Chapter 3
Improved multipolar Hardy inequalities

Cristian Cazacu and Enrique Zuazua

Abstract In this paper we prove optimal Hardy-type inequalities for Schrödinger operators with positive multi-singular inverse square potentials of the form

$$A_\lambda := -\Delta - \lambda \sum_{1\leq i<j\leq n} \frac{|x_i - x_j|^2}{|x-x_i|^2|x-x_j|^2}, \quad \lambda > 0.$$

More precisely, we show that A_λ is non-negative in the sense of L^2 quadratic forms in $\mathbb{R}^N$, if and only if $\lambda \leq (N-2)^2/n^2$, independently of the number n and location of the singularities $x_i \in \mathbb{R}^N$, where $N \geq 3$ denotes the space dimension. This aims to complement some of the results in Bosi et al. [6] obtained by the "expansion of the square" method. Due to the interaction of poles, our optimal result provides a singular quadratic potential behaving like $(n-1)(N-2)^2/(n^2|x-x_i|^2)$ at each pole x_i. Besides, the authors in [6] showed optimal Hardy inequalities for Schrödinger operators with a finite number of singular poles of the type $B_\lambda := -\Delta - \sum_{i=1}^{n} \lambda/|x-x_i|^2$, up to lower order L^2-reminder terms. By means of the optimal results obtained for A_λ, we also build some examples of bounded domains Ω in which these lower order terms can be removed in $H_0^1(\Omega)$. In this way we obtain new lower bounds for the optimal constant in the standard multi-singular Hardy inequality for the operator B_λ in bounded domains. The best lower bounds are obtained when the singularities x_i are located on the boundary of the domain.

Cristian Cazacu
1. BCAM–Basque Center for Applied Mathematics, Mazarredo, 14, E-48009 Bilbao, Basque Country, Spain
2. Departamento de Matemáticas, Universidad Autónoma de Madrid, E-28049 Madrid, Spain
e-mail: cazacu@bcamath.org

Enrique Zuazua
1. BCAM–Basque Center for Applied Mathematics, Mazarredo, 14, E-48009 Bilbao, Basque Country, Spain
2. Ikerbasque, Basque Foundation for Science, Alameda Urquijo 36-5, Plaza Bizkaia, E-48011, Bilbao, Basque Country, Spain
e-mail: zuazua@bcamath.org

Key words: Hardy inequalities, multipolar potentials, Schrödinger operators.

2010 Mathematics Subject Classification: 35J10, 26D10, 46E35, 35Q40, 35J75.

3.1 Introduction

This paper is concerned with a class of Schrödinger operators of the form $-\Delta + V(x)$ with multipolar Hardy-type singular potentials like $V \sim \sum_i \alpha_i/|x-x_i|^2$, $\alpha_i \in \mathbb{R}$, $x_i \in \mathbb{R}^N$, $N \geq 3$.

The study of such singular potentials is motivated by applications to various fields as molecular physics [26], quantum cosmological models such as the Wheeler-de-Witt equation (see e.g. [5]) and combustion models [21].

The singularity of inverse square potentials cannot be considered as a lower perturbation of the Laplacian since it has homogeneity -2, being critical from both a mathematical and a physical viewpoint.

Potentials of type $1/|x|^2$ arise, for instance, in Frank et al. [19] where a classification of singular spherical potentials is given in terms of the limit $\lim_{r\to 0} r^2 V(r)$. When the limit is finite and non-trivial, V is said to be a *transition potential*. This potential also arises in point-dipole interactions in molecular physics (see Lévy-Leblond [26]), where the interaction among the poles depends on their relative partitions and the intensity of the singularity in each of them.

Multipolar potentials of type $V = \sum_{i=1}^{n} \alpha_i/|x-x_i|^2$ are associated with the interaction of a finite number of electric dipoles. They describe molecular systems consisting of n nuclei of unit charge located at a finite number of points $x_1, \ldots, x_n$ and of n electrons. This type of systems are described by the Hartree-Fock model, where Coulomb multi-singular potentials arise in correspondence to the interactions between the electrons and the fixed nuclei, see Catto et al. [9].

Throughout this paper we study the qualitative properties of Schrödinger operators with inverse square potentials V, improving some results already known in the literature. The positivity and coercivity (in the L^2 norm) of such operators are strongly related to Hardy-type inequalities. The first well-known result relies on a 1-d inequality due to G. H. Hardy [22] which claims that

$$\forall u \in H_0^1(0,\infty), \quad \int_0^\infty u_x^2 dx > \frac{1}{4}\int_0^\infty \frac{u^2}{x^2} dx, \tag{3.1}$$

where the constant $1/4$ is optimal and not attained. Later on, this inequality was generalized to the multi-d case by Hardy-Littlewood-Polya [23] showing that for any Ω an open subset of $\mathbb{R}^N$, containing the origin, it holds that

$$\forall u \in H_0^1(\Omega), \quad \int_\Omega |\nabla u|^2 dx \geq \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2} dx, \tag{3.2}$$

and the constant $(N-2)^2/4$ is optimal and not attained. The reader interested in the existing literature on the extensions of the classical Hardy inequality (3.2) with a singular potential is referred, in particular, to the following papers and the references therein: [7], [20], [2], [8], [4], [18], [30], [31], [29], [25], [11], [12], [13].

In the case of a multi-singular potential $V(x) = \sum_{i=1}^{n} \alpha_i/|x-x_i|^2$ with $\alpha_i \in \mathbb{R}$, where $x_i \in \mathbb{R}^N$ are the singular poles assumed to be fixed, the study of positivity of the quadratic functional

$$\mathscr{D}[u] = S_{\alpha_1,\dots,\alpha_n,x_1,\dots,x_n}[u] := \int_\Omega |\nabla u|^2 dx - \sum_{i=1}^{n} \alpha_i \int_\Omega \frac{u^2}{|x-x_i|^2} dx \tag{3.3}$$

is much more intricate since the interaction among the poles and their configuration matters.

Among other results, in [16] it was proved that when $\Omega = \mathbb{R}^N$, $\mathscr{D}$ is positive if and only if $\sum_{i=1}^{n} \alpha_i^+ \leq (N-2)^2/4$ for any configuration of the poles $x_1,\dots,x_n$, where $\alpha^+ = \max\{\alpha, 0\}$. Conversely, if $\sum_{i=1}^{n} \alpha_i^+ > (N-2)^2/4$, there exist configurations $x_1,\dots,x_n$ for which $\mathscr{D}$ is negative. These results have been improved later on by Bosi, Dolbeault, Esteban [6] when deriving lower bounds of the spectrum of the operator $-\Delta - \mu \sum_{i=1}^{n} 1/|x-x_i|^2$, $\mu \in (0,(N-2)^2/4]$, with $x_1,x_2,\dots,x_n \in \mathbb{R}^N$. Rougly speaking, they showed that for any $\mu \in (0,(N-2)^2/4]$ and any configuration $x_1,x_2,\dots,x_n \in \mathbb{R}^N$, $n \geq 2$, there is a nonnegative constant $K_n < \pi^2$ such that

$$u \in C_0^\infty(\mathbb{R}^N), \quad \frac{K_n + (n+1)\mu}{d^2} \int_{\mathbb{R}^N} u^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{u^2}{|x-x_i|^2} dx \geq 0, \tag{3.4}$$

where d denotes $d := \min_{i \neq j} |x_i - x_j|/2$. The original proof of (3.4) in [6] employs a partition of unity technique, the so-called “IMS" (for Ismagilov, Morgan-Simon, Sigal, see [27], [28]), localizing the singular Schrödinger operator. Inequality (3.4) emphasizes that we can reach the critical singular mass $(N-2)^2/(4|x-x_i|^2)$ at any singular pole x_i to the prize of adding a lower order term in L^2-norm.

