Some results and open problems on the controllability of linear and semilinear heat equation

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Abstract

In these Notes we addres the problem of null controllability of heat equations in two different cases: (a) The semilinear heat equation in bounded domains and (b) The linear heat equation in unbounded domains. Concerning the first problem (a) we show that a number of systems in which blow-up arises may be controlled by means of external forces which are localized in an arbitrarily small open set. In the frame of problem (b) we prove that compactly supported initial data may not be driven to zero if the control is supported in a bounded set. This shows that, although the velocity of propagarion in the heat equation is infinite, this is not sufficient to guarantee null controllability properties.

We also include a list of open problems.

1 Introduction

In these notes we present some recent results on the controllability of linear and semilinear heat equations in bounded and unbounded domains. We describe manly the results that have been developed in full detail in joint works with E. Fernández-Cara [FCZ1,2] and S. Micu [MZ1,2].

In order to unify the presentation, let us consider a smooth domain Ω of $\mathbb{R}^n, n \geq 1$. Note that we do not assume Ω to be bounded. In fact, in part of these notes, Ω will be assumed to be the half-line.

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function.

Let ω be an open non-empty subset $\omega \subset \Omega$.

Given $T > 0$, consider the semilinear heat equation:

$$\begin{cases} u_t - \Delta u + f(u) = v 1_\omega & \text{in} \quad \Omega \times (0,T) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

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In (1.1), $u = u(x,t)$ is the *state* which is assumed to be a scalar function, 1_ω denotes the characteristic function of ω so that $1_\omega \equiv 1$ in ω and $1_\omega \equiv 0$ in $\Omega \backslash \bar{\omega}$ and $v = v(x,t)$ is the control.

System (1.1) is a model example of semilinear parabolic equation or system with an exterior source, v, acting on teh system as a control through the subset ω.

We assume ω to be bounded, even when Ω is unbounded. This assumption is natural since we would like to see what is the minimum amount of control that is needed to control the system.

To simplify the presentation we also assume that

$$f(0) = 0. \tag{1.2}$$

Under the assumption (1.2), $u \equiv 0$ is an equilibrium solution of (1.1). Indeed, when (1.2) holds, (1.1) is satisfied with $u \equiv 0$ and $v \equiv 0$.

The *problem of null-controllability* we address is as follows: Given $u_0 \in L^2(\Omega)$ to find a control $v \in L^2(\omega \times (0,T))$ such that the solution u of (1.1) is globally defined in $\Omega \times (0,T)$ and satisfies

$$u(x,T) \equiv o \text{ in } \Omega \tag{1.3}$$

Some comments are in order:

Remark 1.1 (a) Some minimal assumptions are needed on f to guarantee the local well-posedness of system (1.1). All along this paper we shall assume that

$$\mid f'(s) \mid \leq C(| + \mid s \mid^P), a.e.\, s \in \mathbb{R} \tag{1.4}$$

with $C > 0$ and

$$p \leq 1 + 4/n \tag{1.5}$$

so that for every $u_0 \in L^2(\Omega)$ and $v \in L^2(\omega \times (0,T))$ there exists a maximal time of existence $0 < T^* \leq T$ and a unique $u \in C([0,T); L^2(\Omega))$ solution of (1.1) (see [CH1]).

Under these assumptions the following alternative holds:

(i) Either $T^* = T$,

or

(ii) $T^* < T$ and

$$\| u(t) \|_{L^2(\Omega)} \to \infty \text{ as } \nearrow T^*. \tag{1.6}$$

(b) According to comment (a) above we see that the first requirement in the null-controllability problem is guaranteeing that the control v is such that the solution of (1.1) is globally defined in the time interval $[0,T]$.

But, in addition to this, we require the control to be such that (1.3) holds, i.e. the solution reaches the equilibrium configuration at time T.

(c) Note that when null-controllability holds, extending the solution and control for $t \geq T$ as

$$u \equiv 0,\ t \geq T;\ v \equiv 0,\ t \geq T, \tag{1.7}$$

u is a solution of the semilinear heat equation (1.1) for all $t \geq 0$.

(d) If the nonlinearity f satisfies a suitable growth condition ar infinity, the first requirement in the null-controllability property, i.e. the fact that the solution is defined in the whole interval $[0, T]$ is immediately satisfied. More precisely, as shoun in [CH2], if (1.4), (1.5) are satisfied and, moreover, for some $C > 0$,

$$| f(s) | \leq C \mid s \mid (1 + \log \mid s \mid), \tag{1.8}$$

holds for all $s \geq 0$, then, for all $T > 0$, $u_0 \in L^2(\Omega)$ and $v \in L^2(\omega \times (0,T))$, ther exists a unique globally defined $u \in C\left([0,T]; L^2(\Omega)\right)$ solution of (1.1).

■

In there notes we focus on two particular problems:

Problem 1 Null-controllability for the semilinear equation when Ω is bounded.

Problem 2 Null-controllability for the linear equation (with $f \equiv 0$) in one space dimension when $\Omega = (0, \infty)$.

These two choices are justified by the fact that the problem is by now rather well-understood when $f \equiv 0$, i.e. for the linear heat equation, when Ω is bounded.

More precisely. the following is known (see, for instance, [FI] and [LR]): *Let Ω be a bounded smooth domain of $\mathbb{R}^n$, $n \geq 1$ and let ω be any open non-empty subset of Ω. Let $T > 0$. Then, for any $u_0 \in L^2(\Omega)$ there exists $v \in L^2(\omega \times (0,T))$ such that the solution u of*

$$\begin{cases} u_t - \Delta u = v 1_\omega & in \quad \Omega \times (0,T) \\ u = 0 & on \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & in \quad \Omega, \end{cases} \tag{1.9}$$

satisfies (1.3).

