THE RATE AT WHICH ENERGY DECAYS IN A DAMPED STRING

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ABSTRACT: The energy in a string subject to positive viscous damping is known to decay exponentially in time. Under the assumption that the damping is of bounded variation, we identify the best rate of decay with the supremum of the real part of the spectrum of the infinitesimal generator of the underlying semigroup. We analyze the spectrum of this nonselfadjoint operator in some detail. Our bounds on the real eigenvalues and asymptotic form of the large eigenvalues translate into criteria for over/underdamping and a proof that the decay rate achieves its (negative) minimum over those dampings whose total variation does not exceed a prescribed value.

1. **Introduction**

The displacement u of a string of unit length, fixed at its ends, and in the presence of viscous damping $2a$, satisfies

$$\begin{aligned} u_{tt}(x,t) - u_{xx}(x,t) + 2a(x)u_t(x,t) = 0, \quad 0 < x < 1,\ 0 < t, \qquad (1.1)\\ u(0,t) = u(1,t) = 0, \quad 0 < t, \end{aligned}$$

upon being set in motion by the initial disturbance

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \qquad (1.2)$$

assumed an element of the energy space $X = H_0^1(0,1) \times L^2(0,1)$ with inner product

$$\langle [f,g],[u,v] \rangle = \int_0^1 f'\overline{u}' + g\overline{v}\,dx.$$

We assume $a \in L^\infty(0,1)$ is nonnegative and strictly positive on some subinterval. In this case, the energy in the string at time t,

$$E(t) = \int_0^1 u_x^2(x,t) + u_t^2(x,t)\,dx$$

is known to obey $E(t) \leq CE(0)e^{2\omega t}$ for some finite $C > 0$ and $\omega < 0$, independent of the chosen initial data. We define the *decay rate*, as a function of a, as

$$\omega(a) = \inf\{\omega : \exists C(\omega) > 0 \ s.t.\ E(t) \leq CE(0)e^{2\omega t}, \quad \text{for every finite energy solution of (1.1)}\}. \tag{1.3}$$

This work constitutes an attempt to find the largest class of a on which ω achieves a finite minimum. We prove that ω achieves a finite minimum over those nonnegative a whose total variation does not exceed a prescribed value. We accomplish this upon interpreting (1.1) as the system $V_t = AV$ where $V = [u, u_t]$, $A : D(A) \to X$,

$$A = \begin{pmatrix} 0 & I \\ d^2/dx^2 & -2a \end{pmatrix}, \qquad D(A) = (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1).$$

In terms of decay, the relevant measure is the *spectral abscissa* of A,

$$\mu(a) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\}, \tag{1.4}$$

where $\sigma(A)$ denotes the spectrum of A. It follows easily that $\mu(a) \leq \omega(a)$. Our main result establishes the reverse inequality under the assumption that a is of bounded variation. To our knowledge this has yet to be done carefully even in the case of constant damping. In devoting §2 to this case we establish the procedure to be followed in the variable coefficient case. In §3 we provide rough preliminary bounds on $\sigma(A)$. In §4 we establish necessary and sufficient conditions for the presence of real eigenvalues and so sharpen the results, including the proof of a conjecture, of J. Rauch [14]. In §5 we find the asymptotic form of the eigenvalues and eigenfunctions. These asymptotic forms allow us, in §6, to conclude that the root vectors of A constitute a Riesz basis for X. This in turn provides a Parseval equality from which the desired control on the decay rate in terms of the spectral abscissa follows easily. We close in §7 with comments on related problems and methods.

We have previously announced several of these results in [4].

2. **Constant Damping**

We recall the spectrum of A when a is constant. If $V = [y, z] \in D(A)$ is an eigenvector of A with eigenvalue λ then $z = \lambda y$ and $y'' - 2az = \lambda z$, or

$$y'' - \lambda^2 y - 2a\lambda y = 0, \tag{2.1}$$

subject to

$$y(0) = y(1) = 0. \tag{2.2}$$

When a is constant it follows that $\lambda^2 + 2a\lambda = -n^2\pi^2$, that is,

$$\lambda_{\pm n} = -a \pm \sqrt{a^2 - n^2\pi^2}, \quad n = 1, 2, \ldots. \tag{2.3}$$

with the corresponding eigenvector

$$V_{\pm n} = \sin(n\pi x)[1, \lambda_{\pm n}],$$

when n is such that $\lambda_{\pm n}$ are distinct. Should $a = k\pi$ for some integer k, we call such a defective, define $V_k = \sin(k\pi x)[1, -k\pi]$ as above and the generalized eigenvector V_{-k} via $(A - \lambda_k)V_{-k} = V_k$ and $\langle V_k, V_{-k}\rangle = 0$. That is,

$$V_{-k} = \tfrac{1}{2}\sin(k\pi x)[1/(k\pi), 1].$$

Hence, the algebraic multiplicity of $\lambda_k = -a$ is at least two. We pause to show that it is precisely two and that this is the only eigenvalue of algebraic multiplicity greater than one. If the algebraic multiplicity of λ_n is to exceed one then one must be able to solve $(A - \lambda_n)V_{n,1} = V_n$. With $V_{n,1} = [\phi, \psi]$, this requires $\psi = \lambda_n\phi + \sin(n\pi x)$ and

$$\phi'' + n^2\pi^2\phi = 2(\lambda_n + a)\sin(n\pi x), \quad \phi(0) = \phi(1) = 0.$$

This possesses a solution only when $a = -\lambda_k$ for some k, i.e., when $a = k\pi$. For its algebraic multiplicity to exceed two, one must then be able to solve $(A - \lambda_k)V_{k,2} = V_{-k}$. With $V_{k,2} = [f, g]$, we find $g = \lambda_n f + \sin(k\pi x)$ and

$$f'' + k^2\pi^2 f = \sin(k\pi x), \quad f(0) = f(1) = 0.$$

As this equation does not possess a solution, the algebraic multiplicity of λ_n may not exceed two. Hence, in the case of constant a, the algebraic multiplicity of λ_n is its order as a zero of $\lambda^2 + 2a\lambda + n^2\pi^2$. As this remark will not survive the passage to nonconstant damping we turn to the more general characteristic polynomial, $\lambda \mapsto y_2(1, \lambda)$, of the so–called 'shooting method'. Here $x \mapsto y_2(x, \lambda)$ is the solution of (2.1) subject to the initial conditions

$$y(0, \lambda) = 0, \quad y'(0, \lambda) = 1. \tag{2.4}$$

Clearly the zeros of $\lambda \mapsto y_2(1, \lambda)$ are the eigenvalues of A. In addition, we shall see that an eigenvalue's algebraic multiplicity is the order to which

$y_2(1,\lambda)$ vanishes. As expected, this may be checked explicitly when a is constant. For, in this case

$$y_2(x,\lambda) = \frac{\sinh\sqrt{\lambda^2+2\lambda a}x}{\sqrt{\lambda^2+2\lambda a}}. \tag{2.5}$$

Denoting $\partial/\partial\lambda$ by $\dot{}$ we find

$$\dot{y}_2(1,\lambda_n) = \frac{(\lambda_n+a)(-1)^{n+1}}{n^2\pi^2}.$$

This vanishes only when $\lambda_n = -a$, i.e., when $a = n\pi$ for some n. The second derivative at such a root, $\ddot{y}_2(1,-a) = -1/a^2$, is however, nonzero.

We next demonstrate that $\{V_{\pm n}\}_{n=1}^{\infty}$ constitutes a basis for the energy space X. This is done by comparing it with the orthonormal base of eigenfunctions of the undamped problem, $a \equiv 0$, namely

$$\Phi_{\pm n} = \sin(n\pi x)[1/(n\pi), \pm i], \quad n = 1, 2, \ldots. \tag{2.6}$$

For nonreal λ_n, the normalized eigenvector is $\tilde{V}_n = \frac{1}{n\pi}\sin(n\pi x)[1,\lambda_n]$, and

$$\|\Phi_n - \tilde{V}_n\|_X^2 = |i - \lambda_n/(n\pi)|^2 \int_0^1 \sin^2(n\pi x)\,dx = O(1/n^2).$$

Hence $\{\tilde{V}_{\pm n}\}$ is quadratically close to $\{\Phi_{\pm n}\}$, i.e.,

$$\sum_{n=\pm 1}^{\pm\infty} \|\Phi_n - \tilde{V}_n\|_X^2 < \infty.$$

To see that the $V_{\pm n}$ are, in addition, linearly independent, we turn to the eigenvectors of the adjoint of A,

$$A^* = \begin{pmatrix} 0 & -I \\ -d^2/dx^2 & -2a \end{pmatrix}. \tag{2.7}$$

Of course its eigenvalues are precisely those of A, see (2.3), including multiplicities, while the corresponding eigenvector is

$$W_{\pm n} = \sin(n\pi x)[1, -\lambda_{\mp n}]$$

when $\lambda_{\pm n}$ are distinct. Should $a = k\pi$ for some integer k we define $W_k = \sin(k\pi x)[1, k\pi]$ as above and the generalized eigenvector W_{-k} via $(A^* - \lambda_k)W_{-k} = W_k$ and $\langle W_{-k}, V_{-k}\rangle = 0$. That is,

$$W_{-k} = \tfrac{1}{2}\sin(k\pi x)[1/(k\pi), -1].$$

When a is not defective we see that $\langle V_j, W_n\rangle = -2\lambda_j(a+\lambda_j)\delta_{j,n}$ and hence V_n can not be in the closed linear hull of the remaining V_j, i.e., $\{V_n\}_{n=\pm1}^{\pm\infty}$ is a linearly independent set. If, in fact $a = k\pi$ for some k we note that (i) for $j \neq k$, as above, $\langle V_j, W_n\rangle = -2\lambda_j(a+\lambda_j)\delta_{j,n}$, and (ii) $\langle V_{\pm k}, W_n\rangle = (k\pi/2)\delta_{\pm k,-n}$. Hence, even in the defective case, $\{V_n\}_{n=\pm1}^{\pm\infty}$ is a linearly independent set.

