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LOW FREQUENCY ASYMPTOTIC ANALYSIS OF A STRING WITH RAPIDLY OSCILLATING DENSITY*

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Abstract. We consider the eigenvalue problem associated to the vibrations of a string with a rapidly oscillating bounded periodic density. It is well known that when the size of the microstructure ϵ is small enough with respect to the wavelength of the eigenfunctions $1/\sqrt{\lambda^\epsilon}$, eigenvalues and eigenfunctions can be approximated by those of the limit system where the oscillating density is replaced by its average. On the other hand, it has been observed that when the size of the microstructure is of the order of the wavelength of the eigenfunctions ($\epsilon \sim 1/\sqrt{\lambda^\epsilon}$), singular phenomena may occur.

In this paper we study the behavior of the eigenvalues and eigenfunctions when $1/\sqrt{\lambda^\epsilon}$ approaches the critical size ϵ. To do this we use the WKB approximation which allows us to find an explicit formula for eigenvalues and eigenfunctions with respect to ϵ. In particular, our analysis provides all order correction formulas for the limit eigenvalues and eigenfunctions below the critical size.

Key words. string equation, homogenization, spectral analysis, WKB approximation

AMS subject classifications. 35L05, 35P15, 35C20, 35B27

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1. Introduction.

Consider the following eigenvalue problem:

$$\begin{cases} u''(x) + \lambda\, \rho\left(\frac{x}{\epsilon}\right) u(x) = 0 & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $\rho(x)$ is a periodic function with $0 < \rho_m \le \rho(x) \le \rho_M < \infty$ and ϵ is a small parameter which measures the size of the microstructure. To fix ideas and without loss of generality we take ρ of period 1 though our analysis is independent of the period (see Remark 2(b) below).

Let us denote by $\{\lambda_k^\epsilon\}_{k\in\mathbb{N}}$ the set of eigenvalues of (1.1) ordered in an increasing way, i.e.,

$$0 < \lambda_1^\epsilon < \lambda_2^\epsilon < \cdots < \lambda_k^\epsilon < \cdots \to \infty.$$

The associated eigenfunctions $\{\varphi_k^\epsilon\}_{k\in\mathbb{N}}$ can be chosen to constitute an orthonormal basis of $H_0^1(0,1)$. We are interested in the behavior of eigenvalues and eigenfunctions for small values of the parameter ϵ.

The limit of system (1.1) as $\epsilon \to 0$ is given by

$$\begin{cases} u''(x) + \lambda\, \bar\rho u(x) = 0 & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

where we have changed the oscillating coefficient $\rho(x/\epsilon)$ by its average $\bar\rho = \int_0^1 \rho(x)\, dx$.

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The eigenpairs (λ_k, φ_k) of (1.2) can be computed explicitly:

$$\lambda_k = \frac{k^2\pi^2}{\bar{\rho}}, \qquad k \in \mathbb{N}, \tag{1.3}$$

$$\varphi_k(x) = \sin(k\pi x), \qquad k \in \mathbb{N}. \tag{1.4}$$

It is well known that the eigenpairs $(\lambda_k^\epsilon, \varphi_k^\epsilon)$ of system (1.1) converge to those of system (1.2) as $\epsilon \to 0$. More precisely, we have the following classical result (see [13]): For all $k \in \mathbb{N}$, there exist constants $c_1(k)$ and $c_2(k)$ such that

$$|\sqrt{\lambda_k^\epsilon} - \sqrt{\lambda_k}| \leq c_1(k)\epsilon \quad \forall 0 < \epsilon < 1, \tag{1.5}$$

$$\|\varphi_k^\epsilon - \varphi_k\|_{H_0^1(0,1)} \leq c_2(k)\epsilon \quad \forall 0 < \epsilon < 1. \tag{1.6}$$

However, very little is known about the dependence of the constants $c_1(k)$ and $c_2(k)$ with respect to k.

When the size of the microstructure is of the same order as the wavelength of the vibrations ($k \sim \epsilon^{-1}$ or $\lambda_k^\epsilon \sim \epsilon^{-2}$), the eigenfunctions φ_k^ϵ can exhibit a singular behavior (see [4] and section 2 below) and concentrate most of their energy near one of the extremes of the interval $(0, 1)$. This effect has also been observed in a different eigenvalue problem, similar to (1.1), but with oscillating coefficients in the principal part (see [1], [2], and [3]).

Eigenfunctions (1.4) do not exhibit any concentration of energy in the boundary. This means, in particular, as we shall see in section 2, that constants $c_1(k)$ and $c_2(k)$ cannot be chosen uniformly in k when λ approaches the critical size $\lambda \sim \epsilon^{-2}$. In other words, the limit problem may only provide a uniform approximation to (1.1) when the wavelength of the vibrations is large enough with respect to the size of the microstructure.

In this paper we give explicit formulas of the dependence of λ_k^ϵ and φ_k^ϵ with respect to k and ϵ. We prove that when $k \leq C\epsilon^{-2/3}$, the limit system (1.2) provides a uniform approximation to (1.1). As we take larger eigenvalues corrector terms to the limit eigenvalues have to be introduced to obtain a uniform approximation. The correctors and their regions of validity are also computed.

The use of correctors to the limit eigenvalues only provides information of eigenpairs below the critical size $\lambda \sim \epsilon^{-2}$. This allows us to conclude that, below this critical size, eigenfunctions do not exhibit any concentration of energy near the boundary.

This work was motivated by the problem of the uniform boundary controllability of the one-dimensional wave equation with rapidly oscillating density. When the control is at the extreme $x = 1$ the uniform controllability of the wave equation is equivalent to the following uniform observability property: Find a constant $C > 0$ and a time $T > 0$ independent of ϵ such that the solution v of the uncontrolled wave equation

$$\begin{cases} \rho(x/\epsilon)v_{tt}(x,t) - v_{xx}(x,t) = 0, & x \in (0,1), \quad t > 0, \\ v(0,t) = v(1,t) = 0, v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & \end{cases} \tag{1.7}$$

satisfies

$$\frac{1}{C}\int_0^T |v_x(1,t)|^2 dt \leq \int_0^1 \rho(x/\epsilon)|v_0|^2 dx + \int_0^1 |v_{1,x}|^2 dx \leq C\int_0^T |v_x(1,t)|^2 dt. \tag{1.8}$$

The results in [4] show that the above observability property is not uniform due to the existence of eigenfunctions that behave in the above-described singular way for

$k \sim \epsilon^{-1}$. More precisely, for any $T > 0$ the constant C must grow to infinity as $\epsilon \to 0$. The results of the present work combined with the theory of nonharmonic Fourier series allow us to prove sharp uniform observability results. These are observability inequalities of the form (1.8) for $T > 0$ large enough with a constant C independent on ϵ for the solutions v belonging to the space generated by eigenfunctions associated with eigenvalues λ^ϵ such that $\lambda^\epsilon \leq c\epsilon^{-2}$ with C small enough. We refer to [8] for some preliminary results in this direction and to [6] for a more detailed analysis.

Our analysis can be also applied to the system

$$\begin{cases} \left(a(\frac{x}{\epsilon})u'(x)\right)' + \lambda u(x) = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.9}$$

where the periodic oscillating coefficient is in the principal part. The idea is that with a suitable change of variables system (1.9) may be transformed into (1.1). In this case we also prove a complete asymptotic description of the eigenvalues λ_k^ϵ and eigenfunctions φ_k^ϵ below the critical size $\lambda \sim \epsilon^{-2}$.

Going back to system (1.1), in all this work we implicitly use the fact that λ_k^ϵ and k^2 are of the same order. In fact,

$$\frac{k^2\pi^2}{\rho_M} \leq \lambda_k^\epsilon \leq \frac{k^2\pi^2}{\rho_m} \quad \forall\, k \in \mathbb{N}. \tag{1.10}$$

We can easily obtain this result by means of some rough estimates in the Rayleigh formula. Indeed, observe that if $u \in H_0^1(0,1)$ we have

$$\frac{\int_0^1 |u'(x)|^2 dx}{\rho_M \int_0^1 |u(x)|^2 dx} \leq \frac{\int_0^1 |u'(x)|^2 dx}{\int_0^1 \rho(\frac{x}{\epsilon})|u(x)|^2 dx} \leq \frac{\int_0^1 |u'(x)|^2 dx}{\rho_m \int_0^1 |u(x)|^2 dx}. \tag{1.11}$$

Taking into account that

$$\lambda_k^\epsilon = \max_{\substack{E_k \subset H_0^1(0,1) \\ \dim E_k = k}} \min_{u \in E_k^\perp} \frac{\int_0^1 |u'(x)|^2 dx}{\int_0^1 \rho(\frac{x}{\epsilon})|u(x)|^2 dx}$$

and applying (1.11), we obtain (1.10).

The methods of this paper may also be used to obtain high frequency asymptotic expansions for eigenvalues $\lambda^\epsilon \gg \epsilon^{-2}$. We refer to [9] for a detailed analysis. In this respect it is worth mentioning that the limit eigenvalue problem that describes the first term in the asymptotic expansion is $\lambda_k^\epsilon \sim k^2\pi^2/(\int_0^1 \sqrt{\rho(x/\epsilon)dx})^2$. Note that the averaged density $\bar{\rho}$ has been replaced by the square of the average of $\sqrt{\rho}$.

The rest of the paper is organized as follows: In section 2, following [4], we prove the existence of eigenfunctions φ_k^ϵ with $k \sim \epsilon^{-1}$ which localize most of their energy at one extreme of the interval. In section 3 we study the behavior of eigenpairs below the critical size, i.e., $k \leq C\epsilon^{-1}$ with $C > 0$ small enough. Section 4 is devoted to the analysis of the eigenvalue problem (1.9). Some technical results are proved in Appendixes I and II.

2. Singular behavior of eigenfunctions for $\boldsymbol{\lambda \sim \epsilon^{-2}}$. In this section we show that eigenfunctions φ_k^ϵ corresponding to eigenvalues $\lambda_k^\epsilon \sim \epsilon^{-2}$ can exhibit a concentration of most of their energy near one of the extremes of the interval $[0,1]$. We follow the analysis in [4] to prove the following result which is more general than the one stated in [4].

THEOREM 2.1. *Consider $\rho \in L^\infty(\mathbb{R})$ a nonconstant periodic function with $\rho_m \le \rho(x) \le \rho_M$. Then at least one (and possibly both) of the following properties* (a) *or* (b) *holds:*

(a) *There exist a sequence $\epsilon_j \to 0$, a sequence of eigenfunctions φ_{ϵ_j} of* (1.1), *and positive constants $C_1, C_2 > 0$ such that*

$$\frac{\int_0^1 |\varphi'_{\epsilon_j}(x)|^2 dx}{|\varphi'_{\epsilon_j}(1)|^2} \ge C_1 \epsilon_j e^{C_2/\epsilon_j}; \tag{2.1}$$

(b) *there exist a sequence $\epsilon_j \to 0$, a sequence of eigenfunctions φ_{ϵ_j}, and a positive constant $C > 0$ such that*

$$\frac{\int_0^1 |\varphi'_{\epsilon_j}(x)|^2 dx}{|\varphi'_{\epsilon_j}(1)|^2} \le C \epsilon_j. \tag{2.2}$$

Furthermore, there exists a smooth nonconstant periodic function ρ such that (a) *holds. The same is true for* (b).

REMARK 1. (a) *In Theorem* 2.1 *we assume that $\rho \in L^\infty$ instead of $\rho \in C^2$ which is assumed in* [4]. *The other difference between this result and the one stated in* [4] *is that we have included the singular behavior stated in the case* (b) *of the theorem.*

(b) *Theorem* 2.1 *guarantees that when ρ is a nonconstant periodic function there is no constant $C > 0$ such that*

$$\frac{1}{C}|(\varphi_k^\epsilon)'(1)|^2 \le \int_0^1 |(\varphi_k^\epsilon)'(x)|^2 dx \le C|(\varphi_k^\epsilon)'(1)|^2 \tag{2.3}$$

$\forall\, k \in \mathbb{N}$ *and* $0 < \epsilon < 1$. *Indeed, according to Theorem* 2.1, *at least one of the uniform estimates in* (2.3) *fails.*

(c) *Note that the eigenfunctions φ_{ϵ_j} of the statement of Theorem* 2.1 *and which verify either* (2.1) *or* (2.2) *correspond to eigenvalues that are of the order of $\lambda \sim \epsilon^{-2}$.*

(d) *We do not know if for any nonconstant periodic $\rho \in L^\infty(\mathbb{R})$ one can always find a sequence $\epsilon_j \to 0$ verifying* (2.1). *The same can be said about* (2.2).

(e) *Note that* (2.1) *or* (2.2) *holds for particular choices of the sequence $\epsilon_j \to 0$. Thus, Theorem* 2.1 *is not an obstacle for* (2.3) *to hold uniformly along other sequences $\epsilon_j \to 0$.*

(f) *Concerning the last statement of Theorem* 2.1 *much more can be said. In fact, for any ρ as above there exists x_0 such that $\tilde\rho(x) = \rho(x + x_0)$ satisfies* (a). *The same can be said about* (b).