To simplify the notations, here and throughout the paper when writing $\int \cdot dx$ we denote the integral over $\mathbb{R}^N$. Besides, using the so-called "expansion of the square" method, the authors in [6] proved the following inequality without lower order terms

$$\int |\nabla u|^2 dx \geq \frac{(N-2)^2}{4n} \sum_{i=1}^{n} \int \frac{u^2}{|x-x_i|^2} dx + \frac{(N-2)^2}{4n^2} \sum_{1 \leq i < j \leq n} \int \frac{|x_i - x_j|^2}{|x-x_i|^2 |x-x_j|^2} u^2 dx, \tag{3.5}$$

for any $u \in H^1(\mathbb{R}^N)$ and any set of poles $x_1, x_2, \dots, x_n \in \mathbb{R}^N$, $n \geq 2$. Let us denote the singular potentials in (3.5) by

$$V_i(x) := \frac{1}{|x-x_i|^2}, \quad V_{ij}(x) := \frac{|x_i - x_j|^2}{|x-x_i|^2 |x-x_j|^2}, \quad \forall i,j \in \{1,\dots,n\}, \quad i \neq j. \tag{3.6}$$

Observe that both potentials in (3.6) have a quadratic singularity at each pole x_i, i.e.

$$\lim_{x\to x_i} V_i(x)|x-x_i|^2 = 1, \quad \lim_{x\to x_i} V_{ij}(x)|x-x_i|^2 = 1, \quad \forall i,j \in \{1,\dots,n\}, \quad i \neq j. \tag{3.7}$$

Moreover, due to the symmetry we notice that for any $i \in \{1,\dots,n\}$ we have the asymptotic formula as $x \to x_i$:

$$\sum_{1\le i<j\le n} V_{ij}(x) = \frac{1}{|x-x_i|^2}\Big[\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{|x_i-x_j|^2}{|x-x_j|^2} + O(|x-x_i|^2)\Big] \sim \frac{n-1}{|x-x_i|^2}, \quad \text{as } x\to x_i. \tag{3.8}$$

Therefore, due to (3.8) we remark that the total mass arising at a singular pole x_i in (3.5) is proportional to

$$\frac{(N-2)^2}{4n}\sum_{i=1}^{n} V_i(x) + \frac{(N-2)^2}{4n^2}\sum_{1\le i<j\le n} V_{ij}(x) \sim \frac{(N-2)^2}{4}\frac{2n-1}{n^2}\frac{1}{|x-x_i|^2}, \quad \text{as } x\to x_i, \quad \forall i\in\{1,\dots,n\}. \tag{3.9}$$

Note however that the multiplicative factor in each singularity in (3.9) is smaller than the optimal one that (3.4) yields for $\mu = (N-2)^2/4$. This is so because in (3.5) no other corrected terms are added.

We also mention the articles [6], [15], [16], [1], [17] and the references therein for other inequalities with multipolar singularities.

It is also worth mentioning the literature on Hardy-type inequalities of different nature than those studied in this paper. In particular, in [24] (see also the references therein) the authors investigated the so-called Hardy inequalities for m-dimensional particles of the form

$$\sum_{j=1}^{N}\int_{\mathbb{R}^{mN}} |\nabla_j u|^2 dx \ge \mathscr{C}(m,N)\sum_{1\le i<j\le N}\int_{\mathbb{R}^{mN}} \frac{|u|^2}{r_{ij}^2}dx, \tag{3.10}$$

giving explicit lower bounds for the best constant $\mathscr{C}(m,N)$ when applied to test functions $u \in H^1(\mathbb{R}^{mN})$. In (3.10) we have denoted $x = (x_1,\dots,x_N)$ with $x_i = (x_{i,1},\dots,x_{i,m}) \in \mathbb{R}^m$, $r_{ij} = \sqrt{\sum_{k=1}^m (x_{i,k}-x_{j,k})^2}$ for all $i,j \in \{1,\dots,N\}$ and ∇_j for the gradient associated to the j-th particle. Roughly speaking, the singularities in (3.10) occur in uncountable sets given by the diagonals $x_i = x_j$. The optimality of (3.10) is still an open question excepting the case $m=1$ for which $\mathscr{C}(1,N) = 1/2$ provided the test functions u vanish on diagonals $x_i = x_j$.

In this paper we develop new optimal Hardy-type inequalities with multipolar potentials.

In Section 3.2 we present some general strategies to handle the Hardy inequalities.

In Section 3.3 we complement and improve some results in [6] related to inequality (3.5). Our proofs use convenient transformations involving the product of the fundamental solutions E_i of the Laplacian at the poles x_i, $i \in \{1,\dots,n\}$.

In Theorem 3.1 of Section 3.3, we give an optimal inequality for the operator $A_\lambda = -\Delta - \lambda \sum_{1\leq i<j\leq n} V_{ij}(x)$, $\lambda > 0$, showing a better singular behavior of the potential at each pole x_i than pointed out in (3.5)-(3.9). This allows to show the existence of bounded domains in which, for the bipolar Hardy inequality, the L^2-reminder term in (3.4) can be removed. For this to be done, the best situation seems to be the case in which the singularities are localized on the boundary of the domain, as emphasized in Section 3.4, Proposition 3.1.

In Section 3.5 we end up with some further comments and open questions.

3.2 Preliminaries: some strategies to prove Hardy-type inequalities

There are several techniques for proving Hardy inequalities in smooth domains (including the whole space) which are all interlinked by the following integral identity.

Assume $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is an open set and let $x_1, \ldots, x_n \in \overline{\Omega}$. We also consider a distribution $\varphi \in D'(\Omega)$ such that $\varphi(x) > 0$ in $\Omega \setminus \{x_1, \ldots, x_n\}$ and $\varphi \in C^2(\Omega \setminus \{x_1, x_2, \ldots x_n\})$. Then it holds that

$$\int_\Omega \Big[|\nabla u|^2 + \frac{\Delta\varphi}{\varphi} u^2\Big] dx = \int_\Omega \Big|\nabla u - \frac{\nabla\varphi}{\varphi} u\Big|^2 dx = \int_\Omega \varphi^2 |\nabla(u\varphi^{-1})|^2 dx, \tag{3.11}$$
$$\forall u \in C_0^1(\Omega \setminus \{x_1, x_2, \ldots x_n\}).$$

The proof of (3.11) can be done using integration by parts. In particular, (3.11) can be extended to test functions $u \in H_0^1(\Omega)$ since $C_0^1(\Omega \setminus \{x_1, \ldots, x_n\})$ is dense in $H_0^1(\Omega)$, see e.g. [14].

