Unique continuation and control for the heat equation from an oscillating lower dimensional manifold

Carlos CASTRO * and Enrique ZUAZUA †

Abstract

We consider the linear heat equation with Dirichlet boundary conditions in a bounded domain of $\mathbb{R}^n$, $n \geq 1$, and with a control acting on an lower-dimensional time-dependent manifold of dimension $k \leq n-1$. We analyze the approximate controllability problem. This problem is equivalent to a suitable uniqueness or unique-continuation property of solutions of the heat equation without control. More precisely, it consists in proving that the unique solution of the Dirichlet problem vanishing on the time-dependent manifold is identically zero. This uniqueness problem, however, does not fit in the class of classical Cauchy problems and therefore, the existing tools based on power series expansions, Carleman inequalities and doubling properties do not seem to apply. We give sufficient conditions on the time-dependent manifold for this uniqueness property to hold. The techniques we employ combine the Fourier series representation and the time analyticity of solutions and allow to reduce the problem to a uniqueness question for the eigenfunctions of the Laplacian. We then apply well-known results on the nodal sets of these eigenfunctions.

We also analyze the asymptotic behavior of the control when the time-oscillation of the manifold supporting the control increases. When the frequency of oscillation tends to infinity we prove that the controls converge to an approximate control for the same heat equation but on a manifold of dimension $k+1$ which is independent of time. This is done under suitable time-periodicity assumptions on the original manifold and confirms the fact that increasing time-oscillations of the support of the control increases the efficiency of the control mechanism.

1 Introduction

This paper is devoted to study the properties of approximate controllability for the linear, constant coefficients, heat equation in a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with Dirichlet boundary

*Dep. Matemática e Informática, ETSI Caminos, Canales y Puertos, Univ. Politécnica de Madrid, 28040 Madrid, Spain. **ccastro@caminos.upm.es**

†Dep. Matemáticas, Facultad de Ciencias, Univ. Autónoma de Madrid, 28049 Madrid, Spain. **enrique.zuazua@uam.es**.

Supported by grants BFM2002-03345 of the MCYT (Spain) and the TMR projects of the EU "Homogenization and Multiple Scales" and "New materials, adaptive systems and their nonlinearities: modelling, control and numerical simulations".

conditions and with a control acting on a time-dependent lower-dimensional manifold of dimension $k \leq n-1$.

The main novelty of the analysis carried out in this paper lies precisely on the fact that the control acts on a lower-dimensional manifold which moves in time.

The problem under consideration is rather natural. Indeed, in the ultimate goal of optimally controlling a given system with the minimal amount of control it is natural to consider controls located, for instance, in a single point or on a finite collection of points. This is the so-called pointwise control problem (see J.-L. Lions [L3]). However, in that case the system may easily fail to be controllable. That is the case for instance when the support of the control is located on a nodal set of an eigenfunction of the Laplacian. As a consequence of that, the property of pointwise controllability, even if it holds on some geometric configurations (when the support is chosen in an appropiate way) it is extremely sensitive to the location of the controller.

By the contrary, when the control is located on an open subset of the domain, approximate controllability holds as a consequence of Holmgren's uniqueness Theorem without any geometric restriction. In fact, in that case, a much stronger property, the so-called null-controllability property, holds (see [FI], [LR] or [FZ] among others).

The case where the control is supported on a manifold of dimension $k \leq n-1$ is an intermediate situation between the problem of pointwise control and the control on an open subset. However, when the support of the control is independent of time the situation is quite similar to the one encountered when dealing with the pointwise control problem: If the manifold is contained on the nodal set of some eigenfunction, the approximate controllability property fails.

M. Berggrem in [B] pointed out that a possible way of enhancing the control property when the control was supported in a manifold of dimension $k \leq n-1$ was to make this support to oscillate in time more and more. In [B] some numerical evidences of this fact for the one-dimensional case were also given.

This paper is devoted to analytically investigate this issue. We consider, in particular, the following two problems:

Problem 1: To give sufficient conditions on a time-dependent manifold to ensure the approximate controllability of the heat equation.

Problem 2: To analyze the asymptotic limit of the controllers when the time-frequency of the oscillations of its supports tends to infinity.

Let us briefly describe the results we obtain and the techniques we employ. Concerning the first problem, using the time-analyticity of solutions of the heat equation and the Fourier development of solutions, under suitable assumptions on the time-evolution of the manifolds where the control is supported, the problem is reduced to a unique continuation question on the eigenfunctions of the Laplacian that we solve applying classical results on its nodal sets. At this point, it should be noticed that we do not use fully the existing results on the size of the nodal sets of eigenfunctions ([DF], [L], [JL]) whose consequences in the context of approximate control of the heat equation remain to be investigated.

Once the heat equation is known to be approximately controllable (i.e. once problem 1 is solved) the control can be characterized as the minimum of a suitable quadratic functional over the set of solutions of the adjoint heat equation as in [FPZ].

We then address Problem 2. More precisely, assuming that the manifold in which the control is located is time-periodic and that it satisfies the requirements of Problem 1 to guarantee approximate controllability, we investigate the behavior of the controls as the time-frequency (described by a parameter $1/\varepsilon$) tends to infinity. The control then undergoes a *homogenization* process. In the limit we get controls distributed on a time-independent manifold of dimension $k+1$ modulated by a density factor varying along the manifold. This result does show indeed that time oscillations on the support of the control enhance the controllability properties.

The proof of this second result uses the characterization of controls as minima of suitable quadratic functionals, Γ-convergence arguments and a careful analysis of the behavior as $\varepsilon \to 0$ of traces of solutions of the heat equation along rapidly oscillating manifolds.

Let us now state more precisely the problems we shall address and introduce the notation we shall employ.

2 Problem formulation

Let Ω be a bounded smooth domain of $\mathbb{R}^n$ $(n = 1, 2, 3)$, $\partial\Omega$ its boundary, $T > 0$, $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$. We consider the linear heat equation with an interior control $f(x, t)$ which acts in an open subset $\omega \subset \Omega$:

$$\begin{cases} u_t - \Delta u = f(x,t)\chi_\omega(x) & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{2.1}$$

Here χ_ω represents the characteristic function of the region ω.

Given any $T > 0$ and any open subset $\omega \subset \Omega$ the following approximate controllability property is known to hold: *For any initial data $u^0 \in L^2(\Omega)$, any final data $u^1 \in L^2(\Omega)$ and any $\alpha > 0$ there exists a control $f(x, t) \in L^2(\omega \times (0, T))$ such that the solution u of system (2.1) satisfies*

$$\|u(x,T) - u^1\|_{L^2(\Omega)} \leq \alpha. \tag{2.2}$$

Moreover, it is known (see [FPZ]) that the optimal control (the one with minimal L^2-norm) can be obtained by minimizing a suitable continuous, convex and coercive functional over the space of solutions of the following adjoint system endowed with the L^2-norm of its datum at $t = T$:

$$\begin{cases} -\varphi_t - \Delta\varphi = 0, & \text{in } \Omega \times (0,T) \\ \varphi(T) = \varphi^0 & \text{in } \Omega \\ \varphi = 0 \text{ on } \partial\Omega \times (0,T). \end{cases} \tag{2.3}$$

On the other hand, it is well-known that the approximate controllability property above is equivalent to the following uniqueness property of (2.3): If $\varphi = 0$ in $\omega \times (0, T)$, then $\varphi \equiv 0$. In this case, this uniqueness property does hold as a consequence of Holmgren's Theorem.

This paper is devoted to analyze these questions when the open subset ω of Ω is replaced by a lower dimensional continuous manifold $\gamma(t) \subset \Omega$. In fact, in view of the fact that, in practice, the support of the control needs to be very small compared to the total size of the domain Ω it is very natural to consider the control to be located in such lower dimensional manifolds. When the manifold $\gamma \subset \Omega$ is independent of time the corresponding control system reads as follows:

$$\begin{cases} u_t - \Delta u = f(x,t)\delta_\gamma(x) & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.4}$$

where $\delta_\gamma(x)$ represents the Dirac measure on γ. Here γ can be a point, a curve if $n \geq 2$, or a surface if $n = 3$, for instance. When γ is a point we consider it as a manifold of dimension zero.

It turns out that the approximate controllability property of system (2.4) depends on the location of γ. Indeed, the problem of approximate controllability for system (2.4) can be reduced to a uniqueness problem for the adjoint system (2.3):

$$\varphi = 0 \text{ on } \gamma \Rightarrow \quad \varphi \equiv 0.$$

Using the Fourier series representation of solutions of system (2.3) it can be shown that these properties hold if and only if the only eigenfunction of the Laplacian with homogeneous Dirichlet boundary conditions and vanishing on γ is the identically zero one. In the sequel, the manifolds γ for which this spectral property is satisfied will be referred to as *strategic manifolds*.

The property of γ being strategic is difficult to establish in practice since it is extremely unstable. For example, if $\Omega = (0,1)$ with $n = 1$ then $\gamma = x_0 \in \Omega$ is strategic if and only if it is irrational. In general, k-dimensional manifolds in Ω with $k < n$ are generically strategic. But, by the contrary, γ fails to be strategic if it is contained in a nodal set of any of the eigenfunctions of the Laplacian. Consequently, controllability properties over low dimensional manifolds are hard to use in practice. At this point it is worth noting that the strategic property for a k-dimensional manifold is obviously more likely to hold when the dimension k is larger.

To overcome this difficulty one may consider controls supported on moving (in time) manifolds $\{\gamma(t)\}_{0 \leq t \leq T}$.

The main advantage of moving controls is that it is easy to construct families $\{\gamma(t)\}_{0 \leq t \leq T}$ for which the strategic property holds for $\gamma(t)$ a.e. in $t \in [0,T]$. For example, this is the case in the one dimensional example above when we assume that the control is located at a point that moves continuously in time. In this case, $\gamma(t)$ is irrational, and therefore strategic, a.e. in $t \in [0,T]$. Therefore, the approximate controllability is likely to hold for such moving controls. A previous result in this direction is given in [K] where the approximate controllability for the one dimensional case is proved when considering two pointwise moving controls that meet at a time $t_0 > 0$.