In view of this result it is natural to analyze the two problems above. In Problem 1 we intend to analyze to which extent the null-controllability is kept in the presence of non-linear terms. In Problem 2, in view of the infinite speed of propagation involved in the heat equation, we intend to see whether the null-controllability is kept when Ω is an unbounded domain, ω being bounded.

In Section 2 we analyze Problem 1 and we present the main results in [FCZ1, 2, 3]. Section 3 is devoted to Problem 2 and the results in [MZ1, 2] are presented.

There is a large literature on controllability problems for heat equations. Here we focus on two particular aspects of the theory. The interested reader may learn more from the bibliography at the end of the paper.

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2 Null-controllability for the semilinear heat equation

2.1 Main results

The first main result in [EFC2] is a negative one:

Theorem 2.1 *There exist smoothe non-linearities such that* $f(0) = 0$ *and* (1.4)-(1.5) *with*

$$| f(s) | \sim | s | \log^P | s | \text{ as } | s | \to \infty \tag{2.1}$$

and

$$p > 2 \tag{2.2}$$

for which system (1.1) *fails to be null-controllable for all* $T > 0$.

According to this result if one wants to get null-controllability properties for all non-linearities in a suitable class, roughly, one has to impose a growth condition of the form

$$| f(s) | \leq C | s | \log^2 | s |, \text{ as } | s | \to \infty. \tag{2.3}$$

The second main result in [EFC2] guarantees the null-controllability under a stronger restriction on the growth rate.

Theorem 2.2 *Let* f *be a locally Lipschitz function such that* $f(0) = 0$ *and* (1.4)-(1.5) *holds.*
Assume that

$$\limsup_{|s| \to \infty} \frac{f(s)}{| s | \log^{3/2} | s |} = 0. \tag{2.4}$$

Then, for every $T > 0$*, system* (1.1) *is null-controllable.*

Whether system in null-controllable or not when f behaves ar infinity in the range

$$C_1 | s | \log^{3/2} | s | \leq | f(s) | \leq C_2 | s | \log^2 | s |, \text{ as } | s | \to \infty \tag{2.5}$$

is an interesting open problem.

As for as we know, in the blow-up literature there is no evidence of the faillure of null-controllability for f as in (2.3) (see subsection 2.2 below) but the technique of proof we employ for Theorem 2.2 fails when (2.5) holds with $C_1 > 0$ large enough (see subsection 2.3).

2.2 Sketch of the proof of Theorem 2.1

We consider the nonlinearity

$$f(s) = \int_0^{|s|} \log^P(1+\sigma)d\sigma, \quad \forall s \in \mathbb{R} \tag{2.6}$$

with $p > 2$.

We introduce the convex conjugate f^* and we check that

$$f^*(s) \sim p \mid s \mid^{1-1/p} \exp\left(\mid s \mid^{1/p}\right), \text{ as } \mid s \mid \to \infty. \tag{2.7}$$

Let us assume for the moment that there exists a function $\rho \in \mathcal{D}(\Omega)$ such that $\rho = 0$ in a neighborhood of ω, $\rho \geq 0$ $\int_\Omega \rho dx = 1$ and

$$\rho f^*(2 \mid \Delta\rho \mid /\rho) \in L^1(\Omega). \tag{2.8}$$

We shall return to condition (2.8) later on.

Multiplying by ρ in (1.1) and integrating in Ω it follows, after integration by parts, that

$$\frac{d}{dt}\int_\Omega \rho u dx = \int_\Omega \rho \Delta u dx - \int_\Omega \rho f(u) dx. \tag{2.9}$$

Note that in (2.9) the control v does not appear. This is due to the fact that

$$\int_\Omega \rho v 1_\omega dx = 0$$

by the condition that $\rho \equiv 0$ in ω.

Applying Young and Jensen's inequalities and using the fact that $f(s) = f(\mid s \mid)$ we obtain

$$\frac{d}{dt}\left[-\int_\Omega \rho u dx\right] = -k + \frac{1}{2} f\left[-\int_\Omega \rho u dx\right] \tag{2.10}$$

where

$$k = \frac{1}{2}\int_\Omega \rho f^*(2 \mid \Delta\rho \mid /\rho) dx. \tag{2.11}$$

It is easy to see that, if $-\int \rho u_0 dx$ is large enough, the solution of (2.10) blows-up in fimite time. Moreover, given any $0 < T < \infty$, by taking $-\int_\Omega \rho u_0 dx$ one can guarantee that the solution of (2.6) is in the range in which blow-up occurs since $p > 1$ (in fact $p > 2$).

Taking into account that there are initial data u_0 for which, whatever the control v is, the solution blows-up in time $\leq T$, it is clear that the statement of Theorem 2.1 holds.

Note that, at least apparently, we have not used so far the fact that $p > 2$. But this condition is needed to ensure that (2.8) holds. Indeed, let us analyze (2.8) in the one-dimensional case of course, the only difficulty for (2.8) to be true is at the points where ρ vanishes. Assume for instance that ρ vanishes at $x = 0$. If ρ is heat enough, of the order of

$$\rho(x) = \exp(-x-^{m})$$

we have

$$\rho f^*(2 \mid \Delta\rho \mid /\rho) \sim pm^{2(1-1/p)} x^{-(2m+2)(p-1)/p} \exp\left(n^{2/p} x^{-(2m+2)/p}\right) x \exp(-x^{-m})$$

which is bounded as $x \to 0^+$ provided

$$m > (2m+2)/p.$$

Of course, such a choice of $m > 0$ is always possible when $p > 2$, but not otherwise.

