta_0,-1)$ with $\alpha\in I\!R$ as we wanted to prove.

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## 3 Solutions via Fourier series

In this section we will see how solutions of (1.1) can be developed in terms of Fourier series. As a consequence we obtain a theorem of existence and uniqueness of solutions.

Consider the energy space $\mathcal{H}_{1/2}=X_{1/2}\times X_0=(H^2\cap H_0^1(-1,1))\times(L^2(-1,1)\times I\!R)$ and suppose that the eigenfunctions are normalized in $X_0$ such that $(-1)^k\phi_k'(1)>0$. Observe that this choice is possible since $\phi_k'(1)\neq 0$, as we saw in Proposition 1.

An orthonormal basis in $\mathcal{H}$ is constituted by $\bar\phi_k=(\phi_k,i\lambda_k\phi_k)$, $k\in Z\!\!\!Z\backslash\{0\}$ where $\phi_{-k}=\phi_k$ and $\lambda_{-k}=-\lambda_k$. In the expression above $\phi_k$ is identified with the vector $(\phi_k(x),\phi_k(0))$.

The solution of system (1.1) can be written as

$$((u(x,t),u(0,t)),(u_t(x,t),u_t(0,t))=\sum_{k\in Z\!\!\!Z\backslash\{0\}}a_k e^{i\lambda_k t}\bar\phi_k \tag{3.1}$$

where the complex coefficients $a_k$ are determined by the initial data in the following way: Given $(u^0(x),u^0(0))\in X_{1/2}$ and $(u^1(x),z^1)\in X_0$ we have

$$a_k=<((u^0(x),u^0(0)),(u^1(x),z^1)),\bar\phi_k>_{X_{1/2}\times X_0}.$$

Let us denote by $E_{1/2}$ the energy associated to the space $\mathcal{H}_{1/2} = X_{1/2} \times X_0$. We have

$$\begin{aligned} E_{1/2}(t) &= \|u\|^2_{X_{1/2}} + \|u_t\|^2_{X_0} = \left\| \sum_{k\in \mathbb{Z}\setminus\{0\}} a_k e^{i\lambda_k t}\bar{\phi}_k \right\|^2_{\mathcal{H}_{1/2}} = \sum_{k\in \mathbb{Z}\setminus\{0\}} |a_k \lambda_k e^{i\lambda_k t}|^2 = \\ &= \sum_{k\in \mathbb{Z}\setminus\{0\}} |\lambda_k a_k|^2 = E_{1/2}(0) \end{aligned}$$

which establishes the conservation of energy along time. This also shows that when the initial data belong to the energy space then the solution is given by (3.1) and belongs to the energy space for all $t > 0$.

Observe that if we consider the energy space $\mathcal{H}_\alpha = X_\alpha \times X_{\alpha-1/2}$ instead of $\mathcal{H}_{1/2}$ then the energy, which is also conserved, is given by

$$E_\alpha(t) = \|u(t)\|^2_{X_\alpha} + \|u_t(t)\|^2_{X_{\alpha-1/2}} = \sum_{k\in \mathbb{Z}\setminus\{0\}} |a_k \lambda_k^{2\alpha} e^{i\lambda_k t}|^2 = \sum_{k\in \mathbb{Z}\setminus\{0\}} \lambda_k^{4\alpha} |a_k|^2 = E_\alpha(0).$$

We have proved then the following theorem:

**Theorem 2** *Consider $U^0 = ((u^0(x), z^0(0)), (u^1(x), z^1)) \in \mathcal{H}_\alpha$ and $T > 0$ with $\alpha \in \mathbb{R}$. There exists an unique solution $U(x,t) \in C([0,T]; \mathcal{H}_\alpha)$ of system (1.1) given by (3.1). Furthermore, the energy of the system $E_\alpha(t)$ is conserved along the time.*

# 4 Asymptotics of the spectrum

In this section we study the asymptotic behavior of the eigenvalues of problem (2.9). The results of this section will allow us to prove the necessary observability inequalities to obtain the controllability later on.