It now follows from the Fredholm Alternative, see, e.g., [13, App. D, Theorem 3], that a linear independent set that is quadratically close to an orthonormal basis is in fact equivalent to that basis in the sense that there exists a linear isomorphism $\mathcal{I}$ of X under which $V_{\pm n} = \mathcal{I}\Phi_{\pm n}$. We may now prove the desired special case of our main result.

Theorem 2.1. *If a is constant then $\mu(a) = \omega(a)$.*

Proof: We may expand the initial data as $[u_0, v_0] = \sum_{n=\pm1}^{\pm\infty} \gamma_n \tilde{V}_n$, and note that, so long as a is not defective,

$$[u, u_t] = \sum_{n=\pm1}^{\pm\infty} \exp(\lambda_n t)\gamma_n \tilde{V}_n \tag{2.8}$$

satisfies (1.1) and (1.2). Moreover,

$$\begin{aligned} E(t) = \|[u,u_t]\|_X^2 &= \left\| \mathcal{I} \sum_{n=\pm1}^{\pm\infty} \exp(\lambda_n t)\gamma_n \Phi_n \right\|_X^2 \\ &\le \|\mathcal{I}\|^2 \sum_{n=\pm1}^{\pm\infty} |\exp(\lambda_n t)|^2 |\gamma_n|^2 \le \|\mathcal{I}\|^2 \exp(2\mu t) \sum_{n=\pm1}^{\pm\infty} |\gamma_n|^2 \\ &= \|\mathcal{I}\|^2 \exp(2\mu t) \left\| \sum_{n=\pm1}^{\pm\infty} \gamma_n \Phi_n \right\|_X^2 = \|\mathcal{I}\|^2 \exp(2\mu t) \left\| \mathcal{I}^{-1} \sum_{n=\pm1}^{\pm\infty} \gamma_n \tilde{V}_n \right\|_X^2 \\ &\le \|\mathcal{I}\|^2 \|\mathcal{I}^{-1}\|^2 E(0) \exp(2\mu t). \end{aligned}$$

In case $a = k\pi$ we must of course modify (2.8) to

$$[u, u_t] = t\exp(\lambda_k t)\gamma_{-k}\tilde{V}_k + \sum_{n=\pm1}^{\pm\infty} \exp(\lambda_n t)\gamma_n \tilde{V}_n,$$

and so obtain $E(t) \le \|\mathcal{I}\|^2 \|\mathcal{I}^{-1}\|^2 E(0)(1+t)\exp(2\mu t)$. Hence, even in the defective case, $\mu(a)$ is the decay rate, though if $k = 1$, i.e., $a \equiv \pi$, the infimum in (1.3) is not attained, i.e., there exists no finite C for which $E(t) \leq CE(0)\exp(2\mu t)$. ∎

This result allows us to express the decay rate in terms of the easily computed spectral abscissa, $\mu(a) = -a + \operatorname{Re}\sqrt{a^2 - \pi^2}$. This makes precise the notion of under(over)damping when a is less(greater) than π.

A sequence in a Hilbert space H that is the image of an orthonormal base for H under a single linear isomorphism is commonly known as a Riesz basis for H. In the case of the previous Theorem, $\mathcal{I}$ is defined explicitly as

$$\mathcal{I}[f,g] = \sum_{n=\pm 1}^{\pm\infty} \langle [f,g], \Phi_n \rangle V_n.$$

This permits one to estimate $C = \|\mathcal{I}\|^2 \|\mathcal{I}^{-1}\|$.

3. **The General Case**

We now prepare to prove Theorem 2.1 in the variable coefficient case. Here we shall assume only that

$$0 \leq \alpha \leq a(x) \leq \beta < \infty$$

almost everywhere in $(0,1)$. The eigenvalues of A are the poles of the resolvent $(A-\lambda)^{-1}$. To solve $(A-\lambda)[v_1, v_2] = [f_1, f_2]$ is to solve $v_2 = \lambda v_1 + f_1$ and

$$v_1'' - \lambda(\lambda + 2a)v_1 = f_2 + (\lambda + 2a)f_1.$$

Solving the latter via the Green's operator, $v_1 = G(\lambda)(f_2 + (\lambda + 2a)f_1)$, we find

$$(A-\lambda)^{-1} = \begin{pmatrix} G(\lambda)(\lambda + 2a) & G(\lambda) \\ I + \lambda G(\lambda)(\lambda + 2a) & \lambda G(\lambda) \end{pmatrix}. \tag{3.1}$$

This Green's operator is

$$G(\lambda)\phi(\xi) = \int_0^1 g(x,\xi,\lambda)\phi(x)\,dx, \quad \text{where} \tag{3.2}$$

$$g(x,\xi,\lambda) = \begin{cases} \frac{w_2(x,\lambda)y_2(\xi,\lambda)}{y_2(1,\lambda)} & \text{if } 0 \leq \xi \leq x \leq 1 \\ \frac{w_2(\xi,\lambda)y_2(x,\lambda)}{y_2(1,\lambda)} & \text{if } 0 \leq x \leq \xi \leq 1 \end{cases} \tag{3.3}$$

where $x \mapsto y_2(x,\lambda)$ and $x \mapsto w_2(x,\lambda)$ solve (2.1) subject to (2.4) and

$$y(1,\lambda) = 0, \quad y'(1,\lambda) = -1, \tag{3.4}$$

respectively. This has prepared the following rough estimate on the spectrum of A.

Theorem 3.1. *(i) A possesses a compact inverse and so a discrete spectrum $\sigma(A)$ of eigenvalues of finite algebraic multiplicity. (ii) The eigenvalues are the roots of $\lambda \mapsto y_2(1,\lambda)$. If λ_n is such a root then $y_2(x,\lambda_n)[1,\lambda_n]$ spans the corresponding eigenspace and its algebraic multiplicity is the order to which $\lambda \mapsto y_2(1,\lambda)$ vanishes. (iii) $\sigma(A)$ is symmetric about the real axis and is contained in*

$$\{\lambda \in \mathbf{C} : |\lambda| \geq \pi,\ -\beta \leq \operatorname{Re}\lambda \leq -\alpha\} \cup [-\beta-(\beta^2-\pi^2)_+^{1/2}, -\alpha+(\beta^2-\pi^2)_+^{1/2}].$$

(iv) The root vectors of A are complete in X.

Proof: (i) From (3.1) and (3.3) it follows easily that $\|A^{-1}\Phi_n\|_X = O(1/n)$, so, in fact A^{-1} is Hilbert-Schmidt.
(ii) If $AV_n = \lambda_n V_n$ and $V_n = [y,z]$ then, as just sketched, $V_n = y[1,\lambda_n]$ where y satisfies (2.1) (at λ_n) and (2.2). As the initial value problem (2.1)–(2.4) possesses the unique solution $y_2(x,\lambda_n)$ we see that y must be a scalar multiple of $y_2(x,\lambda_n)$ and $y_2(1,\lambda_n)=0$. Hence, the geometric multiplicity of each eigenvalue is one. Its algebraic multiplicity is its order as a pole of the resolvent, which, again via (3.1) and (3.3), we recognize as its order as a zero of $\lambda \mapsto y_2(1,\lambda)$.
(iii) As A is real it follows that $\bar{V}_n = y_2(x,\bar{\lambda}_n)[1,\bar{\lambda}_n]$ is an eigenvector of A corresponding to the eigenvalue $\bar{\lambda}_n$. On integrating each side of (2.1) against $y_2(x,\bar{\lambda}_n)$ we find

$$\lambda_n = \frac{-\int_0^1 a|y_2|^2\,dx \pm \left(\left(\int_0^1 a|y_2|^2\,dx\right)^2 - \int_0^1 |y_2'|^2\,dx \int_0^1 |y_2|^2\,dx\right)^{1/2}}{\int_0^1 |y_2|^2\,dx}. \tag{3.5}$$

Hence, if λ_n is a nonreal eigenvalue, we find

$$\operatorname{Re}\lambda_n = \frac{-\int_0^1 a|y_2|^2\,dx}{\int_0^1 |y_2|^2\,dx}, \quad (\operatorname{Im}\lambda_n)^2 = \frac{\int_0^1 |y_2'|^2\,dx}{\int_0^1 |y_2|^2\,dx} - \left(\frac{-\int_0^1 a|y_2|^2\,dx}{\int_0^1 |y_2|^2\,dx}\right)^2.$$

It follows that $-\beta \leq \operatorname{Re}\lambda_n \leq -\alpha$ and $|\lambda_n|^2 = (\operatorname{Re}\lambda_n)^2 + (\operatorname{Im}\lambda_n)^2 \geq \pi^2$. When λ_n is real we observe that

$$\frac{\left(\left(\int_0^1 a|y_2|^2\,dx\right)^2 - \int_0^1 |y_2'|^2\,dx \int_0^1 |y_2|^2\,dx\right)^{1/2}}{\int_0^1 |y_2|^2\,dx} \leq (\beta^2-\pi^2)_+^{1/2}.$$

(iv) Our A is a bounded perturbation of a skew symmetric (undamped) operator and so this claim follows directly from Theorem 10.1 of chapter 5 of Goberg and Krein [5]. ∎

The upper bound on $\operatorname{Re}\lambda$ in part (iii) is far from sharp. In particular, the upper bound on the largest real eigenvalue may be positive! We rectify this in the next section. Regarding the bound $\operatorname{Re}\lambda \leq -\alpha$ on nonreal eigenvalues we have already noted that so long as a is strictly positive on a subinterval exponential decay is assured.

The lower bound on real eigenvalues expressed in part (iii) corrects the statement of Corollary 8 in [14]. The lowest order term in the first PDE of the proof of Theorem 7 in [14] should have as its coefficient $(\alpha^2 - a\alpha)$ rather than $(\alpha^2 + a\alpha)$.

4. **Low Frequencies**

It is clear from the previous Theorem that $\beta \geq \pi$ is a necessary condition for existence of real eigenvalues. We assume this inequality throughout the section.