This concludes the sketch of the proof of Theorem 2.1. We refer to [FCZ2] for more details.

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Remark 2.1 We did not check that (2.8) as soon as $\mid f(s) \mid \sim \mid s \mid \log p \mid s \mid$ for $p \leq 2$. However, the existing results on the blow-up literature (see e.f. [G] and [GV]) shows that when f is as above and $1 < p < 2$ solutions of (1.1) blow-up everywhere in Ω and when $p = 2$ solutions of (1.1) (with $v \equiv 0$) blow-up in open subsets of Ω. Taking into account that the result we have actually proved is only compatible with single point blow-up we conclude that the growth rate that Theorem 2.1 proves should be sharp.

But further clasification of this issue by a complete analysis of condition (2.8) would be desirable.

2.3 Sketch of the proof of Theorem 2.2

Here we briefly describe the main steps of the proof of Theorem 2.2. We refer to [EFC2] for a complete proof.

Step 1. Description of the fixed point method

To simplify the presentation we assume that $u_0 \in D_0^{0,\alpha}(\bar{\Omega})$ for some $\alpha > 0$ and $f \in C^1(\mathbb{R})$.

We fix the initial datum u_0 and the control time $T > 0$.

We then introduce the function

$$g(s) = \begin{cases} f(s)/s, & \text{if} \quad s \neq 0 \\ f'(0) & \text{if} \quad s = 0. \end{cases} \tag{2.12}$$

We rewrite system (1.1) as

$$\begin{cases} u_t - \Delta u + g(u)u = v 1_\omega & \text{in} \quad \Omega \times (0,T) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{2.13}$$

For any $z \in X = \{z \in C(\bar{\Omega} \times [0,T]) : z = 0 \text{ on } \partial\Omega \times (0,T)\}$ we introduce the linearized control problem:

$$\begin{cases} u_t - \Delta u + g(z)u = v1_\omega & \text{in} \quad \Omega \times (0,T) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{2.14}$$

As we shall see, there exists a control $v \in L^\infty(\omega \times (0,T))$ for system (2.14) such that its solution u satisfies

$$u(T) = 0 \text{ in } \Omega. \tag{2.15}$$

Moreover, the following bound on v holds: There exists $C > 0$ such that

$$\| v \|_{L^\infty(\omega\times(0,T))} \leq C \exp\left(C \left(+ \| g(z) \|_\infty^{2/3} \right) \right) \| u_0 \|_{L^\infty(\Omega)} . \tag{2.16}$$

In this way we build a nonlinear map $\mathcal{N} : X \to X$ such that $u = \mathcal{N}(z)$ where u is the solution u of (2.14) satisfying (2.15) with the control v verifying the bound (2.16).

It is easy to see that the map $\mathcal{N} : X \to X$ is continuous and compact.

On the other hand, we observe that u solves (2.13) when u is a fixed point of $\mathcal{N}$. Thus, it is sufficient to prove that $\mathcal{N}$ has a fixed point.

We apply Schauder's fixed point Theorem. To do this we have to show that

$$\| \mathcal{N}(z) \|_\infty \leq R, \quad \forall z \in X \ : \| z \|_\infty \leq R \tag{2.17}$$

for a suitable R.

In view of (2.17), using classical energy estimates and the fact that, as a consequence of (2.4),

$$\limsup_{|s|\to\infty} \frac{| g(s) |}{\log^{3(2|s|)}} = 0 \tag{2.18}$$

we deduce that (2.18) for $R > 0$ large enough.

Therefore the problem is reduced to prove the existence of the control v for (2.14) satisfying (2.16).

Step 2. Control of the linearized equation

To analyze the controllability of the linearized equation (2.14) and in order to simplify the notation we set

$$a = g(z). \tag{2.19}$$

System (2.14) then takes the form

$$\begin{cases} u_t - \Delta u + au = v1_\omega & \text{in} \quad \Omega \times (0,T) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{2.20}$$

To analyze the controllability of (2.20) we consider the adjoint system

$$\begin{cases} \varphi_t - \Delta\varphi = a\varphi & \text{in} \quad \Omega \times (0,T) \\ \varphi = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ \varphi(T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.21}$$

The following observability inequality holds:

Lemma 2.1 *There exists a constant $C > 0$ such that*

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq \exp\left(C \left(T + \frac{1}{T} + \left(\sqrt{T} + T \right) \| a \|_\infty + \| a \|_\infty^{2/3} \right) \right) \left(\int_0^T \int_\omega | \varphi | \, dxdt \right)^2 \tag{2.22}$$

for every solution of (2.21) *for all $c \in L^\infty(\Omega \times (0,T))$ and all $T > 0$.*

This observability inequality has been proved in [EFC2] as a refinement of those in [EFC1] in which on the right hand side of (2.22) we had $\| \varphi \|^2_{L^2(\omega\times(0,T))}$ insted of $\| \varphi \|^2_{L^1(\omega\times(0,T))}$. The main ingredient of the proof of (2.22) is the *Global Carleman Inequality* in [FI].

As a consequence of Lemma 2.1, by duality, the following holds:

Lemma 2.2 *Given any $T > 0$, $a \in L^\infty(\Omega \times (0,T))$ and $u_0 \in L^2(\Omega)$ there exists a control $v \in L^\infty(\omega \times (0,T))$ such that the solution u of* (2.20) *satisfies* (2.15).

Moreover, we have the following bound on v:

$$\| v \|_{L^\infty(\omega\times(0,T))} \leq \exp\left(C \left(T + \frac{1}{T} + \left(\sqrt{T} + T \right) \| a \|_\infty + \| a \|_\infty^{2/3} \right) \right) \| u_0 \|_{L^2(\Omega)} \tag{2.23}$$

where $C > 0$ is a constant that only depends on Ω and ω.