(g) *The existence of eigenfunctions which concentrate most of their energy near the boundary is also known for multidimensional problems* (*see* [1]).

In the rest of the section we briefly sketch the proof of Theorem 2.1 to show how these singular eigenfunctions appear.

Consider the change of variables $t = \frac{x}{\epsilon}$ which transforms system (1.1) into

$$\begin{cases} \frac{d^2}{dt^2}\varphi(t) + \lambda\epsilon^2\rho(t)\varphi(t) = 0, & t \in (0, \epsilon^{-1}), \\ \varphi(0) = \varphi(\epsilon^{-1}) = 0, \end{cases} \tag{2.4}$$

where, recall, $\rho(t)$ is a 1-periodic function.

Consider now the following equation in the real line:

$$\frac{d^2}{dt^2}\varphi(t) + \nu\rho(t)\varphi(t) = 0, \qquad t \in \mathbb{R}. \tag{2.5}$$

We observe that solutions of (2.4) can be viewed as solutions of (2.5) such that $\varphi(0) = \varphi(\epsilon^{-1}) = 0$.

Equation (2.5) is known as the Hill equation, for which we can apply the classical Floquet theorem (see [10] for a detailed description of this type of equation). Some classical consequences of this theorem are

- there exist values of ν

$$0 = \nu_0 < \nu_1' \le \nu_2' < \nu_1 \le \nu_2 < \nu_3' \le \nu_4' < \nu_3 \cdots$$

  such that, if we take $\bar{\nu}$ in one of the so-called stability intervals $(\nu_0, \nu_1') \cup (\nu_2', \nu_1) \cup (\nu_2, \nu_3') \cup \cdots$ all solutions of (2.5) are bounded in $s \in \mathbb{R}$. On the other hand, if $\bar{\nu}$ belongs to one of the instability intervals $(\nu_1', \nu_2') \cup (\nu_1, \nu_2) \cup (\nu_3', \nu_4') \cup \cdots$, all solutions are unbounded.
  In particular, when $\bar{\nu} \in (0, \nu_1')$ the solutions of (2.5) remain bounded in $\mathbb{R}$. Thus, roughly speaking, ν_1' is the first value for which the instability in (2.5) may appear.
- if ρ is nonconstant at least one of the instability intervals is not empty.
- consider $\bar{\nu}$ in one of the instability intervals. Then there exist two linearly independent solutions of (2.5) of the form $\varphi_1(t) = e^{-\alpha s} p_1(t)$ and $\varphi_2(t) = e^{\alpha s} p_2(t)$, with $\alpha > 0$ real and p_1, p_2 1-periodic functions with an infinite number of zeros.

Take $\bar{\nu}$ in one of the instability intervals and consider φ_1 and φ_2, the two linearly independent solutions of (2.5) described above. Clearly, functions p_1 and p_2 must have a zero in $[0, 1]$. Now we divide our analysis into two alternative cases: $p_1(0) = 0$ or $p_1(0) \neq 0$.

(a) Suppose that $p_1(0) = 0$. Then, φ_1 is a solution of (2.5) which decays exponentially as $t \to \infty$ and verifies $\varphi_1(0) = 0$ (see Figure 2.1). Moreover, it is easy to prove that there exists a constant $C_1 > 0$ such that

$$|\varphi_1(t)| + |\varphi_1'(t)| \le C_1 e^{-\alpha t}. \tag{2.6}$$

Observe that

$$u_1^\epsilon(x) = \varphi_1(x/\epsilon)$$

is a solution of the problem

$$\begin{cases} u'' + \lambda \rho(\frac{x}{\epsilon}) u = 0, \\ u(0) = 0, \end{cases} \tag{2.7}$$

for $\lambda = \epsilon^{-2}\nu$. Consider now a sequence $\epsilon_k \to 0$ such that $\varphi_1(\epsilon_k^{-1}) = 0$, i.e., $p_1(\epsilon_k^{-1}) = 0$. We can take $\epsilon_k^{-1} = k$ since $p_1(0) = 0$ and p_1 is 1-periodic. Then $u_1^{\epsilon_k}(x)$ is a solution of (2.7) which verifies the additional condition $u_1^{\epsilon_k}(1) = 0$ and then it is a sequence of eigenfunctions of (1.1).

We easily check that $u_1^{\epsilon_k}(x)$ verifies (2.1) (see Figure 2.1).

(b) Suppose now that $p_1(0) \neq 0$. We are going to construct a sequence of eigenfunctions of (1.1) which verifies (2.2). Consider a linear combination, φ_3, of φ_1 and φ_2 such that $\varphi_3(0) = 0$. Then

$$\varphi_3(t) = e^{\alpha t} p_2(t) + a e^{-\alpha t} p_1(t), \qquad (a \ge 0).$$

We have the following estimates for φ_3.

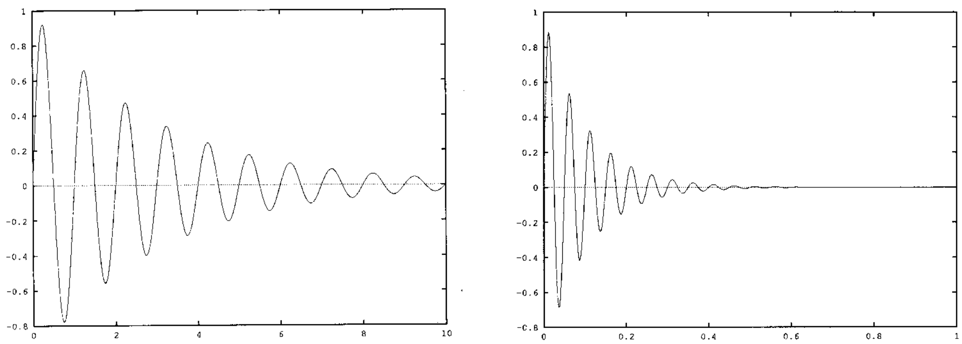

FIG. 2.1. $\varphi_1(t)$ *in* $t \in [0, 10]$ *and* $u_1^\epsilon(x) = \varphi_1(x/\epsilon)$ *with* $\epsilon = 1/20$ *in* $x \in [0, 1]$.

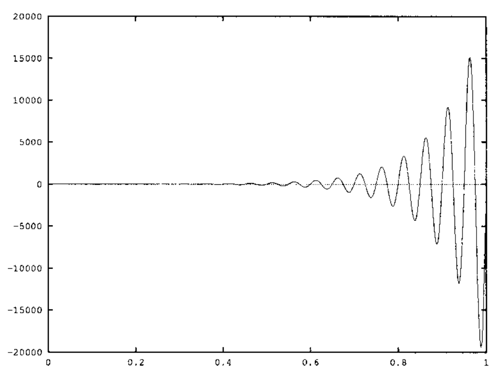

FIG. 2.2. $u_2^{\epsilon_k}(x) = \varphi_3(x/\epsilon_k)$.

LEMMA 2.2. *There exists* $C_1 > 0$ *such that*

$$|\varphi_3(t)| \leq C_1 e^{\alpha t} \quad \forall t > 0.$$

Furthermore, there exists a sequence $\epsilon_k \to 0$ *such that* $\varphi_3(\epsilon_k^{-1}) = 0$ *and*

$$|\varphi_3'(\epsilon_k^{-1})| \geq C_2 e^{\alpha \epsilon_k^{-1}} \tag{2.8}$$

as $\epsilon_k \to 0$, *for some constant* $C_2 > 0$.

Now we choose, as in the previous case, a sequence ϵ_k such that $\varphi_3(\epsilon_k^{-1}) = 0$. Then $u_2^{\epsilon_k}(x) = \epsilon_k^{1/2} \varphi_3(x/\epsilon_k)$ is an eigenfunction of (1.1) and we easily check (2.2) (see Figure 2.2).

The last statement of Theorem 2.1 comes from the fact that if we choose x_0 such that $p_1(x_0) = 0$ and consider the eigenvalue problem (1.1) with density $\rho((x_0 + x)/\epsilon)$, then we are in the case (a) above. On the other hand, if we choose x_1 such that

$p_1(x_1) \neq 0$ and consider the eigenvalue problem (1.1) with density $\rho((x_1+x)/\epsilon)$, then we are in case (b). ☐

3. Eigenvalues and eigenfunctions below the critical size. In this section we derive and discuss explicit formulas for the eigenvalues and eigenfunctions corresponding to low frequencies, those below the critical size ($\lambda_k^\epsilon \sim \epsilon^{-2}$). First we derive two power series expansions for the eigenvalues and eigenfunctions in terms of functions $S^n(t)$ that depend only on ρ, that are defined for $t \in [0, \epsilon^{-1}]$, and that can be computed explicitly by a recursion formula.

The method to find these asymptotic expansions consists basically of four steps: First we use the shooting method to reduce the eigenvalue problem to an initial value one. In the second step we apply the change of variables $x/\epsilon = t$ to remove the dependence of the coefficient $\rho(x/\epsilon)$ on ϵ. After this, we obtain a Cauchy problem in $t \in [0, \epsilon^{-1}]$ whose solutions can be approximated formally using the well-known asymptotic method WKB. Finally, we prove that the asymptotic series provided by the WKB method converges uniformly in the interval $t \in [0, \epsilon^{-1}]$ and we provide completely rigorous formulas for the eigenvalues and eigenfunctions of the original problem.

We organize this section as follows: In the first subsection we state the main result with the formulas for the eigenpairs. In the second subsection we prove this main result. Finally, in the last two subsections we discuss the formulas for the eigenvalues and eigenfunctions and provide some simple examples.

3.1. Power series expansions for the eigenvalues and the eigenfunctions.

THEOREM 3.1. *Consider $\rho \in L^\infty(\mathbb{R})$ a 1-periodic function such that $0 < \rho_m \leq \rho(x) \leq \rho_M < \infty$ almost everywhere (a.e.) in $\mathbb{R}$. Then there exists a constant $C > 0$ such that the eigenvalues λ_k^ϵ of (1.1) with $k \leq C\epsilon^{-1}$ verify the following identity:*

$$\sqrt{\lambda_k^\epsilon \bar{\rho}} + \sum_{n\geq 1} \left(\epsilon^{2n} d_{2n-1} + \epsilon^{2n+1} c_{2n}(\epsilon^{-1})\right) (\lambda_k^\epsilon)^{(2n+1)/2} = k\pi, \tag{3.1}$$

where the constants d_{2n-1} are given by

$$d_{2n-1} = \frac{1}{2\sqrt{\bar{\rho}}} \int_0^1 \sum_{\substack{i+j=2n \\ i,j\geq 1}} S_t^i(t) S_t^j(t) dt, \qquad n \geq 1, \tag{3.2}$$

and $c_{2n}(s)$ are 1-periodic functions given by

$$c_{2n}(s) = i \int_0^s \left[\int_0^{t_1} S_{tt}^{2n}(t_2) dt_2 - \int_0^1 \int_0^t S_{tt}^{2n}(t_2) dt_2 dt\right] dt_1, \quad n \geq 1. \tag{3.3}$$

The functions $S_t^n(t)$ are defined for $t \in [0, \epsilon^{-1}]$. They depend only on ρ and can be computed explicitly from the following recursion formula:

$$S_t^0(t) = -i\sqrt{\bar{\rho}}, \quad S_t^1(t) = -\int_0^t (\rho(t_2) - \bar{\rho}) dt_2 + \gamma_1,$$

$$S_t^n(t) = -\int_0^t \sum_{i+j=n-1} S_t^i(t_2) S_t^j(t_2) dt_2 + \gamma_n, \quad n \geq 2,$$

where

$$\gamma_1 = \int_0^1 \int_0^{t_1} (\rho(t_2) - \bar{\rho}) dt_2 dt_1,$$

$$(3.4)\quad \gamma_n = \frac{1}{2i\sqrt{\bar\rho}} \int_0^1 \sum_{\substack{i+j=n \\ i,j\geq 1}} S_t^i(t_1) S_t^j(t_1) dt_1 + \int_0^1 \int_0^{t_1} \sum_{i+j=n-1} S_t^i(t_2) S_t^j(t_2) dt_2 dt_1.$$

The coefficients S_t^n verify the following properties:

(a) $S_t^n(t)$ *are* 1*-periodic functions,*

(b) $S_t^{2n-1}(t)$ *is a real function while* $S_t^{2n}(t)$ *is purely imaginary.*

For the eigenfunctions we have

$$\varphi_k^\epsilon(x) = A_k^\epsilon e^{\left(\sum_{n=0}^\infty \epsilon^{2n+2}(\lambda_k^\epsilon)^{n+1} S^{2n+1}(\frac{x}{\epsilon})\right)} \sin\left(\operatorname{Im}\left(\sum_{n=0}^\infty \epsilon^{2n+1}(\lambda_k^\epsilon)^{(2n+1)/2} S^{2n}\left(\frac{x}{\epsilon}\right)\right)\right) \tag{3.5}$$

where A_k^ϵ is a normalization constant, $\operatorname{Im}(\cdot)$ *denotes the imaginary part, and* $S^n(t)$ *are given by*

$$S^n(t) = -\int_t^{\epsilon^{-1}} S_t^n(t) dt. \tag{3.6}$$

In (3.5) *the series converges uniformly in $x \in [0,1]$ for k and ϵ such that $k\epsilon \leq C$.*