Remind that, as we proved in Proposition 1, the eigenvalues of (2.9) are given by the positive roots of $\tan x = 0$ and $\tan x = 4/x + \tanh x > 0$. We deduce then that, asymptotically, $\sqrt{\lambda_{2k}} = k\pi$ while $\sqrt{\lambda_{2k-1}}$ are the roots of $\tan x = 1$.

On the other hand, when we remove the mass from system (1.1) the eigenvalues are given by $\sqrt{\lambda_{2k}} = k\pi$ and $\sqrt{\lambda_{2k-1}} = k\pi - \pi/2$. We deduce then that the point mass does not affect the asymptotic behavior of eigenvalues corresponding to odd eigenfunctions while it shifts the fourth order root of the eigenvalues corresponding to even eigenfunctions.

Concerning the spectral gap we have

**Theorem 3** *There exists $\alpha > 0$ such that*

$$\sqrt{\lambda_{2k}} - \sqrt{\lambda_{2k-1}} = k\pi - \mu_{2k-1} = \alpha + o(1), \qquad k \to \infty$$

*where $\mu_{2k-1}$ denotes the k-th positive root of* $\tan x = 1$.

**Corollary 1** *We have*

$$\lambda_{2k} - \lambda_{2k-1} = \mathcal{O}(k), \qquad |k| \to \infty.$$

**Remark 2** *The presence of the mass produces an asymptotic displacement of $\sqrt{\lambda_{2k-1}}$ which reduces the spectral gap. However the quantity that plays a crucial role when analyzing the solutions of (1.1) is $\lambda_{2k} - \lambda_{2k-1}$ and this one remains of order $k$ as $k \to \infty$, i.e. its order does not change.*

**Remark 3** *When we consider the equations (1.3) with the constant of rotational inertia $\gamma > 0$ the presence of the point mass changes the asymptotic behavior of the eigenvalues in an essential way. In this case the aymptotic gap when we do not have the point mass is*

$$\lambda_{2k}(\gamma) - \lambda_{2k-1}(\gamma) = \mathcal{O}(1), \qquad |k| \to \infty$$

*while in the presence of mass is as follows*

$$\lambda_{2k}(\gamma) - \lambda_{2k-1}(\gamma) = \mathcal{O}(1/k), \qquad |k| \to \infty.$$

*In [5] we prove that, due to this loss of spectral gap in the presence of the point mass, system (1.3) is well posed in asymmetric spaces. Of course, this property is not true in the system without mass. We will see that in the case we are considering in this paper, i.e. system (1.1), the presence of the mass is not singular enough to make system (1.1) to be well posed in asymmetric spaces.*

## 5 Observability

In this section we prove some observability results which are consequence of the asymptotic properties of the previous section. The reason to study these properties is that, by means of the so called HUM method, controllability properties can be reduced to suitable observability inequalities for the adjoint system. As (1.1) is a selfadjoint system we are reduced to the same system, without control.

Therefore, consider system (1.1) without control, i.e.

$$\begin{cases} u_{tt}(x,t) + \partial^4 u(x,t) = 0 & x \in (-1,0),\ 0 < t < T \\ u_{tt}(x,t) + \partial^4 u(x,t) = 0 & x \in (0,1),\ 0 < t < T \\ [u](0) = [\partial u](0) = [\partial^2 u](0) = 0 & 0 < t < T \\ u_{tt}(0,t) + [\partial^3 u](0) = 0 & 0 < t < T \\ u(-1,t) = u(1,t) = \partial^2 u(-1,t) = \partial^2 u(1,t) = 0 & 0 < t < T \\ (u(x,0), u(0,0)) = (u^0, z^0) \\ (u_t(x,0), u_t(0,0)) = (u^1, z^1). \end{cases} \tag{5.1}$$

**Proposition 3** *Let be $U^0 = ((u^0, u^0(0)), (u^1, u^1(0))) \in \mathcal{H}_{1/4}$ and consider*

$$U(x,t) = ((u(x,t), u(0,t)), (u_t(x,t), u_t(0,t)))$$

*the solution of system (5.1) with initial data $U^0$. Then for each $T > 0$ there exist constants $C_1, C_2 > 0$ which only depend on $T$ such that*

$$C_1 \|U^0\|^2_{\mathcal{H}_{1/4}} \leq \int_0^T |\partial u(1,t)|^2 \leq C_2 \|U^0\|^2_{\mathcal{H}_{1/4}}, \tag{5.2}$$