In a first approach (2.23) does not imply (2.16). Indeed, in (2.23) the leading term in what concerns the growth rate of the observability constant as $\| a \|_\infty \to \infty$ is of the order of $\exp\left(C \left(\sqrt{T} + T \right) \| a \|_\infty \right)$. However, note that condition (2.15) is also satisfied if u verifies

$$u(\widehat{T}) = 0 \text{ in } \Omega$$

for some $\widehat{T} \leq T$ and the control v is extended by zero to the interval $[\widehat{T}, T]$. Obviously, one always choose $\widehat{T}$ small enough so that

$$\widehat{T} + \frac{1}{\widehat{T}} + \left(\sqrt{\widehat{T}} + \widehat{T} \right) \| a \|_\infty$$

is bounded above by $C \| a \|_\infty^{2/3}$ for all $a \in L^\infty(\Omega \times (0,T))$ with $C > 0$ independent of a.

This is the key remark in the proof of (2.16) and therefore when proving Theorem 2.2. This strategy is in agreement with common sense: In order to avoid the blow-up phenomena to occur, we control the system fast, before the blow-up mechanism is developed.

This concludes the sketch of the proof of Theorem 2.2.

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Remark 2.2

(a) It may be proved that (2.22) may be improved to obtain a global bound on φ provided we introduce a weight vanishing at $t = T$. Indeed, one can get inequalities of the form (2.22) with $\| \varphi(0) \|^2_{L^2(\Omega)}$ on the left hand side replaced by the weighted global quantity

$$\int_0^T \int_\Omega e^{-\gamma/(t-T)} \varphi^2 dx dt. \tag{2.24}$$

We refer to section 4 for a discussion on the best constant $\gamma > 0$ in (2.24).

(b) Analyzing the proof of Theorem 2.2 one sees that the main obstacle to improve the growth condition (2.4) in Theorem 2.2 is the presence of the factor $\exp\left(C \| a \|_\infty^{2/3}\right)$ in the observability inequality. Indeed, if we had $\exp\left(C \| a \|^{1/p}\right)$ instead of $\exp\left(C \| a \|_\infty^{2/3}\right)$ with $p > 3/2$, then one would be able to extend the null-controllability result of Theorem 2.2 to nonlinearities satisfying the weakened growth condition

$$\limsup_{|s| \to \infty} \frac{| f(s) |}{| s | \log^p | s |} = 0.$$

However, this seems to be ont of reach with the L^2-Global Carleman Inequalities in [FI].

We shall return to this open problem in Section 4.

3 Lack of null-controllability for the heat equation on the half line

3.1 Main result

In this section we discuss the following one-dimensional control problem:

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \infty, \quad 0 < t < T \\ u(0,t) = v(t), & 0 < t < T \\ u(x,0) = u_0(x), & 0 < x < \infty. \end{cases} \tag{3.1}$$

Here $u = u(x,t)$ is the state and $v = v(t)$ is the control that acts on the system through the extreme $x = 0$.

The main difference of system (3.1) with respect to those that we discussed above is that the equation holds in the infinite domain $\Omega = (0, \infty) = \mathbb{R}_+$.

We consider the boundary control problem for simplicity. But the same negative results hold when the control acts on a bounded subinterval of $(0, \infty)$.

The null-controllability problem may be formulated as follows: *Given* $u_0 \in L^2(0,\infty)$*, to find* $v \in L^2(0,T)$ *such that the solution* u *of* (3.1) *satisfies*

$$u(x,T) = 0, \quad \forall x > 0. \tag{3.2}$$

In order to make the formulation of the problem precise we have to clearly identify the solution of (3.1) we are working with. We shall analyze the solution defined by transposition that turns out to be in $C\left([0,T];\, H^{-1}\left(\mathbb{R}_+\right)\right)$ (see [L] and [MZ1]). There are however other smooth solutions of (3.3). They grow very fast as $\mid x \mid \to \infty$ and therefore they are not achievable by transposition (see [J] and [MZ1].

Ler us now analyze the observability inequality corresponding to this null-control problem. Consider the adjoint system

$$\begin{cases} \varphi_t + \varphi_{xx} = 0, & 0 < x < \infty, \quad 0 < t < T \\ \varphi(0,t) = 0, & 0 < t < T \\ \varphi(x,T) = \varphi_0(x), & 0 < x < \infty. \end{cases} \tag{3.3}$$

Let us analyze the problem of whether there exists $C > 0$ such that

$$\| \varphi(0) \|^2_{L^2(\mathbb{R}_+)} \leq C \int_0^T \mid \varphi_x(0,t) \mid^2 dt \tag{3.4}$$

holds for every solution of (3.3).

If (3.4) holds it is easy to see that the sytem is null-controllable. However, it is easy to see that (3.4) is not true. Indeed, given $\varphi_0 \in L^2(\mathbb{R}_+)$ we consider the sequence of initial data $\varphi_0^k(x) = \varphi_0(x-k)$. Let φ^k be the solution of (3.3) associated with this datum. It is easy to see that $\| \varphi^k(0) \|_{L^2(\mathbb{R}_+)}$ is bounded below by a positive constant while $\int_0^T \left|\varphi_x^k(0,t)\right|^2 dt$ tende to zero as $k \to \infty$.