However, even if the system can be controlled from a one-parameter family of lower dimensional manifolds $\{\gamma(t)\}_{0 \leq t \leq T}$, the control is expected to be singular because it acts in a very small part of the domain. Indeed, as it was pointed out in [B] for the one-dimensional case, the control exhibits, in general, an highly oscillatory behavior in time.

To improve the efficiency of these moving controls, the possibility of increasing the time oscillations of the curve $\{\gamma(t)\}_{0\le t\le T}$ was suggested in [B]. More precisely, a highly oscillating periodic family of manifolds of the form $\{\gamma(t/\varepsilon)\}_{0\le t\le T}$ was considered, $\{\gamma(t)\}_{0\le t}$ being a $2\pi-$periodic in time family of manifolds. We refer to these controls as *rapidly oscillating controllers*. As $\varepsilon \to 0$ the control acts at any point of the range of $\{\gamma(s)\}_{s\in[0,2\pi)}$ for an increasing number of times. In this way, as $\varepsilon \to 0$, the controls are likely to be close, in some sense, to a control acting on the open set ω defined as the interior set of $P =$ *range* $\{\gamma(s)\}_{s\in[0,2\pi)}$ with respect to the relative topology, which, typically, contains a manifold of dimension $k+1$. As we mentioned above, controls acting on higher dimensional manifolds are likely to be more efficient. In the particular case $k = n-1$, the limit set (as $\varepsilon \to 0$) ω is an open subset of Ω and the limit system is approximately controllable. Thus, in general, rapidly oscillating controllers should provide a more efficient way to control the system.

This paper is devoted to rigourously prove that both ideas presented before are correct. First we provide several controllability results of the heat equation for a large class of lower dimensional *moving controls* and second, we prove the convergence of *rapidly oscillating controllers* as $\varepsilon \to 0$ to a certain class of controllers distributed on a $k+1$-dimensional manifold.

The rest of this paper is divided in four more sections. In Section 3 we give the main approximate controllability results in this paper. We distinguish the case of one space dimension and the multi-dimensional one. We also state the main results on the asymptotic behavior of the controls on rapidly oscillating control regions. Section 4 is devoted to analyze the problem of unique continuation which is equivalent to the approximate controllability one. We distinguish again the 1-d and the multi-dimensional cases. In Section 5 we prove the convergence results for the rapidly oscillating controllers as the oscillation parameter ϵ goes to zero. Finally, in Section 6 we provide some comments and extensions.

3 Main results

In this paper we restrict ourselves to the case where Ω is an open set of $\mathbb{R}^n$ in dimensions $n = 1, 2, 3$ but the techniques we employ and the results we get can be easily generalized to higher dimensions.

We consider system (2.4) with a moving control

$$\left\{\begin{array}{lll} u_t - \Delta u = f(x,t)\delta_{\gamma(t)}(x) & \text{in} & Q \\ u = 0 & \text{on} & \Sigma \\ u(x,0) = u^0(x) & \text{in} & \Omega. \end{array}\right. \tag{3.1}$$

We assume that $\gamma(t)$ satisfies the following hypotheses:

1. $\gamma(t)$ is a Lipschitz-continuous k-dimensional manifold in Ω with $0 \le k \le n-1$, for all $t \in [0,T]$.

2. The set $\gamma = \{\gamma(t)\}_{0\le t\le T}$ is a time-continuous family of manifolds in the sense of the following definition:

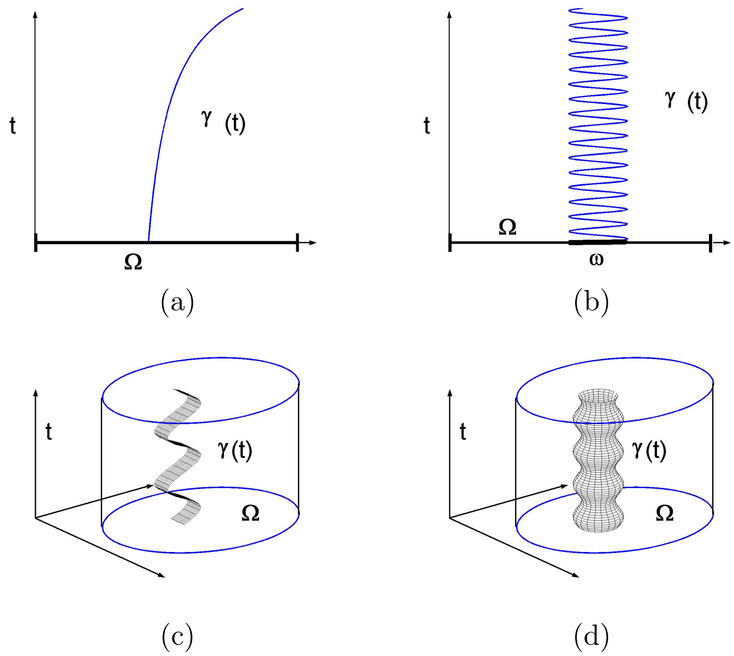


Figure 1: Examples of time-continuous families of manifolds that illustrate the Definition 3.1: (a) and (b), single point ($k = 0$) with non-regular and regular in time trajectories respectively. (c) and (d), curves ($k = 1$) with regular in time trajectories. Note that in case (d) the parametrization of $\gamma(t)$, for each $t \in [0, T]$, requires at least two charts. Thus we have to consider $A = 2$ in the Definition 3.1.

Definition 3.1 *We say that $\{\gamma(t)\}_{0\leq t\leq T}$ is a C^s ($s \geq 0$) (resp. analytic) family of k-dimensional manifolds when $\gamma(t)$ is a Lipschitz-continuous k-dimensional manifold for all $t \in [0, T]$ and there exists a finite family of functions $\{\psi_\alpha(x, t)\}_{\alpha=1}^{A}$, $A \geq 1$ being independent of t, such that*

1. *$\psi_\alpha : V_\alpha \times [0, T] \subset I\!R^k \times [0, T] \to \{\gamma(t)\}_{0\leq t\leq T}$ where V_α are compact sets of $I\!R^k$, for any $\alpha = 1, ..., A$, and*

$$\{\gamma(t)\}_{0\leq t\leq T} \subset \bigcup_{\alpha=1}^{A} \psi_\alpha(V_\alpha, t), \qquad \textit{for all } 0 \leq t \leq T.$$

2. *$\psi_\alpha(y, t) \in C([0, T]; W^{1,\infty}(V_\alpha))$ for all $\alpha = 1, ..., A$.*

3. *For any fixed $y \in V_\alpha$, $\psi_\alpha(y, t)$ is C^s (resp. analytic) in the time variable t.*

Remark 3.1 *1. Note that a C^s family of k-dimensional manifolds may be constituted by manifolds $\gamma(t)$ which are not C^s. More precisely, the condition of being of class C^s only refers to the regularity on the time variable.*

2. The assumption on the Lipschitz-continuity in space of the manifolds $\gamma(t)$ is the minimal one to define the surface measure σ on $\gamma(t)$ (see [N], ch. 4 sec. 7). Thus, the measure of $\gamma(t)$, arising below, and the integral on $\gamma(t)$ are well-defined for any $t \in [0,T]$.

3. The measure of $\gamma(t) \subset \Omega$ is finite for all $t \in [0,T]$. Moreover, hypothesis 2 in Definition 3.1 ensures the time-continuity of the total measure of $\gamma(t)$. Therefore the measure of $\gamma(t)$ is in fact uniformly bounded in $t \in [0,T]$.

4. The case $k = 0$ corresponds to that in which, for each $t \in [0,T]$, $\gamma(t)$ is reduced to a single point or a finite number of points. For instance, when $\gamma(t) = \{x(t)\}$ for all $t \in [0,T]$ the control in (3.1) takes the form $f(t)\delta_{x=\gamma(t)}$ and f is independent of x.

The control $f(x,t)$ (resp. $f(t)$, if $k = 0$) in (3.1) is assumed to belong to $L^2(0,T;L^2(\gamma(t)))$ (resp. $L^2(0,T)$), i.e.

$$\int_0^T \int_{\gamma(t)} |f(x,t)|^2 d\sigma \, dt < \infty \qquad \text{(resp. } \int_0^T |f(t)|^2 dt < \infty\text{).} \tag{3.2}$$

Then, system (3.1) is well defined in different Sobolev spaces depending on the space dimension n, and the dimension k of the manifold $\gamma(t)$. We introduce the spaces:

$$H_{-1} = \begin{cases} H^{-1}(\Omega) \text{ if } n-k=1, \\ L^2(\Omega) \text{ if } n-k>1, \end{cases} \qquad H_0 = \begin{cases} L^2(\Omega) \text{ if } n-k=1, \\ H_0^1(\Omega) \text{ if } n-k>1, \end{cases}$$
$$H_1 = \begin{cases} H_0^1(\Omega) \text{ if } n-k=1, \\ H^2 \cap H_0^1(\Omega) \text{ if } n-k>1, \end{cases} \tag{3.3}$$

and we denote by H_i', $i = 1,2,3$, their duals. Recall that we are assuming the space dimension $n = 1,2,3$. Similar results can be proved for higher dimensions. But the choice of H_i, $i = 1,2,3$, has to be suitably modified for $n \geq 4$.