We first exploit the observation that the real eigenvalues of A are the fixed points of a parametrized self–adjoint eigenvalue problem. Regarding $\lambda \leq 0$ as a parameter, the problem,

$$\psi'' - \lambda^2\psi = 2\nu a\psi, \quad \psi(0) = \psi(1) = 0, \tag{4.1}$$

admits the simple eigenvalues

$$0 > \nu_1(\lambda, a) > \nu_2(\lambda, a) > \rightarrow -\infty, \tag{4.2}$$

and a corresponding base of eigenfunctions $\{\psi_k\}$. Clearly, λ_0 is a real eigenvalue of A if and only if it is a fixed point of $\lambda \mapsto \nu_k(\lambda, a)$ for some k. Consequently, we examine the dependence of ν_k on a and λ.

¿From the well known characterization

$$\nu_k(\lambda, a) = -\min_{E_k} \max_{\psi \in E_k} \frac{\int_0^1 |\psi'|^2\,dx + \lambda^2 \int_0^1 \psi^2\,dx}{2\int_0^1 a\psi^2\,dx}, \tag{4.3}$$

where the E_k vary over the k–dimensional subspaces of $H_0^1(0,1)$, come the rough estimates

$$\frac{-k^2\pi^2 - \lambda^2}{2\alpha} \leq \nu_k(\lambda, a) \leq \frac{-k^2\pi^2 - \lambda^2}{2\beta}. \tag{4.4}$$

¿From the ordering in (4.2), the real eigenvalues of A must lie between the fixed points of $\lambda \mapsto \nu_1(\lambda, a)$. Hence, (4.4) provides the following improvement of Theorem 3.1 part (iii),

$$\sigma(A) \cap \mathbf{R} \subset [-\beta - \sqrt{\beta^2 - \pi^2}, -\beta + \sqrt{\beta^2 - \pi^2}]. \tag{4.5}$$

As $-\beta + \sqrt{\beta^2 - \pi^2} \le -\alpha$ when $\alpha(2\beta - \alpha) < \pi^2$ we have found a simple proof of Theorem 6 of [14].

Theorem 4.1. *If $\alpha(2\beta - \alpha) < \pi^2$ then $\sigma(A) \subset \{z \in \mathbf{C} : \operatorname{Re} z \le -\alpha\}$.*

¿From the upper bound we see that if $\beta < k\pi$ then A has at most $k-1$ real eigenvalues larger than $-\beta + \sqrt{\beta^2 - (k-1)^2\pi^2}$. This, together with the fact that no nonreal eigenvalue has real part greater than $-\alpha$, gives the following affirmative reply to a conjecture of Rauch [14].

Theorem 4.2. *If $\beta < k\pi$ then A has at most $k-1$ eigenvalues with real parts in excess of $-\alpha$.*

To guarantee the presence of a real eigenvalue greater than $-\alpha$ one finds from the lower bound in (4.4) (with $k = 1$) that it suffices to assume $\alpha > \pi$.

We require $\dot{\nu}_k$, the derivative of ν_k with respect to λ. Formally, this is simply the derivative of the Rayleigh quotient in (4.3) evaluated at the corresponding eigenfunction,

$$\dot{\nu}_k(\lambda, a) = \frac{-\lambda \int_0^1 \psi_k^2 \, dx}{\int_0^1 a\psi_k^2 \, dx}. \tag{4.6}$$

For a precise derivation we note that $\lambda \mapsto (d^2/dx^2 - \lambda^2)$ is a holomorphic family of type (A), in the sense of Kato [9]. As each eigenvalue is simple, (4.6) follows directly from formula (VII.6.29) on page 422 of [9].

Theorem 4.3. *Assume that A has j distinct real eigenvalues. (i) If j is even then each of these eigenvalues is of algebraic multiplicity one, they may be ordered*

$$\lambda_{-1} < \lambda_{-2} < \cdots < \lambda_{-j/2} < \lambda_{j/2} < \cdots < \lambda_2 < \lambda_1,$$

and $x \mapsto y_2(x, \lambda_{\pm k})$ has precisely $k-1$ zeros in $(0,1)$, for $k = 1, 2, \ldots, j/2$. (ii) If j is odd then one has the ordering

$$\lambda_{-1} < \cdots < \lambda_{-(j-1)/2} < \lambda_{-(j+1)/2} = \lambda_{(j+1)/2} < \lambda_{(j-1)/2} < \cdots < \lambda_1,$$

and $x \mapsto y_2(x, \lambda_{\pm k})$ *has precisely* $k-1$ *zeros in* $(0,1)$*, for* $k = 1, 2, \ldots (j+1)/2$*. For* $k \leq (j-1)/2$*, the algebraic multiplicity of* $\lambda_{\pm k}$ *is one while that of* $\lambda_{(j+1)/2}$ *is at least two.*

Proof: Assume that A has but one real eigenvalue, λ_1. In this case, λ_1 is in fact a multiple root of $\nu_1(\lambda, a) = \lambda$. For, if not, the monotonicity of $\lambda \mapsto \nu(\lambda, a)$ and the fact that it behaves like $-\lambda^2$ for large λ would together produce a second distinct root. Hence,

$$1 = \dot{\nu}_1(\lambda_1, a) = \frac{-\lambda_1 \int_0^1 \psi_1^2\, dx}{\int_0^1 a\psi_1^2\, dx}, \quad \text{i.e.,} \quad \lambda_1 = \frac{-\int_0^1 a\psi_1^2\, dx}{\int_0^1 \psi_1^2\, dx}.$$

We now show this to be a necessary and sufficient condition for $\dot{y}_2(1, \lambda_1) = 0$. Differentiate (2.1) with respect to λ

$$\dot{y}_2''(x, \lambda) - \lambda^2 \dot{y}_2(x, \lambda) - 2a(x)\lambda \dot{y}_2(x, \lambda) - 2(\lambda + a(x))y_2(x, \lambda) = 0, \qquad (4.7)$$

multiply by y_2, subtract the result from the product of (2.1) and $\dot{y}_2$, and conclude

$$\dot{y}_2 y_2'' - \dot{y}_2'' y_2 = 2(\lambda + a)y_2^2.$$

Upon integration this yields

$$\dot{y}_2(1, \lambda)y_2'(1, \lambda) = 2\int_0^1 (\lambda + a)y_2^2(x, \lambda)\, dx.$$

Now $y_2'(1, \lambda_1) \neq 0$ by the uniqueness of the initial value problem for (2.1). Hence,

$$\dot{y}_2(1, \lambda_1) = \frac{2}{y_2'(1, \lambda_1)} \int_0^1 (\lambda_1 + a)\psi_1^2\, dx = 0.$$

so the algebraic multiplicity of λ_1 is at least two. As $y_2(x, \lambda_1) = \psi_1(x)$ it follows that $x \mapsto y_2(x, \lambda_1)$ has no interior zeros.

If A has two distinct real eigenvalues then by the monotonicity of $\lambda \mapsto \nu_k(\lambda, a)$ and the ordering of the ν_k they are simple roots of $\lambda \mapsto \nu_1(\lambda, a)$ and hence of algebraic multiplicity one. At each root y_2 is the first eigenfunction of (4.1) and so has no interior zeros in x.

If A has three distinct real eigenvalues we recognize the outer pair to be simple roots of $\nu_1(\lambda, a) - \lambda$ while the center is necessarily a multiple root of $\nu_2(\lambda, a) - \lambda$. The pattern of multiplicities and interior zeros is now established. ∎

When $\alpha \geq \pi$, from (4.4) comes a simple lower bound on the greatest real eigenvalue,

$$-\alpha + \sqrt{\alpha^2 - \pi^2} \leq \lambda_1(a). \tag{4.8}$$

We now relax the dependence on α in (4.4) and so in (4.8). This is accomplished by addressing

$$\check{\nu}_1(\lambda) \equiv \inf\{\nu_1(\lambda, a) : 0 \leq a(x) \leq \beta,\ \int_0^1 a\,dx = a_0\}. \tag{4.9}$$

Krein states without proof, [10, §4.3], that this infimum is attained, regardless of the value of λ, at

$$a(x) = a(1-x) = \begin{cases} \beta & \text{if } 0 < x < a_0/(2\beta) \\ 0 & \text{if } a_0/(2\beta) < x < 1/2. \end{cases}$$

This is incorrect. Unfortunately, we discovered his error only after propagating a number of its consequences throughout an earlier draft and into a preliminary announcement. In particular, parts (v) and (vi) of Théorème 3 in Cox and Zuazua [4] are incorrect. We now remedy Krein's assertion.

Theorem 4.4. *The infimum in* (4.9) *is attained at*

$$\check{a}(x) = \check{a}(1-x) = \begin{cases} \beta & \text{if } 0 < x < x_1 \\ \gamma & \text{if } x_1 < x < 1/2, \end{cases} \qquad \gamma = \frac{\beta}{1 + \left(\frac{\pi}{2\lambda x_1}\right)^2},$$

where $x_1 = \dfrac{\beta\pi^2}{4\lambda^2(\beta - a_0)}(\sqrt{1 + 4(\lambda/\pi)^2(\beta - a_0)a_0/\beta^2} - 1)$. *The infimum is*

$$\check{\nu}_1(\lambda) = \frac{-1}{2a_0}\left(\lambda^2 + \frac{\pi^2}{2x_1}\right).$$

Proof: We compare an admissible a with its even part

$$a_*(x) = \tfrac{1}{2}(a(x) + a(1-x)).$$

If ψ and ψ_* are solutions of (4.1) corresponding to $\nu_1(\lambda, a)$ and $\nu_1(\lambda, a_*)$ then ψ_* is even and, recalling (4.3)

$$\begin{aligned} \nu_1(\lambda, a_*) &= \frac{-\int_0^1 |\psi_*'|^2\,dx - \lambda^2 \int_0^1 \psi_*^2\,dx}{2\int_0^1 a_*\psi_*^2\,dx} \\ &= \frac{-\int_0^1 |\psi_*'|^2\,dx - \lambda^2 \int_0^1 \psi_*^2\,dx}{2\int_0^1 a\psi_*^2\,dx} \leq \nu_1(\lambda, a). \end{aligned}$$

Hence, it suffices to restrict ourselves to even a.