However, one could think that a variant of (3.4) in which the $L^2(\mathbb{R}_+)-$norm of $\varphi(x,0)$ is replaced by a suitable weighted norm could be true. But the situation is much wose then that. Indeed, assume that there exists a positive smooth function ρ such that the following weighted observability inequality holds

$$\| \varphi(0) \|^2_{L^2(\mathbb{R}_+;\rho)} \leq C \int_0^T \mid \varphi_x(0,t) \mid^2 dt \tag{3.5}$$

for every solution of (3.3). Here we are using the notation

$$\| f \|^2_{L^2(\mathbb{R}_+;\rho)} = \int_0^\infty f^2 \rho(x) dx.$$

If (3.5) were true, we would deduce by duality that for any $u_0 \in L^2(\mathbb{R}_+;\rho^{-1})$ there exists a control $v \in L^2(0,T)$ such that the solution of (3.1) satisfies (3.2).

But, as the following result from [MZ1] shows, there is no positive weight ρ for which null-controllability holds in $L^2(\mathbb{R}_+;\rho^{-1})$. Consequently the observability inequality (3.5) does not hold:

Theorem 3.1 *There is no $u_0 \in \mathcal{D}(\mathbb{R}_+)$, $u_0 \not\equiv 0$ for which there exists $v \in L^2(0,T)$ such that the solution of* (3.1) *satisfies* (3.1).

This result is in strong contrast with those of the case where the domain Ω is bounded. Even if the information propagates at infinite speed on the heat equation the null-control condition is not achievable for any nice smooth and compactly supported initial data.

In a first reading this result might seen to be in contradiction with those in [J] that, in particular, guarantee that: *For any bounded and continuous u_0 there exists a continuous control $v \in C([0,T])$ and a smooth solution of* (3.1) *such that* (3.2) *holds.*

However, both results are compatible. The solution in [J] is not the "physical" solution in the sense of transposition but another one that grows extremely fast as $| x | \to \infty$.

In the following sections we give a sketch of the proof of Theorem 3.1. We refer the interested reader to [MZ1] for the details. We first reformulate the control problem as a moment problem. When doing so we use in an essential way the similarity variables and the weighted Sobolev spaces introduced in [Ek] when analyzing asymptotic properties of solutions of semilinear heat equations. We show that moment problem is critical and, in fact, that one may not expect a L^2-solution for "nice" data.

More precisely, the control problem turns out to be equivalent to the following moment problem: Given $\mathcal{S} > 0$ and a sequence $\{a_m\}_{m\geq 1}$ to find $v \in L^2(0, \mathcal{S})$ such that

$$\int_0^{\mathcal{S}} v(s) e^{ms} ds = a_m, \ \forall_m \geq 1. \tag{3.6}$$

As we shall see, it terms out that (3.6) may only have a non-trivial solution $v \in L^2(0,T)$ when the coefficients $\{a_m\}_{m\geq 1}$ grow exponentially fast at infinity. In fact, the following holds:

Theorem 3.2 *Let $S > 0$ be given. Assume that*

$$\lim_{m\to\infty} \frac{| a_m |}{e^{m\delta}} = 0 \tag{3.7}$$

for all $\delta > 0$. Then, if (3.6) *admits a non-trivial solution $v \in L^2(0, \mathcal{S})$, necessarily $a_m = 0$, for all $m \geq 1$.*

As a consequence of Theorem 3.2, Theorem 3.1 holds immediately since, roughly speaking, $\{a_m\}$ are the Fourier coefficients of the datum to be controlled and, when the initial datum is C^∞ and of compact support, the sequence $\{a_m\}_{m\geq 1}$ is bounded and (3.7) holds for all $\delta > 0$.

In [MZ1], using Pailey-Wiener's Theorem we prove that there are exponentially growing coefficients $\{a_m\}$ for which (3.6) admits a solution $v \in L^2(0, S)$. As an immediate consequence we deduce the existence of initial data with exponentially growing Fourier coefficients that are null-controllable.

In [MZ2] we have extended these results to the case of the boundary control in the half space of $\mathbb{R}^n$. The similarity variables apply in any conical domain but our analysis does not apply in such a general setting. We shall return to this question in section 4.

The rest of this section is organized as follows. In subsection 3.2 we describe the similarity variables and reduce the control problem to the moment problem (3.6). In subsection 3.3 we prove the negative result of Theorem 3.2 on the moment problem.

3.2 Reduction of the control problem

We introduce the similarity variables

$$y = x \Big/ \sqrt{t+1}\ ,\ s = \log(t+1) \tag{3.8}$$

and the new unknown:

$$w(y,s) = e^{s/2} u \left(e^{s/2}, e^s - 1 \right). \tag{3.9}$$

Then, u solves (3.1) if and only if w solves

$$\begin{cases} w_s - w_{yy} - \frac{1}{2} y w_y - \frac{1}{2} w = 0, & y > 0, \quad 0 < s < S \\ w(0,s) = \widetilde{v}(s), & 0 < s < S \\ w(y,0) = u_0(y), & y > 0 \end{cases} \tag{3.10}$$

with

$$\widetilde{v}(s) = e^{s/2} v(e^s - 1) \tag{3.11}$$

and

$$S = \log(T+1). \tag{3.12}$$

Obviously $v \in L^2(0,T)$ if and only if $\widetilde{v} \in L^2(0,S)$. Consequently, the null controllability of system (3.1) and (3.10) are equivalent properties. Therefore, in the sequel we shall analyze the null controllability of system (3.10).

Let us now analyze the elliptic operator involved in (3.10). We set

$$Lw = -w_{yy} - \frac{1}{2} y w_y. \tag{3.13}$$

Observe that the operator L may also be written in the symmetric form

$$Lw = -\frac{1}{k}(k w_y) y, \tag{3.14}$$

where

$$K(y) = \exp(y^2/4). \tag{3.15}$$

Therefore, in order to analyze (3.10), it is natural to introduce the following weighted spaces:

- $L^2(\mathbb{R}_+; k) = \left\{ f : \mathbb{R}_+ \to \mathbb{R} : \int_0^\infty f^2 k(y) dy < \infty \right\}$
- $H`1_0(\mathbb{R}_+; k) = \left\{ f \in L^2(\mathbb{R}_+; k) : f_y \in L^2(\mathbb{R}_+; k)\,,\ f(0) = \right\}$.