The term $f(x,t)\delta_{\gamma(t)}$ on the right hand side of (3.1) clearly satisfies

$$f(x,t)\delta_{\gamma(t)} \in L^2(0,T;H_1'). \tag{3.4}$$

Indeed, if we denote by $< \cdot, \cdot >$ the duality pairing between $L^2(0,T;H_1)$ and its dual we have,

$$< f(x,t)\delta_{\gamma(t)}, \varphi >= \begin{cases} \displaystyle\int_0^T \int_{\gamma(t)} f(x,t)\varphi(x,t) d\sigma \, dt, & \text{if } k \geq 1, \\ \displaystyle\int_0^T f(t)\varphi(\gamma(t),t) dt, & \text{if } k = 0, \end{cases} \tag{3.5}$$

which is well-defined under the assumptions above on $\gamma(t)$, since

$$\left| \int_0^T \int_{\gamma(t)} f(x,t)\varphi(x,t) d\sigma \, dt \right| \leq \|f\|_{L^2(0,T;L^2(\gamma(t)))} \|\varphi\|_{L^2(0,T;L^2(\gamma(t)))}, \qquad \text{if } k \geq 1,$$
$$\left| \int_0^T f(t)\varphi(\gamma(t),t) dt \right| \leq \|f\|_{L^2(0,T)} \|\varphi\|_{L^2(0,T;L^\infty(\Omega))}, \qquad \text{if } k = 0,$$

and

$$\|\varphi\|_{L^2(0,T;L^2(\gamma(t)))} \leq \begin{cases} C(\gamma)\|\varphi\|_{L^2(0,T;L^\infty)} \leq C(\gamma)\|\varphi\|_{L^2(0,T;H^2\cap H_0^1)}, & \text{if } n=3;\ k=1, \\ C(\gamma)\|\varphi\|_{L^2(0,T;H_0^1(\Omega))}, & \text{if } n=2,3;\ n-k=1, \end{cases}$$

$$\|\varphi\|_{L^2(0,T;L^\infty(\Omega))} \leq \begin{cases} C(\gamma)\|\varphi\|_{L^2(0,T;H_0^1)}, & \text{if } n=1, \\ C(\gamma)\|\varphi\|_{L^2(0,T;H^2\cap H_0^1)}, & \text{if } n=2,3. \end{cases} \tag{3.6}$$

Here we have used the Sobolev embeddings

$$\begin{aligned} &H_0^1(\Omega) \subset C(\bar{\Omega}),\ \text{if } n=1, \\ &H^2 \cap H_0^1(\Omega) \subset C(\bar{\Omega}),\ \text{if } n=2,3 \text{ and } n-k>1, \end{aligned}$$

and the trace Theorem (see [N], Ch. 4 Sec. 7), in the case $n=2,3$ and $n-k=1$, guaranteeing that

$$\| \varphi|_{\gamma(t)}(\cdot,t)\|_{L^2(\gamma(t))} \leq C(\gamma(t))\|\varphi(\cdot,t)\|_{H_0^1(\Omega)} \qquad \text{for } n=2,3 \text{ and } n-k=1.$$

The constant $C(\gamma(t))$ only depends on the measure of $\gamma(t)$. Thus, in view of the assumptions on γ, it can be chosen to be uniformly bounded in $t \in [0,T]$. Consequently (3.6) holds and therefore (3.4) holds too.

It is at this point where defining H_1 distinguishing the value of $n-k$ is needed. The same can be said about the assumptions we have made on the manifold γ.

We now define the weak solution of system (3.1) by transposition (see [LM]). Let $\psi \in L^2(0,T;H_{-1})$ and consider the adjoint heat equation

$$\begin{cases} -\varphi_t - \Delta\varphi = \psi & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{3.7}$$

Note that the unique solution of (3.7) belongs to the class

$$\varphi \in C([0,T];H_0) \cap L^2(0,T;H_1).$$

Multiplying the equation in (3.1) by φ and integrating by parts we obtain formally the following identity

$$\begin{aligned} &\int_0^T \int_{\gamma(t)} f\varphi d\sigma\, dt + \int_\Omega u^0\varphi(0)dx = \int_0^T \int_\Omega u\psi dx\, dt \quad \text{if } k \geq 1, \\ &\int_0^T f(t)\varphi(\gamma(t),t)dt + \int_\Omega u^0\varphi(0)dx = \int_0^T \int_\Omega u\psi dx\, dt \quad \text{if } k=0. \end{aligned}$$

This motivates the following definition: *We say that u is a solution of (3.1), in the sense of transposition, if*

$$< f\delta_{\gamma(t)}, \varphi > + < u^0, \varphi(0) >_0 = \int_0^T < u, \psi >_{-1} dt, \qquad \forall \psi \in L^2(0,T;H_{-1}), \tag{3.8}$$

where $< \cdot,\cdot >$ (resp. $< \cdot,\cdot >_0$, $< \cdot,\cdot >_{-1}$) is the duality pairing between $L^2(0,T;H_1)$ (resp. H_0, H_{-1}), and its dual space $L^2(0,T;H_1')$ (resp. H_0', H_{-1}).

It is easy to see that for any initial data $u^0 \in H_0'$ there exists an unique solution of (3.1) in the sense of transposition in the class

$$u \in C\left([0,T]\,; H_0'\right). \tag{3.9}$$

For example, if we consider the case $n-k>1$ (the case $n-k=1$ is even simpler), then $H_{-1} = L^2(\Omega)$ and $\psi \in L^2(0,T;L^2(\Omega))$. The solution φ of system (3.7) is in the class $\varphi \in L^2(0,T;H^2 \cap H_0^1(\Omega)) \cap C([0,T];H_0^1(\Omega))$ and then, the map

$$\psi \to (\varphi(x,t), \varphi(x,0))$$

is linear and continuous from $L^2(0,T;L^2(\Omega))$ to $L^2(0,T;H^2 \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. Therefore, for any initial data $u^0 \in H^{-1}(\Omega)$, the left hand side of (3.8) is linear and continuous from $\psi \in L^2(0,T;L^2(\Omega))$ to $I\!R$. Then, there exists an unique $u \in L^2(0,T;L^2(\Omega))$ satisfying (3.8), i.e. a solution of (3.1) in the sense of transposition. Moreover, it is easy to see that this solution u satisfies the first equation in system (3.1) in the sense of distributions and then we deduce that $u_t \in L^2(0,T;(H^2 \cap H_0^1(\Omega))')$. Consequently,

$$u \in C\Big([0,T]\,; H^{-1}(\Omega)\Big),$$

and (3.9) holds for $n-k>1$.

We consider the following approximate controllability problem for system (3.1): *Given u^0, $u^1 \in H_0'$ and $\alpha > 0$, to find a control $f \in L^2(0,T;L^2(\gamma(t)))$ such that the solution $u = u(x,t)$ of (3.1) satisfies*

$$\|u(T) - u^1\|_{H_0'} \leq \alpha. \tag{3.10}$$

For the sake of clarity we divide the rest of this section in two subsections where we state separately the results for the one-dimensional case and those for higher space dimensions.

3.1 The one dimensional case

We assume that $\Omega = (0,L)$ with $L > 0$. Let $\gamma : [0,T] \to \Omega$ be any continuous curve. The following holds:

Theorem 3.1 *Let $\gamma(t) : [0,T] \to \Omega$ be a non-constant continuous curve satisfying at least one of the following conditions:*

1. *There exists an open subinterval $\mathcal{U} \subset [0,T]$ where γ is not analytic at any point $t \in \mathcal{U}$.*
2. *There exists $t_1 \in (0,T)$ where $t \to \gamma(t)$ is not analytic and a subinterval $(t_1,t_2) \subset (0,T)$ where γ is analytic and can be extended analytically to a subinterval (t_0,t_2) with $t_1 \in (t_0,t_2)$.*

3. *γ can be extended analytically to a curve $\bar{\gamma}(t) : (-\infty, T] \to I\!R$ satisfying one of the following conditions:*

   (a) *$\bar{\gamma}(t)$ meets the boundary of Ω, i.e. there exits $t_0 \in (-\infty, T]$ such that $\bar{\gamma}(t_0) \in \partial\Omega$.*

   (b) *The set of accumulation points of $\bar{\gamma}(t)$ as $t \to -\infty$ contains at least one point $x_0 \in \Omega$ such that x_0/L is irrational.*

   (c) *The set of accumulation points of $\bar{\gamma}(t)$ as $t \to -\infty$ is reduced to an unique $x_0 \in \Omega$ and there exists a sequence $t_n \to -\infty$ such that*

   $$\lim_{t_n \to -\infty} (\bar{\gamma}(t_n) - x_0)^{-1} e^{t_n(\lambda_3 - \lambda_2)} = 0, \tag{3.11}$$

   where $\lambda_3 - \lambda_2 = 5\pi^2/L^2$ is the difference between the third and second eigenvalues of the Laplace operator in Ω.

Then, for any $T > 0$, system (3.1) is approximately controllable.

Remark 3.2 *The conditions on γ in Theorem 3.1 above do not characterize all possible curves for which approximate controllability holds. However, they may be considered sharp in different senses:*

1. *Condition 1 is sharp in the sense that if γ is analytic in one interval $(t_0, t_1) \subset [0, T]$ then the approximate controllability may fail. We can consider for example a curve $\gamma(t) = x_0$ with x_0 nonstrategic, i.e. x_0/L rational.*

2. *Condition 2 is sharp in the sense that if $\gamma : (t_1, t_2) \to \Omega$ is analytic but cannot be extended analytically to a subinterval (t_0, t_2) with $t_1 \in (t_0, t_2)$, then approximate controllability may fail (see example 1 in section 4.1 below).*

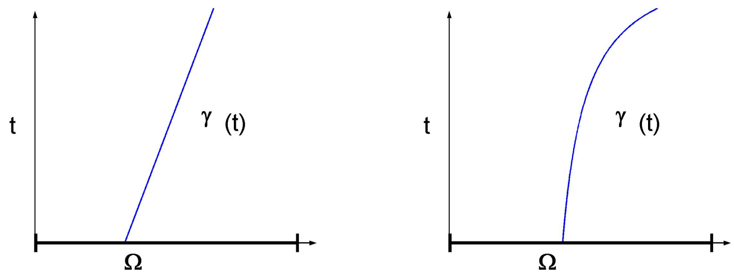
3. *Condition 3 is sharp in the sense that if $\gamma : [0, T] \to \Omega$ is analytic but cannot be extended analytically to $t \in (-\infty, T]$ then approximate controllability may fail (see example 2 in section 4.1 below).*

4. *Condition 3 is also sharp in the sense that if $\gamma : [0, T] \to \Omega$ is analytic and can be extended analytically to $\bar{\gamma} : (-\infty, T] \to \Omega$ in such a way that the set of accumulation points of $\bar{\gamma}(t)$ as $t \to -\infty$ is reduced to an unique nonstrategic point $x_0 \in \Omega$ that does not satisfy (3.11), then approximate controllability may fail (see example 3 in section 4.1 below).*

We give now a number of examples showing that the conditions in Theorem 3.1 cover a large class of curves.

Examples:

1. *Weierstrass type functions.* Let $\gamma(t) : [0, T] \to \Omega$ be a C^1 function with nowhere defined second derivative. Then $\gamma(t)$ is Lipschitz and satisfies condition 1 above. A function with this property can be constructed integrating the classical Weierstrass example of a continuous function that is nowhere differentiable (see [C], Ch. 4, Sec. 7, for example).