The identity (3.11) could be extended to more general classes of distributions φ depending on the applications that we have in mind. Here we are interested in applications to Hardy inequalities with multipolar potentials located at the poles $x_1, \ldots, x_n$, with $x_i \neq x_j$ for all $i \neq j$ and $i, j \in \{1, \ldots, n\}$.

Various aspects of the identities involved in (3.11) have been used in the literature to prove and analyze Hardy inequalities in different contexts. But, as far as we know, (3.11) has not been stated explicitly as it stands before.

Identity (3.11) could be directly applied to obtain Hardy inequalities with potentials of the form $-\Delta\varphi/\varphi$, i.e.

$$\int_\Omega |\nabla u|^2 dx \geq \int_\Omega \Big(-\frac{\Delta\varphi}{\varphi}\Big) u^2 dx, \quad \forall u \in H_0^1(\Omega). \tag{3.12}$$

In order to derive inequalities for a concrete potential $V = V(x) \in L^1_{loc}(\Omega)$, one needs to look for a corresponding φ such that

$$-\frac{\Delta\varphi}{\varphi} \geq V(x), \quad \forall x \in \Omega. \tag{3.13}$$

Some of the existing techniques to prove Hardy-type inequalities use "the expansion of the square" method (e.g. [6]) or suitable functional transformations (e.g. [7], [3]). In view of (3.11), all these techniques are actually equivalent and the problem can always be reduced to checking pointwise inequalities for a potential V and a corresponding φ as in (3.13).

Optimality. For a general φ satisfying (3.11), we cannot say anything about the optimality of (3.12). To argue in that sense, next we give a counterexample by means of the standard Hardy inequality. Assume $\Omega = \mathbb{R}^N$, $N \geq 3$, and let $\lambda < \lambda_\star := (N-2)^2/4$. Then we consider

$$\varphi = |x|^{-(N-2)/2+\sqrt{\lambda_\star - \lambda}},$$

and observe that $\varphi > 0$ in $\mathbb{R}^N$, $\varphi \in C^\infty(\mathbb{R}^N \setminus \{0\})$ and therefore φ satisfies the identity (3.11) before. Then for such φ, inequality (3.12) becomes

$$\int |\nabla u|^2 dx \geq \lambda \int \frac{u^2}{|x|^2} dx, \quad \forall u \in H^1(\mathbb{R}^N),$$

which is not optimal as follows from (3.2).

3.3 Multipolar Hardy inequalities

Assume $N \geq 3$ and consider n poles $x_1, \ldots x_n \in \mathbb{R}^N$, $n \geq 2$, such that $x_i \neq x_j$ for any $i \neq j$, and $i, j \in \{1, 2, \ldots, n\}$. In the sequel we improve the result (3.5) by Bosi et al. [6]. The main result of this section is as follows.

Theorem 3.1. *It holds that*

$$\int |\nabla u|^2 dx \geq \frac{(N-2)^2}{n^2} \sum_{1 \leq i < j \leq n} \int \left| \frac{x - x_i}{|x - x_i|^2} - \frac{x - x_j}{|x - x_j|^2} \right|^2 u^2 dx, \quad \forall u \in H^1(\mathbb{R}^N), \tag{3.14}$$

or equivalently

$$\int |\nabla u|^2 dx \geq \frac{(N-2)^2}{n^2} \sum_{1 \leq i < j \leq n} \int \frac{|x_i - x_j|^2}{|x - x_i|^2 |x - x_j|^2} u^2 dx, \quad \forall u \in H^1(\mathbb{R}^N). \tag{3.15}$$

Moreover, the constant $(N-2)^2/n^2$ is optimal.

In the sequel, we prove Theorem 3.1 applying identity (3.11) before.

Proof of Theorem 3.1.

By density arguments it is sufficient to prove (3.14) for any function $u \in C_0^1(\mathbb{R}^N \setminus \{x_1, \ldots, x_n\})$. Then, according to (3.11)-(3.13), it is enough to find a proper φ satisfying

$$-\frac{\Delta\varphi}{\varphi} \geq \frac{(N-2)^2}{n^2} \sum_{1\leq i<j\leq n} \left| \frac{x-x_i}{|x-x_i|^2} - \frac{x-x_j}{|x-x_j|^2} \right|^2, \quad \forall x \in \mathbb{R}^N \setminus \{x_1,\dots,x_n\}.$$

Let us choose

$$\varphi = E^{1/n} = \prod_{i=1}^{n} E_i^{1/n}, \tag{3.16}$$

where $E = \prod_{i=1}^{n} E_i$ and E_i is the fundamental solution of the Laplacian at the singular pole x_i, $i \in \{1,\dots,n\}$, i.e.

$$E_i = \frac{|x-x_i|^{2-N}}{\omega_N(N-2)}. \tag{3.17}$$

Here ω_N denotes the $(N-1)$-Hausdorff measure of the unit sphere S^{N-1} in $\mathbb{R}^N$. Note that φ chosen in (3.16) verifies the integrability conditions to validate the identity (3.11). On the other hand, we have

$$\nabla E = \Big(\sum_{i=1}^{n} \frac{\nabla E_i}{E_i}\Big) E, \quad \forall x \in \mathbb{R}^N \setminus \{x_1,\dots,x_n\}. \tag{3.18}$$

Due to the fact that $-\Delta E_i = \delta_{x_i}$ for all $i \in \{1,\dots,n\}$ we obtain

$$\begin{aligned} \Delta E &= \Big(\sum_{i=1} \frac{\Delta E_i}{E_i} + 2\sum_{1\leq i<j\leq n} \frac{\nabla E_i \cdot \nabla E_j}{E_i E_j}\Big) E \\ &= 2\Big(\sum_{1\leq i<j\leq n} \frac{\nabla E_i \cdot \nabla E_j}{E_i E_j}\Big) E, \quad \forall x \in \mathbb{R}^N \setminus \{x_1,\dots,x_n\}. \end{aligned} \tag{3.19}$$

Combining (3.16), (3.17), (3.18) and (3.19) we notice that φ satisfies precisely the equation

$$-\Delta\varphi - \frac{(N-2)^2}{n^2} \sum_{1\leq i<j\leq n} \left| \frac{x-x_i}{|x-x_i|^2} - \frac{x-x_j}{|x-x_j|^2} \right|^2 \varphi = 0, \quad \forall x \in \mathbb{R}^N \setminus \{x_1,\dots,x_n\}. \tag{3.20}$$

Then, identity (3.11) becomes

$$\begin{aligned} \int \Big[|\nabla u|^2 - \frac{(N-2)^2}{n^2} \sum_{1\leq i<j\leq n} \left| \frac{x-x_i}{|x-x_i|^2} - \frac{x-x_j}{|x-x_j|^2} \right|^2 u^2\Big] dx \\ = \int \left|\nabla u - \frac{\nabla(E^{1/n})}{E^{1/n}} u\right|^2 dx = \int |\nabla(uE^{-1/n})|^2 E^{2/n} dx \geq 0. \end{aligned} \tag{3.21}$$

This concludes the proof of (3.14).