These spaces, endowed with their canonical norms are Hilbert spaces. We also introduce the dual of $H_0^1(\mathbb{R}_+; k)$ that we shall denote by $H^{-1}(\mathbb{R}_+; k)$.

The following properties are well known (see [Ek] and [MZ1]):

- $\int_0^\infty f^2 k(y) dy \leq 16 \int_0^\infty \mid f_y \mid^2 k(y) dy,\ \forall f \in H_0^1(\mathbb{R}_+; k)$;

- the embedding $H_0^1(\mathbb{R}_+;k) \hookrightarrow L^2(\mathbb{R}_+;k)$ is compact;
- $L : H_0^1(\mathbb{R}_+;k) \hookrightarrow H^{-1}(\mathbb{R}_+;k)$ is an isomorphism;
- the domain of L as unbounded operator in $L^2(\mathbb{R}_+;k)$ is $D(L) = H^2(\mathbb{R}_+;k) \cap H_0^1(\mathbb{R}_+;k)$;
- the eigenvalue of L are $\lambda_j = j$, $j \geq 1$ and the corresponding eigenfunctions

  $$\phi_j = d^{2j-1}[\phi]/dy^{2j-1},\ j \geq 1$$

  where $\phi = k^{-1}$.
- The eigenfunctions may be normalized to constitute an orthonormal basis of $L^2(\mathbb{R}_+;k)$.

$$\begin{cases} \xi_s + \xi_{yy} + \frac{y}{2}\xi_y + \frac{1}{2}\xi = h, & y > 0, \quad 0 < s < S \\ \xi(0,s) = 0, & 0 < s < S \\ \xi(y,\mathcal{S}) = \xi_0(y), & y > 0. \end{cases} \tag{3.16}$$

Multiplying in (3.10) by ξk and integrating by parts (formally by now), we deduce that

$$\int_0^{\mathcal{S}} \int_{\mathbb{R}_+} whkdyds = \int_{\mathbb{R}_+} w(y,\mathcal{S})\xi_0(y)dy - \int_{\mathbb{R}_+} w_0(y)\xi(y,0)k(y)dy + \int_0^{\mathcal{S}} \xi_y(0,s)v(s)ds. \tag{3.17}$$

We adopt as definition of solution of (3.10) in the sense of transposition. More precisely, given $w_0 \in L^2(\mathbb{R}_+;k)$ and $v \in L^2(0,s)$, $w \in C\left([0,S]; H^{-1}(\mathbb{R}_+;k)\right)$ is said to be solution of (3.10) in the sense of transposition if (3.17) hold for every solution ξ of (3.16) with $\xi_0 \in H_0^1(\mathbb{R}^+;k)$ and $h \in L^1\left(0,\mathcal{S}; H_0^1(\mathbb{R}_+;k)\right)$. We refer to [MZ1] for the proof of the existence and uniqueness of this solution.

In the sequel when discussing the null-controllability of system (3.10) we will be studying the null-controllability of the solution of (3.10) defined, as above, by transposition.

With the precise definition of solution in mind we can transform the control problem into a moment problem. Indeed, according to (3.17), if $w(s) \equiv 0$ we have

$$\int_0^{\mathcal{S}} \int_{\mathbb{R}_+} whkdyds = \int_0^{\mathcal{S}} \xi_y(0,s)v(s)ds - \int_{\mathbb{R}_+} w_0(y)\xi(y,0)k(y)dy \tag{3.18}$$

for all solution of (3.16) with $\xi_0 \in H_0^1(\mathbb{R}_+;k)$ and $h \in L^1\left(0,\mathcal{S};\, H_0^1(\mathbb{R}_+,k)\right)$. By taking $h \equiv 0$ and $\xi_0 = \phi_j$ and taking into account that the solution of ξ is then

$$\xi(y,s) = e^{-(j-1/2)(\mathcal{S}-s)}\phi_j(y)$$

we deduce that, if

$$w_0 = \sum_{j \geq 1} a_j \phi_j(y),$$

then (3.18) if and only if

$$\int_0^{\mathcal{S}} v(s)e^{(j-1/2)s}ds = a_j \Big/ \phi_j(0), \forall_j \geq 1. \tag{3.19}$$

On the other hand, it can be seen that

$$| \phi_j'(0) | \sim \frac{1}{\sqrt{\pi}} \sqrt[4]{j}, \text{ as } j \to \infty.$$

Consequently, in what concerns the statement of Theorem 3.2, the moment problems (3.6) and (3.18) are equivalent. The functions v in (3.19) and (3.6) are related by the multiplicative factor $e^{s/2}$.

We conclude this section proving the main result on the moment problem.

3.3 Proof of the main result on the moment problem

This subsection is devoted to prove Theorem 3.2. We follow very closely a method by Rill-Nardzewski described in [Y, p. 166-167].

First of all we observe that for any $g \in L^2(0, \mathcal{S})$ the following identity holds:

$$\lim_{x\to\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_0^{\mathcal{S}} e^{kx(s-u)} g(u)du = \int_0^s g(u)du. \tag{3.20}$$

To prove (3.20) ir is sufficient to observe that

$$\sum_{k=1}^{\infty} \frac{(1-)^{k-1}}{k!} \int_0^{\mathcal{S}} e^{kx(s-u)} g(u)du = \int_0^s \left\{1 - \exp\left[-e^{x(s-u)}\right]\right\} g(u)du$$

and to pass to the limit on the last integral as $x \to \infty$ by means of the dominated convergence Theorem.