(a) (b)

Figure 2: Trajectories of γ for which the approximate controllability property holds.

2. *Piecewise analytic curves.* Let $\gamma_1, \gamma_2 : [0,T] \to \Omega$ be two analytic curves that meet at time t_0, i.e. there exists $t_0 \in (0,T)$ where $\gamma_1(t_0) = \gamma_2(t_0)$ and $\gamma_1'(t_0) \neq \gamma_2'(t_0)$. Then $\gamma : [0,T] \to \Omega$ defined as follows

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0,t_0] \\ \gamma_2(t) & \text{if } t \in [t_0,T] \end{cases}$$

satisfies condition 2 in Theorem 3.1 (see Figure 1 (a)).

3. *Constant velocity curves.* Let $\gamma(t) = x_0 + \alpha t$ with $x_0 \in \Omega$ (see Figure 2 (a)). Then clearly γ satisfies condition 3 (a) in Theorem 3.1.

4. *Periodic analytic curves.* Let $\gamma : \mathbb{R} \to \Omega$ be any nonconstant periodic analytic curve (see Figure 1 (b)). In this case, the set of accumulation points of $\gamma(t)$ as $t \to -\infty$ is the range of γ over a period, which is an interval, and therefore it contains infinitely many points such that x_0/L is irrational. Then γ satisfies condition 3 (b) in Theorem 3.1.

5. Curves that converge to an unique point as $t \to -\infty$ and satisfy (3.11). For example, let $\gamma(t) = x_0 + \beta/t$ with $x_0 \in \Omega$ and β to be chosen in order to have $\gamma(t) \in \Omega$ for all $t \in [0,T]$. Then γ satisfies condition 3 (c) in Theorem 3.1 (see Figure 2 (b)).

3.2 The higher dimensional case

The situation is now more complex. We only consider nonconstant (in time) analytical families of k-dimensional manifolds $\{\gamma(t)\}_{0\leq t\leq T}$ satisfying, in particular, the following hypothesis:

$$\begin{array}{ll} \{\gamma(t)\}_{0\leq t\leq T} & \text{can be extended to an analytical family } \{\bar{\gamma}(t)\}_{-\infty<t\leq T} \\ & \text{s.t. } \bar{\gamma}(t) \subset \Omega, \quad \forall t \in (-\infty,T). \end{array} \tag{3.12}$$

Thus we are only considering the analogue of case 3 in the statement of Theorem 3.1.

In order to state our result let us introduce the eigenvalue problem associated to system (2.4):

$$\begin{cases} -\Delta w(x) = \lambda w(x), & x \in \Omega \\ w(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.13}$$

The associated eigenvalues will be denoted by

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_j < \ldots$$

each one with finite multiplicity $l(j) \geq 1$.

We also introduce the following sets associated to the families $\{\gamma(t)\}_{0\leq t\leq T}$ satisfying (3.12).

Definition 3.2 *Let $\{\gamma(t)\}_{0\leq t\leq T}$ be a family satisfying (3.12). We define the set of 'accumulation points' of $\{\gamma(t)\}_{0\leq t\leq T}$ as*

$$P = \left\{ x \in \overline{\Omega},\ s.t.\ \exists t_n \to -\infty,\ x_n \in \bar{\gamma}(t_n) \text{ with } x_n \to x \right\}, \tag{3.14}$$

and for each $x \in P$, the set of 'accumulation directions'

$$D_x = \left\{ \begin{array}{ll} v \in S^{n-1},\ s.t. & \exists t_n \to -\infty,\ x_n \in \bar{\gamma}(t_n) \text{ with } x_n \to x, \frac{x_n - x}{\|x_n - x\|} \to v \\ & \text{and } \lim_{t_n \to -\infty} \|x_n - x\|^{-1} e^{ct_n} = 0,\ \forall c > 0 \end{array} \right\}. \tag{3.15}$$

Remark 3.3 *Observe that D_x only contains those directions v for which $\|x_n - x\|$ converges to zero slower than any exponential as $t_n \to -\infty$. As indicated in Remark 3.2, in the one dimensional case approximate controllability fails for some curves that converge to an unique accumulation point more rapidly than a certain exponential.*

The following result reduces the approximate controllability property of system (3.1) to a suitable unique continuation property for the eigenfunctions of (3.13).

Theorem 3.2 *Assume that $\{\gamma(t)\}_{0\leq t\leq T}$ satisfies the hypothesis (3.12) and consider the sets P and D_x introduced in Definition 3.2. Assume also that the following spectral unique continuation property holds for the eigenfunctions of (3.13):*

> *The only eigenfunction w of (3.13) that satisfies*
> $w(x_0 + \mu v) = 0, \forall x_0 \in P,\ \forall v \in D_{x_0},$ *and* $\forall \mu \in I\!R$ *s.t.* $x_0 + \bar{\mu} v \in \bar{\Omega},\ \forall \bar{\mu} \in [0, \mu]$ (3.16)
> *is the trivial one.*

Then, for any $T > 0$ system (2.4) is approximately controllable.

We observe that there are two special situations where the spectral unique continuation property (3.16) holds trivially:

1. P is not included in the zero set of any of the eigenfunctions of (3.13).

2. P is included in the zero set of any of the eigenfunctions of (3.13) but there exists $x_0 \in P$ for which D_{x_0} contains an open set of S^{n-1}. Indeed, in this case if

$$w(x_0 + \mu v) = 0,\ \forall x_0 \in P,\ \forall v \in D_{x_0},\ \text{and } \forall \mu \in I\!R \text{ s.t. } x_0 + \bar{\mu} v \in \overline{\Omega},\ \forall \bar{\mu} \in [0, \mu]$$

then $w = 0$ on an open set of Ω and, by classical unique continuation for eigenfunctions (Holmgren's Theorem), $w \equiv 0$.

Now, we give some examples where one of the above situations holds and therefore the approximate controllability property is satisfied.

Examples:

1. *Stationary control.* Assume that the control is located on a time-independent k-dimensional manifold γ. In this case, property (3.12) holds trivially. The set P coincides with γ and D_x is empty for all $x \in P$. Therefore, the unique continuation property (3.16) is satisfied if and only if γ is not included in the nodal set of any of the eigenfunctions of (3.13).

2. *Time-periodic $n-1$-dimensional manifolds.* Assume that $\{\gamma(t)\}_{0 \le t \le T}$ is a non-constant periodic (in time) analytic (in time) family of $n-1$-dimensional manifolds. Then, $\{\gamma(t)\}_{0 \le t \le T}$ satisfies property (3.12) and the set P is the range of $\gamma(t)$ over a period $[0, p]$, i.e.

$$\bigcup_{t \in [0,p]} \gamma(t) = P \subset \Omega.$$

We observe that the dimension of P is n generically. However this is not always the case. For example if $n = 2$ we may consider $\Omega = (-1, 1) \times (-1, 1)$ and $\gamma(t)$ the subinterval $[0, \frac{1}{2} + \frac{1}{4}\sin(t)]$ in the axis $y = 0$.

When the dimension of P is n, it contains an open set of Ω. In this case, the unique continuation property (3.16) is a consequence of the unique continuation property for the eigenfunctions of (3.13).

We observe that, in general, the dimension of the set P is larger for periodic moving controls than for stationary ones. Therefore, the uniqueness property is much more likely to hold for moving controls.

3. *Pointwise control in 2-d.* We assume that $n = 2$, for instance, and $k = 0$. Here the control is localized on a continuous curve $\gamma(t) \subset \Omega \times [0, T]$. Consider the spiral curve around any point $x_0 \in \Omega$:

$$\gamma(t) = x_0 + \frac{\beta}{T + 1 - t}(\cos t, \sin t), \quad \beta > 0.$$

Here the parameter β must be chosen small enough to guarantee that $\gamma(t) \in \Omega$ for all $t \in (-\infty, T]$. In this case, $\gamma(t)$ is analytic, satisfies (3.12), and we have $P = \{x_0\}$ and $D_{x_0} = S^1$. Therefore, the unique continuation property (3.16) holds.

3.3 Rapidly oscillating controllers

Finally, we consider the case of rapidly oscillating controllers, i.e. where the control is located on a k-dimensional manifold $\{\gamma(t/\varepsilon)\}_{0\leq t\leq T}$ with $\varepsilon > 0$ a small parameter and $\{\gamma(t)\}_{0\leq t}$ a time-dependent periodic and analytic family of k-dimensional manifolds. To simplify the presentation, we restrict ourselves to the case $k = n - 1$.

We assume without loss of generality that $\gamma(t)$ is periodic of period 2π.

We also assume that the range of $\gamma(t)$ over a period $[0, p]$, i.e. $\bigcup_{t\in[0,p]} \gamma(t) = P \subset \Omega$ has dimension n. As we pointed out in Example 2 above (section 3.2) this is not always the case. However, for the purposes of this section, it is natural to consider the case where the supports of the controls scan a larger area.

In this case, system (3.1) reads as follows:

$$\begin{cases} u_t - \Delta u = f(x,t)\delta_{\gamma(t/\epsilon)}(x) & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{3.17}$$

We observe that, by hypothesis, the dimension of P, the range of $\gamma(t)$ over a period, is n and it contains an open set of Ω. Therefore, the unique continuation property (3.16) is a consequence of the unique continuation property for the eigenfunctions of the Laplace operator, and the approximate controllability of system (2.4) holds.

Let us introduce the limit problem

$$\begin{cases} u_t - \Delta u = f(x,t)m_\gamma(x) & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega, \end{cases} \tag{3.18}$$

where $m_\gamma(x)$ is the limit of $\delta_{\gamma(t/\varepsilon)}(x)$ in the $L^\infty(0,T;H_1')$ weak-* topology as $\varepsilon \to 0$. As we will see, this limit measure $m_\gamma(x)$ is supported in the nonempty open set $\omega \subset \Omega$ defined as the interior set of P.