REMARK 2. (a) *As described in Theorem* 2.1, *for any $\nu \in (0, \nu_1')$ system* (2.5) *is stable. It would be interesting to see if constant ν_1' and the constant $C > 0$ for which the asymptotic development* (3.1) *is valid for $k\epsilon \leq C$ are related. In view of* (1.10) *whenever $k\epsilon \leq C$ we have $\nu_k^\epsilon = \lambda_k^\epsilon \epsilon^2 \leq C^2\pi^2/\rho_m$. Thus, one may expect* (3.1) *to be valid in the range $k\epsilon \leq C$ for any $C < \sqrt{\rho_m \nu_1'}/\pi$. This is an open problem. Note that the proof of Theorem* 3.1 *provides some bounds on the constant $C > 0$ for which* (3.1) *is valid.*

(b) *In Theorem* 3.1 *we have assumed that the periodicity of the density ρ is* 1*. This is not essential in our analysis. In fact, if ρ is of period T then, setting $\hat\rho(y) = \rho(Ty)$ and $\hat\epsilon = \epsilon T$, the problem is reduced to the* 1*-periodic case. As Theorem* 3.1 *is valid for any sequence ϵ it applies particularly to $\hat\epsilon = \epsilon T$.*

3.2. Proof of Theorem 3.1. We use the shooting method which consists first in solving the Cauchy problem

$$\begin{cases} y'' + \lambda\rho(\frac{x}{\epsilon})y = 0, \\ y(1) = 0,\ y'(1) = 1. \end{cases} \tag{3.7}$$

Once we have solved system (3.7) the eigenvalues and eigenfunctions are characterized as the pairs (y, λ) such that $y(0) = 0$.

To solve system (3.7) we consider the change of variables $x/\epsilon = t$ which transforms (3.7) into

$$y''(t) + \epsilon^2\lambda\rho(t)y(t) = 0 \quad \text{in } (0, \epsilon^{-1}) \tag{3.8}$$

with the boundary conditions

$$y(\epsilon^{-1}) = 0, \qquad y'(\epsilon^{-1}) = \epsilon. \tag{3.9}$$

We divide the rest of the proof into three steps. First we obtain a formal asymptotic expansion of the solutions of (3.8)–(3.9). After this we prove the uniform convergence of the asymptotic expansion and finally, in the third step, we come back to the original variable x.

Step 1: *The formal asymptotic expansion.* To approximate the solutions of (3.8) we use the classical asymptotic method WKB (see [5, Chap. 10]).

REMARK 3. *The WKB method is usually applied to singular perturbation problems, i.e., problems where the small parameter is in the principal part of the differential operator. In general, the WKB method is formal when applied to singular perturbation problems.*

In (3.8), *depending on whether* $\epsilon^2\lambda$ *is small or large, we have a regular or a singular perturbation problem. In this work we focus on the case where* $\epsilon^2\lambda$ *is small which corresponds to the regular perturbation problem. This will allow us to validate all the calculations. We refer to* [9] *for the analysis of the singular perturbation problem corresponding to eigenvalues* λ *such that* $\lambda >> \epsilon^{-2}$.

Assume that solutions of (3.8) can be written in the form

$$y_\epsilon(t) = \mathrm{Im}\left(\exp\left(\sum_{n=0}^{\infty} \epsilon^{n+1}\lambda^{(n+1)/2} S^n(t)\right)\right) \tag{3.10}$$

for suitable complex coefficients S^n depending on t. Substituting this expression in (3.8), we obtain

$$\sum_{n=0}^{\infty} \epsilon^{n+1}\lambda^{(n+1)/2} S^n_{tt} + \left(\sum_{n=0}^{\infty} \epsilon^{n+1}\lambda^{(n+1)/2} S^n_t\right)^2 + \epsilon^2\lambda\rho(t) = 0.$$

Equating the terms with the same powers of $\epsilon\sqrt{\lambda}$ we easily obtain the following system of equations:

$$\begin{cases} S^0_{tt} = 0, \\ S^1_{tt} + (S^0_t)^2 + \rho(t) = 0, \\ S^n_{tt} + \sum_{i+j=n-1} S^i_t S^j_t = 0, \quad n \geq 2. \end{cases} \tag{3.11}$$

To integrate this system we need a boundary condition for each S^n_t. We assume that

$$S^n_t \text{ is 1-periodic.} \tag{3.12}$$

From the first equation in (3.11) we deduce that S^0_t is constant. On the other hand, integrating the second equation in (3.11) over a period and taking into account that S^1_t is periodic we obtain

$$(S^0_t)^2 + \int_0^1 \rho(t)\, dt = 0$$

and then

$$S^0_t(t) = \pm i\sqrt{\bar{\rho}}. \tag{3.13}$$

We take the $-$ sign for the moment.

Integrating the second equation in (3.11) we obtain

$$S^1_t(t) = -\int_0^t (\rho(t_2) - \bar{\rho})dt_2 + \gamma_1. \tag{3.14}$$

The constant γ_1 will be chosen below to guarantee the periodicity of S^2_t.

Integrating the third equation in (3.11) over a period and taking into account that S_t^2 is periodic we have

$$2\int_0^1 S_t^0 S_t^1 = -2i\sqrt{\bar\rho}\int_0^1\left[-\int_0^{t_1}(\rho(t_2)-\bar\rho)dt_2+\gamma_1\right]dt_1 = 0.$$

Therefore

$$\gamma_1 = \int_0^1\int_0^{t_1}(\rho(t_2)-\bar\rho)dt_2dt_1. \tag{3.15}$$

Now S_t^2 can be computed integrating the third equation in (3.11). Following this process we easily obtain the following formulas for the coefficients S_t^n:

$$S_t^n(t) = \int_0^t S_{tt}^n + \gamma_n = -\int_0^t \sum_{i+j=n-1} S_t^i(t_2)S_t^j(t_2)dt_2 + \gamma_n, \qquad n\ge 2, \tag{3.16}$$

where

$$\gamma_n = \frac{1}{2i\sqrt{\bar\rho}}\int_0^1 \sum_{\substack{i+j=n\\ i,j\ge 1}} S_t^i(t_1)S_t^j(t_1)dt_1 + \int_0^1\left(\int_0^{t_1}\sum_{i+j=n-1}S_t^i(t_2)S_t^j(t_2)dt_2\right)dt_1,\ n\ge 2. \tag{3.17}$$

Using an induction argument in formulas (3.16) and (3.17) it is not difficult to check the following two properties of the coefficients S_t^n:

1. S_t^n is real if n is odd and purely imaginary if n is even.
2. S_t^n is the conjugate of the function $\overline{S_t^n}$ we would obtain taking the + sign in (3.13).

The coefficients $S^n(t)$ are obtained by integrating, i.e.,

$$S^n(t) = -\int_t^{\epsilon^{-1}} S_t^n(t)dt. \tag{3.18}$$

Note that S^n is chosen such that $S^n(\epsilon^{-1}) = 0$ which implies, in particular, that $y(\epsilon^{-1}) = 0$. We have then found two conjugate solutions of (3.11). Imposing the boundary conditions (3.9) we get the formula

$$y_\epsilon(t) = A_k^\epsilon \exp\left(\sum_{n=0}^\infty \epsilon^{2n+2}\lambda^{(n+1)}S^{2n+1}(t)\right)\sin\left(\operatorname{Im}\sum_{n=0}^\infty \epsilon^{2n+1}\lambda^{(2n+1)/2}S^{2n}(t)\right) \tag{3.19}$$

for the solution of (3.8)–(3.9). In view of (3.18), the equation (3.19) satisfies the first boundary condition in (3.9) while the normalization constant A_ϵ given by

$$A_k^\epsilon = \left[\operatorname{Im}\left(\sum_{n=0}^\infty \epsilon^{2n}\lambda^{(2n+1)/2}S_t^{2n}(\epsilon^{-1})\right)\right]^{-1}$$

is chosen to fulfill the second condition, i.e., $y_\epsilon'(\epsilon^{-1}) = \epsilon$.

Step 2: *Uniform convergence of the series.* The analysis we have done above is completely formal. To justify all the computations we have to prove that the series

$\sum_{n=0}^{\infty} \epsilon^{n+1}\lambda^{(n+1)/2}S^n(t)$ converges uniformly in $t \in (0, \epsilon^{-1})$ and that it can be differentiated term by term twice. Using classical results of differentiation of series it is enough to prove that the series

$$\sum_{i=0}^{\infty} \epsilon^{i+1}\lambda^{(i+1)/2}S^i, \qquad \sum_{i=0}^{\infty} \epsilon^{i+1}\lambda^{(i+1)/2}S^i_t, \qquad \text{and} \qquad \sum_{i=0}^{\infty} \epsilon^{i+1}\lambda^{(i+1)/2}S^i_{tt} \tag{3.20}$$

converge uniformly in $t \in (0, \epsilon^{-1})$ for k and ϵ fixed such that $k\epsilon \le C$ with $C > 0$ sufficiently small.

The following lemma provides the necessary estimates on the coefficients to guarantee the convergence of these series.

LEMMA 3.2. *There exists a constant $D_1 > 0$ such that*

$$\|S^0_t\|_\infty \le \frac{D_1}{2^6},$$
$$\|S^n_t\|_\infty \le \frac{(D_1)^{n+1}}{2^6(n+1)^2}, \qquad \frac{\|S^n_t\|_\infty}{\sqrt{\rho}} \le \frac{(D_1)^n}{2^6(n+1)^2} \quad \forall n \ge 1,$$

$\forall\ 0 < \epsilon \le 1$ *and* $\forall\ \lambda > 0$, *where* $\|\cdot\|_\infty$ *represents the norm in the space* $L^\infty(0, \epsilon^{-1})$.

We leave the proof of this lemma to Appendix I.

Now we are going to see that, as a consequence of this lemma, the series in (3.20) are uniformly convergent in $t \in (0, \epsilon^{-1})$.

Observe that

$$\sum_{i=0}^{\infty} \epsilon^{i+1}\lambda^{(i+1)/2}S^i = -\sum_{i=0}^{\infty} \epsilon^{i+1}\lambda^{(i+1)/2}\int_t^{\epsilon^{-1}} S^i_t.$$

On the other hand,

$$\left|\sum_{i=0}^{\infty} \epsilon^{i+1}\lambda^{(i+1)/2}\int_t^{\epsilon^{-1}} S^i_t\right| \le \sum_{i=0}^{\infty} \epsilon^{i+1}\lambda^{(i+1)/2}\epsilon^{-1}\|S^i_t\|_\infty \tag{3.21}$$
$$= \sqrt{\lambda}\sum_{i=0}^{\infty} \epsilon^i\lambda^{i/2}\|S^i_t\|_\infty \le \sqrt{\lambda}\left(\sum_{i=0}^{\infty} \frac{D_1(D_1\epsilon\sqrt{\lambda})^i}{2^6(i+1)^2}\right),$$

which is uniformly convergent if we fix ϵ and λ such that $\lambda \le D_1^{-2}\epsilon^{-2}$. We have proved that the series in (3.19) makes sense for λ and ϵ such that $\lambda \le C\epsilon^{-2}$ with $C = D_1^{-2}$.

The second series in (3.20) converges uniformly in view of the estimates of Lemma 3.2 and an argument similar to the one used above.

Concerning the third series in (3.20) we have

$$\sum_{n=0}^{\infty} \epsilon^{n+1}\lambda^{(n+1)/2}\|S^n_{tt}\|_\infty = \sum_{n=0}^{\infty} \epsilon^{n+1}\lambda^{(n+1)/2}\|\sum_{i+j=n-1} S^i_t S^j_t\|_\infty \tag{3.22}$$
$$\le \sum_{n=0}^{\infty}\left(\left(\epsilon\sqrt{\lambda}\right)^{n+1}\sum_{i+j=n-1}\|S^i_t\|_\infty\|S^j_t\|_\infty\right)$$
$$\le \sum_{n=0}^{\infty}\left(\left(\epsilon\sqrt{\lambda}\right)^{n+1}\sum_{i+j=n-1}\frac{(D_1)^{i+1}}{2^6(i+1)^2}\frac{(D_1)^{j+1}}{2^6(j+1)^2}\right)$$
$$= \sum_{n=0}^{\infty}\left(\frac{(D_1\epsilon\sqrt{\lambda})^{n+1}}{2^{12}}\sum_{i+j=n-1}\frac{1}{(i+1)^2(j+1)^2}\right)$$

which can be bounded, using Lemma 5.2 in Appendix I, by

$$\sum_{n=0}^{\infty} \left(\frac{(D_1 \epsilon \sqrt{\lambda})^{n+1}}{2^8 (n+1)^2} \right).$$

The latter converges in the same range of λ and ϵ.