Optimality of the constant.

Next we complete the proof of Theorem 3.1 by showing the optimality of the constant $(N-2)^2/n^2$ in (3.14).

According to (3.21), we actually showed that for all $u \in H^1(\mathbb{R}^N)$ we have

$$\begin{aligned} \int |\nabla u|^2 dx - \frac{(N-2)^2}{n^2} \sum_{1\leq i<j\leq n} \int \left| \frac{x-x_i}{|x-x_i|^2} - \frac{x-x_j}{|x-x_j|^2} \right|^2 u^2 dx \\ = \int |\nabla (uE^{-1/n})|^2 E^{2/n} dx. \end{aligned} \tag{3.22}$$

Here $B_r(x) \subset \mathbb{R}^N$, for some fixed $r>0$ and $x \in \mathbb{R}^N$, denotes the ball of radius r centered at x.

For $\varepsilon > 0$ aimed to be small ($\varepsilon < \min\{1, d/2\}$), we consider the cut-off functions $\theta_\varepsilon \in C_0(\mathbb{R}^N)$ defined by

$$\theta_\varepsilon(x) = \begin{cases} 0, & |x-x_i| \leq \varepsilon^2, \quad \forall i \in \{1,\dots,n\}, \\ \frac{\log |x-x_i|/\varepsilon^2}{\log 1/\varepsilon}, & \varepsilon^2 \leq |x-x_i| \leq \varepsilon, \quad \forall i \in \{1,\dots,n\}, \\ 1, & x \in B_{1/\varepsilon}(0) \setminus \cup_{i=1}^n B_\varepsilon(x_i), \\ \varepsilon\left(\frac{2}{\varepsilon} - |x|\right), & 1/\varepsilon \leq |x| \leq 2/\varepsilon, \\ 0, & |x| \geq 2/\varepsilon. \end{cases} \tag{3.23}$$

Then we consider the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ defined by

$$u_\varepsilon := E^{1/n}\theta_\varepsilon, \quad \varepsilon > 0,$$

which belongs to $C_0(\mathbb{R}^N)$ $(\subset H^1(\mathbb{R}^N))$ since θ_ε belongs to $C_0(\mathbb{R}^N)$ and is supported far from the poles x_i.

In the sequel we show that $\{u_\varepsilon\}_{\varepsilon>0}$ is an approximating sequence for $(N-2)^2/n^2$, that is

$$\lim_{\varepsilon \searrow 0} \frac{\int |\nabla u_\varepsilon|^2 dx}{\sum_{1\leq i<j\leq n} \int \left| \frac{x-x_i}{|x-x_i|^2} - \frac{x-x_j}{|x-x_j|^2} \right|^2 u_\varepsilon^2 dx} = \frac{(N-2)^2}{n^2}. \tag{3.24}$$

Firstly, we can easily notice that there exists a constant $C>0$ depending on d (uniformly in ε) such that

$$\sum_{1\leq i<j\leq n} \int \left| \frac{x-x_i}{|x-x_i|^2} - \frac{x-x_j}{|x-x_j|^2} \right|^2 u_\varepsilon^2 dx > C, \quad \forall \varepsilon > 0. \tag{3.25}$$

On the other hand, taking into account where $\nabla\theta_\varepsilon$ is supported, we split in two parts the right hand side of (3.22) as

$$
\begin{aligned}
\int |\nabla(u_\varepsilon E^{-1/n})|^2 E^{2/n} dx \\
&= \sum_{i=1}^{n} \int_{B_\varepsilon(x_i)\setminus B_{\varepsilon^2}(x_i)} |\nabla\theta_\varepsilon|^2 E^{2/n} dx + \int_{B_{2/\varepsilon}(0)\setminus B_{1/\varepsilon}(0)} |\nabla\theta_\varepsilon|^2 E^{2/n} dx \\
&:= I_1 + I_2. \qquad (3.26)
\end{aligned}
$$

Next we obtain

$$
I_1 = \frac{1}{\omega_N^2 (N-2)^2} \sum_{i=1}^{n} \int_{B_\varepsilon(x_i)\setminus B_{\varepsilon^2}(x_i)} \frac{1}{\log^2(1/\varepsilon)} \frac{1}{|x-x_i|^2} \prod_{j=1}^{n} |x-x_j|^{2(2-N)/n} dx. \qquad (3.27)
$$

Since

$$
|x-x_j| \geq \frac{d}{2}, \quad \forall x \in B_\varepsilon(x_i), \quad \forall j \neq i, \quad \forall i,j \in \{1,\ldots,n\}, \qquad (3.28)
$$

from (3.27) we deduce that

$$
\begin{aligned}
I_1 &\leq \frac{\left(\frac{d}{2}\right)^{2(n-1)(2-N)/n}}{\omega_N^2 (N-2)^2} \frac{1}{\log^2(1/\varepsilon)} \sum_{i=1}^{n} \int_{B_\varepsilon(x_i)\setminus B_{\varepsilon^2}(x_i)} |x-x_i|^{2(2-N)/n-2} dx \\
&= \frac{n\left(\frac{d}{2}\right)^{2(n-1)(2-N)/n}}{\omega_N^2 (N-2)^2} \frac{1}{\log^2(1/\varepsilon)} \int_{\varepsilon^2}^{\varepsilon} r^{N-1} \int_{S^{N-1}} r^{2(2-N)/n-2} d\sigma dr \\
&= \frac{n\left(\frac{d}{2}\right)^{2(n-1)(2-N)/n}}{\omega_N (N-2)^2} \frac{1}{\log^2(1/\varepsilon)} \int_{\varepsilon^2}^{\varepsilon} r^{(N-2)(1-2/n)-1} dr. \qquad (3.29)
\end{aligned}
$$

From (3.27) and (3.29) we obtain that

$$
I_1 = \begin{cases} O\left(\frac{1}{\log(1/\varepsilon)}\right), & n = 2, \\ O(\varepsilon^{(N-2)(1-2/n)}), & n \geq 3. \end{cases} \qquad (3.30)
$$

Taking $\varepsilon > 0$ small enough such that $\varepsilon < 1/2m$, where $m = \max_{i=1,\ldots,n} |x_i|$, it holds

$$
|x-x_i| \geq \frac{1}{2\varepsilon}, \quad \forall x \in B_{2/\varepsilon}(0) \setminus B_{1/\varepsilon}(0), \quad \forall i \in \{1,\ldots,n\}. \qquad (3.31)
$$