The following holds:

Theorem 3.3 *Let us assume that $\{\gamma(t)\}_{t\leq T}$ is a non-constant periodic analytic family of $(n-1)$-dimensional manifolds and $\varepsilon > 0$ a small parameter.*

Given $T > 0$; $u^0, u^1 \in H_0'$ and $\alpha > 0$ there exists a sequence of approximate controls $f_\varepsilon \in L^2(0,T;L^2(\gamma(t/\varepsilon)))$ (resp. $f_\varepsilon \in L^2(0,T)$, if $n = 1$ and $k = 0$) of system (3.17), satisfying (3.10), which is uniformly bounded in $L^2(0,T;L^2(\gamma(t/\varepsilon)))$ (resp. $L^2(0,T)$).

Moreover, the controls f_ε can be chosen such that they strongly converge in the following sense:

$$f_\varepsilon(x,t)\delta_{\gamma(t/\varepsilon)}(x) \to f(x,t)m_\gamma(x) \text{ in } L^2(0,T;H_1') \text{ as } \varepsilon \to 0, \tag{3.19}$$

where f is an approximate control for the limit system (3.18) so that (3.10) holds.

The measure $m_\gamma(x)$ is the limit of $\delta_{\gamma(t/\varepsilon)}$ in the $L^\infty(0,T;H_1')$ weak- topology as $\varepsilon \to 0$. This limit measure m_γ is supported in in the nonempty subset $\omega \subset \Omega$, the interior set of P, and it is characterized by*

$$\begin{aligned}
\int_\omega \varphi(x) m_\gamma(x)\, d\sigma &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\gamma(s)} \varphi(x)\, d\sigma\, ds, \qquad \forall \varphi \in H_1,\ \text{if } k \geq 1,\ \text{i.e. } n \geq 2 \\
\int_\omega \varphi(x) m_\gamma(x)\, d\sigma &= \frac{1}{2\pi}\int_0^{2\pi} \varphi(\gamma(s))\, ds, \qquad \forall \varphi \in H_1,\ \text{if } k = 0.
\end{aligned} \tag{3.20}$$

Here σ denotes the surface measure of ω. In the case $k = n-1$, ω is an open subset of Ω and $d\sigma = dx$.

On the other hand, with the above controls, the solutions u_ε of (2.4) converge strongly in $C([0,T];H_0')$ as $\varepsilon \to 0$ to the solution u of the limit problem (3.18). This solution u satisfies (3.10).

Remark 3.4 *1. As a consequence of the statement in Theorem 3.3, system (3.18) is approximately controllable. In fact, Theorem 3.3 guarantees that the control of (3.18) may be achieved as the limit when $\varepsilon \to 0$ of the controls of (2.4) in the sense of (3.19).*

Note however that one could prove directly the approximate controllability of (3.18) since the limit control f acts on an open set of Ω.

2. It is worth to note that for the limit system (3.18), where the control is located on an open subset of Ω, a stronger controllability property is known to hold: the so-called null controllability property. This means that if $u^1 = 0$ then the control can be chosen to be exact, i.e. the solution of system (3.18) will satisfy $u(x,T) \equiv 0$. Whether this null controllability property holds for the approximate systems (2.4) and the possible convergence of the controls as $\varepsilon \to 0$ are open problems.

3. When $k < n-1$ the set P will not contain an open subset of Ω but, generically, a $k+1$-dimensional manifold. Therefore, the approximate controllability property of both the approximate systems and the limit system will depend on wether the set ω, the interior of P with respect to the relative topology, belongs to the nodal set of any of the eigenfunctions of the Laplace operator or not.

If this approximate controllability property holds then the analogous to Theorem 3.3 still holds. The limit density m_γ is then supported on the manifold ω of dimension $k+1 < n$ and it takes the form

$$m_\gamma(x) = g(x)\delta_\omega(x)$$

where $g(x)$ is a density function defined on ω by (3.20).

4. In [B] the numerical approximation of the one dimensional case $\Omega = (0,1)$ and $\gamma(t) = x_0 + \delta\cos(t)$ is addressed. Using the standard formal asymptotic expansion method in homogenization and the limit in the sense of distributions of the measure $\delta_{\gamma_\varepsilon(t)}$, the author obtains the first order approximation of a limit control when both ϵ and δ tend to zero ($\epsilon << \delta$). This first order *approximation consists on two steady distributed controls, concentrated at the points $x = x_0 \pm \delta$, which only depend on the time variable t.*

Here we prove that, when $\varepsilon \to 0$, $\delta > 0$ being fixed, the sequence of controls can be chosen to converge, in the sense stated in the above theorem, to a non-steady distributed control (see (3.22) in the example 1 below) acting on the range of $\gamma(t)$ over a period. Namely, the control may vary in x and t. The control obtained in [B] corresponds to the first order approximation (as $\delta \to 0$) of the limit control, given in (3.22) below, obtained as limit when $\varepsilon \to 0$ (see [B] for details).

The result above proves that, to some extent, the approximate controls depend continuously on ε. As $\varepsilon \to 0$ the transition from controls supported on a manifold of dimension $n-1$ to controls with support in an open subset of Ω is made.

We finish this section showing several examples of limit densities $m_\gamma(x)$ in some particular cases.

Examples:

1. Pointwise control, case 1. We assume that $\gamma(t)$ is reduced to an unique point at each time $t \in [0, T]$. The trajectory of the control $x = x(t)$ is included in a simple curve $\omega \subset \Omega$ and the control moves forward and backward scanning the curve ω.

In this case, the integral in (3.20) can be simplified studying separately the closed intervals where $\gamma(s)$ is one-to-one $\{I_h\}_{h=1}^H$. Note that the whole interval $[0, 2\pi]$ is divided in the subintervals I_h. Indeed, if there is a subinterval $I \subset (0, 2\pi)$ such that I is not included in $\bigcup_{h=1}^H I_h$ then $\gamma(s)$ must be constant on I and then constant everywhere because of the analyticity of γ.

Note also that the number H of subintervals $I_h \subset [0, 2\pi]$ must be finite because of the analyticity of γ. Indeed, at the extremes of I_h, the control has a returning point where its trajectory changes direction and γ' vanishes. If there are infinitely many intervals I_h, there are infinitely many points in a period $[0, 2\pi]$ where γ' vanishes. Thus, $\gamma(t)$ must be constant and this is in contradiction with the hypotheses on $\gamma(t)$.

Let $\omega_h = \gamma(I_h) \subset \Omega$ and $\gamma_h^{-1} : \omega_h \to I_h$ the inverse function of ψ in each one of the sets ω_h. Then,

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \varphi(\gamma(s))\, ds &= \frac{1}{2\pi}\sum_{h=1}^{H}\int_{I_h} \varphi(\gamma(s))\, \frac{1}{|\gamma'(s)|}\, |\gamma'(s)|\, ds \\ &= \frac{1}{2\pi}\sum_{h=1}^{H}\int_{\omega_h} \varphi(x) \frac{1}{\left|\gamma'(\gamma_h^{-1}(x))\right|} d\omega_h. \end{aligned} \tag{3.21}$$

Then,

$$m_\gamma(x) = \begin{cases} \frac{1}{2\pi}\sum_{h=1}^{H} \frac{1}{|\gamma'(\gamma_h^{-1}(x))|}\delta_{\omega_h}, & \text{if } n \geq 2, \\ \frac{1}{2\pi}\sum_{h=1}^{H} \frac{1}{|\gamma'(\gamma_h^{-1}(x))|}\chi_{\omega_h}, & \text{if } n = 1 \end{cases}.$$

Note that m_γ is defined over the whole curve ω since $\cup_h \omega_h = \omega$.

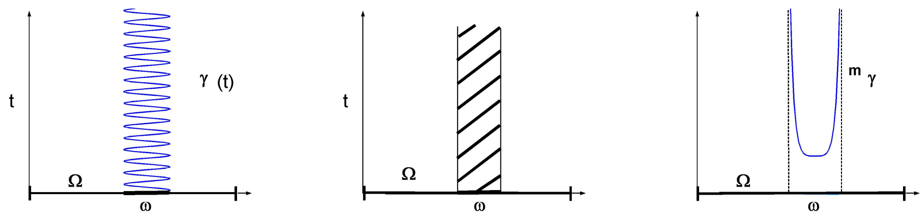


Figure 3: The sequence of controls acting on curves $\gamma(t/\varepsilon)$, with $\gamma(s)$ periodic (see (a)), converge as $\varepsilon \to 0$ to a control acting in the whole interval ω (see (b)) with a density $m_\gamma(x)$ which is singular at the extremes of ω (see (c)).

The function $m_\gamma(x)$ is singular at the extremes of the intervals I_h since $\gamma'(s) = 0$ for some points $s \in [0, 2\pi]$. For example, in the one dimensional case studied in [B], $\Omega = (0, 1)$, $\gamma(t) = x_0 + \delta \cos(t)$ and

$$m_\gamma(x) = \begin{cases} \frac{1}{\pi\sqrt{\delta^2-(x-x_0)^2}} \text{ if } |x - x_0| < \delta, \\ 0 \text{ otherwise,} \end{cases} \tag{3.22}$$

which is singular at $x = x_0 \pm \delta$. Observe however that $m_\gamma(x) \in L^1(\Omega)$ since the integral in (3.21) is well-defined for all $\varphi \in H_1$.

2. Pointwise control, case 2. Now we assume that the trajectory of the control follows a simple closed curve without any returning point. This is only possible if $n \geq 2$. In this case, $\gamma : [0, 2\pi) \to \Omega$ is one-to-one and $\gamma'(s) \neq 0$ for any $s \in [0, 2\pi)$. Then, we have

$$m_\gamma(x) = \frac{1}{2\pi} \frac{1}{|\gamma'(\gamma^{-1}(x))|} \delta_\omega(x).$$

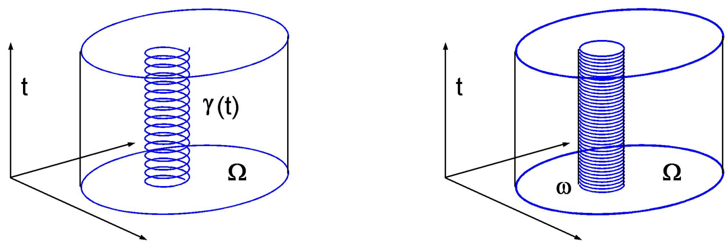
In particular, if γ follows a circular trajectory of radius R around $(x_0, y_0) \in \Omega \subset I\!R^2$, $\gamma(t) = (x_0, y_0) + R(\cos(t), \sin(t))$, then $|\gamma'(\gamma^{-1}(x))| = R$ and m_γ is given by $m_\gamma(x) = R\delta_\omega(x)$.