Step 3: *Coming back to the original variable* x**.** We have proved that the solution $y_\epsilon(t)$ of (3.8)–(3.9) is given by (3.10) when $\lambda \leq C\epsilon^{-2}$, and we have calculated the functions $S^n(t)$ from the coefficient ρ by a recursion formula. Coming back to the original variable $x = t\epsilon$ we see that $y_\epsilon(x/\epsilon)$ is the solution of (3.7) and then, according to the shooting method, the eigenvalues λ_k^ϵ of (1.1) are given by the solutions of

$$\text{Im} \left(\sum_{n=0}^{\infty} \epsilon^{2n+1} \lambda^{(2n+1)/2} S^{2n}(0) \right) = k\pi, \tag{3.23}$$

and the eigenfunctions are given by (3.10) with $\lambda = \lambda_k^\epsilon$, i.e., they are given by the formula (3.5).

To conclude the proof of the theorem we are going to see that the formula (3.23) is equivalent to (3.1). Observe that

$$S^0(0) = i\epsilon^{-1}\sqrt{\bar{\rho}},$$

$$S^2(0) = -\int_0^{\epsilon^{-1}} S_t^2(t)\, dt = -\int_0^{\epsilon^{-1}} \left[\int_0^{t_1} S_{tt}^2(t_2) dt_2 + \gamma_2 \right] dt_1. \tag{3.24}$$

This last expression is uniformly bounded in ϵ if and only if the term inside the brackets can be written as a periodic function minus its average (see Lemma 5.1 in Appendix I). Now, by definition, we have (see formula (3.17))

$$\gamma_2 = -\int_0^1 \int_0^t S_{tt}^2(t_1) dt_1 dt + \frac{1}{2i\sqrt{\bar{\rho}}} \int_0^1 (S_t^1(t))^2 dt, \tag{3.25}$$

and we deduce that (3.24) is uniformly bounded if and only if the second term in (3.25) vanishes, i.e., $S_t^1 \equiv 0$. On the other hand, as we showed in (3.14),

$$S_t^1(t) = -\int_0^t (\rho(t_2) - \bar{\rho}) dt_2 + \gamma_1$$

which is nonzero if and only if ρ is constant. This proves in particular that S^2 is uniformly bounded if and only if ρ is constant. Otherwise S^2 grows linearly in t.

So, we have that $S^2(0) = i\left(c_2(\epsilon^{-1}) + \epsilon^{-1} d_1\right)$ where

$$d_1 = \frac{1}{2\sqrt{\bar{\rho}}} \int_0^1 (S_t^1(t))^2 dt$$

and

$$c_2(s) = i \int_0^s \left[\int_0^{t_1} S_{tt}^2(t_2) dt_2 - \int_0^1 \int_0^t S_{tt}^2(t_1) dt_1 dt \right] dt_1.$$

The term inside the brackets is a periodic function minus its average and then $c_2(s)$ is 1-periodic.

With the same arguments we deduce that

$$S^{2n}(0) = i\left(c_{2n}(\epsilon^{-1}) + \epsilon^{-1} d_{2n-1}\right), \tag{3.26}$$

where $c_{2n}(s)$ are the 1-periodic functions given in (3.3) and d_{2n-1} are the constants given in (3.2). Substituting these expressions of $S^{2n}(0)$ in (3.23) we obtain (3.1). This concludes the proof of Theorem 3.1.

3.3. Analysis of the eigenvalues. From formula (3.1) we can derive some interesting properties of the eigenvalues. In particular, we show that we can improve the convergence of the approximation to the eigenvalues introducing correctors. We consider separately the case of the first order approximation, the second order one, and the general case where we obtain a weaker but more explicit formula for the eigenvalues than the one stated in (3.1).

To complete this section we study the gap between two consecutive eigenvalues. This property is essential in the uniform controllability of the wave equation with oscillating density which is our main motivation in this work.

3.3.1. First order approximation. Consider the series (3.1) truncated in the first term. We have

$$\sqrt{\lambda_k^\epsilon \bar{\rho}} + \mathcal{O}(\epsilon^2 (\lambda_k^\epsilon)^{3/2}) = k\pi$$

and then

$$\sqrt{\lambda_k^\epsilon} = \frac{k\pi}{\sqrt{\bar{\rho}}} + \mathcal{O}(\epsilon^2 k^3). \tag{3.27}$$

This means, in particular, that the eigenvalues λ_k^ϵ are uniformly close to those of the limit problem (1.2) if $k \leq C\epsilon^{-2/3}$ with $C > 0$ small enough.

3.3.2. Second order approximation. If we truncate the series (3.1) in the second term we obtain

$$\sqrt{\lambda_k^\epsilon \bar{\rho}} + \epsilon^2 (\lambda_k^\epsilon)^{3/2} d_1 + \epsilon^3 (\lambda_k^\epsilon)^{3/2} c_2(\epsilon^{-1}) + \mathcal{O}(\epsilon^4 (\lambda_k^\epsilon)^{5/2}) = k\pi. \tag{3.28}$$

Then,

$$\begin{aligned} \sqrt{\lambda_k^\epsilon} &= \frac{k\pi}{\sqrt{\bar{\rho}}} - \frac{d_1}{\sqrt{\bar{\rho}}}\epsilon^2 (\lambda_k^\epsilon)^{3/2} - \frac{c_2(\epsilon^{-1})}{\sqrt{\bar{\rho}}}\epsilon^3 (\lambda_k^\epsilon)^{3/2} + \mathcal{O}(\epsilon^4 (\lambda_k^\epsilon)^{5/2}) \\ &= \frac{k\pi}{\sqrt{\bar{\rho}}} + F(\sqrt{\lambda_k^\epsilon}) + \mathcal{O}(\epsilon^4 k^5). \end{aligned} \tag{3.29}$$

From formula (3.29) and the mean value theorem we deduce

$$\begin{aligned} &\sqrt{\lambda_k^\epsilon} - \frac{k\pi}{\sqrt{\bar{\rho}}} - F\left(\frac{k\pi}{\sqrt{\bar{\rho}}}\right) = F(\sqrt{\lambda_k^\epsilon}) - F\left(\frac{k\pi}{\sqrt{\bar{\rho}}}\right) + \mathcal{O}(\epsilon^4 k^5) \\ &\leq |F'(\xi)| \left|\sqrt{\lambda_k^\epsilon} - \frac{k\pi}{\sqrt{\bar{\rho}}}\right| + \mathcal{O}(\epsilon^4 k^5) = |F'(\xi)| \mathcal{O}(\epsilon^2 k^3) + \mathcal{O}(\epsilon^4 k^5), \end{aligned} \tag{3.30}$$

where ξ is a real number between $\sqrt{\lambda_k^\epsilon}$ and $k\pi/\sqrt{\bar{\rho}}$. On the other hand,

$$F'(\xi) = -3d_1 \frac{\epsilon^2 \xi^2}{\sqrt{\bar{\rho}}} - 3\epsilon^3 \xi^2 \frac{c_2(\epsilon^{-1})}{\sqrt{\bar{\rho}}} = \mathcal{O}(\epsilon^2 \xi^2) = \mathcal{O}(\epsilon^2 k^2). \tag{3.31}$$

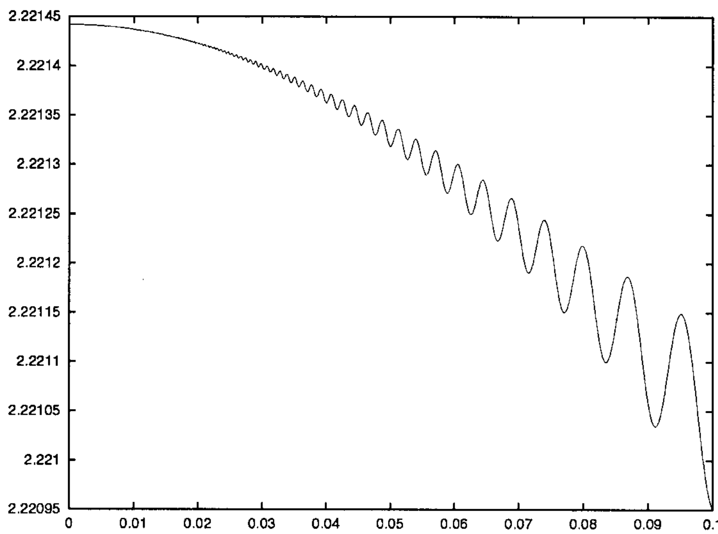

FIG. 3.1. $\sqrt{\lambda_1^\epsilon}$ *as a function of* ϵ *when* $\rho(t) = 2 + \sin 2\pi t$.

Combining (3.30) and (3.31) we deduce the following behavior of λ_k^ϵ:

$$\sqrt{\lambda_k^\epsilon} = \frac{k\pi}{\sqrt{\bar\rho}} - \epsilon^2 (k\pi)^3 \frac{d_1}{\bar\rho^2} - \epsilon^3 (k\pi)^3 \frac{c_2(\epsilon^{-1})}{\bar\rho^2} + \mathcal{O}(\epsilon^4 k^5). \tag{3.32}$$

We observe that $c_2(\epsilon^{-1})$ oscillates as $\epsilon \to 0$ because $c_2(t)$ is 1-periodic. In particular, this means that the third order differential of none of the eigenvalues λ_k^ϵ with respect to ϵ at $\epsilon = 0$ exists.

In Figure 3.1 we show this behavior when $\rho(t) = 2 + \sin 2\pi t$ for $k = 1$ fixed. In this case we can easily compute the functions $S_t^n(t)$ and obtain the formulas (3.32) for the eigenvalues. Indeed, we have

$$S_t^0(t) = -i\sqrt{2}, \qquad S_t^1(t) = \frac{\cos 2\pi t}{2\pi}, \quad S_t^2(t) = i\frac{16\sin 2\pi t - 1}{16\sqrt{2}\pi^2}.$$

This allows us to compute d_1 and $c_2(s)$:

$$d_1 = \frac{1}{16\sqrt{2}\pi^2}, \qquad c_2(s) = \sqrt{2}\frac{\cos 2s\pi - 1}{4\pi^3}.$$

Therefore, substituting in formula (3.32) for $k = 1$ we have

$$\sqrt{\lambda_1^\epsilon} = \frac{\pi}{\sqrt{2}} - \epsilon^2 \frac{\pi}{64\sqrt{2}} - \epsilon^3 \frac{\sqrt{2}}{16}(\cos(2\pi/\epsilon) - 1) + \mathcal{O}(\epsilon^4). \tag{3.33}$$

Summarizing, we see that for k fixed, the constant d_1 gives us the first order corrector. This constant is given by

$$d_1 = \frac{1}{2\sqrt{\bar\rho}} \int_0^1 \left(\int_0^t (\rho(t_2) - \bar\rho) dt_2 - \int_0^1 \int_0^{t_1} (\rho(t_2) - \bar\rho) dt_2 dt_1 \right)^2 dt, \tag{3.34}$$

and it is always nonzero for any nonconstant ρ. This means that the limit eigenvalues $k\pi$ never provide a good approximation of λ_k^ϵ when $k \gg \epsilon^{-2/3}$. The third term in (3.32) is always small below the critical size $k \leq C\epsilon^{-1}$. However, it produces an oscillatory behavior of the eigenvalues λ_k^ϵ as a function of $\epsilon \to 0$. Observe that we can avoid this behavior taking particular sequences $\epsilon_n \to 0$. For instance, in the example above, if we consider $\epsilon_n = 1/n$ with $n \in \mathbb{N}$ the third term in (3.33) disappears.

In general, the oscillatory terms in the expansion of the eigenvalues (3.1) disappear when we consider the sequence $\epsilon_n = 1/n$ because, in view of (3.3), $c_{2k}(n) = 0 \forall\ k, n \in \mathbb{N}$.

3.3.3. Higher order approximations. Taking more terms in the asymptotic expansion (3.1) we can obtain higher order correctors in (3.32). More precisely, cutting off the series in (3.1) at the level $n = N$ we obtain an approximation of $\sqrt{\lambda_k^\epsilon}$ with an error of the order of $\epsilon^{2N+2}k^{2N+3}$, i.e.,

$$\sqrt{\lambda_k^\epsilon} = \frac{k\pi}{\sqrt{\bar\rho}} - \sum_{n=1}^{N} \epsilon^{2n}(k\pi)^{2n+1}\frac{d_{2n-1}}{\bar\rho^{n+1}} - \sum_{n=1}^{N} \epsilon^{2n+1}(k\pi)^{2n+1}\frac{c_{2n}(\epsilon^{-1})}{\bar\rho^{n+1}} + \mathcal{O}(\epsilon^{2N+2}k^{2N+3}), \tag{3.35}$$

$\forall\ N \in \mathbb{N}$. Note, however, that, with a finite number of terms of the series in (3.1), we never reach the optimal region $\epsilon k \leq C$ with C small enough.