We note that $\check{\psi}$, the first eigenfunction associated with $\check{a}$, is

$$\check{\psi}(x) = \check{\psi}(1-x) = \begin{cases} \sin\sqrt{-\lambda^2 - 2\beta\check{\nu}_1(\lambda)}x & \text{if } 0 < x < x_1 \\ \check{\psi}(x_1) & \text{if } x_1 < x < 1/2. \end{cases}$$

Matching to first order at x_1 we find

$$\sqrt{-\lambda^2 - 2\beta\check{\nu}_1(\lambda)}x_1 = \pi/2, \tag{4.10}$$

while in order for $\check{\psi}$ to be constant between x_1 and $1-x_1$ we must satisfy

$$\lambda^2 + 2\gamma\nu_1(\lambda, \check{a}) = 0. \tag{4.11}$$

Furthermore, $\check{a}$ is not admissible unless

$$\beta x_1 + \gamma(\tfrac{1}{2} - x_1) = a_0/2. \tag{4.12}$$

Our solution, for x_1, γ, and $\check{\nu}_1$, to these three algebraic equations lies in the statement of the Theorem.

Finally, if a is even and admissible then

$$\begin{aligned}\int_0^1 a\check{\psi}^2\,dx - \int_0^1 \check{a}\check{\psi}^2\,dx &= 2\int_0^{1/2}(a-\check{a})\check{\psi}^2\,dx \\ &= 2\int_0^{x_1}(a-\beta)\check{\psi}^2\,dx + \check{\psi}^2(x_1)\int_{x_1}^{1/2}(a-\gamma)\,dx \\ &\ge 2\check{\psi}^2(x_1)\left(\int_0^{1/2} a\,dx - \beta x_1 - \gamma(\tfrac{1}{2} - x_1)\right) = 0.\end{aligned}$$

As a result

$$\begin{aligned}\nu_1(\lambda, \check{a}) &= \frac{-\int_0^1 |\check{\psi}'|^2\,dx - \lambda^2\int_0^1 \check{\psi}^2\,dx}{2\int_0^1 \check{a}\check{\psi}^2\,dx} \\ &\le \frac{-\int_0^1 |\check{\psi}'|^2\,dx - \lambda^2\int_0^1 \check{\psi}^2\,dx}{2\int_0^1 a\check{\psi}^2\,dx} \le \nu_1(\lambda, a),\end{aligned}$$

that is, $\check{a}$ is a minimizer. ∎

As a lower bound on x_1 produces a lower bound on $\check{\nu}_1(\lambda)$ we note that from $\sqrt{1+t} \ge 1 + 2t/(4+t)$ for $t \ge 0$ it follows that

$$x_1 \ge \frac{\beta a_0\pi^2}{2\beta^2\pi^2 + 2\lambda^2(\beta - a_0)a_0}.$$

This translates immediately into

$$\nu_1(\lambda, a) \geq \frac{-1}{2a_0}\left(\lambda^2\left(2 - \frac{a_0}{\beta}\right) + \frac{2\pi^2\beta^2}{a_0}\right).$$

This right–handside has a fixed point (with respect to λ) when $4\beta\pi^2 \leq a_0(a_0^2 + 2\pi^2)$. In particular,

Theorem 4.5. *If $a_0^2 \geq \sqrt{2}\pi\beta$ then A possesses a real eigenvalue $\lambda_1(a)$ and*

$$\frac{-\beta a_0}{2\beta - a_0} \leq \lambda_1(a). \tag{4.13}$$

We now assume the existence of $\lambda_1(a)$ and work to improve the upper bound in (4.5), i.e., $\lambda_1(a) \leq -\beta + \sqrt{\beta^2 - \pi^2}$.

With the knowledge, gained in Theorem 4.3, that $y_2(x, \lambda_1(a))$ is of one sign, we are able to bound λ_1 from above strictly in terms of a_0, the average of a.

Theorem 4.6. $\lambda_1(a) \leq -2/a_0$.

Proof: We compare $\lambda_1(a)$ to $\gamma_1(a)$, the first eigenvalue of

$$\phi'' = 2a\gamma\phi, \quad \phi \in H_0^1(0,1).$$

The first eigenfunctions $v_1(x) \equiv y_2(x, \lambda_1(a))$ and ϕ_1 satisfy

$$\int_0^1 \phi_1 v_1''\, dx = \int_0^1 v_1 \phi_1''\, dx$$

and so

$$\lambda_1^2 \int_0^1 v_1\phi_1\, dx + 2\lambda_1 \int_0^1 a v_1 \phi_1\, dx = 2\gamma_1 \int_0^1 a v_1\phi_1\, dx.$$

As a, v_1, and ϕ_1 are positive, it follows that

$$\lambda_1(a) \leq \gamma_1(a). \tag{4.14}$$

This inequality also follows from the more general considerations of Krein and Langer [11, Thm. 2.5]. It remains to show that $\gamma_1(a) \leq -2/a_0$. This inequality is a consequence of Krein's discovery, [10, Thm. 1], that $a \mapsto \gamma_1(a)$ attains its maximum, over functions of mean a_0 that take values between 0 and β, at

$$\hat{a}(x) = \begin{cases} \beta & \text{if } 1/2 - a_0/2\beta < x < 1/2 + a_0/2\beta \\ 0 & \text{otherwise.} \end{cases}$$

To compute $\gamma_1(\hat{a})$ one simply matches x and $c\cos\sqrt{2\beta\mu}(\frac{1}{2}-x)$ to first order at $x = \frac{1}{2}(1-\frac{a_0}{\beta})$. In particular, $\gamma_1(\hat{a}) = -(2\beta/a_0^2)\xi_1$ where ξ_1 is the least positive root of

$$\sqrt{\xi}\tan\sqrt{\xi} = \frac{a_0}{\beta - a_0}.$$

Krein's lower bound [10, (1.19)], $\xi_1 > a_0/\beta$, completes our proof. ∎

We pause now in order to indicate an extension of the previous bound to the damped wave equation on an open bounded connected subset of $\mathbf{R}^n$. If Ω is such a set, a is bounded and nonnegative on Ω, and $\lambda_1(a)$ is the greatest negative number for which $\Delta v = \lambda^2 v + 2\lambda a v$ has a nontrivial solution in $H_0^1(\Omega)$, then it follows, precisely as in Theorem 4.3, that this solution, v_1, may be chosen nonnegative. Furthermore, as in the previous proof, $\lambda_1(a) \le \gamma_1(a)$, where $\gamma_1(a)$ is the first eigenvalue of $\Delta\phi = \gamma 2a\phi$ in $H_0^1(\Omega)$. In the case of planar Ω we now produce an upper bound on $\gamma_1(a)$ in terms of $a_0 \equiv \|a\|_{L^1(\Omega)}$, $\beta \equiv \|a\|_{L^\infty(\Omega)}$, the area of Ω, $|\Omega|$, and α_1, the first zero of the Bessel function J_0.

Theorem 4.7. *If Ω is an open bounded connected subset of $\mathbf{R}^2$ then*

$$\lambda_1(a) \le \frac{-\pi}{a_0}\frac{\alpha_1^2}{2-\alpha_1^2\log\left(\frac{a_0}{\beta|\Omega|}\right)}.$$

Proof: The maximum of $a \mapsto \gamma_1(a)$ over those a that take values in $[0,\beta]$ and integrate to a_0 is attained at a function $\hat{a}$ that is characterized, see Cox [3], in terms of its associated positive eigenfunction, $\hat{\phi}$, and a scalar ℓ, as

$$\hat{a}(x) = \begin{cases} \beta & \text{if } \hat{\phi}(x) \ge \ell, \\ 0 & \text{otherwise.} \end{cases}$$

We now denote by Ω_* the disk centered at the origin for which $|\Omega_*| = |\Omega|$ and by $\hat{\phi}_*$ and $\hat{a}_*$ the respective Schwarz (replace each super level set by a disk of its area) rearrangements of $\hat{\phi}$ and $\hat{a}$. In this way,

$$\hat{a}_*(x) = \begin{cases} \beta & \text{if } |x| < R_1 = \sqrt{a_0/\beta\pi}, \\ 0 & \text{if } R_1 < |x| < R_2 = \sqrt{|\Omega|/\pi}, \end{cases}$$

and one may argue as in [3, (2.5)] that

$$\gamma_1(\hat{a}) = -\frac{\int_\Omega |\nabla\hat{\phi}|^2}{2\int_\Omega \hat{a}\hat{\phi}^2\,dx} \le -\frac{\int_{\Omega_*} |\nabla\hat{\phi}_*|^2\,dx}{2\int_{\Omega_*} \hat{a}_*\hat{\phi}_*^2\,dx} \le \gamma_1(\hat{a}_*).$$

To compute $\gamma_1(\hat{a}_*)$ one must match $\log(r/R_2)$ and $cJ_0(\sqrt{2\beta\mu}r)$ to first order at $r = R_1$. We note that Krein's calculation of $\gamma_1(\hat{a}_*)$ in [10, §4.2.2] is incorrect. For, in place of $\log(r/R_2)$ he has taken $R_2 - r$.

We find that

$$\gamma_1(\hat{a}_*) = \frac{-\pi}{2a_0}\xi_1 \tag{4.15}$$

where ξ_1 is the least positive root of

$$\begin{aligned} Q(\xi) &\equiv J_0(\sqrt{\xi}) - t\sqrt{\xi}J_0'(\sqrt{\xi}), \qquad t = \log(R_1/R_2) \\ &= 1 + (t - \tfrac{1}{4})\xi + \tfrac{1}{8}(\tfrac{1}{4} - t)\xi^2 + \cdots . \end{aligned} \tag{4.16}$$

As $t < 0$ it is not hard to see that ξ_k, the kth zero of Q, obeys

$$\beta_k^2 < \xi_k < \alpha_k^2, \tag{4.17}$$

where α_k (β_k) is the kth zero of J_0 (J_0') and $\beta_1 = 0$. Furthermore, Q being an entire function of order 1/2, we have

$$Q(\xi) = \prod_{k=1}^{\infty}\left(1 - \frac{\xi}{\xi_k}\right). \tag{4.18}$$

Equating the first derivatives of (4.16) and (4.18), we find

$$\frac{1}{4} - t = \sum_{k=1}^{\infty}\frac{1}{\xi_k}.$$

The second inequality in (4.17) gives

$$\frac{1}{\xi_1} \le \frac{1}{4} - t - \sum_{k=2}^{\infty}\frac{1}{\alpha_k^2}.$$

As α_k^2 is the kth eigenvalue of $-(rz')' = \lambda rz,\ rz'(0) = 0,\ z(1) = 0$ the sum of their reciprocals is the trace of the associated Green's operator. That is

$$\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2} = -\int_0^1 x\log x\,dx = \frac{1}{4}.$$

As a result

$$\frac{1}{\xi_1} \le \frac{1}{\alpha_1^2} - t, \quad \text{i.e.,} \quad \xi_1 \ge \frac{2\alpha_1^2}{2 - \alpha_1^2\log\left(\frac{a_0}{\beta|\Omega|}\right)}.$$

This bound in (4.15) completes the proof. ∎

In our third and final approach to the real eigenvalues of A we address the positivity of the discriminant

$$F(y) \equiv \left(\int_0^1 ay^2\,dx\right)^2 - \int_0^1 |y'|^2\,dx \int_0^1 y^2\,dx$$

appearing in (3.5).