Using this identity it is easy to deduce that if $v \in L^2(0, \mathcal{S})$ is such that

$$\left| \int_0^{\mathcal{S}} v(u)e^{ju}du \right| \leq C_\delta e^{j^\delta}, \forall_j \geq 1 \tag{3.21}$$

then, necessarily, the support of v is contained in $[0, \delta]$. Indeed, according to (3.20), we have

$$\lim_{j\to\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_0^{\mathcal{S}} e^{k_j(s-u)} v(\mathcal{S}-u)du = \int_0^s v(\mathcal{S}-u)du \tag{3.22}$$

but, in view of (3.21),

$$\left| \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_0^{\mathcal{S}} e^{k_j(s-u)} v(\mathcal{S}-u) \right| \leq \sum_{k=1}^{\infty} \frac{1}{k!} e^{k_j(s-u)} \left| \int_0^{\mathcal{S}} e^{k_j\tau} v(\tau)d\tau \right|$$

$$\leq C_\delta \sum_{k=1}^{\infty} \frac{1}{k!} e^{k_j(s-\mathcal{S})} e^{k_j\delta} = C_\delta \left[\exp\left[e^{j(s-\mathcal{S}+\delta)}\right] - 1\right].$$

Taking into account that the last expression tends to zero as $j \to \infty$ for $s \leq \mathcal{S} - \delta$ we deduce that

$$\int_0^s v(\mathcal{S} - u) = 0, \, \forall s \leq \mathcal{S} - \delta$$

and consequently

$$v(S - s) = 0, \, \forall s \leq \mathcal{S} - \delta$$

i.e.

$$v(s) = 0, \, \forall s \geq \delta. \tag{3.23}$$

According to (3.23), we deduce that if (3.21) holds for all $\delta > 0$, then $v(s) = 0$ for all $s \geq 0$.

4 Open problem

In this section we present some open problems related to the issues we have addressed above.

4.1 Sharp constants on the observability inequality for the constant coefficient heat equation

Consider the linear, constant coefficient heat equation

$$\begin{cases} \varphi_t - \Delta\varphi = 0 & \text{in} \quad \Omega \times (0,T) \\ \varphi = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ \varphi(x,0) = \varphi^0(x) & \text{in} \quad \Omega \end{cases} \tag{4.1}$$

where Ω is a bounded smooth domain, and $T > 0$. Let ω be an open non-empty subset of Ω.

As mentioned above (see section 2), there exist positive constant $\gamma > 0$ and $C > 0$ such that

$$\int_0^T \int_\Omega e^{-\gamma/t} \varphi^2 dx dt \leq C \int_0^T \int_\omega \varphi^2 dx dt \tag{4.2}$$

for every solution of (4.1). It would be interesting to characterize the best constant γ in (4.2). This problem might be easier to solve when $T = 0$. Note that (4.2) still holds for $T = 0$ and that both integrals converge since the L^2-norm of solutions of (4.2) tends exponentially to zero as time foes to infinity. The inequality then reads as

$$\int_0^\infty \int_\Omega e^{-\gamma/t} \varphi^2 dx dt \leq C \int_0^\infty \int_\omega \varphi^2 dx dt, \tag{4.3}$$

and characterizing the best constant $\gamma > 0$ in (4.3) is an interesting open problem.

Note that solutions of (4.1) may be written in Fourier series as

$$\varphi = \sum_{k \geq 1} a_k e^{-\lambda_k t} \phi_k(x)$$

where $\{a_k\}$ are the Fourier coefficients, $\{\lambda_k\}_{k\geq 1}$ the eigenvalues of $-\Delta$ with Dirichlet boundary conditions and $\{\phi_k\}_{k\geq 1}$ an orthonormal basis of $L^2(\Omega)$ constituted by eigenfunctions. Then

$$\| \varphi(t) \|^2_{L^2(\Omega)} = \sum_{k\geq 1} a_k^2 e^{-2\lambda_k t}$$

and

$$\int_0^\infty \int_\Omega e^{-\gamma/t}\varphi^2 dxdt = \sum_{k\geq 1} a_k^2 \int_0^\infty e^{-2\lambda_k t-\gamma/t}dt.$$

On the other hand

$$\int_\omega \varphi^2 dxdt = \sum_{k,j\geq 1} a_k a_j \int_0^\infty e^{-(\lambda_k+\lambda_j)t}dt \int_\omega \phi_k\phi_j dx = \sum_{k,j\geq 1} \frac{a_k a_j}{\lambda_k+\lambda_j}\int_\omega \phi_k\phi_j dx.$$

Thus, getting the best constant $\gamma > 0$ in (4.3) is equivalent to finding it so that the inequality

$$\sum_{k\geq 1} a_k^2 \int_0^\infty e^{-2\lambda_k t-\gamma/t}dt \leq C \int_{k,j\geq 1} \frac{a_k a_j}{\lambda_k+\lambda_j}\int_\omega \phi_k\phi_j dx \tag{4.4}$$

holds for all $\{a_k\} \in \ell^2$.

It is not difficult to get a lower bound on the constant $\gamma > 0$ in (4.2) and (4.3). Indeed, as shoun in [FCZ1], if we set

$$\varphi = \cos\left(\frac{\rho x_1}{2t}\right)\exp\left(\rho^2/4t\right)G(x,t), \tag{4.5}$$

where G is the fundamental solution of the heat equation

$$G(x,t) = (4\pi t)^{-n/2}\exp\left(- \mid x \mid^2 /4t\right),$$

then φ is a very singular solution of the heat equation in $\mathbb{R}^n \times (0,\infty)$.