3.3.4. Estimates on the gap between two consecutive eigenvalues. As we mentioned in the introduction, the first motivation of this work was the problem of the uniform boundary observability and controllability of solutions of the wave equation with rapidly oscillating coefficients. This problem can be handled by means of the theory of nonharmonic Fourier series (see, for instance, [8]). However, to do that one needs uniform lower bounds on the gap or distance between consecutive eigenvalues. The following proposition shows that this uniform bound can be obtained in the whole region $k\epsilon \leq C$.

PROPOSITION 3.3. *Given $\delta > 0$, there exists a constant $C(\delta) > 0$ such that*

$$\sqrt{\lambda_{k+1}^\epsilon} - \sqrt{\lambda_k^\epsilon} \geq \frac{\pi}{\sqrt{\bar\rho}} - \delta,$$

$\forall\ k$ and ϵ with $\epsilon k \leq C(\delta)$.

REMARK 4. *Note that, even if the uniform convergence of the eigenvalues toward their limits is only valid for $k \leq C\epsilon^{-2/3}$, the uniform gap condition holds up to the critical level $k \leq C\epsilon^{-1}$. Observe that, as $\delta \to 0$, the gap in Proposition 3.3 converges to the gap $\pi/\sqrt{\bar\rho}$ of the limit constant coefficient problem.*

When $\epsilon = 1/n$ with $n \in \mathbb{N}$ the uniform gap condition holds below the critical level $k \leq C\epsilon^{-1}$ with $C < \sqrt{\rho_m \nu_1'}/\pi$ which is the constant in Remark 2 (see [7]). When k approaches $\sqrt{\rho_m \nu_1'}/\pi\epsilon^{-1}$ the distance between two consecutive eigenvalues becomes smaller and the uniform gap condition is lost.

Proof. From formula (3.1) we deduce that

$$\sqrt{\lambda_k^\epsilon} = \frac{k\pi}{\sqrt{\bar\rho}} - \frac{1}{\sqrt{\bar\rho}}\sum_{n\geq 1}\epsilon^{2n}(\lambda_k^\epsilon)^{(2n+1)/2}d_{2n-1} + \mathcal{O}((\epsilon k)^3). \tag{3.36}$$

Then,

$$\sqrt{\lambda_{k+1}^\epsilon} - \sqrt{\lambda_k^\epsilon} = \frac{(k+1)\pi}{\sqrt{\bar\rho}} - \frac{1}{\sqrt{\bar\rho}}\sum_{n\geq 1}\epsilon^{2n}\left(\sqrt{\lambda_{k+1}^\epsilon}\right)^{2n+1} d_{2n-1} \tag{3.37}$$

$$-\frac{k\pi}{\sqrt{\bar\rho}}+\frac{1}{\sqrt{\bar\rho}}\sum_{n\ge1}\epsilon^{2n}(\sqrt{\lambda_k^\epsilon})^{2n+1}d_{2n-1}+\mathcal{O}((\epsilon k)^3)$$
$$=\frac{\pi}{\sqrt{\bar\rho}}-\frac{1}{\sqrt{\bar\rho}}\sum_{n\ge1}d_{2n-1}\epsilon^{2n}\left[\left(\sqrt{\lambda_{k+1}^\epsilon}\right)^{2n+1}-(\sqrt{\lambda_k^\epsilon})^{2n+1}\right]+\mathcal{O}((\epsilon k)^3).$$

By the mean value theorem we have

$$\left(\sqrt{\lambda_{k+1}^\epsilon}\right)^{2n+1}-(\sqrt{\lambda_k^\epsilon})^{2n+1}=g\left(\sqrt{\lambda_{k+1}^\epsilon}\right)-g(\sqrt{\lambda_k^\epsilon})$$
$$=g'(\xi_n)\left(\sqrt{\lambda_{k+1}^\epsilon}-\sqrt{\lambda_k^\epsilon}\right)=(2n+1)\,\xi_n^{2n}\left(\sqrt{\lambda_{k+1}^\epsilon}-\sqrt{\lambda_k^\epsilon}\right)\quad\forall n\ge1, \tag{3.38}$$

where $\xi_n\in(\sqrt{\lambda_k^\epsilon},\sqrt{\lambda_{k+1}^\epsilon})$. So, combining (3.37) and (3.38),

$$\sqrt{\lambda_{k+1}^\epsilon}-\sqrt{\lambda_k^\epsilon}=\frac{\pi}{\sqrt{\bar\rho}}-\frac{\sqrt{\lambda_{k+1}^\epsilon}-\sqrt{\lambda_k^\epsilon}}{\sqrt{\bar\rho}}\sum_{n\ge1}d_{2n-1}\epsilon^{2n}\,(2n+1)\,\xi_n^{2n}+\mathcal{O}((\epsilon k)^3),$$

and then

$$\sqrt{\lambda_{k+1}^\epsilon}-\sqrt{\lambda_k^\epsilon}=\frac{\pi+\mathcal{O}((\epsilon k)^3)}{\sqrt{\bar\rho}+\sum_{n\ge1}d_{2n-1}\epsilon^{2n}\,(2n+1)\,\xi_n^{2n}}.$$

So, to finish the proof of Proposition 3.3 we only have to check that

$$\sum_{n\ge1}d_{2n-1}\epsilon^{2n}\,(2n+1)\,\xi_n^{2n}=\mathcal{O}((\epsilon k)^2).$$

Observe that

$$\left|\sum_{n\ge1}d_{2n-1}\epsilon^{2n}\,(2n+1)\,\xi_n^{2n}\right|=\left|\sum_{n\ge1}\frac{\epsilon^{2n}\,(2n+1)\,\xi_n^{2n}}{2\sqrt{\bar\rho}}\int_0^1\sum_{\substack{i+j=2n\\ i,j\ge1}}S_t^i(t)S_t^j(t)dt\right|$$
$$\le\sum_{n\ge1}\frac{\epsilon^{2n}\,(2n+1)\,\xi_n^{2n}}{2\sqrt{\bar\rho}}\sum_{\substack{i+j=2n\\ i,j\ge1}}\left\|S_t^i(t)\right\|_\infty\left\|S_t^j(t)\right\|_\infty. \tag{3.39}$$

Here the last sum can be estimated as in (3.22) and then (3.39) can be bounded by

$$\sum_{n\ge1}\frac{\epsilon^{2n}\,(2n+1)\,\xi_n^{2n}}{2\sqrt{\bar\rho}}\times\frac{(D_1)^{2n+2}}{2^8(2n+2)^2} \tag{3.40}$$

which converges as long as $\epsilon\xi_nD_1<1\ \forall\ n\ge1$. On the other hand, as $\xi_n\in(\sqrt{\lambda_k^\epsilon},\sqrt{\lambda_{k+1}^\epsilon})$, $\xi_n=\mathcal{O}(k)$ and (3.40) is of the order of $\mathcal{O}(\epsilon^2k^2)$ as we wanted to prove.

3.4. Analysis of the eigenfunctions. In this section we analyze the formula for the eigenfunctions stated in Theorem 3.1. We start proving a uniform observability property for the eigenfunctions. This property and the spectral gap stated in section 3.3.4 constitute the two ingredients to prove the uniform controllability property which motivates our work.

In the second and third parts we discuss formula (3.5) and we analyze the first order terms in the asymptotic formula for the eigenfunctions.

3.4.1. Uniform observability of eigenfunctions. From Theorem 3.1 we deduce that the eigenfunctions φ_k^ϵ are given by formula (3.5).

We observe that this function can grow exponentially when $\lambda << \epsilon^{-2}$, $\epsilon \to 0$, if the function $\mathrm{Re}\sum_{i=0}^{\infty} \epsilon^{i+1} (\lambda_k^\epsilon)^{(i+1)/2} S^i(t)$ is not periodic but, for example, grows linearly with respect to the variable t. This would give us eigenfunctions with most of their energy localized in one of the extremes of the interval of the same type as those we have constructed in section 2. Our purpose in this section is to prove that this is not the case. So, we deduce that the singular behavior of eigenfunctions described in section 2 does not occur when we consider eigenfunctions with small $\lambda_k^\epsilon \epsilon^2$, i.e., below the critical range.

PROPOSITION 3.4. *The function*

$$\mathrm{Re}\sum_{i=0}^{\infty} \epsilon^{i+1} (\lambda_k^\epsilon)^{(i+1)/2} S^i(t)$$

is 1*-periodic with respect to* t $\forall$ $k \in \mathbb{N}$ *and* $\epsilon > 0$ *such that* $k\epsilon \le C$.

Proof of Proposition 3.4. First of all observe that

$$\mathrm{Re}\left(\sum_{i=0}^{\infty} \epsilon^{i+1} (\lambda_k^\epsilon)^{(i+1)/2} S^i\right) = \epsilon^2 \lambda_k^\epsilon S^1 + \epsilon^4 (\lambda_k^\epsilon)^2 S^3 + \epsilon^6 (\lambda_k^\epsilon)^3 S^5 + \cdots. \tag{3.41}$$

In this expression S^1 is a periodic function since

$$S^1(t) = -\int_t^{\epsilon^{-1}} \left[-\int_0^t (\rho(t_2) - \bar{\rho}) dt_2 + \gamma_1\right] dt_1$$

which is the primitive of a periodic function minus its average (see Lemma 5.1 in Appendix I). Therefore, it is bounded uniformly in $t \in [0, \epsilon^{-1}]$. Concerning S^3 we have

$$S^3(t) = -\int_t^{\epsilon^{-1}} S_t^3(t)\, dt = -\int_t^{\epsilon^{-1}} \left[-\int_0^{t_1} S_{tt}^3(t_2) dt_2 + \gamma_3\right] dt_1, \tag{3.42}$$

which is uniformly bounded in $t \in (0, \epsilon^{-1})$ when $\epsilon \to 0$ if the expression inside the brackets is a periodic function minus its average. Recall that

$$\gamma_3 = \int_0^1 \int_0^{t_1} S_{tt}^3(t_1) dt_1 dt + \frac{1}{2i\sqrt{\bar{\rho}}} \int_0^1 2 S_t^1 S_t^2 dt,$$

so that the condition for (3.42) to be uniformly bounded is that

$$\int_0^1 S_t^1 S_t^2 dt = 0. \tag{3.43}$$

In general, the condition for S^{2n+1} to be uniformly bounded in $t \in [0, \epsilon^{-1}]$, when $\epsilon \to 0$, is

$$\int_0^1 \sum_{\substack{i+j=2n+1\\ i,j\ge 1}} S_t^i S_t^j dt = 0. \tag{3.44}$$

Indeed, in this case

$$S^{2n+1}(t) = -\int_t^{\epsilon^{-1}} \left[-\int_0^{t_1} S_{tt}^{2n+1}(t_2)dt_2 + \gamma_{2n+1} \right] dt_1$$

and the constant γ_{2n+1} is exactly the average of the function $\int_0^t S_{tt}^{2n+1}(t_1)dt_1$. Then, in view of Lemma 5.1 in Appendix I we deduce that S^{2n+1} is a 1-periodic function.

In Appendix II we give a proof of formulas (3.43) and (3.44). This completes the proof of Proposition 3.4. □

As a consequence of Proposition 3.4 eigenfunctions φ_k^ϵ with $k \leq C\epsilon^{-1}$ cannot exhibit an accumulation of energy near the boundary of the interval $(0,1)$ of the type we described in section 2. In fact, the uniform bound of S^{2n+1} in $t \in [0, \epsilon^{-1}]$ allows us to prove the following observability result.