3. Control on a curve in dimension $n = 2$. Let $\gamma(t)$ be a curve for any $t \in [0, T]$. To simplify the presentation, we assume that $\{\gamma(t)\}_{0 \leq t \leq T}$ can be described by an unique chart $\psi : V \times [0, T] \to \{\gamma(t)\}_{0 \leq t \leq T}$, where V is an open and bounded subinterval of $I\!R$.

The measure m_γ satisfies (3.20). It can be computed studying separately the closed time intervals $\{I_h\}_{h=1}^H$ where $\psi(y, s) : V \times I_h \to \Omega$ is one-to-one. Let $\omega_h = \psi(V, I_h) \subset \Omega$ and $\psi_h^{-1} : \omega_h \to V \times I_h$ be the inverse function of ψ. Then,

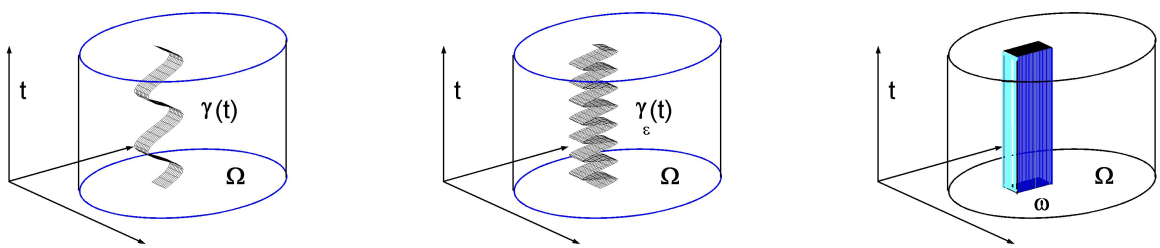
$$\int_0^{2\pi} \int_{\gamma(s)} \varphi(x)\, d\sigma\, ds = \sum_{h=1}^H \int_{I_h} \int_V \varphi(\psi(y, s)) \frac{\left|\frac{\partial \psi}{\partial y}\right|}{\left|\frac{\partial \psi}{\partial y} \times \frac{\partial \psi}{\partial s}\right|} \left|\frac{\partial \psi}{\partial y} \times \frac{\partial \psi}{\partial s}\right| dy\, ds$$

$$= \sum_{h=1}^H \int_{\omega_h} \varphi(x) \frac{\left|\frac{\partial \psi}{\partial y}(\psi_h^{-1}(x))\right|}{\left|\frac{\partial \psi}{\partial y}(\psi_h^{-1}(x)) \times \frac{\partial \psi}{\partial s}(\psi_h^{-1}(x))\right|} dx. \tag{3.23}$$



(a) (b)

Figure 4: The pointwise control located on a periodic circular trajectory in time (see (a)) converges, as the period goes to zero, to a control acting on the lateral boundary of the space-time cylinder $\omega \times [0, T]$ (see (b)).



(a) (b) (c)

Figure 5: The control located on a curve with periodic trajectory in time (see (a)) converges, as the period goes to zero (see (b)), to a control acting on a 2−d domain ω for all time (see (c)).

Then,

$$m_\gamma(x) = \frac{1}{2\pi} \sum_{h=1}^{H} \frac{\left| \frac{\partial \psi}{\partial y}(\psi_h^{-1}(x)) \right|}{\left| \frac{\partial \psi}{\partial y}(\psi_h^{-1}(x)) \times \frac{\partial \psi}{\partial s}(\psi_h^{-1}(x)) \right|} \chi_{\omega_h}.$$

Note that m_γ is defined over the whole ω since $\cup_h \omega_h = \omega$, which is a two-dimensional set.

4 Unique continuation for the adjoint system

It is well-known that the approximate controllability of system (2.4) is a consequence of the following unique continuation property for the adjoint system: *If* $\varphi \in C([0,T]; H_0)$ *solves*

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

can we guarantee that

$$\varphi(x,t) = 0, \quad \forall t \in [0,T] \text{ and } \forall x \in \gamma(t) \quad \Rightarrow \quad \varphi \equiv 0? \tag{4.2}$$

In fact, as we will see later on, when this unique-continuation property is satisfied, the control can be built as a minimizer of a quadratic, convex, continuous and coercive functional in a Hilbert space, associated with the adjoint system (4.1).

This section is devoted to analyze this uniqueness problem. We divide it in two subsections where we study separately the one-dimensional case and the higher dimensional one.

4.1 The one-dimensional case

Lemma 4.1 *Assume that* $\Omega = (0,L)$ *and* $\gamma : [0,T] \to \Omega$ *is a continuous curve which satisfies the hypotheses in the statement of Theorem 3.1. Then the unique continuation property (4.2) holds for the solutions of the adjoint problem (4.1).*

As we stated in Remark 3.2, the conditions in Theorem 3.1 are sharp in different senses. This is so because these conditions are sharp in the context of Lemma 4.1. We present now three examples that illustrate this fact. After these examples we give the proof of Lemma 4.1.

Example 1. This example shows that condition 2 in Theorem 3.1 is sharp in the sense that if $\gamma : (t_1, T] \to \Omega$ cannot be extended analytically for $t < t_1$ then the unique continuation property (4.2) for the solutions of the adjoint problem (4.1) may fail.

We consider $\Omega = (0,1)$ and the solution of the backward system (4.1) given by

$$\varphi(x,t) = e^{\lambda_2(t-T)} \sin(\sqrt{\lambda_2}x) + e^{\lambda_4(t-T)} \sin(\sqrt{\lambda_4}x) = e^{4\pi^2(t-T)} \sin(2\pi x) + e^{16\pi^2(t-T)} \sin(4\pi x).$$

For any $t \in [0,T]$ the points $x = x(t) \in \Omega$ for which $\varphi(x,t) = 0$ are characterized by

$$0 = \sin(2\pi x) + e^{12\pi^2(t-T)} \sin(4\pi x) = \sin(2\pi x)\left(1 + e^{12\pi^2(t-T)} 2\cos(2\pi x)\right).$$

Thus, $\varphi(x(t),t) = 0$ if and only if $x(t) = 1/2 \in \Omega$ (for any $t \in [0,T]$) or $x(t) \in \Omega$ is any of the two roots of the equation

$$\cos(2\pi x(t)) = -\frac{1}{2} e^{-12\pi^2(t-T)}, \text{ for } t \in (t_1, T], \tag{4.3}$$

where t_1 is defined as $t_1 = T - (\log 2)/(12\pi^2)$. Note that for $t < t_1$ the right hand side term in (4.3) is smaller than -1 and therefore (4.3) has no roots for $t < t_1$.

We define $\gamma(t)$ as follows:

$$\gamma(t) = \begin{cases} 1/2 & \text{if } t \in [0, t_1], \\ x(t) = \frac{1}{2\pi} \arccos\left(-\frac{1}{2} e^{-12\pi^2(t-T)}\right) \text{ with } x(t) \in (0, \frac{1}{2}] & \text{if } t \in [t_1, T], \end{cases}$$

Clearly $\gamma(t)$ is a continuous curve for which $\varphi(\gamma(t), t) = 0$ for all $t \in [0, T]$. Moreover, $\gamma(t)$ is analytic in $t \in [0, t_1) \cup (t_1, T]$ and it is easy to see (differentiating in (4.3)) that

$$\lim_{\substack{t \to t_1 \\ t > t_1}} \gamma'(t) = \infty.$$

This means that $\gamma : (t_1, T] \to \Omega$ cannot be analytically extended for $t < t_1$.

Thus, we have obtained a curve $\gamma(t)$ for which the unique continuation property (4.2) fails and that shows that condition 2 in Theorem 3.1 is sharp in the sense described above.

Example 2. This example shows that condition 3 is sharp in the sense that if γ cannot be extended analytically to a curve $\bar{\gamma} : (-\infty, T] \to \Omega$ then the unique continuation property (4.2) may fail.

We consider $\Omega = (0, 1)$ and the solution of the backwards system (4.1) given by

$$\varphi(x, t) = c_1 e^{\lambda_1(t-T)} \sin(\sqrt{\lambda_1} x) + e^{\lambda_2(t-T)} \sin(\sqrt{\lambda_2} x) = c_1 e^{\pi^2(t-T)} \sin(\pi x) + e^{4\pi^2(t-T)} \sin(2\pi x),$$

with c_1 a constant to be chosen later.

For any $t \in [0, T]$ the points $x = x(t) \in \Omega$ for which $\varphi(x, t) = 0$ are characterized by

$$0 = c_1 \sin(\pi x) + e^{3\pi^2(t-T)} \sin(2\pi x) = \sin(\pi x) \left(c_1 + e^{3\pi^2(t-T)} 2\cos(\pi x)\right).$$

Thus, $\varphi(x(t), t) = 0$ if and only if $x(t) \in \Omega$ is a root of

$$\cos(\pi x(t)) = -\frac{c_1}{2} e^{-3\pi^2(t-T)}. \tag{4.4}$$

We choose $c_1 \in (0, 2)$ in such a way that (4.4) has a unique root $x(t) \in \Omega$ for $t \in [0, T]$, i.e. the right hand side of (4.4) is greater than -1.

Clearly $\gamma(t)$ is a continuous curve for which $\varphi(\gamma(t), t) = 0$ for all $t \in [0, T]$. Moreover, $\gamma(t)$ is analytic in $t \in [0, T]$ but cannot be extended to an analytic curve $\bar{\gamma} : (-\infty, T] \to \Omega$. In fact, its analytic extension is defined implicitly by (4.4) that has no roots if t very (negative) large.

Thus, we have obtained a curve $\gamma(t)$ for which the unique continuation property (4.2) fails and that shows that Condition 3 in Theorem 3.1 is sharp in the sense described above.