Theorem 4.8. *A has a real eigenvalue iff there exists a nontrivial $\psi \in H_0^1(0,1)$ for which $F(\psi) \geq 0$.*

Proof: If A has a real eigenvalue λ_0 then, recalling (3.5), we find $F(y_2(\cdot,\lambda_0)) \geq 0$.

¿From (4.3) comes

$$-\frac{\int_0^1 |\psi'|^2\,dx + \lambda^2 \int_0^1 \psi^2\,dx}{2\int_0^1 a\psi^2\,dx} \leq \nu_1(\lambda), \quad \forall\,\psi \in H_0^1(0,1). \tag{4.19}$$

Hence, ν_1 will have a fixed point when there exists a ψ for which

$$\lambda \mapsto \int_0^1 |\psi'|^2\,dx + \lambda^2 \int_0^1 \psi^2\,dx + 2\lambda \int_0^1 a\psi^2\,dx$$

possesses a real and negative root, i.e., a ψ for which $F(\psi) \geq 0$. ∎

Lemma 4.9. *If $y \in H_0^1(0,1)$ then*

$$\int_0^1 ay^2\,dx \leq a_0 \left(\int_0^1 |y'|^2\,dx\right)^{1/2} \left(\int_0^1 y^2\,dx\right)^{1/2}.$$

Proof:

$$\begin{aligned}
\int_0^1 ay^2\,dx &= \int_0^1 a(x)\left(\int_0^x y(s)y'(s)\,ds - \int_x^1 y(s)y'(s)\,ds\right)dx \\
&= \int_0^1\int_s^1 a(x)y(s)y'(s)\,dx\,ds - \int_0^1\int_0^s a(x)y(s)y'(s)\,dx\,ds \\
&= \int_0^1 y(s)y'(s)\left(\int_s^1 a(x)\,dx - \int_0^s a(x)\,dx\right)ds \\
&\leq a_0 \int_0^1 |yy'|\,dx \leq a_0 \left(\int_0^1 y^2\,dx\right)^{1/2}\left(\int_0^1 |y'|^2\,dx\right)^{1/2}
\end{aligned}$$ ∎

Corollary 4.10. *If A has a real eigenvalue then $a_0 \geq 1$.*

Proof: There exists a nontrivial $y \in H_0^1(0,1)$ for which

$$\int_0^1 |y'|^2\,dx \int_0^1 y^2\,dx \leq \left(\int_0^1 ay^2\,dx\right)^2 \leq a_0^2 \int_0^1 |y'|^2\,dx \int_0^1 y^2\,dx. \blacksquare$$

We can sharpen both our necessary and sufficient conditions upon recalling Krein's bounds on Λ_1, the least eigenvalue of

$$-\phi'' = \Lambda a \phi, \quad \phi(0) = \phi(1) = 0. \tag{4.20}$$

In particular, see [10, eq. (0.2)],

$$\frac{4}{a_0} \leq \Lambda_1 \leq \frac{\pi^2 \beta}{a_0^2}. \tag{4.21}$$

Theorem 4.11. *If A has a real eigenvalue then $\beta a_0 \geq 4$.*

Proof: We have

$$\int_0^1 ay^2\,dx \leq \beta \int_0^1 y^2\,dx \quad \text{and} \quad \int_0^1 ay^2\,dx \leq \frac{1}{\Lambda_1} \int_0^1 |y'|^2\,dx,$$

and so, for that y for which $F(y) \geq 0$ we find,

$$\begin{aligned}\int_0^1 |y'|^2\,dx \int_0^1 y^2\,dx \leq \left(\int_0^1 ay^2\,dx\right)^2 &\leq \frac{\beta}{\Lambda_1} \int_0^1 |y'|^2\,dx \int_0^1 y^2\,dx \\ &\leq \frac{\beta}{4} a_0 \int_0^1 |y'|^2\,dx \int_0^1 y^2\,dx. \blacksquare\end{aligned}$$

Theorem 4.12. *If $a_0^2 \geq \pi\beta$ then A has a real eigenvalue greater than $-\pi$.*

Proof: Let ϕ_1 denote the first eigenfunction of (4.20), and from

$$\pi^2 \int_0^1 \phi_1^2\,dx \leq \int_0^1 |\phi_1'|^2\,dx \quad \text{and} \quad \int_0^1 |\phi_1'|^2\,dx = \Lambda_1 \int_0^1 a\phi_1^2\,dx$$

deduce that

$$\int_0^1 \phi_1^2\,dx \int_0^1 |\phi_1'|^2\,dx \leq \frac{\Lambda_1^2}{\pi^2} \left(\int_0^1 a\phi_1^2\,dx\right)^2 \leq \frac{\pi^2\beta^2}{a_0^4} \left(\int_0^1 a\phi_1^2\,dx\right)^2.$$

Hence, if $a_0^2 \geq \pi\beta$ then $F(\phi_1) \geq 0$ and so A possesses a real eigenvalue. Recalling (4.19) we see that this eigenvalue is in fact to the right of the largest root of

$$\lambda \mapsto \int_0^1 |\phi_1'|^2\,dx + \lambda^2 \int_0^1 \phi_1^2\,dx + 2\lambda \int_0^1 a\phi_1^2\,dx,$$

i.e., to the right of

$$-\int_0^1 a\phi_1^2\,dx + \left(\left(\int_0^1 a\phi_1^2\,dx\right)^2 - \int_0^1 |\phi_1'|^2\,dx\right)^{1/2}.$$

Whether this value is greater than $-\pi$ is equivalent to whether

$$-\int_0^1 |\phi_1'|^2\,dx \geq \pi^2 - 2\pi \int_0^1 a\phi_1^2\,dx,$$

that is, to whether

$$\left(\frac{2\pi}{\Lambda_1} - 1\right) \frac{\int_0^1 |\phi_1'|^2\,dx}{\int_0^1 \phi_1^2\,dx} \geq \pi^2,$$

which in turn is equivalent to $\Lambda_1 \leq \pi$ which indeed is true when $a_0^2 \geq \pi\beta$. ∎

We close this section with a natural complement to Corollary 4.10.

Theorem 4.13. *If $a_0 < 1$ then the spectrum of A is contained in the sector*

$$re^{i\theta} : 0 \leq r,\ \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} + \arcsin a_0,$$

and its conjugate.

Proof: Sectorial bounds such as these follow from the general remarks of Krein and Langer [11, 2.1]. For $\lambda \in \sigma(A)$ we write $\operatorname{Re}\lambda = |\lambda|\cos\theta$ and $\operatorname{Im}\lambda = |\lambda|\sin\theta$. Arguing as in the proof of Theorem 3.1 we find

$$\frac{\cos^2\theta}{\sin^2\theta} = \frac{(\operatorname{Re}\lambda)^2}{(\operatorname{Im}\lambda)^2} = \frac{\left(\int_0^1 a|y_2|^2\,dx\right)^2}{\int_0^1 |y_2|^2\,dx \int_0^1 |y_2'|^2\,dx - \left(\int_0^1 a|y_2|^2\,dx\right)^2} \leq \frac{a_0^2}{1-a_0^2},$$

where the inequality follows from Lemma 4.7. It follows that $|\cos\theta| \leq a_0$, i.e., that θ is within $\arcsin a_0$ of either $\pi/2$ or $3\pi/2$. ∎

5. **High Frequencies**

There exist a number of means by which one may study the large eigenvalues of A. Henry, [7], in the context of functional differential equations, argues that if $T_0(t)$ is a semigroup whose asymptotic behavior is determined by the spectrum of its infinitesimal generator then the same may be said of any compact perturbation of T_0. Neves, et al., [12], have extended Henry's findings to systems of hyperbolic equations in one space dimension. Our asymptotic form for the spectrum, under the hypothesis that $a \in C^1(0,1)$, follows from Theorem B of [12].

A second, more classical, approach that yields asymptotics for both the eigenvalues and eigenfunctions is the shooting method. Here one studies $\lambda \mapsto y_2(1,\lambda)$, where $y_2(x,\lambda)$ solves (2.1) subject to (2.2), for λ of large magnitude. Such an approach has been systematically studied by Birkoff and Langer [1]. Chen et al., [2], argue, without proof, under the assumption that $a \in C^1(0,1)$, that (2.1) is indeed amenable to the methods of [1] and proceed to claim the asymptotic form for the eigenvalues found below in our Theorem 5.3.

We too shall adopt the shooting method, though in guise perhaps closer in spirit to Henry than to Birkoff and Langer. In particular, we shall use an ansatz of Horn [8] to find an exact solution to an equation that differs from (2.1) only by a potential term. We then develop y_2 as a power series in this fake potential. Via this explicit elementary approach we shall see that it suffices to assume $a \in BV(0,1)$. We were guided in this application of the shooting method by the elegant exposition of Poschel and Trubowitz [13].