Due to (3.31) we obtain

$$
\begin{aligned}
I_2 &= \frac{1}{\omega_N^2 (N-2)^2} \int_{B_{2/\varepsilon}(0)\setminus B_{1/\varepsilon}(0)} \varepsilon^2 \prod_{i=1}^{n} |x-x_i|^{2(2-N)/n} dx \\
&\leq \frac{1}{\omega_N^2 (N-2)^2} \int_{B_{2/\varepsilon}(0)\setminus B_{1/\varepsilon}(0)} \varepsilon^2 \prod_{i=1}^{n} \left(\frac{1}{2\varepsilon}\right)^{2(2-N)/n} dx \\
&= \frac{1}{\omega_N^2 (N-2)^2} \left(\frac{1}{2}\right)^{2(2-N)} \int_{B_{2/\varepsilon}(0)\setminus B_{1/\varepsilon}(0)} \varepsilon^{2(N-1)} dx \\
&= \frac{2^{2(N-2)}}{\omega_N (N-2)^2} \varepsilon^{2(N-1)} \int_{1/\varepsilon}^{2/\varepsilon} r^{N-1} dr \\
&= O(\varepsilon^{N-2}), \text{ as } \varepsilon \to 0.
\end{aligned}
\tag{3.32}
$$

In conclusion, according to (3.26), (3.30) and (3.32) we get

$$
\lim_{\varepsilon \searrow 0} \int |\nabla (u_\varepsilon E^{-1/n})|^2 E^{2/n} dx = 0, \quad \forall n \geq 2. \tag{3.33}
$$

Combining (3.22), (3.25) and (3.33) we end up with the optimality of $(N-2)^2/n^2$ as in (3.24), and the proof of Theorem 3.1 is complete.

□

Remark 3.1.
Our optimal result in Theorem 3.1 provides an inequality with a positive singular quadratic potential which behaves asymptotically like

$$
\frac{(N-2)^2}{n^2} \sum_{1\leq i<j\leq n} V_{ij}(x) \sim \frac{(N-2)^2}{4} \frac{4n-4}{n^2} \frac{1}{|x-x_i|^2} \qquad \text{as } x \to x_i, \quad \forall i \in \{1,\ldots,n\}, \tag{3.34}
$$

at each pole x_i. In particular, for any $n \geq 2$, Theorem 3.1 represents an improvement of (3.5), in the sense that the multiplication factor in (3.34) which corresponds to the quadratic singularity is larger than that one obtained in inequality (3.5) as emphasized in (3.9).

Remark 3.2.
The proof of (3.5) in [6] was obtained by expanding the square

$$
\int \left| \nabla u + \alpha \sum_{i=1}^{n} \frac{x-x_i}{|x-x_i|^2} u \right|^2 dx \geq 0, \quad \alpha \in \mathbb{R}, \tag{3.35}
$$

which gives

$$
\begin{aligned}
0 \leq \int |\nabla u|^2 dx + [n\alpha^2 - (N-2)\alpha] \sum_{i=1}^{n} \int \frac{u^2}{|x-x_i|^2} dx \\
- \alpha^2 \sum_{1\leq i<j\leq n} \int \frac{|x_i - x_j|^2}{|x-x_i|^2 |x-x_j|^2} u^2 dx.
\end{aligned}
\tag{3.36}
$$

More precisely, (3.5) is a consequence of (3.36) when $\alpha = (N-2)/(2n)$. Moreover, we remark that the expansion (3.36) also applies to derive the inequality of Theorem 3.1 with a different choice of α, that is $\alpha = (N-2)/n$.

The quadratic term in (3.21) is given by the formula

$$\int \left|\nabla u - \frac{\nabla(E^{1/n})}{E^{1/n}} u\right|^2 dx = \int \left|\nabla u + \frac{N-2}{n} \sum_{i=1}^{n} \frac{x-x_0}{|x-x_0|^2} u\right|^2 dx,$$

which motivates the use of the "expansion of the square" emphasized above for $\alpha = (N-2)/n$. This was not observed in [6]. In fact we got to this point indirectly as a consequence of the direct application of identity (3.11).

Remark 3.3.
Adimurthi et al. proved in particular in [3] that, whenever E satisfies $-\Delta E = \sum_{1\le i\le n} \delta_{x_i}$ for some given poles $x_1, \ldots, x_n \in \mathbb{R}^N$, the following inequality holds

$$\int |\nabla u|^2 dx \geq \frac{1}{4} \int \left|\frac{\nabla E}{E}\right|^2 u^2 dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N). \tag{3.37}$$

A direct application of (3.37) in the context of multipolar Hardy inequalities would consist on taking $E = E_1 + \ldots + E_n$. If $N \geq 3$ we then get (cf. [1])

$$\int |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int \left|\frac{\sum_{i=1}^n (x-x_i)|x-x_i|^{-N}}{\sum_{i=1}^n |x-x_i|^{2-N}}\right|^2 u^2 dx, \quad \forall u \in H^1(\mathbb{R}^N). \tag{3.38}$$

Observe that the potential V in (3.38), given by

$$V_{N,n,x_1,\ldots,x_n}(x) := \left|\frac{\sum_{i=1}^n (x-x_i)|x-x_i|^{-N}}{\sum_{i=1}^n |x-x_i|^{2-N}}\right|^2, \tag{3.39}$$

is non-negative and moreover has a quadratic singularity at each pole x_i. More precisely, $V_{N,n,x_1,\ldots,x_n}$ satisfies

$$V_{N,n,x_1,\ldots,x_n}(x) = \frac{1}{|x-x_i|^2} + O(|x-x_i|^{N-4}), \text{ as } x \to x_i, \quad \forall i \in \{1,\ldots,n\}, \tag{3.40}$$

respectively

$$V_{N,n,x_1,\ldots,x_n}(x) = O(\frac{1}{|x|^2}), \quad \text{as } |x| \to \infty. \tag{3.41}$$

From (3.40) and (3.41) we can easily deduce that

$$V_{N,n,x_1,\ldots,x_n}(x) = \sum_{i=1}^{n} \frac{1}{|x-x_i|^2} + O(1), \quad \forall x \in \mathbb{R}^N, \tag{3.42}$$

where $O(1)$ denotes a changing sign quantity, uniformly bounded in $\mathbb{R}^N$. For $N \geq 4$, the identification (3.42) shows that inequality (3.38) allows to deduce an inequality in the spirit of (3.4) in which the same critical singular potential is obtained, paying the prize of adding a lower order term in L^2-norm. The multiplication factor of the lower order term obtained through (3.38), remains to be compared with that one which corresponds to (3.4).

On the contrary, inequality (3.38) does not allow to get optimal results as in Theorem 3.1 when removing the corrected lower order terms in L^2-norm.

We point out, that the key role for showing Theorem 3.1 was played by identity (3.11) applying for suitable distributions involving the product of the fundamental solutions of the Laplacian at each singular pole x_i. This allows to prove optimal Hardy inequalities for singular quadratic potentials of the form

$$W(x) = \sum_{i=1}^{n} \frac{\lambda_i(x)}{|x-x_i|^2}, \quad \forall x \in \mathbb{R}^N,$$

where

$$\lambda_i(x) > 0 \text{ in } \mathbb{R}^N, \quad \lim_{x \to x_i} \lambda_i(x) = (n-1)\frac{(N-2)^2}{2n^2}, \quad \forall i \in \{1,\dots,n\}.$$

The weights λ_i in Theorem 3.1 are given by $\lambda_i(x) = (N-2)^2/(2n^2)\sum_{\substack{j=1 \\ j\neq i}}^{n} |x_i - x_j|^2/|x-x_j|^2$.