Proposition 3.5. *There exist C, $c > 0$ such that the following estimates hold for the eigenfunctions φ_k^ϵ of* (1.1) *with $k \leq c\epsilon^{-1}$:*

$$\frac{1}{C}\left(|(\varphi_k^\epsilon)'(0)|^2 + |(\varphi_k^\epsilon)'(1)|^2\right) \leq \int_0^1 |(\varphi_k^\epsilon)'(x)|^2 \leq C\left(|(\varphi_k^\epsilon)'(0)|^2 + |(\varphi_k^\epsilon)'(1)|^2\right). \tag{3.45}$$

Remark 5. *As we mentioned in section* 3.3.4, *in order to obtain uniform observability estimates for the solutions of the wave equation we need a uniform lower bound on the gap between consecutive eigenvalues. The uniform observability inequality* (3.45) *for eigenfunctions is the second main ingredient with which to apply the tools from nonharmonic Fourier series* (*see* [8], [15]). *Observe that, as pointed out in section* 2, (3.45) *is false if $\lambda\epsilon^2$ is large enough.*

Proof of Proposition 3.5. Due to the symmetry of the problem we can reduce the proof of formula (3.45) to prove

$$\frac{1}{C}|(\varphi_k^\epsilon)'(1)|^2 \leq \int_0^1 |(\varphi_k^\epsilon)'(x)|^2 \leq C|(\varphi_k^\epsilon)'(1)|^2. \tag{3.46}$$

We normalize the eigenfunctions to have $(\varphi_k^\epsilon)'(1) = 1$. Then, φ_k^ϵ are given by formula (3.5) where constant A_k^ϵ is

$$A_k^\epsilon = \left[\left.\frac{d}{dx}\right|_{x=1} \mathrm{Im}\left(\sum_{n=0}^{\infty} \epsilon^{n+1} (\lambda_k^\epsilon)^{(n+1)/2} S^n\left(\frac{x}{\epsilon}\right) \right) \right]^{-1}.$$

Define $S(x) = \sum_{n=0}^{\infty} \epsilon^{n+1}(\lambda_k^\epsilon)^{(n+1)/2} S^n(\frac{x}{\epsilon})$. Note that $S(x)$ depends on k and ϵ but we do not make this fact explicit in the notation to simplify it. Then

$$\varphi_k^\epsilon(x) = A_k^\epsilon e^{\mathrm{Re}(S(x))} \sin\left[\mathrm{Im}(S(x))\right]. \tag{3.47}$$

We have

$$(\varphi_k^\epsilon)'(x) = A_k^\epsilon \mathrm{Re}(S') e^{\mathrm{Re}(S)} \sin[\mathrm{Im}(S)] + A_k^\epsilon e^{\mathrm{Re}(S)} \mathrm{Im}(S') \cos[\mathrm{Im}(S)]. \tag{3.48}$$

Now we are going to estimate these terms. We claim that

$$|\mathrm{Re}(S(x))| \leq C\epsilon k, \qquad |\mathrm{Re}(S'(x))| \leq C\epsilon k^2. \tag{3.49}$$

The second claim can be easily obtained as in (3.21) taking into account that the sum starts at $i = 1$ since we are only considering the real part of $S'(x)$. Concerning the first claim, we first observe that $\|S^{2i+1}\|_\infty \leq \|S_t^{2i+1}\|_\infty$ since S_t^{2i+1} is of zero average (see Lemma 5.1). Applying Lemma 3.2 we have

$$\|S^{2i+1}\|_\infty \leq \frac{(D_1)^{2i+1}}{2^6(2i+1)^2}.$$

Using this inequality, arguing as in (3.21), and taking into account that $\epsilon\sqrt{\lambda}$ is a common factor of all the terms appearing in the series we deduce the first claim.

Now, we estimate the term ImS for which we need the following formula:

$$S^{2n}(t) = i\left(c_{2n}(1/\epsilon) - c_{2n}(t) + (1/\epsilon - t)d_{2n-1}\right) \quad \forall t \in [0, \epsilon^{-1}]. \tag{3.50}$$

In $t = 0$ this formula coincides with formula (3.26). This identity can be proved in a similar way as we did for (3.26).

Using (3.50) and the formula (3.1) for the eigenvalues, we obtain

$$\begin{aligned}
(3.51)\ \mathrm{Im}(S(x)) &= \sum_{n=0}^{\infty} \epsilon^{2n+1}(\lambda_k^\epsilon)^{(2n+1)/2} S^{2n}\left(\frac{x}{\epsilon}\right) \\
&= \sqrt{\lambda_k^\epsilon \bar{\rho}}(1-x) + \sum_{n=1}^{\infty} \epsilon^{2n+1}(\lambda_k^\epsilon)^{(2n+1)/2}\left(c_{2n}(1/\epsilon) - c_{2n}(x/\epsilon) + \frac{1-x}{\epsilon} d_{2n-1}\right) \\
&= (1-x)\left[\sqrt{\lambda_k^\epsilon \bar{\rho}} + \sum_{n=1}^{\infty} \epsilon^{2n}(\lambda_k^\epsilon)^{(2n+1)/2}(d_{2n-1} + \epsilon c_{2n}(\epsilon^{-1}))\right] \\
&\quad + \sum_{n=1}^{\infty} \epsilon^{2n+1}(\lambda_k^\epsilon)^{(2n+1)/2}\left(c_{2n}(1/\epsilon) - c_{2n}(x/\epsilon) - (1-x)c_{2n}(\epsilon^{-1})\right) \\
&= (1-x)k\pi + \epsilon^3(\lambda_k^\epsilon)^3 \sum_{n=1}^{\infty} \epsilon^{2n-2}(\lambda_k^\epsilon)^{(n-1)}\left(c_{2n}(1/\epsilon) - c_{2n}(x/\epsilon) - (1-x)c_{2n}(\epsilon^{-1})\right) \\
&= (1-x)k\pi + \mathcal{O}(\epsilon^3 k^3).
\end{aligned}$$

Analogously, we have

$$\mathrm{Im}(S'(x)) = -k\pi + \mathcal{O}(\epsilon^2 k^3), \quad A_k^\epsilon = \frac{1}{-k\pi + \mathcal{O}(\epsilon^2 k^3)}. \tag{3.52}$$

From (3.48)–(3.52) we can estimate the two terms in (3.48):

$$\begin{aligned}
&A_k^\epsilon \mathrm{Re}(S') e^{\mathrm{Re}(S)} \sin[\mathrm{Im}(S)] \\
&\quad = \frac{\mathcal{O}(\epsilon k^2)}{-k\pi + \mathcal{O}(\epsilon^2 k^3)}(1 + \mathcal{O}(\epsilon k)) \sin\left(k\pi(1-x) + \mathcal{O}(\epsilon^3 k^3)\right) = \mathcal{O}(\epsilon k), \\
&A_k^\epsilon e^{\mathrm{Re}(S)} \mathrm{Im}(S') \cos[\mathrm{Im}(S)] \\
&\quad = \frac{1 + \mathcal{O}(\epsilon k)}{-k\pi + \mathcal{O}(\epsilon^2 k^3)}(-k\pi + \mathcal{O}(\epsilon^2 k^3)) \cos\left(k\pi(1-x) + \mathcal{O}(\epsilon^3 k^3)\right) \\
&\quad = \cos\left(k\pi(1-x)\right) + \mathcal{O}(\epsilon k)
\end{aligned}$$

for ϵk sufficiently small.

Then, clearly

$$|(\varphi_k^\epsilon)'(x)|^2 = \cos^2(k\pi(1-x)) + \mathcal{O}(\epsilon k).$$

Finally,

$$\int_0^1 |(\varphi_k^\epsilon)'(x)|^2 = \frac{1}{2} + \mathcal{O}(\epsilon k),$$

which is equivalent to (3.46) due to the normalization we have chosen for the eigenfunctions. □

3.4.2. First order approximation. Formula (3.5) allows us to give an approximate expression for the eigenfunctions φ_k^ϵ below the critical size. From (3.47)–(3.52) we obtain

$$\left\| \varphi_k^\epsilon(x) - \frac{\sin(k\pi(1-x))}{-k\pi} \right\|_{W^{1,\infty}(0,1)} = \left\| \varphi_k^\epsilon(x) - (-1)^k \frac{\sin(k\pi x)}{k\pi} \right\|_{W^{1,\infty}(0,1)} = \mathcal{O}(k\epsilon), \tag{3.53}$$

i.e., the eigenfunctions φ_k^ϵ are close to those of the limit problem (1.2) if $k\epsilon \leq C$ with C small enough, i.e., below the critical size.

Remark 6. *In view of* (3.53), *the eigenfunctions of the limit problem* (1.2) *provide a good approximation up to the critical size. This is not the case for the eigenvalues, where the limit eigenvalues only provide a good approximation when* $k \leq C\epsilon^{-2/3}$ *with* C *small enough.*

3.4.3. Correctors. In this section we show how we can obtain correctors to the limit eigenfunctions from formula (3.5) when we fix the parameter k. We center our attention in the first corrector of the first eigenfunction. Proceeding as in the proof of Proposition 3.5 we get

$$\varphi_1^\epsilon = -\exp\left(\frac{\epsilon^2\pi^2}{\bar{\rho}} S^1(x/\epsilon)\right) \frac{\sin(\pi x)}{\pi} + \mathcal{O}(\epsilon^3) \quad \text{in } W^{1,\infty}(0,1). \tag{3.54}$$

Observe that the correction to the limit eigenfunction has an oscillatory behavior with respect to x. In Figure 3.2 we show this behavior in the above example, i.e., with $\rho^\epsilon(x) = 2 + \sin(2\pi x/\epsilon)$.

4. The case where the oscillating coefficient is in the principal part. In this section we study the system

$$\begin{cases} (a(x/\epsilon)u')' + \lambda u = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{4.1}$$

where $a(x) \in L^\infty(R)$ is a periodic function with $0 < a_m \leq a(x) \leq a_M < \infty$ and ϵ is small. We assume, without loss of generality, that the period of a is 1.

We observe that the oscillating coefficient is now in the principal part of the operator.

Let us denote by $\{\lambda_k^\epsilon\}_{k\in N}$ the eigenvalues of (4.1) ordered in an increasing way.

As in system (1.1) the eigenvalues and eigenfunctions of (4.1) converge, as $\epsilon \to 0$, to those of the limit system

$$\begin{cases} \widehat{a}u'' + \lambda u = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{4.2}$$

where $\widehat{a} = 1/\int_0^1 \frac{dr}{a(r)}$. The eigenpairs of (4.2) can be also computed explicitly:

$$\begin{aligned} \lambda_k &= \widehat{a}k^2\pi^2, & k \in N, \\ \varphi_k(x) &= \sin(k\pi x), & k \in N. \end{aligned} \tag{4.3}$$

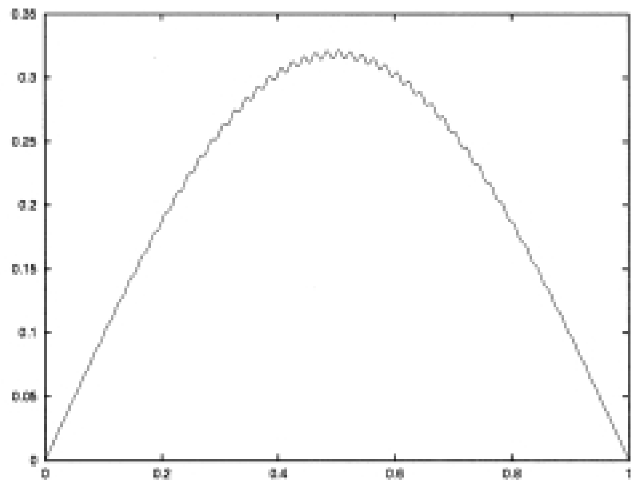

FIG. 3.2. $\varphi_1^\epsilon(x)$ *when* $\rho^\epsilon(x) = 2 + \sin 2\pi x/\epsilon$ *and* $\epsilon = 1/50$. *In this figure, we have multiplied the amplitude of the small oscillation around the main curve* $\sin \pi x$ *by the factor* $4\epsilon^{-1}$. *Otherwise the small oscillation could not be appreciated.*

We show that the analysis of the previous section can be also applied to this system to obtain a full description of the convergence of the spectrum. The idea is to reduce system (4.1) to one in the form (1.1) for which we can apply Theorem 3.1. We refer to [11] for some further results in this direction.

Consider the following change of variables:

$$\begin{aligned} & y(x) = \frac{\int_0^{x/\epsilon} \frac{dr}{a(r)}}{\int_0^{\epsilon^{-1}} \frac{dr}{a(r)}}, \quad \delta(\epsilon) = \frac{\int_0^1 \frac{dr}{a(r)}}{\int_0^{\epsilon^{-1}} \frac{dr}{a(r)}}, \\ & b(s) = a(t(s)), \quad v(y) = u(x(y)), \\ & \mu = \lambda \left(\epsilon \int_0^{\epsilon^{-1}} \frac{dr}{a(r)} \right)^2, \quad s(t) = \frac{\int_0^t \frac{dr}{a(r)}}{\int_0^1 \frac{dr}{a(r)}}, \end{aligned} \tag{4.4}$$

where $x(y)$ represents the inverse function of $y(x)$, i.e., $x(y) = x$ if and only if $y(x) = y$, and $t(s)$ is the inverse function of $s(t)$. Obviously, this change of variables depends on ϵ but, for the sake of simplicity, we do not make this fact explicit in the notation.

This change of variables transforms system (4.1) into

$$\left\{ \begin{array}{l} v_{yy}(y) + \mu b(y/\delta) v(y) = 0, \quad y \in (0,1), \\ v(0) = v(1) = 0. \end{array} \right. \tag{4.5}$$

Remark that the function $b(s)$ is 1-periodic, as is $a(t)$, and then system (4.5) is equivalent to system (1.1).