*for all solution of (5.1).*

**Remark 4** *The second inequality in (5.2) establishes a regularity result which is not a consequence of the fact $U \in C([0,T]; \mathcal{H}_{1/4})$. Indeed observe that when $U \in C([0,T]\,; \mathcal{H}_{1/4})$, $u \in C([0,T]\,; H^{-1}(-1,1))$ but this is not sufficient to guarantee that $\partial u(1,t) \in L^2(0,T)$.*

*The first inequality in (5.2) is an observability result which says that the norm of the solution in $\mathcal{H}_{1/4}$ can be measured continuously by the quantity $\partial u(1,t)$ in $L^2(0,T)$.*

**Remark 5** *Due to the fact that the characteristic lines of the equation are horizontal the speed of propagation is infinity. Therefore, the observability time in (5.2) can be chosen to be arbitrarily small.*

*Thanks to both inequalities (5.2) we deduce that $\mathcal{H}_{1/4}$ is the optimal space of observability, i.e. the largest space of initial data for which the solutions of system (5.1) can be estimated by means of the $L^2$-norm of $\partial u(1,t)$.*

**Remark 6** *Observe that, in contrast with the result of [5] related to the system (1.3) in which the term of rotational inertia does not vanish, the optimal space of observability for system (1.1) is symmetric, i.e. the regularity to both sides of $x = 0$ is the same.*

**Remark 7** *The symmetry of the optimal space of observability when one observes from one extreme allows to conclude that there are not asymmetric spaces in which the solutions remain along the time. In fact the second inequality in (5.2) is of local nature and provides a regularity result for $\partial u(1,t)$ which only depends on the regularity of the solution to the right of the point mass, i.e. only depends on the smoothness of solutions near $x = 1$. So, suppose that there are spaces with different regularity to both sides of $x = 0$ in which system (1.1) is well posed. Without loss of generality we can suppose that, in fact, they are less regular to the left of $x = 0$. Then, the second inequality in (5.2) would hold for a larger space, i.e. a space less regular to the left of $x = 0$ which is in contradiction with the first inequality of (5.2).*

**Proof of Proposition 3.-** Le us write the inequalities (5.2) in terms of Fourier series:

$$C_1 \sum_{k \in \mathbb{Z} \setminus \{0\}} \lambda_k |a_k|^2 \leq \int_0^T \left| \sum_{k \in \mathbb{Z} \setminus \{0\}} a_k e^{i\lambda_k t} \phi_k'(1) \right|^2 dt \leq C_2 \sum_{k \in \mathbb{Z} \setminus \{0\}} \lambda_k |a_k|^2. \tag{5.3}$$

We will need the following result due to A. Haraux [7]:

**Theorem 4** *Let $\{\lambda_n\}$ be a sequence of real numbers such that*

$$\lambda_{n+1} - \lambda_n \geq \alpha > 0, \qquad \forall |n| \geq N$$

*and*

$$\lambda_{n+1} - \lambda_n \geq \beta > 0.$$

*Consider also $T > \pi/\alpha$. Then there exist constants $C_1(T)$ and $C_2(T)$ which only depend on $\alpha$, $N$ and $\beta$ such that if $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{-i\lambda_n t}$ we have*

$$C_1(T) \sum_{n\in\mathbb{Z}} |a_n|^2 \leq \int_{-T}^{T} |f(t)|^2 \leq C_1(T) \sum_{n\in\mathbb{Z}} |a_n|^2$$

*for all $(a_n) \in \ell^2$.*

Taking into account this result and the behavior of the eigenvalues given in Corollary 1, we deduce that

$$C_1 \sum_{k\in\mathbb{Z}\setminus\{0\}} \lambda_k |a_k|^2 |\phi_k'(1)|^2 \leq \int_0^T \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} a_k e^{i\lambda_k t} \phi_k'(1) \right|^2 dt \leq C_2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \lambda_k |a_k|^2 |\phi_k'(1)|^2. \quad (5.4)$$

As we are supposing the eigenfunctions to be normalized in $X_0 = L^2(-1,1) \times \mathbb{R}$, it is easy to see that there exist constants $D_1$ and $D_2$ such that

$$D_1 \lambda_k^{1/2} \leq |\phi_k'(1)| \leq D_2 \lambda_k^{1/2}. \quad (5.5)$$

Combining (5.4) and (5.5) we easily conclude (5.3).