Example 3. This example shows that condition 3 is also sharp in the sense that it is possible to construct solutions of the adjoint heat equation (4.1) that vanish at curves $\gamma(t)$ that do not satisfy (3.11). We consider $\Omega = (0, 1)$ and the solution of the backward system (4.1) given by

$$\varphi(x, t) = e^{\lambda_2(t-T)} \sin(\sqrt{\lambda_2} x) - e^{\lambda_3(t-T)} \sin(\sqrt{\lambda_3} x) = e^{4\pi^2(t-T)} \sin(2\pi x) - e^{9\pi^2(t-T)} \sin(3\pi x).$$

It is easy to check that φ satisfies the following properties

$$\varphi(1/2,t) > 0 \text{ if } t \leq T,$$
$$\varphi(2/3,t) < 0 \text{ if } t \leq T,$$
$$\frac{\partial \varphi}{\partial x}(x,t) < 0 \text{ if } x \in [1/2,2/3] \text{ and } t \leq T,$$

Therefore, for any $t \leq T$, $\varphi(x,t)$ has a unique zero in $x \in (1/2,2/3)$ at the point $\gamma(t)$, i.e.

$$\varphi(\gamma(t),t) = 0, \text{ for all } t \leq T.$$

On the other hand, by the analytic version of the Implicit Function Theorem (see [H] p. 43, for example), the function $t \to \gamma(t)$ is analytic in $t \in (-\infty, T]$.

Moreover, as $\varphi(\gamma(t),t) = 0$ for all $t \in (-\infty, T]$,

$$\sin(2\pi\gamma(t)) = e^{5\pi^2(t-T)} \sin(3\pi\gamma(t)) \to 0, \text{ as } t \to -\infty.$$

Therefore $\gamma(t) \to 1/2$ as $t \to -\infty$ and, by the Taylor expansion of $\sin(2\pi x)$ and $\sin(3\pi x)$ near $x = 1/2$, we easily deduce that

$$\gamma(t) \sim \frac{1}{2} - \frac{1}{2\pi} e^{5\pi^2(t-T)}, \text{ as } t \to -\infty.$$

In this way we have constructed an example of a non-trivial solution of (4.1) that vanishes at a non-constant curve $\gamma(t)$ that does not satisfy (3.11).

Proof of Lemma 4.1: We divide the proof in different parts depending on the hypothesis we make on γ.

Part 1 Assume that γ satisfies hypothesis 1 in the statement of Theorem 3.1. Then there exists an open set $\mathcal{U} \subset (0,T)$ such that $\gamma : \mathcal{U} \to \Omega$ is not analytic at any point $t \in \mathcal{U}$. In this case, we prove the unique continuation property (4.2) by contradiction.

Assume that there exists a nontrivial solution $\varphi \in C([0,T]; H_0)$ of the heat equation (4.1) such that $\varphi(\gamma(t),t) = 0$ for all $t \in \mathcal{U}$. As the Laplace operator generates an analytic semigroup (see for example [P], p. 211), the solution $\varphi : \Omega \times (-\infty, T) \to I\!R$ of system (4.1) is analytic. By the analytic version of the Implicit Function Theorem either $\frac{\partial \varphi}{\partial x}(\gamma(t_0), t_0) = 0$ or there exists a unique solution $x = \hat{\gamma}(t)$ of $\varphi(x,t) = 0$ near $(x,t) = (\gamma(t_0), t_0)$ with $\hat{\gamma}$ analytic. The latter is in contradiction with the hypothesis on $x = \gamma(t)$. Therefore, $\frac{\partial \varphi}{\partial x}(\gamma(t_0), t_0) = 0$ and this is true for all $t_0 \in \mathcal{U}$.

As $\gamma(t)$ is a non-characteristic curve of the heat equation, Holmgren's uniqueness theorem asserts that the only solution φ of the heat equation with Cauchy data $\varphi(\gamma(t_0), t_0) = \frac{\partial \varphi}{\partial x}(\gamma(t_0), t_0) = 0$ on γ is $\varphi \equiv 0$. This is in contradiction with the hypothesis.

Part 2 Assume that γ satisfies hypothesis 2 in the statement of Theorem 3.1. Then there exists $t_1 \in (0,T)$ where $\gamma(t)$ is not analytic and a subinterval $(t_1, t_2) \subset (0,T)$ where γ is analytic and can be extended analytically to a subinterval (t_0, t_2) with $t_1 \in (t_0, t_2)$.

We continue $\gamma(t)$ analytically to $\bar{\gamma} : (t_0, t_2) \to I\!R$. Observe that the composition $\varphi(\bar{\gamma}(t), t)$ is analytic and therefore, in view of the fact that $\varphi(\bar{\gamma}(t), t) = \varphi(\gamma(t), t) = 0$ for $t \in (t_1, t_2)$,

$$\varphi(\bar{\gamma}(t), t) = 0, \qquad \forall t \in (t_0, t_2). \tag{4.5}$$

Let ω be the open region of $(x, t) \in I\!R \times (t_0, t_1)$ limited by the curves $x = \gamma(t)$, $x = \bar{\gamma}(t)$ and the horizontal line $t = t_0$. Note that φ vanishes at those parts of the boundary of ω constituted by γ and $\bar{\gamma}$.

Multiplying the adjoint heat equation (4.1) by the solution φ and integrating we easily obtain the following

$$0 = \int_{\bar{\gamma}(t)}^{\gamma(t)} (-\varphi_t - \varphi_{xx})\varphi \, dx = -\frac{1}{2}\frac{d}{dt}\int_{\bar{\gamma}(t)}^{\gamma(t)} |\varphi|^2 dx + \int_{\bar{\gamma}(t)}^{\gamma(t)} |\varphi_x|^2 dx \geq -\frac{1}{2}\frac{d}{dt}\int_{\bar{\gamma}(t)}^{\gamma(t)} |\varphi|^2 dx \tag{4.6}$$

Thus, the function $\Phi(t) = \int_{\bar{\gamma}(t)}^{\gamma(t)} |\varphi|^2 dx$ is increasing, positive and vanishes at $t = t_1$ because γ and $\bar{\gamma}$ coincide for $t = t_1$. Therefore,

$$\int_{\bar{\gamma}(t)}^{\gamma(t)} |\varphi|^2 dx = 0, \quad \forall t \in [t_0, t_1],$$

and we deduce that φ must be zero in the open non-empty set ω. By Holmgren's uniqueness theorem we deduce that $\varphi \equiv 0$.

Part 3 Assume that γ satisfies hypothesis 3 (a) in the statement of Theorem 3.1. Then, γ is analytic and can be extended analytically to a curve $\bar{\gamma}(t) : (-\infty, T] \to I\!R$ such that $\bar{\gamma}(t)$ meets the boundary of Ω, i.e. there exits $t_0 \in (-\infty, T]$ such that $\bar{\gamma}(t_0) \in \partial\Omega$.

The solution φ of (4.1) can be extended to all $x \in I\!R$ by odd extension and periodicity. The resulting φ satisfies the adjoint system (4.1) on $I\!R \times (-\infty, T)$. Observe that the composition $\varphi(\bar{\gamma}(t), t)$ is analytic and therefore

$$\varphi(\bar{\gamma}(t), t) = 0, \qquad \forall t \in (-\infty, T]. \tag{4.7}$$

By hypothesis, $\bar{\gamma}$ meets the boundary of Ω at $t_0 \in (-\infty, T]$ and $\gamma(t_0)$ must be one of the extremes of Ω, say $x = 0$. Let ω be the open region of $(x, t) \in I\!R \times (t_0 - \alpha, t_0)$ with $\alpha > 0$, bounded by $\bar{\gamma}$ and the axis $x = 0$. Note that ω cannot be an empty set because, in this case, $\bar{\gamma}$ and the axis $x = 0$ would coincide over the interval $(t_0 - \alpha, t_0)$ and, by the analyticity of $\bar{\gamma}$, they would coincide everywhere. This situation is excluded in our hypothesis on $\bar{\gamma}$.

Note also that φ vanishes on the subset of the boundary of ω constituted by $x = 0$ and $\bar{\gamma}$. Then we can argue as in Part 2 above to prove that $\varphi \equiv 0$ on ω and therefore, $\varphi \equiv 0$.

Part 4 Assume that γ satisfies either hypothesis 3 (b) or 3 (c) in the statement of Theorem 3.1.

Let $\varphi \in C([0, T]; H_0)$ be a solution of (4.1) with $\varphi(\gamma(t), t) = 0$ for all $t \in [0, T]$. Obviously, this solution can be extended naturally to all $t \leq T$ and the extension is analytic.

On the other hand, by hypothesis, γ can be also extended to an analytic curve $\bar{\gamma} : (-\infty, T] \to \Omega$. Therefore, the composition $\varphi(\bar{\gamma}(t), t)$ is still analytic in $t \in (-\infty, T]$.

Now observe that $\bar{\gamma}$ and γ coincide in $t \in [0, T]$. Therefore, $\varphi(\bar{\gamma}(t), t) = \varphi(\gamma(t), t) = 0$ for any $t \in [0, T]$ and by the unique continuation of analytic functions we deduce that

$$\varphi(\bar{\gamma}(t), t) = 0 \qquad \forall t \in (-\infty, T]. \tag{4.8}$$

Let us introduce the Fourier representation of φ

$$\varphi(x, t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j (T-t)} w_j(x)$$

where

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots$$

are the eigenvalues of (3.13), $w_j(x)$ is the eigenfunction associated to λ_j, and c_j the Fourier coefficients. We choose $\{w_j(x)\}_{j \geq 1}$ to be orthonormal in $H_0^1(\Omega)$. Observe that $\varphi(T) = \varphi^0 \in H_0 = H_0^1(\Omega)$ and this implies that

$$\sum_{j \geq 1} |c_j|^2 < \infty.$$

From (4.8) we have

$$0 = \varphi(\bar{\gamma}(t), t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j (T-t)} w_j(\bar{\gamma}(t)), \quad \forall t \in (-\infty, T]. \tag{4.9}$$

We have to prove that this implies that $c_j = 0$ for all $j \geq 1$. We proceed by induction in j. We consider the case $j = 1$. Multiplying the series in (4.9) by $e^{\lambda_1 (T-t)}$ we obtain

$$c_1 w_1(\bar{\gamma}(t)) + \sum_{j=2}^{\infty} c_j e^{(\lambda_1 - \lambda_j)(T-t)} w_j(\bar{\gamma}(t)) = 0, \quad \forall t \in (-\infty, T]. \tag{4.10}$$

The second term on the left hand side converges to zero as $t \to -\infty$. Indeed,

$$\begin{aligned} \left| \sum_{j=2}^{\infty} c_j e^{(\lambda_1 - \lambda_j)(T-t)} w_j(\bar{\gamma}(t)) \right|^2 &\leq \left\| \sum_{j=2}^{\infty} c_j e^{(\lambda_1 - \lambda_j)(T-t)} w_j \right\|^2_{L^\infty(\Omega)} \qquad (4.11) \\ \leq \ C \left\| \sum_{j=2}^{\infty} c_j e^{(\lambda_1 - \lambda_j)(T-t)} w_j \right\|^2_{H_0^1(\Omega)} &= C \sum_{j=2}^{\infty} e^{2(\lambda_1 - \lambda_j)(T-t)} |c_j|^2 \end{aligned}$$

which converges to zero as $t \to -\infty$. Consequently, the first term in (4.10) converges to zero, as $t \to -\infty$, too.