The ansatz for (2.1) suggested by Horn is

$$y(x,\lambda) = \phi(x)e^{\lambda\zeta(x)}\sum_{n=0}^{\infty} f_n(x)\lambda^{-n}, \quad f_0(x) \equiv 1.$$

Its application in (2.1), upon equating like powers of λ, produces, as a first term

$$z_1(x,\lambda) = e^{\lambda x + \int_0^x a\,dt}.$$

This of course does not satisfy (2.1) but rather

$$-z'' + \lambda^2 z + 2\lambda a z + (a^2 + a')z = 0. \tag{5.1}$$

Though this makes sense as an equation in $H^{-1}(0,1)$ (write $a'z = (az)' - az'$), by requiring slightly more of a we shall gain sufficient control of z. We shall assume that a is of bounded variation. In this way, a' is a measure and the standard weak form of (5.1) has sense. Via reduction of order (5.1) possesses the second solution

$$z_2(x,\lambda) = z_1(x,\lambda)\int_0^x z_1^{-2}(t,\lambda)\,dt = e^{\lambda x + \int_0^x a\,dt}\int_0^x e^{-2\lambda t - 2\int_0^t a\,ds}\,dt. \quad (5.2)$$

Note that z_2 satisfies (2.4) and, upon integrating by parts, that

$$z_2(x,\lambda) = \frac{1}{\lambda}\sinh(\lambda x + \textstyle\int_0^x a\,dt) + \frac{1}{\lambda}e^{\lambda x + \int_0^x a\,dt}\displaystyle\int_0^x a e^{-2\lambda t - 2\int_0^t a\,ds}\,dt. \quad (5.3)$$

In addition, we see easily in (5.2) that $\lambda \mapsto z_2(1,\lambda)$ has *no* real zeros and, of course, $y_2 = z_2$ when $a' = -a^2$, i.e., when $a(x) = (x+c)^{-1}$ for some positive c. In this case one finds the spectrum of A to be the zeros of

$$\lambda \mapsto \int_0^1 e^{-2\lambda x}(x+c)^{-2}\,dx.$$

We now show through careful estimation that, in general, the roots of $y_2(1,\cdot)$ are asymptotically close to those of $z_2(1,\cdot)$.

¿From (5.3) comes our first estimate of z_2.

$$|z_2(x,\lambda)| \le \frac{e^{3\beta}(1+\beta)}{|\lambda|}, \quad (5.4)$$

when $0 < x < 1$ and $-\beta \le \operatorname{Re}\lambda \le 0$. Integrating by parts in (5.3) produces

$$z_2(x,\lambda) = \frac{1}{\lambda}\sinh(\lambda x + \textstyle\int_0^x a\,dt) - \displaystyle\frac{e^{\lambda x + \int_0^x a\,dt}}{2\lambda^2}\Bigg(a(0) - a(x)e^{-2\lambda x - 2\int_0^x a\,dt} + \int_0^x (a' - 2a^2)e^{-2\lambda t - 2\int_0^t a\,ds}\,dt\Bigg).$$

Hence, where $0 < x < 1$ and $-\beta \le \operatorname{Re}\lambda \le 0$, we see that

$$|z_2(x,\lambda) - \frac{1}{\lambda}\sinh(\lambda x + \textstyle\int_0^x a\,dt)| \le \displaystyle\frac{e^{5\beta}(T_a + \beta^2 + \beta)}{|\lambda|^2}, \quad (5.5)$$

where T_a denotes the total variation of a. We demonstrate that y_2 may replace z_2 in the above. Compare [13, Theorem 1.1].

Theorem 5.1. *If $a \in BV(0,1)$ then there exist constants C_0 and C_1 such that*

$$\left|y_2(x,\lambda) - \frac{\sinh(\lambda x + \int_0^x a\,dt)}{\lambda}\right| \le \frac{C_0(\beta, T_a)}{|\lambda|^2} \quad (5.6)$$

and

$$|y_2'(x,\lambda) - \cosh(\lambda x + \textstyle\int_0^x a\,dt)| \le \frac{C_1(\beta, T_a)}{|\lambda|} \tag{5.7}$$

uniformly for $0 < x < 1$ *and* $-2\beta \le \operatorname{Re}\lambda \le 0$.

Proof: Note that y_2 is the solution of

$$-z'' + \lambda^2 z + 2\lambda a z + (a^2 + a')z = (a^2 + a')z, \quad z(0) = 0,\ z'(0) = 1.$$

Hence, with $K(x,t,\lambda) = z_1(x,\lambda)z_2(t,\lambda) - z_2(x,\lambda)z_1(t,\lambda)$, y_2 solves

$$y_2(x,\lambda) = z_2(x,\lambda) + \int_0^x K(x,t,\lambda)(a^2(t) + a'(t))y_2(t,\lambda)\,dt \tag{5.8}$$

We solve this integral equation in series form

$$y_2(x,\lambda) = z_2(x,\lambda) + \sum_{n=1}^{\infty} S_n(x,\lambda),$$

where $S_0 = z_2$ and

$$\begin{aligned} S_n(x,\lambda) &= \int_0^x K(x,t,\lambda)(a^2(t) + a'(t))S_{n-1}(t,\lambda)\,dt \\ &= \int_{0 \le t_1 \le \cdots \le t_{n+1} = x} z_2(t_1,\lambda) \prod_{i=1}^{n} [K(t_{i+1}, t_i, \lambda)(a^2(t_i) + a'(t_i))]\,dt_1 \cdots dt_n. \end{aligned}$$

Having estimated z_2 we turn to K. In particular,

$$\begin{aligned} K(x,t,\lambda) &= z_1(x,\lambda)z_1(t,\lambda)\int_t^x z_1^{-2}(s,\lambda)\,ds \\ &= \frac{-1}{\lambda}\sinh(\lambda(x-t) + \textstyle\int_t^x a\,ds) + \\ &\quad \frac{1}{\lambda} e^{\lambda(x+t) + \int_0^x a\,dt + \int_0^t a\,ds} \int_t^x a e^{-2\lambda s - 2\int_0^s a\,dr}\,ds. \end{aligned}$$

As a result

$$|K(x,t,\lambda)| \le \frac{e^{3\beta(x-t)}(1+\beta)}{|\lambda|}.$$

With this we find

$$\begin{aligned} |S_n(x,\lambda)| &\le \frac{e^{6\beta}(1+\beta)^{n+1}}{|\lambda|^{n+1}} \int_{0 \le t_1 \le \cdots \le t_{n+1} = x} \prod_{i=1}^{n} |a^2(t_i) + a'(t_i)|\,dt_1 \cdots dt_n. \\ &= \frac{e^{6\beta}(1+\beta)^{n+1}}{|\lambda|^{n+1}} \frac{1}{n!} \left(\int_0^x |a^2(t) + a'(t)|\,dt \right)^n \\ &\le \frac{e^{6\beta}(1+\beta)^{n+1}}{|\lambda|^2} \frac{(\beta^2 + T_a)^n}{n!}, \end{aligned}$$

when $|\lambda| \geq 1$. Hence, the S_n are summable and

$$\begin{aligned}
&\left| y_2(x,\lambda) - \frac{\sinh(\lambda x + \int_0^x a\,dt)}{\lambda} \right| \\
&\qquad \leq \left| z_2(x,\lambda) - \frac{\sinh(\lambda x + \int_0^x a\,dt)}{\lambda} \right| + \left| \sum_{n=1}^{\infty} S_n(x,\lambda) \right| \\
&\qquad \leq \frac{e^{5\beta}(T_a + \beta^2 + \beta)}{|\lambda|^2} + \frac{(1+\beta)e^{6\beta + (1+\beta)(T_a+\beta^2)}}{|\lambda|^2}.
\end{aligned}$$

This establishes (5.6).

Regarding the estimate of y_2' we simply differentiate (5.8),

$$y_2'(x,\lambda) = z_2'(x,\lambda) + \int_0^x K_x(x,t,\lambda)(a^2(t) + a'(t))y_2(t,\lambda)\,dt.$$

and proceed to bound each of these terms. First, from

$$\begin{aligned}
z_2'(x,\lambda) &= (\lambda + a)z_2(x,\lambda) + e^{-\lambda x - \int_0^x a\,dt} \\
&= \cosh(\lambda x + \textstyle\int_0^x a\,dt) + \frac{a(x)}{\lambda}\sinh(\lambda x + \int_0^x a\,dt) - \\
&\quad \frac{e^{\lambda x + \int_0^x a\,dt}}{2\lambda^2}\Bigg(a(0) - a(x)e^{-2\lambda x - 2\int_0^x a\,dt} + \\
&\quad \int_0^x (a' - 2a^2)e^{-2\lambda t - 2\int_0^t a\,ds}\,dt \Bigg),
\end{aligned}$$

comes the estimate

$$|z_2'(x,\lambda) - \cosh(\lambda x + \textstyle\int_0^x a\,dt)| \leq \frac{e^{3\beta}}{|\lambda|}(\beta^2 + \beta + T_a).$$

Next,

$$\begin{aligned}
|K_x(x,t,\lambda)| &= |z_1'(x,\lambda)z_2(t,\lambda) - z_2'(x,\lambda)z_1(t,\lambda)| \\
&= |\lambda + a(x)||z_1(x,\lambda)||z_2(t,\lambda)| + |\lambda + a(x)||z_2(x,\lambda)||z_1(t,\lambda)| + \\
&\quad |z_1^{-1}(x,\lambda)||z_1(t,\lambda)| \\
&\leq e^{3\beta}(3 + 2\beta).
\end{aligned}$$

And finally,

$$|y_2(x,\lambda)| \leq \frac{e^{3\beta}}{|\lambda|}(1+\beta)(1 + e^{3\beta + (1+\beta)(\beta^2 + T_a)}).$$

Together, these three estimates produce

$$|y_2'(x,\lambda) - \cosh(\lambda x + \int_0^x a\,dt)| \leq \frac{e^{6\beta}}{|\lambda|}(\beta^2+\beta+T_a)(4+5\beta+2\beta^2)(1+e^{3\beta+(1+\beta)(\beta^2+T_a)}),$$

and the proof is complete. ∎

Via Rouchés Theorem this result will force the (large) zeros of $\lambda \mapsto y_2(1,\lambda)$ to lie in a neighborhood of the roots of $\lambda \mapsto \lambda^{-1}\sinh(\lambda + a_0)$, these being

$$-a_0 \pm in\pi, \quad n = \pm 1, \pm 2, \ldots.$$

To make this precise we choose N, the least integer greater than $4C_0/\pi$, and, with respect to

$$\Gamma_N \equiv \{z : |z+a_0| = N\pi + \pi/2\}, \quad \text{and}$$
$$\Gamma_{\pm n} \equiv \{z : |z + a_0 \mp in\pi| = 2C_0/(n\pi)\}, \quad n > N,$$

prove the following preliminary estimate

Lemma 5.2. *If $z \in \Gamma_n$ and $n \geq N$ then $|\sinh(z+a_0)| > C_0/|z|$.*

Proof: If $z \in \Gamma_N$ then $z = -a_0 + (N\pi + \pi/2)e^{i\theta}$ where $\theta \in [0, 2\pi)$. Hence,

$$|\sinh(z+a_0)|^2 = \sinh^2((N\pi+\pi/2)\cos\theta) + \sin^2((N\pi+\pi/2)\sin\theta).$$

As this function achieves its minimum at $\theta = \pi/2$, we see that

$$|\sinh(z+a_0)| \geq 1, \quad z \in \Gamma_N.$$

As $C_0/|z| < 1/4$ for $z \in \Gamma_N$ our claim follows for $n = N$.