■

# 6 Non-homogeneous problems

In this section we study the solvability of our system when we introduce non-homogeneous equations and non-homogeneous boundary conditions.

## 6.1 System with non-homogeneous right hand side

Consider the following system:

$$\begin{cases} u_{tt}(x,t) + \partial^4 u(x,t) = f(x,t) & x \in (-1,0),\ 0 < t < T \\ u_{tt}(x,t) + \partial^4 u(x,t) = f(x,t) & x \in (0,1),\ 0 < t < T \\ [u](0,t) = [\partial u](0,t) = [\partial^2 u](0,t) = 0, & 0 < t < T \\ z_{tt}(t) + [\partial^3 u](0,t) = g(t) & 0 < t < T \\ u(0,t) = z(t), & 0 < t < T \\ u(-1,t) = u(1,t) = \partial^2 u(-1,t) = \partial^2 u(1,t) = 0 & 0 < t < T \\ (u(x,0), u(0,0)) = (u^0, z^0) \\ (u_t(x,0), u_t(0,0)) = (u^1, z^1). \end{cases} \quad (6.1)$$

We have the following result:

**Theorem 5** *Let be* $U^0 = ((u^0, z^0), (u^1, z^1)) \in \mathcal{H}_{1/4}$ *and* $(f(x,t), g(t)) \in L^1(0,T; X_{-1/4})$. *There exists an unique function* $u(x,t)$ *in the class*

$$((u(x,t), z(t)), (u_t(x,t), z_t(t))) \in C([0,T]; \mathcal{H}_{1/4})$$

*solution of system (6.1). Furthermore, we have*

$$\int_0^T |\partial u(1,t)|^2 \leq C \left( \|(f,g)\|_{L^1(0,T;X_{-1/4})} + \|((u^0,z^0),(u^1,z^1))\|_{\mathcal{H}_{1/4}} \right) \tag{6.2}$$

*with* $C$ *independent of the initial data and* $(f,g)$.

**Remark 8** *Inequality (6.2) provides a regularity result for the solutions at* $x = 1$ *which can not be deduced from the class in which the solutions are.*

**Proof of Theorem 5.-** Due to the linearity of the system and Proposition 3, by superposition it is enough to prove the result for the case in which the initial data $U^0$ is zero.

It is easy to check that the solution of system (6.1) with zero initial data is given by

$$u(x,t) = \int_0^t v(x, t-s, s)\, ds,$$

where $v(x,t,s)$ is the unique solution of system (1.1)-(1.2) with initial data

$$\left\{ \begin{array}{l} (v(x,0,s), v(0,0,s)) = 0 \\ (v_t(x,0,s), v_t(0,0,s) = (f(x,s), g(s)). \end{array} \right. \tag{6.3}$$

By Theorem 2, $((v, v(0)), (v_t, v_t(0))) \in C([0,T]\,; \mathcal{H}_{-1/4})$ and we deduce that

$$((u, u(0)), (u_t, u_t(0))) \in C([0,T]\,; \mathcal{H}_{-1/4}).$$

Furthermore, by Proposition 3 we have

$$\begin{aligned} \int_0^T |\partial v(1,t,s)|^2\, dt \leq C \|((0,0),(f(\cdot,s),g(s)))\|^2_{\mathcal{H}_{-1/4}} = \\ = C\|(f(\cdot,s),g(s))\|^2_{X_{-1/4}}. \end{aligned}$$

By Minkowski's inequality we deduce that

$$\|\partial u(1,t)\|_{L^2(0,T)} = \left\| \int_0^T \int_0^t \partial v(1,t-s,s)\, ds \right|_{L^2(0,T)} \leq C\|(f(\cdot,s),g(s))\|_{L^1(0,T,X_{-1/4})}. \quad \blacksquare$$

We will also need the following result:

**Theorem 6** *If we take* $(f,g) = \left(\frac{\partial F}{\partial t}, \frac{\partial G}{\partial t}\right)$ *in Theorem 5 where* $(F,G) \in L^1(0,T;X_{1/4})$ *and zero initial data we have*

$$\int_0^T |\partial u(1,t)|^2 \, dt \le C \|(f,g)\|^2_{L^1(0,T;X_{1/4})}$$

*with* $C$ *independent on* $(F,G)$.