As we mentioned before, the potential $V_{N,n,x_1,\dots x_n}$ cannot be compared with W on any bounded connected domain Ω with $x_1,\dots,x_n \in \overline{\Omega}$. Indeed, next we emphasize this in two concrete examples.

Firstly, for $N=3$, $n=2$, we consider the singular poles $0, x_0 \in \overline{\Omega} \subset \mathbb{R}^3$ and we obtain $V_{3,2,0,x_0}(x_0/2) = 0$ while $W(x_0/2) > 0$.

Secondly, let us consider a configuration with three singular poles $x_1, x_2, x_3 \in \mathbb{R}^3$ determining an equilateral triangle such that

$$|x_1| = |x_2| = |x_3| > 0, \quad x_1 + x_2 + x_3 = 0,$$

and let $\Omega \subset \mathbb{R}^3$ be a connected bounded open set with $x_1, x_2, x_3 \in \overline{\Omega}$.
Then $V_{3,3,x_1,x_2,x_3}(0) = 0$ while $W(0) > 0$.

3.4 New bounds for the bipolar Hardy inequality in bounded domains

We now present some consequences of the previous multipolar Hardy inequality in Theorem 3.1 to bounded domains in $H_0^1(\Omega)$.

In this subsection we present some applications of Theorem 3.1 to bounded domains in the case of a bipolar potential

$$V(x) = \frac{1}{|x-x_1|^2} + \frac{1}{|x-x_2|^2}, \tag{3.43}$$

for some $x_1, x_2 \in \mathbb{R}^N$, $N \geq 3$ with $x_1 \neq x_2$. In consequence, we derive new lower bounds for the bipolar Hardy inequality, which turn out to be optimal in the case where the poles are located on the boundary of the domain.

We have seen that Theorem 3.1 provides an inequality involving a bipolar potential which behaves asymptotically like

$$\frac{(N-2)^2}{4} V_{12}(x) \sim \frac{(N-2)^2}{4} \frac{1}{|x-x_i|^2}, \quad \text{as } x \to x_i, \quad \forall i \in \{1,2\}. \tag{3.44}$$

On the other hand, inequality (3.5) provides a bipolar potential with a weaker quadratic singularity which is asymptotically given by

$$\frac{(N-2)^2}{8}(V_1+V_2) + \frac{(N-2)^2}{16} V_{12}(x) \sim \frac{3(N-2)^2}{16} \frac{1}{|x-x_i|^2}, \quad \text{as } x \to x_i, \quad \forall i \in \{1,2\}. \tag{3.45}$$

Theorem 3.1 may give better lower bounds than inequality (3.5) for the Hardy inequality with the bipolar potential V as in (3.43). The main results of this section are as follows.

As a consequence of Theorem 3.1 we have

Proposition 3.1. *Assume* $0 \leq \alpha, \beta \leq 1$.

For any $x_1 \neq x_2$ *and for all* $u \in C_0^\infty(B_{r(x_1,x_2)}(C(x_1,x_2)))$ *we have*

$$\int_{B_{r(x_1,x_2)}(C(x_1,x_2))} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{B_{r(x_1,x_2)}(C(x_1,x_2))} \Big[\frac{\alpha}{|x-x_1|^2} + \frac{\beta}{|x-x_2|^2}\Big] u^2 dx, \tag{3.46}$$

where $B_{r(x_1,x_2)}(C(x_1,x_2))$ *is the ball centered at the point*

$$C(x_1,x_2) = \frac{\beta}{\alpha+\beta} x_1 + \frac{\alpha}{\alpha+\beta} x_2,$$

of radius

$$r(x_1,x_2) = \frac{\sqrt{\alpha+\beta-\alpha\beta}}{\alpha+\beta} |x_1 - x_2|,$$

as shown in Fig. 3.1.

As a consequence of inequality (3.5) we have

Proposition 3.2. *Assume* $0 \leq \alpha, \beta \leq 1$.

For any $x_1 \neq x_2$ *and for all* $u \in C_0^\infty(B_{r(x_1,x_2)}(C(x_1,x_2)))$ *we have*

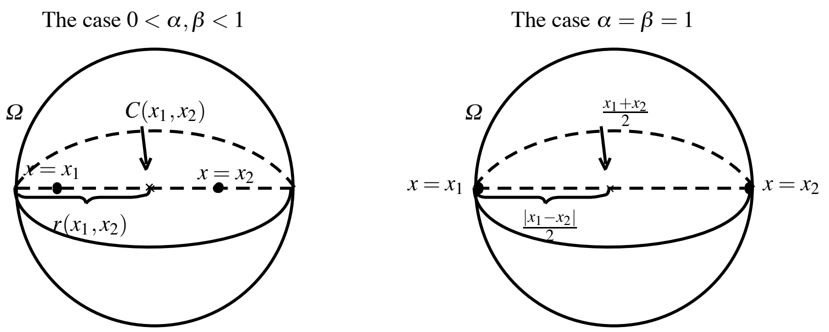

Fig. 3.1 Domains where improved bipolar inequalities hold. The best results are obtained when both singularities are located on the boundary, case which corresponds to $\alpha = \beta = 1$.

$$\int_{B_{r(x_1,x_2)}(C(x_1,x_2))} |\nabla u|^2 dx \geq \Big[\frac{(N-2)^2}{8} + \frac{(N-2)^2}{16}\alpha\Big] \int_{B_{r(x_1,x_2)}(C(x_1,x_2))} \frac{u^2}{|x-x_1|^2} dx + \Big[\frac{(N-2)^2}{8} + \frac{(N-2)^2}{16}\beta\Big] \int_{B_{r(x_1,x_2)}(C(x_1,x_2))} \frac{u^2}{|x-x_2|^2} dx, \tag{3.47}$$

where $B_{r(x_1,x_2)}(C(x_1,x_2))$ is the ball centered at the point

$$C(x_1,x_2) = \frac{\beta}{\alpha+\beta} x_1 + \frac{\alpha}{\alpha+\beta} x_2,$$

of radius

$$r(x_1,x_2) = \frac{\sqrt{\alpha+\beta-\alpha\beta}}{\alpha+\beta} |x_1 - x_2|,$$

as shown in Fig. 3.1.

Remark 3.4. The constraints $\alpha, \beta \leq 1$ impose to the singular poles x_1, x_2 to belong to $\in \overline{B_{r(x_1,x_2)}(C(x_1,x_2))}$.

Remark 3.5. We observe that for α, β getting closer to 1, the result of Proposition 3.1 is better than the one of Proposition 3.2.

Next we prove only Proposition 3.1 since the proof of Proposition 3.2 follows the same steps.

Proof of Proposition 3.1.