By Theorem 3.1 there exists a constant $C > 0$ such that the eigenvalues μ_k^δ of

(4.5) with $k \leq C\delta^{-1}$ verify the following identity:

$$\sqrt{\mu_k^\delta \overline{b}} + \sum_{n\geq 1} \left(\delta^{2n} d_{2n-1} + \delta^{2n+1} c_{2n}(\delta^{-1})\right) (\mu_k^\delta)^{(2n+1)/2} = k\pi. \tag{4.6}$$

Recall that the constants d_n and the periodic functions $c_n(x)$ can be computed from the coefficient b and

$$\overline{b} = \int_0^1 b(y)dy = \int_0^1 a(x)y'(x)dx = \int_0^1 \frac{1}{\int_0^1 \frac{dr}{a(r)}} dx = \widehat{a}. \tag{4.7}$$

Let us see how we can derive asymptotic formulas for the eigenvalues from identity (4.6). Consider the first order approximation which we obtain truncating the series in the first term:

$$\sqrt{\mu_k^\delta} = \frac{k\pi}{\sqrt{\overline{b}}} + \mathcal{O}(\delta^3 k^3). \tag{4.8}$$

Coming back to the original variables λ and ϵ we have

$$\sqrt{\lambda_k^\epsilon}\epsilon \int_0^{\epsilon^{-1}} \frac{dr}{a(r)} = \frac{k\pi}{\sqrt{\widehat{a}}} + \mathcal{O}(\delta^3 k^3). \tag{4.9}$$

Due to the definition of $\delta(\epsilon)$ we have that

$$\epsilon a_m/\hat{a} \leq \delta(\epsilon) \leq \epsilon a_M/\hat{a} \tag{4.10}$$

and therefore,

$$\sqrt{\lambda_k^\epsilon} = \frac{k\pi}{\epsilon \int_0^{\epsilon^{-1}} \frac{dr}{a(r)} \sqrt{\widehat{a}}} + \mathcal{O}(\epsilon^2 k^3). \tag{4.11}$$

We observe that the eigenvalues λ_k^ϵ show an oscillatory behavior as a function of ϵ in the first term instead of the third one which was where the oscillation appeared in the case of the density. So, this case exhibits a less regular behavior of the eigenvalues in the sense that here the first derivative of the eigenvalues λ_k^ϵ with respect to ϵ is not well defined as $\epsilon \to 0$.

This lack of regularity of the eigenvalues with respect to the oscillation parameter was proved for the first time in [14] (see also [12]). The analysis in [14] (which also holds for particular higher dimensional domains) uses classical homogenization techniques and provides a less explicit description of the dependence of λ_k^ϵ with respect to ϵ. They prove that depending on the sequence $\epsilon_j \to 0$ the limit $\lim_{\epsilon_j \to 0}(\lambda_k^{\epsilon_j} - \lambda_k)/\epsilon_j$ can take different values, while our analysis provides a complete description of the dependence of λ_k^ϵ with respect to ϵ and can be applied to obtain all order correction formulas to the homogenized eigenvalues and eigenfunctions.

In Figure 4.1 we show λ_1^ϵ as a function of ϵ in the particular case $a(t) = 1/(2 + \sin(2\pi t))$.

5. Appendix I.

In this section we are going to prove Lemma 3.2.

LEMMA 3.2. *There exists a constant $D_1 > 0$ such that*

$$\left\|S_t^0\right\|_\infty \leq \frac{D_1}{2^6}, \tag{5.1}$$

$$\|S_t^n\|_\infty \leq \frac{(D_1)^{n+1}}{2^6(n+1)^2}, \quad \frac{\|S_t^n\|_\infty}{\sqrt{\rho}} \leq \frac{(D_1)^n}{2^6(n+1)^2}, \quad \forall\, n \geq 1, \tag{5.2}$$

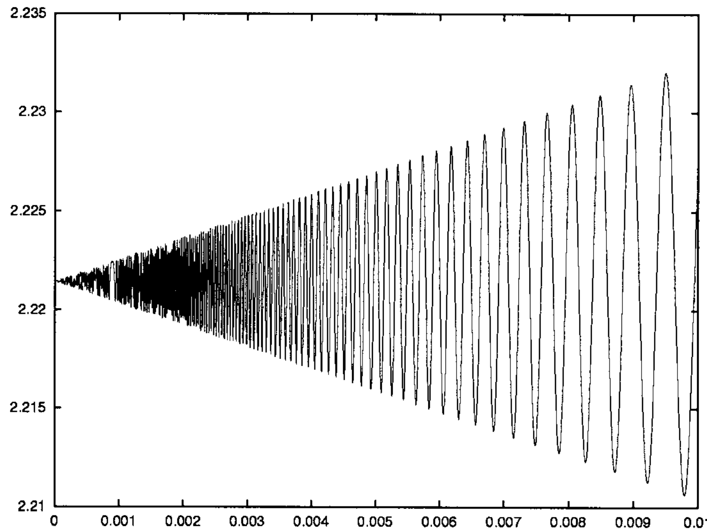

FIG. 4.1. λ_1^ϵ *as a function of* ϵ *when* $a(t) = 1/(2 + \sin(2\pi t))$.

$\forall\ 0 < \epsilon \leq 1$ *and* $\forall\ \lambda > 0$*, where* $\|\cdot\|_\infty$ *represents the norm in* $L^\infty(0, \epsilon^{-1})$.

Proof. First of all we observe that $|S_t^0| = \sqrt{\rho}$ and we deduce (5.1) with $D_1 \geq \sqrt{\rho}\,(4 \times 16)$. The inequalities (5.2) will be proved by induction. Observe that for each $n \geq 1$ the first inequality in (5.2) can be derived from the second one because we are supposing $D_1 \geq \sqrt{\rho}$. So we will focus on proving the second one.

We need the following lemma.

LEMMA 5.1. *Consider* $f(t)$ *a* 1*-periodic function and denote by* $\bar{f} = \int_0^1 f(s)\, ds$ *its average. Then* $g(t) = \int_0^t (f(s) - \bar{f})ds$ *is a* 1*-periodic function. Moreover* $\|g(t)\|_\infty \leq \|f\|_\infty$*. On the other hand, if we define* $g_\alpha(t) = \int_0^t (f(s) - \alpha)ds$ *with* $\alpha \in \mathbb{R}$*, then* g_α *is* 1*-periodic if and only if* $\alpha = \bar{f}$.

Proof of Lemma 5.1. It is clear that $g(t)$ is 1-periodic. Concerning the bound for $g(t)$, it is enough to prove it for $t \in [0, 1]$. Suppose that it is not true. Then, there exists $t_0 \in [0, 1]$ such that $|g(t_0)| > \|f\|_\infty$, but then

$$|g(t)| = \left| \int_0^t (f(s) - \bar{f})ds \right| \leq \left\| f - \bar{f} \right\|_\infty t \leq 2 \|f\|_\infty t,$$

and t_0 must be greater than $1/2$. However,

$$0 = \int_0^1 (f(s) - \bar{f})ds = \int_0^{t_0} (f(s) - \bar{f})ds + \int_{t_0}^1 (f(s) - \bar{f})ds = g(t_0) + \int_{t_0}^1 (f(s) - \bar{f})ds$$

and we have a contradiction because $|g(t_0)| > \|f\|_\infty$ and

$$\left| \int_{t_0}^1 (f(s) - \bar{f})ds \right| \leq 2 \|f\|_\infty (1 - t_0) < \|f\|_\infty$$

because $t_0 > 1/2$.

Concerning the last part of the lemma we observe that

$$g_\alpha(t) = \int_0^t (f(s) - \bar{f})ds + \int_0^t (\bar{f} - \alpha)ds = \int_0^t (f(s) - \bar{f})ds + t(\bar{f} - \alpha).$$

Then, by the first part of the lemma, g_α can be written as the sum of a 1-periodic function and a linear function which vanishes when $\bar{f} = \alpha$. At this point the proof is finished. □

Consider $n = 1$. Then, by definition of γ_1 and using Lemma 5.1,

$$|S_t^1| = \left| \int_t^{\epsilon^{-1}} (\bar{\rho} - \rho(s))ds - \gamma_1 \right| \leq 2 \left\| \int_t^{\epsilon^{-1}} (\bar{\rho} - \rho(s))ds \right\|_\infty \leq 2\|\rho\|_\infty$$

so that the second inequality in (5.2) holds taking

$$D_1 \geq 2^6 \max \left\{ \frac{2\|\rho\|_\infty}{\sqrt{\bar{\rho}}}, \sqrt{\bar{\rho}} \right\}. \tag{5.3}$$

Consider now $n > 1$ and suppose we have chosen $D_1 > 0$ in such a way that (5.1) and (5.2) hold for any $i \leq n - 1$, i.e.,

$$\left\| S_t^i \right\|_\infty \leq \frac{(D_1)^{i+1}}{2^6(i+1)^2} \quad \text{and} \quad \frac{\left\| S_t^i \right\|_\infty}{\sqrt{\bar{\rho}}} \leq \frac{(D_1)^i}{2^6(i+1)^2}, \qquad 1 \leq i \leq n-1,$$

with D_1 as above. We are going to see that

$$\frac{\|S_t^n\|_\infty}{\sqrt{\bar{\rho}}} \leq \frac{(D_1)^n}{2^6(n+1)^2}. \tag{5.4}$$

Taking into account formulas (3.16) and (3.4), we have

$$\begin{aligned}
\frac{|S_t^n|}{\sqrt{\bar{\rho}}} &= \frac{\left| -\int_t^{\epsilon^{-1}} S_{tt}^n + \gamma_n \right|}{\sqrt{\bar{\rho}}} \leq \frac{1}{\sqrt{\bar{\rho}}} \left| \int_t^{\epsilon^{-1}} S_{tt}^n \right| + \frac{1}{\sqrt{\bar{\rho}}} |\gamma_n| \\
&= \frac{1}{\sqrt{\bar{\rho}}} \left| \int_t^{\epsilon^{-1}} \left(\sum_{\substack{i+j=n-1 \\ i,j \geq 1}} S_t^i S_t^j + 2S_t^0 S_t^{n-1} \right) \right| + \frac{1}{\sqrt{\bar{\rho}}} |\gamma_n| \\
&= \frac{1}{\sqrt{\bar{\rho}}} \left| \int_t^{\epsilon^{-1}} \left(\sum_{\substack{i+j=n-1 \\ i,j \geq 1}} S_t^i S_t^j - 2i\sqrt{\bar{\rho}} \int_t^{\epsilon^{-1}} S_{tt}^{n-1} + 2i\sqrt{\bar{\rho}}\gamma_{n-1} \right) \right| + \frac{1}{\sqrt{\bar{\rho}}} |\gamma_n| .
\end{aligned}$$

We observe that the term inside the brackets is a periodic function minus its average due to the definitions of γ_{n-1}. So, we can apply Lemma 5.1 and the above expression can be bounded by

$$\begin{aligned}
&\frac{1}{\sqrt{\bar{\rho}}} \left\| \sum_{\substack{i+j=n-1 \\ i,j \geq 1}} S_t^i S_t^j - 2i\sqrt{\bar{\rho}} \int_t^{\epsilon^{-1}} S_{tt}^{n-1} \right\|_\infty + \frac{1}{\sqrt{\bar{\rho}}} |\gamma_n| \\
&\leq \frac{1}{\sqrt{\bar{\rho}}} \sum_{\substack{i+j=n-1 \\ i,j \geq 1}} \left\| S_t^i S_t^j \right\|_\infty + 2 \left\| \int_t^{\epsilon^{-1}} S_{tt}^{n-1} \right\|_\infty + \frac{1}{\sqrt{\bar{\rho}}} |\gamma_n| = A_1 + A_2 + A_3.
\end{aligned} \tag{5.5}$$

We will analyze separately all these terms. The following lemma will be needed.

LEMMA 5.2. *Consider $n > 0$. Then*

$$\sum_{i+j=n} \frac{1}{(i+1)^2(j+1)^2} \leq \frac{2^4}{(n+2)^2}.$$

Proof. Let $[\cdot]$ be the integer part function. Then

$$\sum_{i+j=n} \frac{1}{(i+1)^2(j+1)^2} \leq \sum_{i=0}^{[n/2]} \frac{1}{(i+1)^2(n-i+1)^2} + \sum_{j=0}^{[n/2]} \frac{1}{(j+1)^2(n-j+1)^2}$$
$$\leq \frac{2}{(n+2)^2} \sum_{i=0}^{[n/2]} \frac{1}{(i+1)^2} \left(\frac{n+2}{n-i+1}\right)^2 \leq \frac{2}{(n+2)^2} \sum_{i=0}^{[n/2]} \frac{1}{(i+1)^2} 2^2 \leq \frac{2^3}{(n+2)^2} \sum_{i=1}^{\infty} \frac{1}{i^2}$$
$$\leq \frac{2^4}{(n+2)^2}$$

because, if $i \leq \frac{n}{2}$, then

$$\frac{n+2}{n-i+1} = 1 + \frac{i+1}{n-i+1} \leq 1 + \frac{\frac{n}{2}+1}{\frac{n}{2}+1} = 2. \qquad \square$$

We are now ready to estimate the terms A_i given in (5.5). By the hypothesis of induction (5.3) we have

$$A_1 = \frac{1}{\sqrt{\rho}} \sum_{\substack{i+j=n-1 \\ i,j\geq 1}} \left\| S_t^i S_t^j \right\|_\infty = \sum_{\substack{i+j=n-1 \\ i,j\geq 1}} \frac{\left\| S_t^i \right\|_\infty}{\sqrt{\rho}} \left\| S_t^j \right\|_\infty$$
$$\leq \sum_{\substack{i+j=n-1 \\ i,j\geq 1}} \frac{(D_1)^i}{2^6 (i+1)^2} \frac{(D_1)^{j+1}}{2^6 (j+1)^2} = \frac{(D_1)^n}{2^{12}} \sum_{\substack{i+j=n-1 \\ i,j\geq 1}} \frac{1}{(i+1)^2 (j+1)^2}$$
$$\leq \frac{(D_1)^n}{2^8 (n+1)^2}.$$

The term A_2 can be estimated as follows:

$$A_2 = 2 \left\| \int_t^{\epsilon^{-1}} S_{tt}^{n-1} \right\|_\infty = 2 \left\| S_t^{n-1}(\epsilon^{-1}) - S_t^{n-1}(t) \right\| \leq 4 \left\| S_t^{n-1} \right\|_\infty$$
$$= 4\sqrt{\rho} \frac{\left\| S_t^{n-1} \right\|_\infty}{\sqrt{\rho}} \leq 4\sqrt{\rho} \frac{(D_1)^{n-1}}{2^6 n^2} \leq 16\sqrt{\rho} \frac{(D_1)^{n-1}}{2^6 (n+1)^2}$$
$$\leq \frac{(D_1)^n}{2^8 (n+1)^2},$$

since $D_1 \geq 2^6\sqrt{\rho}$ in view of (5.3).