**Proof of Theorem 6.-** Observe that $u = v_t$ where $v$ is the solution of system

$$\begin{cases} v_{tt}(x,t) + \partial^4 v(x,t) = F(x,t) & x \in (-1,0),\ 0 < t < T \\ v_{tt}(x,t) + \partial^4 v(x,t) = F(x,t) & x \in (0,1),\ 0 < t < T \\ [u](0,t) = [\partial v](0,t) = [\partial^2 u](0,t) = 0, & 0 < t < T \\ w_{tt}(t) + [\partial^3 v](0,t) = G(t) & 0 < t < T \\ v(0,t) = w(t), & 0 < t < T \\ v(-1,t) = v(1,t) = \partial^2 v(-1,t) = \partial^2 v(1,t) = 0 & 0 < t < T \end{cases} \tag{6.4}$$

with zero initial data. This is due to the fact that in this proof we may assume $(F,G)$ to be of compact support in time. Indeed, if Theorem 6 is proved for those $(F,G)$, it can then be extended by density to all $(F,G) \in L^1(0,T;X_{1/4})$.

With this in mind we see that the appropriate initial conditions for $v$ are as follows:

$$v(x,0) = u_t(x,0) = 0;\ v_t(x,0) = u_{tt}(x,0) = F(x,0) - \partial^4 u(x,0) = 0.$$

To complete the proof of Theorem 6 it is sufficient to prove that the following lemma holds:

**Lemma 1** *Assume that* $U^0 = 0$ *and* $(f,g) \in L^1(0,T;X_{1/4})$. *Then, there exists* $C > 0$ *such that the solution of (6.1) satisfies*

$$\int_0^T |\partial u_t(1,t)|^2 \, dt \le C \parallel (f,g) \parallel^2_{L^1(0,T;X_{1/4})} \tag{6.5}$$

*for all* $(f,g)$ *as above.*

**Proof of Lemma 1.-** As in Theorem 5 above $u = \int_0^t v(x,t-s;s)ds$ where $v$ solves (1.1)-(1.2) with initial data (6.3). Then $\omega = v_t$ verifies (1.1)-(1.2) with initial data

$$\begin{cases} \omega(0;s) = f(s), \\ \omega_t(0;s) = 0. \end{cases} \tag{6.6}$$

In view of Proposition 3 we have

$$\int_0^T |\partial \omega(1,t)|^2 \, dt \le C \parallel (f,0) \parallel^2_{\mathcal{H}_{1/4}} = C \parallel f \parallel^2_{X_{1/4}} . \tag{6.7}$$

On the other hand

$$\partial u_t = \int_0^t \partial v_t(t-s;s)ds = \int_0^t \partial \omega(t-s;s)ds$$

and therefore

$$\int_0^T |\partial u_t(1,t)|^2\, dt = \left\| \int_0^t \partial\omega(1, t-s; s) ds \right\|^2_{L^2_t(0,T)}. \tag{6.8}$$

Now, by Minkowski's inequality and (6.7) we deduce that

$$\left\| \int_0^t \partial\omega(1, t-s; s) ds \right\|_{L^2_t(0,T)} \leq C \parallel f \parallel_{L^1(0,T;X_{1/4})}. \tag{6.9}$$

Combining (6.8)-(6.9) we deduce that (6.5) holds.

This concludes the proof of Lemma 1 and Theorem 6.