If $z \in \Gamma_n$ then $z = -a_0 + in\pi + \rho_n e^{i\theta}$ where $\rho_n = 2C_0/(n\pi)$ and $\theta \in [0, 2\pi)$. Hence,

$$|\sinh(z+a_0)|^2 = \sinh^2(\rho_n\cos\theta) + \sin^2(\rho_n\sin\theta).$$

This too achieves its minimum at $\theta = \pi/2$. Hence, via the mean value theorem,

$$|\sinh(z+a_0)| \geq \sin(\rho_n) = \rho_n - \tfrac{1}{2}\rho_n^2\sin\xi$$

for some $\xi \in (0, \rho_n)$. As

$$\frac{C_0}{|z|} = \frac{C_0}{|-a_0 + in\pi + \rho_n e^{i\theta}|} \leq \frac{C_0}{n\pi - \rho_n},$$

it remains only to check that

$$\frac{C_0}{n\pi - \rho_n} \le \rho_n - \tfrac{1}{2}\rho_n^2 \sin\xi,$$

that is, that

$$\sin\xi \le \frac{n\pi}{2C_0}\left(2 - \frac{1}{1 - \frac{2C_0}{n^2\pi^2}}\right).$$

As $C_0 \ge 1$ the right hand side in fact is larger than one when $n > 4C_0/\pi$. ∎

Theorem 5.3. *If $a \in BV(0,1)$ then A has exactly $2N$ eigenvalues, including multiplicity, in Γ_N and one simple eigenvalue in Γ_n for each $n > N$. This exhausts the spectrum of A.*

Proof: For $\lambda \in \Gamma_n$ we see that

$$\left| y_2(1,\lambda) - \frac{\sinh(\lambda + a_0)}{\lambda} \right| \le \frac{C_0}{|\lambda|^2} < \left| \frac{\sinh(\lambda + a_0)}{\lambda} \right|.$$

Hence, by Rouchés Theorem, $y_2(1,\lambda)$ possesses the same number of zeros in Γ_n, and in the complement of their union, as $\lambda^{-1}\sinh(\lambda + a_0)$. ∎

This affords an immediate comparison with the constant case.

Corollary 5.4. *If $a \in BV(0,1)$ then $\omega(a) \ge -a_0$. In particular, over all such a with $a_0 \le \pi$, the constant $a \equiv \pi$ achieves the greatest rate of decay.*

Proof: From the Theorem we find the spectral abscissa, $\mu(a)$, to be no less than $-a_0$. As $\omega(a) \ge \mu(a)$ the result follows. ∎

The Theorem also provides us with a means to order the large eigenvalues of A. We write

$$\sigma(A) = \{\lambda_n\}_{n=\pm 1}^{\pm\infty}$$

where

$$|\lambda_n + a_0| \le N\pi + \pi/2, \quad |n| \le N, \quad \text{and} \quad |\lambda_n + a_0 - in\pi| \le \frac{2C_0}{|n|\pi}, \quad |n| > N. \tag{5.9}$$

These eigenvalue estimates may now be used to refine the eigenfunction estimates. In particular, (5.6) and (5.9) yield

$$\begin{aligned} y_2(x,\lambda_n) &= \frac{\sinh(\int_0^x a\,dt - a_0 x + in\pi x + O(1/n))}{-a_0 + in\pi + O(1/n)} + O(1/n^2) \\ &= \frac{\sinh(\int_0^x a\,dt - a_0 x + in\pi x)}{-a_0 + in\pi} + O(1/n^2). \end{aligned}$$

A similar estimate is true of y_2'. We collect these for future use in.

Theorem 5.5. *If $a \in BV(0,1)$ then*

$$\begin{aligned} y_2(x,\lambda_n) &= \frac{\sinh(\xi(x)+in\pi x)}{-a_0+in\pi} + O(1/n^2), \quad \textit{and} \\ y_2'(x,\lambda_n) &= \cosh(\xi(x)+in\pi x) + O(1/|n|), \quad \textit{where} \\ \xi(x) &= \int_0^x a\,dt - x\int_0^1 a\,dx \end{aligned}$$

measures the deviation of a from constant.

6. **The Root Vectors**

We now address the extent to which the root vectors of A constitute a basis for X. We must first fix some notation. Denoting the algebraic multiplicity of λ_n by m_n, to λ_n is associated the Jordan Chain of root vectors, $\{V_{n,j}\}_{j=0}^{m_n-1}$,

$$\begin{aligned} V_{n,0}(x) &= y_2(x,\lambda_n)[1,\lambda_n], \\ AV_{n,j} &= \lambda_n V_{n,j} + V_{n,j-1}, \quad \langle V_{n,j}, V_{n,0}\rangle = 0, \quad j=1,\ldots,m_n-1. \end{aligned} \tag{6.1}$$

$V_{n,0}$ is an eigenvector and the chain is a basis for the root subspace

$$\mathcal{L}_n \equiv \{V : (A-\lambda_n)^{m_n} V = 0\}.$$

Our work in the last section permits us to conclude that $m_n = 1$ when $|n| > N$. Now it is not difficult to show that, unless a is constant, the $V_{n,0}$ are *not* quadratically close to the Φ_n (the base with $a \equiv 0$). Hence, a less constructive method than that used in §2 must be invoked. In particular, we shall exploit the following characterization.

Theorem 6.1. *(Bari, see [3, Theorem 2.1, Chapter VI]). $\{\phi_n\}$ is a Riesz basis of H if and only if $\{\phi_n\}$ is complete in H and there corresponds to it a complete biorthogonal sequence $\{\psi_n\}$, and for any $f \in H$ one has*

$$\sum_n |\langle \phi_n, f\rangle|^2 < \infty, \quad \sum_n |\langle \psi_n, f\rangle|^2 < \infty.$$

To construct a sequence biorthogonal to the $\{V_{n,j}\}$ we naturally look to the rootvectors of the adjoint of A, see (2.7). It follows that $\sigma(A) = \sigma(A^*)$, including multiplicities, and to $\bar{\lambda}_n$ is associated the Jordan Chain of root vectors, $\{W_{n,j}\}_{j=0}^{m_n-1}$,

$$\begin{aligned} &W_{n,0}(x) = y_2(x, \bar{\lambda}_n)[1, -\bar{\lambda}_n], \\ &A^* W_{n,j} = \bar{\lambda}_n W_{n,j} + W_{n,j-1}, \quad \langle W_{n,j}, V_{n,m_n-1}\rangle = 0, \quad j = 1, \dots, m_n - 1. \end{aligned} \tag{6.2}$$

Note that $W_{n,0}$ is an eigenvector for A^* and that the subsequent $W_{n,j}$ are uniquely determined so long as $\langle W_{n,0}, V_{n,m_n-1}\rangle \neq 0$. In addition, the chain $\{W_{n,j}\}_{j=0}^{m_n-1}$ is a basis for the root subspace

$$\mathcal{L}_n^* \equiv \{W : (A^* - \bar{\lambda}_n)^{m_n} W = 0\}.$$

Lemma 6.2. *There exists a $c > 0$ such that*

$$\langle V_{n,p}, W_{j,k}\rangle = \langle V_{n,p}, W_{n,m_n-1-p}\rangle \delta_{n,j}\delta_{m_n-1-p,k} \geq c\, \delta_{n,j}\delta_{m_n-1-p,k}.$$

Proof: We first check that $\mathcal{L}_j \perp \mathcal{L}_k^*$ when $j \neq k$. Taken together

$$\begin{aligned} &\langle AV_{j,0}, W_{k,0}\rangle = \lambda_j \langle V_{j,0}, W_{k,0}\rangle, \quad \text{and} \\ &\langle AV_{j,0}, W_{k,0}\rangle = \langle V_{j,0}, A^* W_{k,0}\rangle = \lambda_k \langle V_{j,0}, W_{k,0}\rangle, \end{aligned}$$

predict that $(\lambda_j - \lambda_k)\langle V_{j,0}, W_{k,0}\rangle = 0$ and so $\langle V_{j,0}, W_{k,0}\rangle = 0$. Next,

$$\begin{aligned} &\langle AV_{j,0}, W_{k,1}\rangle = \lambda_j \langle V_{j,0}, W_{k,1}\rangle, \quad \text{and} \\ &\langle AV_{j,0}, W_{k,1}\rangle = \langle V_{j,0}, A^* W_{k,1}\rangle = \lambda_k \langle V_{j,0}, W_{k,1}\rangle + \langle V_{j,0}, W_{k,0}\rangle, \end{aligned}$$

predict that $(\lambda_j - \lambda_k)\langle V_{j,0}, W_{k,1}\rangle = 0$. Proceeding in this way one finds the two chains to be orthogonal.