Let us consider an open bounded subset $\Omega \subset \mathbb{R}^N$, $N \geq 3$. Applying Theorem 3.1 we have that

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\Omega} \Big| \frac{x-x_1}{|x-x_1|^2} - \frac{x-x_2}{|x-x_2|^2} \Big|^2 u^2 dx, \quad \forall u \in H_0^1(\Omega). \tag{3.48}$$

In the sequel, we are seeking for domains $\Omega \subset \mathbb{R}^N$ such that $x_1, x_2 \in \overline{\Omega}$ and

$$\Big| \frac{x-x_1}{|x-x_1|^2} - \frac{(x-x_2)}{|x-x_2|^2} \Big|^2 \geq \frac{\alpha}{|x-x_1|^2} + \frac{\beta}{|x-x_2|^2}, \quad \forall x \in \overline{\Omega}. \tag{3.49}$$

Using the identity $2(x-x_1)(x-x_2) = |x-x_1|^2 + |x-x_2|^2 - |x_1-x_2|^2$, then (3.49) is equivalent to

$$|x_1-x_2|^2 \geq \alpha |x-x_2|^2 + \beta |x-x_1|^2, \quad \forall x \in \overline{\Omega}. \tag{3.50}$$

Expanding the squares in (3.50) and dividing by $\alpha+\beta$ we obtain

$$\frac{1-\beta}{\alpha+\beta}|x_1|^2 + \frac{1-\alpha}{\alpha+\beta}|x_2|^2 - \frac{2}{\alpha+\beta} x_1 \cdot x_2 \geq |x|^2 - 2x \cdot \Big(\frac{\alpha}{\alpha+\beta} x_2 + \frac{\beta}{\alpha+\beta} x_1 \Big). \tag{3.51}$$

Coupling the squares we rewrite (3.51) as

$$\begin{aligned} \frac{1-\beta}{\alpha+\beta}|x_1|^2 + \frac{1-\alpha}{\alpha+\beta}|x_2|^2 - \frac{2}{\alpha+\beta} x_1 \cdot x_2 + \Big| \frac{\alpha}{\alpha+\beta} x_2 + \frac{\beta}{\alpha+\beta} x_1 \Big|^2 \\ \geq \Big| x - \Big(\frac{\alpha}{\alpha+\beta} x_2 + \frac{\beta}{\alpha+\beta} x_1 \Big) \Big|^2. \end{aligned} \tag{3.52}$$

After some computations on the left hand side of (3.52) we get

$$\frac{\alpha+\beta-\alpha\beta}{(\alpha+\beta)^2}|x_1-x_2|^2 \geq \Big| x - \Big(\frac{\alpha}{\alpha+\beta} x_2 + \frac{\beta}{\alpha+\beta} x_1 \Big) \Big|^2, \quad \forall x \in \overline{\Omega}. \tag{3.53}$$

Due to this, the proof is finished by identifying properly the set Ω.

□

We notice that, as far as α and β get closer to 1, the poles $x = x_1$ respectively $x = x_2$, are pushed to the boundary of the domain as drawn in Fig. 3.1. Indeed, if $\alpha = \beta = 1$ then x_1 and x_2 are located on the boundary of $B_{r(x_1,x_2)}(C(x_1,x_2)) = B_{|x_1-x_2|/2}((x_1+x_2)/2)$. Moreover, we will have the non trivial inequality

$$\int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} \Big[\frac{1}{|x-x_1|^2} + \frac{1}{|x-x_2|^2} \Big] u^2 dx. \tag{3.54}$$

for all $u \in C_0^1(B_{|x_1-x_2|/2}((x_1+x_2)/2))$.

As we said before, inequality (3.54) is new and not trivial. In fact, it provides an improved result in higher dimensions as follows.

Applying Hardy inequalities with boundary singularities (see e.g. [10]) we have that

$$\int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} |\nabla u|^2 dx \geq \frac{N^2}{4} \int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} \frac{u^2}{|x-x_1|^2} dx,$$

and

$$\int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} |\nabla u|^2 dx \geq \frac{N^2}{4} \int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} \frac{u^2}{|x-x_2|^2} dx,$$

the constant $N^2/4$ being optimal in both cases. Thus

$$\int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} |\nabla u|^2 dx \geq \frac{N^2}{8} \int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} \left[\frac{1}{|x-x_1|^2} + \frac{1}{|x-x_2|^2}\right] u^2 dx.$$

Note that inequality (3.54) is better for $N \geq 7$ since

$$\frac{(N-2)^2}{4} \geq \frac{N^2}{8}, \quad \forall N \geq 7.$$

3.5 Further comments and open problems

- In the spirit of (3.11) we may look for potentials V with an infinite number of singularities for which the Hardy inequality holds true. Besides, as such V are given as an infinite series one needs to make sure that they are well-defined. For instance, a potential of the form

$$V(x) = \sum_{(i,j,k)\in\mathbb{Z}^3} \frac{1}{|x_1-i|^2+|x_2-j|^2+|x_3-k|^2}, \quad x=(x_1,x_2,x_3)\in\mathbb{R}^3,$$

  diverges at every point $x \in \mathbb{R}^3$ and therefore the corresponding Hardy inequality does not make sense.
  The issue of potentials with an infinite number of singularities will be the subject of a forthcoming work.

- The optimality of the result in Proposition 3.1 remains to be analyzed.
- Identity (3.11) could be also applied for other choices of φ that we performed in Theorem 3.1. In particular, If we choose $\varphi = E^{1/2} = \prod_{i=1}^n E_i^{1/2}$ then we deduce the following inequality

$$\int |\nabla u|^2 dx$$

$$\geq \frac{(N-2)^2}{4} \sum_{i=1}^n \int \frac{u^2}{|x-x_i|^2} dx - \frac{(N-2)^2}{2} \sum_{1\leq i<j\leq n} \int \frac{(x-x_i)(x-x_j)}{|x-x_i|^2|x-x_j|^2} u^2 dx, \tag{3.55}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$. Next we may wonder the question if we can reprove inequality (3.4) through (3.55) since the potential in (3.55) has a critical quadratic singularity at any pole $x_i \in \mathbb{R}^N$. Firstly, we note that this is possible for the subcritical case $\mu < (N-2)^2/4$ which allows to get lower order terms in L^2-norm. However, (3.55) does not provide better L^2 lower order term than (3.4) does.
In the critical case $\mu = (N-2)^2/4$ we cannot obtain L^2 reminder terms from (3.55) unless the lower order term in inequality (3.55) has positive sign in a small neighborhood of the singular poles x_i. More precisely, the question to answer is whether for any configuration of the poles $x_1, \dots, x_n$, there exists $\varepsilon > 0$ small enough such that

$$-\sum_{1\le i<j\le n} \frac{(x-x_i)(x-x_j)}{|x-x_i|^2|x-x_j|^2} \ge 0, \quad \forall x \in \cup_{i=1}^n B_\varepsilon(x_i)? \tag{3.56}$$

Unfortunately, (3.56) is not true. We give a counterexample below (Fig. 3.2).
Let us consider a configuration of three poles x_1, x_2, x_3 determining an equilateral triangle with the vertices at x_i, $i \in \{1,2,3\}$ such that

$$|x_i - x_j| = l > 0, \quad \forall i \neq j, \quad \forall i, j \in \{1,2,3\}.$$

Given $\varepsilon > 0$, we also consider $x_\varepsilon \in \mathbb{R}^3$ located on the line determined by x_1 and x_3 such that $|x_\varepsilon - x_1| = \varepsilon$, $|x_\varepsilon - x_3| = \varepsilon + l$ (as in Fig. 3.2). Then we have

$$|x_\varepsilon - x_2|^2 = \varepsilon^2 + l^2 + \varepsilon l, \quad |x_\varepsilon - x_3| = (\varepsilon + l)^2.$$

In view of this, we can easily obtain that

$$-\sum_{1\le i<j\le 3} \frac{(x_\varepsilon-x_i)(x_\varepsilon-x_j)}{|x_\varepsilon-x_i|^2|x_\varepsilon-x_j|^2} < 0,$$

for $\varepsilon > 0$ small enough, fact which contradicts (3.56).