■

## 6.2 Non-homogeneous boundary conditions

Consider the following system:

$$\begin{cases} u_{tt}(x,t) + \partial^4 u(x,t) = 0 & x \in (-1,0),\ 0 < t < T \\ u_{tt}(x,t) + \partial^4 u(x,t) = 0 & x \in (0,1),\ 0 < t < T \\ [u](0,t) = [\partial u](0,t) = [\partial^2 u](0,t) = 0, & 0 < t < T \\ z_{tt}(0,t) + [\partial^3 u](0,t) = 0 & 0 < t < T \\ u(0,t) = z(t) & 0 < t < T \\ u(-1,t) = u(1,t) = \partial^2 u(-1,t) = 0 & 0 < t < T \\ \partial^2 u(1,t) = q(t) & 0 < t < T \\ (u(x,0), u(0,0)) = (u^0, z^0) & \\ (u_t(x,0), u_t(0,0)) = (u^1, z^1). & \end{cases} \tag{6.10}$$

Here $q(t) \in L^2(0,T)$ and $((u^0, y^0), (u^1, z^1)) \in \mathcal{H}_{1/4} = X_{1/4} \times X_{-1/4}$.

We are considering now a problem with a non-homogeneous boundary term $q$ which plays the role of the control.

Before anything we must give a precise notion of solution. We will adopt the notion of solution by transposition. First we introduce the following adjoint system:

$$\begin{cases} \varphi_{tt}(x,t) + \partial^4 \varphi(x,t) = f(x,t) & x \in (-1,0),\ 0 < t < T \\ \varphi_{tt}(x,t) + \partial^4 \varphi(x,t) = f(x,t) & x \in (0,1),\ 0 < t < T \\ [\varphi](0,t) = [\partial \varphi](0,t) = [\partial^2 \varphi](0,t) = 0, & 0 < t < T \\ \zeta_{tt}(0,t) + [\partial^3 \varphi](0,t) = g(t) & 0 < t < T \\ \varphi(0,t) = \zeta(t) & 0 < t < T \\ \varphi(-1,t) = \varphi(1,t) = \partial^2 \varphi(-1,t) = \partial^2 \varphi(1,t) = 0 & 0 < t < T \\ (\varphi(x,T), \varphi(0,T)) = (0,0) & \\ (\varphi_t(x,T), \varphi_t(0,T)) = (0,0), & \end{cases} \tag{6.11}$$

where $(f(x,t), g(t)) \in C([0,T]; X_{-1/4})$. Thanks to the reversibility in time of system (6.11) we can apply Theorem 2 and deduce that the unique solution is in the class

$$((\varphi(x,t), \varphi(0,t)), (\varphi_t(x,t), \zeta_t(t))) \in C([0,T]; \mathcal{H}_{1/4}).$$

Multiplying by $\varphi$ system (6.10) and integrating by parts we obtain

$$\int_0^T \int_{-1}^1 u(x,t) f(x,t)\, dx\, dt + \int_0^T z(t)\, g(t)\, dt = \int_{-1}^1 u^1(x)\varphi(x,0)\, dx - \int_{-1}^1 u^0(x)\varphi_t(x,0)\, dx +$$
$$+u^1(0)\varphi(0,0) - u^0(0)\varphi_t(0,0) - \int_0^T q(t)\varphi_x(1,t)\, dt.$$

This motivates the following definition: *We will say that $(u,z) \in C([0,T];X_{1/4})$ is a weak solution in the sense of transposition of (6.10) if for all $(f,g) \in X_{-1/4}$ the following identity holds:*

$$< (u,z),(f,g) >_{X_{1/4}\times X_{-1/4}} = < (u^1,z^1),(\varphi(\cdot,0),\varphi(0,0)) >_{X_{-1/4}\times X_{1/4}} -$$
$$- < (u^0,u^0(0)),(\varphi_t(\cdot,0),\zeta_t(0)) >_{X_{1/4}\times X_{-1/4}} - \int_0^T q(t)\varphi_x(1,t)\, dt. \qquad (6.12)$$

**Proposition 4** *For any $q \in L^2(0,T)$, $(u^0,z^0) \in X_{1/4}$ and $(u^1,z^1) \in X_{-1/4}$ there exists an unique solution in the sense of transposition of system (6.10) in the class*

$$(u,z) \in C([0,T];X_{1/4}) \cap C^1([0,T];X_{-1/4}). \qquad (6.13)$$

*Moreover, there exists $C(T)$ such that*

$$\|(u,z)\|_{L^\infty(0,T;X_{1/4})} + \|(u_t,z_t)\|_{L^\infty(0,T;X_{-1/4})}$$
$$\leq C(T)\left(\|q\|_{L^2(0,T)} + \|(u^0,z^0)\|_{X_{1/4}} + \|(u^1,z^1)\|_{X_{-1/4}}\right)$$