Now we will estimate the last term A_3. Taking into account the definition of γ_n, we have

$$A_3 = \frac{|\gamma_n|}{\sqrt{\rho}} \leq \frac{1}{2\rho} \left| \int_{\epsilon^{-1}-1}^{\epsilon^{-1}} \left(\sum_{\substack{i+j=n \\ i,j\geq 1}} S_t^i S_t^j + 2i\sqrt{\rho} \int_0^t S_{tt}^n \right) \right| \leq \frac{1}{2} \sum_{\substack{i+j=n \\ i,j\geq 1}} \frac{\left\| S_t^i \right\|_\infty}{\sqrt{\rho}} \frac{\left\| S_t^j \right\|_\infty}{\sqrt{\rho}}$$

$$+\frac{\|S^n_{tt}\|_\infty}{\sqrt{\rho}} \leq \frac{1}{2}\sum_{\substack{i+j=n\\ i,j\geq 1}} \frac{(D_1)^i}{2^6\,(i+1)^2}\frac{(D_1)^j}{2^6\,(j+1)^2} + \frac{1}{\sqrt{\rho}}\sum_{i+j=n-1}\left\|S^i_t S^j_t\right\|_\infty \tag{5.6}$$

$$\leq \frac{1}{2}\frac{(D_1)^n}{2^{12}}\sum_{\substack{i+j=n\\ i,j\geq 1}}\frac{1}{(i+1)^2\,(j+1)^2} + \sum_{\substack{i+j=n-1\\ i,j\geq 1}}\frac{\|S^i_t\|_\infty}{\sqrt{\rho}}\left\|S^j_t\right\|_\infty + 2\left\|S^0_t\right\|_\infty\frac{\|S^{n-1}_t\|_\infty}{\sqrt{\rho}}$$

$$\leq \frac{1}{2}\frac{(D_1)^n}{2^8(n+2)^2} + \frac{(D_1)^n}{2^{12}}\left(\sum_{\substack{i+j=n-1\\ i,j\geq 1}}\frac{1}{(i+1)^2\,(j+1)^2} + \frac{2}{n^2}\right)$$

$$= \frac{(D_1)^n}{2^8(n+2)^2}\left(\frac{1}{2} + \sum_{i+j=n-1}\frac{1}{(i+1)^2\,(j+1)^2}\right) \leq 2\frac{(D_1)^n}{2^8(n+1)^2}.$$

From the estimates $A_1,\ A_2$, and A_3 we easily conclude that

$$A_1 + A_2 + A_3 \leq 4\frac{(D_1)^n}{2^8(n+1)^2} = \frac{(D_1)^n}{2^6(n+1)^2}$$

as we wanted to prove. ∎

6. Appendix II. In this section we prove the following result which was needed in section 3.4.1.

Lemma 6.1. *The coefficients* S^{2n+1} *are* 1*-periodic* $\forall n \geq 0$. *Therefore* $S^{2n+1} \in L^\infty(0,\epsilon^{-1})$ *for each* $n \geq 0$.

Proof. We proceed by induction. First of all observe that the result for S^1 was proved in section 3.4.1. Now suppose that it is true for $n \leq N-1$. We have to check that it is also true for $n = N$. As we pointed out in section 3.4.1 it is enough to prove the following formula:

$$\int_0^1 \sum_{\substack{i+j=2N+1\\ i,j\geq 1}} S^i_t S^j_t = 0.$$

Integrating by parts we have to check

$$\int_0^1 \sum_{n=1}^N S^{2n}_{tt} S^{2(N-n)+1} = 0.$$

Indeed the terms in the boundary which appear when integrating by parts vanish due to the periodicity of S^{2n}_{tt} and S^{2n+1} for $n \leq N-1$. We observe that, using formulas (3.11), we have

$$\int_0^1 \sum_{n=1}^N S^{2n}_{tt} S^{2(N-n)+1} = -\int_0^1 \sum_{n=1}^N S^{2(N-n)+1}\sum_{h_1+h_2=2n-1} S^{h_1}_t S^{h_2}_t \tag{6.1}$$

$$= -\int_0^1 \sum_{n=1}^N \sum_{h=0}^{n-1} 2S^{2(N-n)+1} S^{2h}_t S^{2(n-h)-1}_t$$

$$= -\int_0^1 \sum_{h=0}^{N-1} \sum_{n=h+1}^{N} 2S^{2(N-n)+1} S^{2h}_t S^{2(n-h)-1}_t$$

$$= -2\int_0^1 \sum_{h=0}^{N-1} S_t^{2h} \sum_{\substack{i_1+i_2=2(N-h)\\ i_1,i_2 \text{odd}}} S^{i_1} S_t^{i_2}.$$

Now we need the following lemma.

LEMMA 6.2. *The following identity holds for each $k \geq 1$:*

$$\frac{d}{dt}\left(\sum_{\substack{i_1+\cdots+i_k=N\\ i_1,\dots,i_k \text{odd}}} S^{i_1}\cdots S^{i_k}\right) = k \sum_{\substack{i_1+\cdots+i_k=N\\ i_1,\dots,i_k \text{odd}}} S^{i_1}\cdots S_t^{i_k}.$$

Proof.

$$\frac{d}{dt}\left(\sum_{\substack{i_1+\cdots+i_k=N\\ i_1,\dots,i_k \text{odd}}} S^{i_1}\cdots S^{i_k}\right) = \sum_{\substack{i_1+\cdots+i_k=N\\ i_1,\dots,i_k \text{odd}}} S_t^{i_1} S^{i_2}\cdots S^{i_k} + \cdots$$
$$+ \sum_{\substack{i_1+\cdots+i_k=N\\ i_1,\dots,i_k \text{odd}}} S^{i_1}\cdots S_t^{i_k} = k \sum_{\substack{i_1+\cdots+i_k=N\\ i_1,\dots,i_k \text{odd}}} S^{i_1}\cdots S_t^{i_k}$$

because we can change the order of the indexes i_k without changing the sum. ∎

Coming back to (6.1) we have, by Lemma 6.2, the following:

$$\int_0^1 \sum_{n=1}^{N} S_{tt}^{2n} S^{2(N-n)+1} = \int_0^1 \sum_{h=0}^{N-1} S_t^{2h} \left(\sum_{\substack{i_1+i_2=2(N-h)\\ i_1,i_2 \text{odd}}} S^{i_1} S^{i_2}\right)_t \tag{6.2}$$
$$= \int_0^1 \sum_{h=0}^{N-1} S_{tt}^{2h} \sum_{\substack{i_1+i_2=2(N-h)\\ i_1,i_2 \text{odd}}} S^{i_1} S^{i_2} = \int_0^1 \sum_{n=1}^{N-1} S_{tt}^{2n} \sum_{\substack{i_1+i_2=2(N-n)\\ i_1,i_2 \text{odd}}} S^{i_1} S^{i_2},$$

where the last identity comes from the fact that $S_{tt}^0 = 0$.

We observe that the last term of this expression is similar to the first one but its order has been reduced in the sense that the sum of the indexes i_k is smaller. Now we are going to iterate this process to make this order as small as possible.

We need the following lemma.

LEMMA 6.3. *Consider $k \geq 1$. Then,*

$$\int_0^1 \sum_{n=1}^{N-k} S_{tt}^{2n} \sum_{\substack{i_1+\cdots+i_{k+1}=2(N-n)+1-k\\ i_1,\dots,i_{k+1} \text{odd}}} S^{i_1}\cdots S^{i_{k+1}}$$
$$= \frac{2}{k+2}\int_0^1 \sum_{n=1}^{N-k-1} S_{tt}^{2n} \sum_{\substack{i_1+\cdots+i_{k+2}=2(N-n)-k\\ i_1,\dots,i_{k+2} \text{odd}}} S^{i_1}\cdots S^{i_{k+2}}.$$

Proof.

$$\begin{aligned}
&\int_0^1 \sum_{n=1}^{N-k} S_{tt}^{2n} \sum_{\substack{i_1+\cdots+i_{k+1}=2(N-n)+1-k \\ i_1,\ldots,i_{k+1}\text{odd}}} S^{i_1}\cdots S^{i_{k+1}} \\
&= -\int_0^1 \sum_{n=1}^{N-k} \sum_{h_1+h_2=2n-1} S_t^{h_1} S_t^{h_2} \sum_{\substack{i_1+\cdots+i_{k+1}=2(N-n)+1-k \\ i_1,\ldots,i_{k+1}\text{odd}}} S^{i_1}\cdots S^{i_{k+1}} \\
&= -2\int_0^1 \sum_{n=1}^{N-k} \sum_{h=0}^{N-1} S_t^{2h} S_t^{2(n-h)-1} \sum_{\substack{i_1+\cdots+i_{k+1}=2(N-n)+1-k \\ i_1,\ldots,i_{k+1}\text{odd}}} S^{i_1}\cdots S^{i_{k+1}} \\
&= -2\sum_{h=0}^{N-k-1} \sum_{n=h+1}^{N-k} \sum_{\substack{i_1+\cdots+i_{k+1}=2(N-n)+1-k \\ i_1,\ldots,i_{k+1}\text{odd}}} S_t^{2h} S_t^{2(n-h)-1} S^{i_1}\cdots S^{i_{k+1}} \\
&= -2\sum_{h=0}^{N-k-1} S_t^{2h} \sum_{\substack{i_1+\cdots+i_{k+1}+i_{k+2}=2(N-h)-k \\ i_1,\ldots,i_{k+1},i_{k+2}\text{odd}}} S^{i_1}\cdots S_t^{i_{k+2}} \\
&= -\frac{2}{k+2}\sum_{h=0}^{N-k-1} S_t^{2h} \left(\sum_{\substack{i_1+\cdots+i_{k+1}+i_{k+2}=2(N-h)-k \\ i_1,\ldots,i_{k+1},i_{k+2}\text{odd}}} S^{i_1}\cdots S^{i_{k+2}} \right)_t \\
&= \frac{2}{k+2}\sum_{h=0}^{N-k-1} S_{tt}^{2h} \sum_{\substack{i_1+\cdots+i_{k+1}+i_{k+2}=2(N-h)-k \\ i_1,\ldots,i_{k+1},i_{k+2}\text{odd}}} S^{i_1}\cdots S^{i_{k+2}}. \qquad \square
\end{aligned}$$

Applying successively Lemma 6.3 to the expression (6.2), we obtain

$$\begin{aligned}
&\int_0^1 \sum_{n=1}^{N} S_{tt}^{2n} S^{2(N-n)+1} \\
&= \int_0^1 \sum_{n=1}^{N-1} S_{tt}^{2n} \sum_{\substack{i_1+i_2=2(N-n) \\ i_1,i_2\text{odd}}} S^{i_1}S^{i_2} = \frac{2}{3}\int_0^1 \sum_{n=1}^{N-2} S_{tt}^{2n} \sum_{\substack{i_1+i_2+i_3=2(N-n)-1 \\ i_1,i_2,i_3\text{odd}}} S^{i_1}S^{i_2}S^{i_3} \\
&= \cdots = \frac{2^{N-1}}{N!}\int_0^1 S_{tt}^2 \sum_{\substack{i_1+\cdots+i_N=N \\ i_1,\cdots,i_N\text{odd}}} S^{i_1}\cdots S^{i_N} = \frac{2^{N-1}}{N!}\int_0^1 S_{tt}^2 \left(S^1\right)^N \\
&= -\frac{2^N}{N!}\int_0^1 S_t^0 S_t^1 \left(S^1\right)^N = -\frac{2^N}{(N+1)!} S_t^0 \int_0^1 \left(\left(S^1\right)^{N+1}\right)_t = 0.
\end{aligned}$$

Note that the last integral vanishes due to the periodicity of S^1. This concludes the proof of Lemma 6.1. $\square$

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