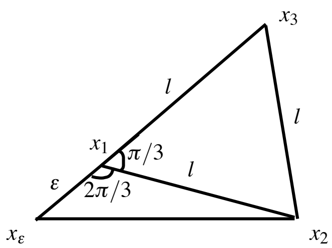

Fig. 3.2 Counterexample to (3.56)

More general, one can show that there is no configuration $x_1,\dots,x_n$ for which (3.56) is true. The condition (3.56) is violated at the singular poles x_{k_i}, with $\{k_i \mid i \in \{1,\dots,n\}\} \subset \{1,\dots,n\}$, which are located on the boundary of the smallest convex set containing all the poles $x_1,\dots,x_n$.

Acknowledgements Partially supported by the Grant MTM2011-29306-C02-00 of the MICINN (Spain), project PI2010-04 of the Basque Government, the ERC Advanced Grant FP7-246775 NUMERIWAVES, the ESF Research Networking Program OPTPDE, the grant PN-II-ID-PCE-2011-3-0075 of CNCS-UEFISCDI Romania and a doctoral fellowship from UAM (Universidad Autónoma de Madrid)

References

1. Adimurthi: Best constants and Pohozaev identity for Hardy-Sobolev type operators, preprint.
2. Adimurthi, Sandeep, K.: Existence and non-existence of the first eigenvalue of the perturbed Hardy-Sobolev operator. Proc. Roy. Soc. Edinburgh Sect. A, **132**, 1021Ű-1043, (2002)
3. Adimurthi, Sekar, A.: Role of the fundamental solution in Hardy-Sobolev-type inequalities. Proc. Roy. Soc. Edinburgh Sect. A, **136**, 1111–1130, (2006)
4. Barbatis, G., Filippas, S., Tertikas, A.: A unified approach to improved L^p Hardy inequalities with best constants. Trans. Amer. Math. Soc., **356**, 2169–2196, (2004)
5. Berestycki, H., Esteban, M. J.: Existence and bifurcation of solutions for an elliptic degenerate problem. J. Differential Equations, **134**, 1–25, (1997)
6. Bosi, R., Dolbeault, J., Esteban, M. J.: Estimates for the optimal constants in multipolar Hardy inequalities for Schrödinger and Dirac operators. Commun. Pure Appl. Anal., **7**, 533–562, (2008)
7. Brezis, H., Vázquez, J. L.: Blow-up solutions of some nonlinear elliptic problems. Rev. Mat. Univ. Complut. Madrid, **10**, 443–469, (1997)
8. Caffarelli, L., Kohn, R., Nirenberg, L.: First order interpolation inequalities with weights. Compositio Math., **53**, 259–275, (1984)
9. Catto, I., Le Bris, C., Lions, P.-L.: On the thermodynamic limit for Hartree-Fock type models. Ann. Inst. H. Poincaré Anal. Non Linéaire, **18**, 687–760, (2001)
10. Cazacu, C.: On Hardy inequalities with singularities on the boundary. C. R. Acad. Sci. Paris, Ser. I, **349**, 273–277, (2011)
11. Dolbeault, J., Duoandikoetxea, J., Esteban, M. J., Vega, L.: Hardy-type estimates for Dirac operators. Ann. Sci. École Norm. Sup. (4), **40**, 885–900, (2007)
12. Duyckaerts, T.: A singular critical potential for the Schrödinger operator. Canad. Math. Bull., **50**, 35–47, (2007)
13. Duyckaerts, T.: Inégalités de résolvante pour l'opérateur de Schrödinger avec potentiel multipolaire critique. Bull. Soc. Math. France, **134**, 201–239, (2006)
14. Evans, L. C., Gariepy, R. F.: Measure theory and fine properties of functions. CRC Press, Boca Raton, FL (1992)
15. Felli, V., Marchini, E. M., Terracini, S.: On Schrödinger operators with multipolar inverse-square potentials. J. Funct. Anal., **250**, 265–316, (2007)
16. Felli, V., Terracini, S.: Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity. Comm. Partial Differential Equations, **31**, 469–495, (2006)
17. Felli, V., Terracini, S.: Nonlinear Schrödinger equations with symmetric multi-polar potentials. Calc. Var. Partial Differential Equations, **27**, 25–58, (2006)
18. Filippas, S., Maz′ya, V., Tertikas, A.: On a question of Brezis and Marcus. Calc. Var. Partial Differential Equations, **25**, 491–501, (2006)

19. Frank, W. M., Land, D. J., Spector, R. M.: Singular potentials. Rev. Modern Phys. **43**, 36–98, (1971)
20. García Azorero, J. P., Peral, I.: Hardy inequalities and some critical elliptic and parabolic problems. J. Differential Equations, **144**, 441–476, (1998)
21. Gel′fand, I. M.: Some problems in the theory of quasi-linear equations. Uspehi Mat. Nauk, **14**, 87–158, (1959)
22. Hardy, G. H.: An inequality between integrals. Messenger of Math., **54**, 150–156, (1925)
23. Hardy, G. H., Littlewood, J. E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1988)
24. Hoffmann-Ostenhof, M., Hoffmann-Ostenhof, T., Laptev, A., Tidblom, J.: Many-particle Hardy inequalities. J. Lond. Math. Soc. (2), **77**, 99–114, (2008)
25. Krejčiřík, D., Zuazua, E.: The Hardy inequality and the heat equation in twisted tubes. J. Math. Pures Appl. (9), **94**, 277–303, (2010)
26. Lévy-Leblond, J. M.: Electron capture by polar molecules. Phys. Rev., **153**, 1–4, (1967)
27. Morgan, J. D.: Schrödinger operators whose operators have separated singularities. J. Operator Theory. **1**, 109–115, (1979)
28. Simon B.: Semiclassical analysis of low lying eigenvalues. I. Nondegenerate minima: asymptotic expansions. Ann. Inst. H. Poincaré Sect. A (N.S.), **38**, 295–308, (1983)
29. Tertikas, A.: Critical phenomena in linear elliptic problems. J. Funct. Anal., **154**, 42–66, (1998)
30. Tintarev, K., Fieseler, K. H.: Concentration compactness. Imperial College Press, London, (2007)
31. Vázquez, J. L., Zuazua, E.: The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. J. Funct. Anal., **173**, 103–153, (2000)