*holds for all solution.*

**Proof.-** In view of Theorem 5 the right hand side of (6.12) defines a linear continuous operator in $L^1(0,T;X_{-1/4})$. Therefore, by duality, we deduce that there exists an unique $(u,z) \in L^\infty\left(0,T;X_{1/4}\right)$ solution of (6.10). Moreover, there exists $C>0$ such that

$$\| (u,z) \|_{L^\infty\left(0,T;X_{1/4}\right)} \leq C\left(\| q \|_{L^2(0,T)} + \|(u^0,z^0)\|_{X_{1/4}} + \|(u^1,z^1)\|_{X_{-1/4}}\right). \qquad (6.14)$$

Furthermore, as a consequence of Theorem 6 we deduce that $(u_t,z_t) \in L^\infty\left(0,T;X_{-1/4}\right)$. In fact, we have the estimate

$$\|(u_t,z_t)\|_{L^\infty\left(0,T;X_{-1/4}\right)} \leq C\left(\| q \|_{L^2(0,T)} + \|(u^0,z^0)\|_{X_{1/4}} + \|(u^1,z^1)\|_{X_{-1/4}}\right) \qquad (6.15)$$

holds for every $q$. The continuity with respect to time in (6.13) can be proved by density. To see this it is sufficient to observe that when $q$ is smooth enough and of compact support, solutions of (6.10) belong to $C([0,T];\mathcal{H}_{1/4})$.

■

## 7 Controllability

In this section we address the control problem when we act at the right extreme of the system. Consider the system:

$$\left\{\begin{array}{ll} u_{tt}(x,t)+\partial^4 u(x,t)=0 & x\in(-1,0),\ 0<t<T \\ u_{tt}(x,t)+\partial^4 u(x,t)=0 & x\in(0,1),\ 0<t<T \\ [u](0,t)=[\partial u](0,t)=[\partial^2 u](0,t)=0, & 0<t<T \\ z_{tt}(0,t)+[\partial^3 u](0,t)=0 & 0<t<T \\ u(0,t)=z(t) & 0<t<T \\ u(-1,t)=u(1,t)=\partial^2 u(-1,t)=0 & 0<t<T \\ \partial^2 u(1,t)=q(t) & 0<t<T \\ (u(x,0),u(0,0))=(u^0,z^0) & \\ (u_t(x,0),u_t(0,0))=(u^1,z^1), & \end{array}\right. \tag{7.1}$$

where $q$ is the control.

The main result of this section is as follows:

**Theorem 7** *Let be $T>0$ and consider the initial data $U^0=((u^0,u^0(0)),(u^1,z^1))\in\mathcal{H}_{1/4}$, i.e.*

$$(u^0,u^1,z^1)\in H_0^1(-1,1)\times L^2(-1,1)\times I\!R.$$

*There exists a control $q\in L^2(0,T)$ such that the solution of the system (7.1) verifies:*

$$\begin{array}{l} u(x,T)=u_t(x,T)=0,\quad x\in(-1,0)\cup(0,1), \\ z(T)=z_t(T)=0. \end{array}$$

**Remark 9** *As system (7.1) is reversible in time this result of control to zero is equivalent to the exact controllability.*

**Proof.-** Applying HUM the control problem is reduced to the obtention of the following observability inequalities for the uncontrolled system, i.e. for (7.1) with $q=0$: There exist positive constants $C_1$ and $C_2$ such that

$$C_1\|U^0\|^2_{\mathcal{H}_{1/4}}\le\int_0^T|\partial u(1,t)|^2\le C_2\|U^0\|^2_{\mathcal{H}_{1/4}}.$$

As we saw in Proposition 3 these inequalities do hold for all $T>0$.

**Acknowledgements.** This work was done while the first author was supported by a doctoral fellowship of the "Universidad Complutense de Madrid". The authors were also partially supported by grants PB93-1203 of the DGICYT (Spain) and CHRX-CT94-0471 of the European Union.

The authors acknowledge G. leugering for informing them about the work of S. W. Taylor [11].

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