Let $\bar{B}(x_0,\rho)$ a ball contained in $\Omega\backslash\omega$ and consider $\psi(x) = \varphi(x-x_0,t)$, with φ as in (4.5) for this value of ρ. It can be checked that ψ is bounded on $\partial\Omega \times (0,\infty)$ and in $\omega \times (0,\infty)$ but $\| \psi(t) \|_{L^2(\Omega)}$ increases exponentially as $t \to 0^+$. This solution ψ may be modified on the boundary so that it satisfies the Dirichlet homogeneous boundary condition, without changing the other properties. In this way we see that the constant $\gamma > 0$ in (4.2) and/or (4.3) necessarily satisfies $\gamma > \rho^2/4$.

Therefore, we have that the following lower bound on γ:

$$\gamma \geq \sup_{\bar{B}(x_0,\rho)C\Omega\backslash\bar{\omega}} \rho^2/4.$$

This lower bound may be sharp in radially symmetric geometries. But this remains to be proved.

4.2 Sharp observability constant for the heat equation plus a potential

Let us consider now the linear heat equation with a bounded potential

$$\begin{cases} \varphi_t - \varphi - xx + a\varphi = 0 & \text{in} \quad \Omega \times (0,T) \\ \varphi = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ \varphi(x,0) = \varphi_0(x) & \text{in} \quad \Omega, \end{cases} \tag{4.6}$$

where $a = a(x,t) \in L^\infty(\Omega \times (0,T))$.

As we have shown above, given a non-empty open subset $\omega \subset \Omega$ there exists a positive constant $C = C(\omega, \Omega) > 0$ such that the following observability inequality holds

$$\| \varphi(T) \|^2_{L^2(\Omega)} \leq \exp\left[C \left(T + \frac{1}{T} + T \| a \|_\infty + \| a \|_\infty^{2/3} \right) \right] \int_0^T \int_\omega \varphi^2 dxdt \tag{4.7}$$

for every solution of (4.6), for all $T > 0$ and all potential $a \in L^\infty(\Omega \times (0,T))$.

We have also seen that the exponential factor $\exp\left(C \| a \|_\infty^{2/3} \right)$ in the observability inequality (4.7) is the main obstacle for proving the null-controllability of the semilinear heat equation for nonlinearities that grow at infinity faster that $| s | \log^{3/2} | s |$. However, there is no evidence of the lack of null controllability in the class of non-linearity that grow at infinity shower that $| s | \log^2(s)$.

This suggests that, the factor $\exp\left(C \| a \|_\infty^{2/3} \right)$ could possibly be replaced by $\exp\left(C \| a \|_\infty^{1/2} \right)$. However, as we have shown above, this does not seem possible to do with the global Carleman inequalities we have used. Note however that these are Carleman inequalities in $L^2(\Omega \times (0,T))$. One could think on Carleman inequalities in spaces $L^p(0,T; L^q(\Omega))$. This could lead to an improvement of the factor $\exp\left(C \| a \|_\infty^{2/3} \right)$ and consequently to a relaxation of the growth condition on the nonlinearity ensuring null-controllability.

4.3 Relaxing the growth condition on the nonlinearity ensuring null-controllability

As we have mentioned in problem 4.2 above, there is no evidence that null-controllability fails for non-linearities that grow at infinity less that $| s | \log^2 | s |$. In fact there is no evidence of faillure of null-controllability for nonlinearities f such that $\sqrt{| r(s) |}$ is not integrable at infinity, F being the primitive of f, i.e. $F(z) = \int_0^z f(s)ds$.

However, by now, we only have a proof of null-controllability under the condition that

$$| f(s) | << | s | \log^{3/2} | s | \quad \text{as} \quad | s | \to \infty \tag{4.8}$$

or, under the slightly weaker condition that

$$\limsup_{|s| \to \infty} \frac{| f(s) |}{| s | \log^{3/2} | s |} = \ell \tag{4.9}$$

is small enough.

Relaxing the growth rates (4.8)-(4.9) is an interesting open problem that is closely related to the improvement of the observability inequality (4.8), as we mentioned in problem 4.2 above.

4.4 On the lack of null controllability on general unbounded domains

In section 3 we have proved that "nice" initial data (compactly supported and smooth) are not null-controllable for the heat equation on the half-line $\mathbb{R}^+$ with a Dirichlet boundary control at $x = 0$.

The same results can be proved on the half-space $\mathbb{R}^n_+$ (see [MZ2]).

On the other hand, the similarity variables apply in any conical domain Ω. However, the methods in [MZ2] do not apply in such a general geometric setting because of the lack of ortoghonality of the traces of the normal derivatives of the eigenfunctions on $\partial\Omega$.

It would be interesting to know if the negative result of section 3, extended to $\mathbb{R}^n_+$ in [MZ2], holds in any conical domain.

4.5 Lack of null-controllability for general $1-d$ equations on the half line

Let us consider the $1-d$, bariable coefficient, linear heat equation on the half-line:

$$\begin{cases} \rho(x)u_t - (a(x)u_x)_x = 0, & x > 0, t > 0 \\ u(0,t) = v(t), & t > 0 \\ u(x,0) = u_0(x), & x > 0. \end{cases} \tag{4.10}$$

It would be interesting to see if the negative results of section 3 may be extended to this more general equation (with, say, a and ρ positive, bounded and smooth functions).

Of course, the same problem arises for more general eqations: coefficients depending both on x and t, equations with lower order potentials, equations involving semilinear terms,. . . .

References

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