We now address the orthogonality between $\mathcal{L}_n$ and $\mathcal{L}_n^*$. Regarding (6.1) the Fredholm Alternative requires that $\langle V_{n,j}, W_{n,0}\rangle = 0,\ j = 0, 1, \dots, m_n - 2$. By completeness it then follows that $\langle V_{n,m_n-1}, W_{n,0}\rangle \neq 0$. Likewise, $\langle V_{n,0}, W_{n,j}\rangle = 0$ for $j = 0, 1, \dots, m_n - 2$ and $\langle V_{n,0}, W_{n,m_n-1}\rangle \neq 0$. On comparing $\langle AV_{n,1}, W_{n,m_n-k}\rangle$ and $\langle V_{n,1}, A^* W_{n,m_n-k}\rangle$ we find

$$\langle V_{n,1}, W_{n,m_n-k-1}\rangle = \langle V_{n,0}, W_{n,m_n-k}\rangle$$

and so $V_{n,1}$ is orthogonal to each $W_{n,j}$ save when $j = m_n - 2$. Continuing in this way we find $\langle V_{n,2}, W_{n,m_n-k-1}\rangle = \langle V_{n,1}, W_{n,m_n-k}\rangle$ and so $V_{n,2}$ is orthogonal to each $W_{n,j}$ save when $j = m_n - 3$. The pattern is now established.

It remains to show that these two systems may be binormalized. This is tedious though straightforward (only a finite number of our chains have length greater than one) for the low frequencies, while for the high frequencies,

$$\begin{aligned}\langle V_{n,0}, W_{n,0}\rangle &= \langle y_2(x,\lambda_n)[1,\lambda_n], y_2(x,\bar{\lambda}_n)[1,-\bar{\lambda}_n]\rangle \\ &= \int_0^1 (y_2'(x,\lambda_n))^2 - \lambda_n^2 y_2^2(x,\lambda_n)\,dx \\ &= \int_0^1 \cosh^2(\lambda_n x + \textstyle\int_0^x a\,dt) - \sinh^2(\lambda_n x + \int_0^x a\,dt)\,dx + O\left(\dfrac{1}{|\lambda_n|}\right) \\ &= 1 + O(1/|n|).\end{aligned}$$

This establishes the c of the claim. ∎

We introduce the normalized eigenvectors

$$\begin{aligned}\tilde{V}_{n,0}(x) &= \langle V_{n,0}, W_{n,0}\rangle^{-1/2} V_{n,0}(x) = V_{n,0}(x) + O(1/|n|), \quad \text{and} \\ \tilde{W}_{n,0}(x) &= \langle V_{n,0}, W_{n,0}\rangle^{-1/2} W_{n,0}(x) = W_{n,0}(x) + O(1/|n|)\end{aligned}$$

for $|n| > N$. It remains only to establish, for each $[f,g] \in X$, the convergence of

$$\begin{aligned}&\sum_{n>N} |\langle \tilde{V}_{n,0}, [f,g]\rangle|^2 \\ &= \sum_{n>N} |\langle V_{n,0}, W_{n,0}\rangle|^{-1} \left| \int_0^1 y_2'(x,\lambda_n)\bar{f}'(x) + \lambda_n y_2(x,\lambda_n)\bar{g}(x)\,dx \right|^2 \\ &= \sum_{n>N} (1 + O(\tfrac{1}{n})) \left| \int_0^1 \cosh(\lambda_n x + \textstyle\int_0^x a\,dt)\bar{f}'(x) \right. \\ &\qquad\qquad \left. + \sinh(\lambda_n x + \textstyle\int_0^x a\,dt)\bar{g}(x)\,dx \right|^2 \\ &= \sum_{n>N} (1 + O(\tfrac{1}{n})) \left| \int_0^1 (\cosh\xi(x)\bar{f}'(x) + \sinh\xi(x)\bar{g}(x))\cos n\pi x\,dx \right|^2 + \\ &\qquad (1 + O(\tfrac{1}{n})) \left| \int_0^1 (\sinh\xi(x)\bar{f}'(x) + \cosh\xi(x)\bar{g}(x))\sin n\pi x\,dx \right|^2.\end{aligned}$$

We have used Theorem 5.5 in the last step. As ξ is bounded, the coefficients of $\cos n\pi x$ and $\sin n\pi x$ belong to $L^2(0,1)$, and therefore these series converge. The sum over negative n is handled identically. Having fulfilled the conditions of Theorem 6.1, we find

Theorem 6.3. $\{\tilde{V}_{n,j} : n = \pm 1, \dots, \pm\infty; j = 0, \dots, m_n - 1\}$ *is a Riesz basis for* X.

Now there exists a linear isomorphism $\mathcal{I}$ of X and an orthonormal base $\{e_{n,j}\}$ for X for which $\tilde{V}_{n,j} = \mathcal{I}e_{n,j}$. We proceed exactly as in the proof of Theorem 2.1. We expand the initial data in

$$[u_0, v_0] = \sum_{n=\pm 1}^{\pm\infty} \sum_{j=0}^{m_n-1} \gamma_{n,j} \tilde{V}_{n,j},$$

and note that

$$[u, u_t] = \sum_{n=\pm 1}^{\pm\infty} \exp(\lambda_n t) \sum_{j=0}^{m_n-1} \gamma_{n,j} \sum_{k=0}^{j} \tfrac{t^{(j-k)}}{(j-k)!} \tilde{V}_{n,k}$$

satisfies our initial value problem, (1.1), (1.2). On recalling from Theorem 5.3. that at most $2N$ eigenvalues may be of algebraic multiplicity greater than one and that $2N$ is the maximum such multiplicity we may conclude the existence of a finite C for which

$$E(t) \leq CE(0)(1 + t^{2N}) \exp 2\mu t.$$

We have established our main result.

Theorem 6.4. *If* $a \in BV(0, 1)$ *then* $\mu(a) = \omega(a)$.

With this identification we may now establish the existence of an optimal choice of a.

Theorem 6.5. $a \mapsto \omega(a)$ *attains its minimum over those nonnegative* a *satisfying* $T(a) \leq M$.

Proof: If $\omega(a_n) \to -\infty$ then the asymptotic results require that $\|a_n\|_1 \to \infty$. This together with $T(a_n) \leq M$ requires that $\alpha_n \equiv \min\{a_n(x) : x\} \to \infty$. In light of (4.8), i.e., $\lambda_1(a_n) \geq -\alpha_n + \sqrt{\alpha_n^2 - \pi^2}$, the latter implies that $\lambda_1(a_n) \to 0$, contrary to $\omega(a_n) \to -\infty$.

As a result, we may denote ω's *finite* infimum by $\check{\omega}$ and choose a minimizing sequence $\{a_n\}$ for which $T(a_n) \leq M$ and $\|a_n\|_1 \leq C$ for some finite C independent of n. Helly's Theorem, see Graves [6, Thm. XII.33], now guarantees the existence of a function $\breve{a}$ for which $T(\breve{a}) \le M$ and $a_n \to \breve{a}$ in $L^2(0,1)$. We examine the two possibilities, whether or not equality holds in $\omega(\breve{a}) \ge -\|\breve{a}\|_1$.

If $\omega(\breve{a}) = -\|\breve{a}\|_1$ we simply note that $\omega(a_n) \ge -\|a_n\|_1$, $\omega(a_n) \to \breve{\omega}$, and $\|a_n\|_1 \to \|\breve{a}\|_1$. Hence, $\breve{\omega} \ge -\|\breve{a}\|_1 = \omega(\breve{a})$, i.e., $\breve{a}$ is a minimizer.

If $\omega(\breve{a}) > -\|\breve{a}\|_1$ we define the half–plane

$$H \equiv \{z : \tfrac{1}{2}(\omega(\breve{a}) - \|\breve{a}\|_1) \le \operatorname{Re} z\},$$

and note the existence of finite N_1 and N_2 such that for $|n| > N_1$ one finds exactly N_2 points in $\sigma(a_n) \cap H$. That $\omega(a_n) \to \omega(\breve{a})$ now follows from the continuity of a finite system of eigenvalues under generalized convergence of $A(a_n)$ to $A(\breve{a})$, see Kato [9, §IV.3.5]. To show that $A(a_n) \to A(\breve{a})$ in the generalized sense it suffices, see [9, Thm. 2.23.b], to show that $A(a_n)^{-1}$ converges to $A(\breve{a})^{-1}$ in the uniform operator topology of $L(X)$, the space of bounded linear operators on X. With respect to (3.1),

$$\begin{aligned}
\|A(a_n)^{-1} - A(\breve{a})^{-1}\|_{L(X)} &= \sup_{\|[f,g]\|_X=1} \|[G(0)(a_n - \breve{a})f, 0]\|_X \\
&= \sup_{\|f'\|_2=1} \left\| \int_0^1 g_x(x,\xi,0)(a_n(\xi) - \breve{a}(\xi))f(\xi)\, d\xi \right\|_2 \\
&\le \sup_{\|f'\|_2=1} \|g_x\|_\infty \|f\|_2 \|a_n - \breve{a}\|_2 \\
&\le \pi \|a_n - \breve{a}\|_2.
\end{aligned}$$

The result now follows from the $L^2(0,1)$ convergence of a_n to $\breve{a}$. ∎

7. Comments

In Theorem 6.4 we have expressed the decay rate in terms of the spectral abscissa. It is of practical importance so long as one has a full characterization of the latter. We have characterized the real and large eigenvalues though remain fairly ignorant of those nonreal eigenvalues in the disk bounded by Γ_N. Can their algebraic multiplicities indeed exceed one? May the real part of one of them exceed the real part of each of the real eigenvalues? The parametrized eigenvalue problem (4.1) continues to make sense for complex λ. In this case however one is merely trading one nonselfadjoint problem for another. Though the latter indeed corresponds to a spectral operator, very little, of a quantitative nature, appears known regarding the nonasymptotic region of its spectra. Our success in §4 was almost entirely dependent on the available variational structure.

Regarding issues of optimal design, we have yet to determine whether $a \mapsto \omega(a)$ is even bounded below over those a of finite total variation. Regarding attempts to characterize minimizers recall that we remarked at the close of §2 that ω is not Lipschitz near $a = \pi$.

The arguments of §§3 and 4, as noted, extend to a variety of problems in several variables. The higher dimensional version of the functional F of Theorem 4.8 provides an interesting test for the presence of real eigenvalues. The shooting method, a one–dimensional tool, constrains the arguments of §5 to such generalizations as

$$\rho u_{tt} - (\sigma u_x)_x + 2au_t - qu = 0,$$

and their fourth order counterparts.

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