Assume that γ satisfies hypothesis 3 (b) or 3 (c) in the statement of Theorem 3.1. Let $x_0 \in \Omega$ be an accumulation point of $\bar{\gamma}(t)$ as $t \to -\infty$, and consider $t_n \to -\infty$ such that

$$\bar{\gamma}(t_n) \to x_0.$$

Passing to the limit as $(x_n, t_n) \to (x_0, -\infty)$ in (4.10) we obtain

$$c_1 w_1(x_0) = 0. \tag{4.12}$$

As the zeroes of the first eigenfunction w_1 lie on the boundary of Ω we deduce that $c_1 = 0$.

Now we complete the induction argument in j. Assume that $c_j = 0$ for all $j < J$, $J \geq 2$, and let us prove that $c_J = 0$. First of all, we observe that (4.9) reads as follows

$$0 = \varphi(\bar{\gamma}(t), t) = \sum_{j=J}^{\infty} c_j e^{-\lambda_j (T-t)} w_j(\bar{\gamma}(t)), \quad \forall t \in (-\infty, T]. \tag{4.13}$$

Multiplying now the series in (4.13) by $e^{\lambda_J (T-t)}$ we obtain

$$c_J w_J(\bar{\gamma}(t)) + \sum_{j=J+1}^{\infty} c_j e^{(\lambda_J - \lambda_j)(T-t)} w_j(\bar{\gamma}(t)) = 0, \quad \forall t \in (-\infty, T]. \tag{4.14}$$

Once again, it is easy to see that the second term on the left hand side converges to zero as $t \to -\infty$.

Let $x_0 \in \Omega$ be an accumulation point of $\bar{\gamma}(t)$, as $t \to -\infty$ and consider $t_n \to -\infty$ such that

$$\bar{\gamma}(t_n) \to x_0.$$

Passing to the limit as $(x_n, t_n) \to (x_0, -\infty)$ in (4.10) we obtain

$$c_J w_J(x_0) = 0. \tag{4.15}$$

Now we distinguish the cases where γ satisfies hypothesis 3 (b) or 3 (c) in the statement of Theorem 3.1.

Assume that γ satisfies hypothesis 3 (b). Then we can choose x_0 in such a way that x_0/L is irrational. As the zeroes of the eigenfunctions w_j lie on points $x \in \Omega$ such that x/L is rational, we deduce that $c_J = 0$.

On the other hand, if γ satisfies hypothesis 3 (c) in the statement of Theorem 3.1 then (4.15) is not enough to guarantee that $c_J = 0$ because the eigenfunction w_J may vanish at x_0.

Multiplying (4.14) by $(\bar{\gamma}(t) - x_0)^{-1}$ we obtain,

$$0 = \frac{c_J w_J(\bar{\gamma}(t_n))}{\bar{\gamma}(t_n) - x_0} + \sum_{j=J+1}^{\infty} \frac{e^{(\lambda_J - \lambda_j)(T-t_n)}}{\bar{\gamma}(t_n) - x_0} c_j w_j(\bar{\gamma}(t_n)). \tag{4.16}$$

Due to (4.15), the first term on the right hand side of (4.16) converges to $c_J w_J'(x_0)$ as $t_n \to -\infty$.

On the other hand, following the argument in (4.11) we easily find the following bound for the second term in (4.16)

$$\begin{aligned}\left|\sum_{j=J+1}^{\infty}\frac{e^{(\lambda_J-\lambda_j)(T-t_n)}}{\bar\gamma(t_n)-x_0}c_jw_j(\bar\gamma(t_n))\right|^2 &\leq \sum_{j=J+1}^{\infty}\frac{e^{-2(T-t_n)(\lambda_{J+1}-\lambda_J)}}{|\bar\gamma(t_n)-x_0|^2}|c_j|^2\\ &\leq \frac{e^{-2(T-t_n)(\lambda_3-\lambda_2)}}{|\bar\gamma(t_n)-x_0|^2}\sum_{j=J+1}^{\infty}|c_j|^2,\end{aligned}$$

that converges to zero as $t_n\to-\infty$ by the hypothesis (3.11).

Therefore, passing to the limit as $t_n\to-\infty$ in (4.16) we obtain

$$c_Jw_J'(x_0)=0. \tag{4.17}$$

From (4.15), (4.17) and the fact that the eigenfunction w_J satisfies a linear second order ordinary differential equation we deduce that $c_J=0$.

■

4.2 The higher dimensional case

The following lemma reduces the unique continuation problem (4.2) to a certain unique continuation property for the eigenfunctions of (3.13).

Lemma 4.2 *Assume that $\{\gamma(t)\}_{0\le t\le T}$ is an analytic family of k-dimensional manifolds on Ω which satisfies the hypothesis (3.12). Let us consider the set of accumulation points P, and for each $x\in P$, the set of accumulation directions D_x, as defined in Definition 3.2. If the spectral unique continuation property (3.16) holds, then the unique continuation property (4.2) for the solutions of the adjoint problem (4.1) holds as well.*

Proof: Let $\varphi\in C([0,T];H_0)$ be a solution of (4.1) with $\varphi(x,t)=0$ for all $(x,t)\in\{\gamma(t)\}_{0\le t\le T}$. Obviously, this solution can be extended naturally to all $t\le T$ by solving (4.1). As the Laplace operator generates an analytic semigroup (see for example [P], p. 211), the solution $\varphi:\Omega\times(-\infty,T)\to I\!R$ of system (4.1) is analytic.

On the other hand, the family $\{\gamma(t)\}_{0\le t\le T}$ can be extended to an analytic family of manifolds $\{\bar\gamma(t)\}_{-\infty<t\le T}$. Let $\{\psi_\alpha(y,t)\}_{\alpha=1}^A$ be a family of charts for $\{\bar\gamma(t)\}_{-\infty<t\le T}$, as in Definition 3.1. Then, for any $y\in V_\alpha$, $\psi_\alpha(y,t)$ is analytic in t and therefore, the composition $\varphi(\psi_\alpha(y,t),t)$ is still analytic in t.

Thus, the fact that $\varphi(x,t)=0$ for all $(x,t)\in\{\gamma(t)\}_{0\le t\le T}$ implies that $\varphi(\psi_\alpha(y,t),t)$ vanishes for $t\in[0,T]$ and $y\in V_\alpha$ with $\alpha\in A$. Therefore, by the analyticity of $\varphi(\psi_\alpha(y,t),t)$ we have that

$$\varphi(\psi_\alpha(y,t),t)=0\quad\forall t\in(-\infty,T],\quad\text{and }\forall y\in V_\alpha\text{ with }\alpha=1,...,A, \tag{4.18}$$

i.e.

$$\varphi(x,t)=0\quad\forall(x,t)\in\{\bar\gamma(t)\}_{-\infty<t\le T}. \tag{4.19}$$

Let us introduce the Fourier representation of φ

$$\varphi(x,t) = \sum_{j=1}^{\infty} e^{-\lambda_j (T-t)} \sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(x)$$

where

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots$$

are the eigenvalues of (3.13), $\{l(j)\}_{j\geq 1}$ their multiplicity and $\{w_{j,k}(x)\}_{k=1,...,l(j)}$ is a system of linear independent eigenfunctions associated to λ_j. We choose $\{w_{j,k}(x)\}_{j,k\geq 1}$ to be an orthonormal basis of H_0. Observe that $\varphi(T) = \varphi^0 \in H_0$ and this implies that

$$\sum_{j,k} |c_{j,k}|^2 < \infty.$$

From (4.19) we have

$$0 = \varphi(x,t) = \sum_{j=1}^{\infty} e^{-\lambda_j (T-t)} \sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(x), \quad \forall (x,t) \in \{\gamma(t)\}_{-\infty<0\leq T}. \tag{4.20}$$

We have to prove that this implies that $c_{j,k} = 0$ for all $j \geq 1$ and $k = 1, ..., l(j)$. We proceed by induction in j.

First, we consider the case $j = 1$. Observe that the first eigenvalue of the Laplace operator λ_1 is simple. Then $l(1) = 1$ and we only have to prove that $c_{1,1} = 0$. Observe also that, from the spectral unique continuation property (3.16), it suffices to prove that

$$c_{1,1} w_{1,1}(x_0 + \mu v) = 0 \quad \forall x_0 \in P,\ \forall v \in D_{x_0}, \quad \text{and } \forall \mu \in I\!R \text{ s.t. } x_0 + \bar{\mu} v \in \Omega \text{ for all } \bar{\mu} \in [0,\mu]. \tag{4.21}$$

Moreover, by the analyticity of the eigenfunction $c_{1,1}w_{1,1}$, this is equivalent to prove that the function $G_{x_0,v,1}(\mu) = c_{1,1}w_{1,1}(x_0 + \mu v)$ satisfies

$$G_{x_0,v,1}(0) = \frac{d^r G_{x_0,v,1}}{d\mu^r}(0) = 0, \quad \forall r \geq 1,\ \forall x_0 \in P \text{ and } \forall v \in D_{x_0}. \tag{4.22}$$

We use an induction argument in r. We start proving that $G_{x_0,v,1}(0) = c_{1,1}w_{1,1}(x_0) = 0$.

Multiplying the series in (4.20) by $e^{\lambda_1 (T-t)}$ and taking into account that λ_1 is simple we obtain

$$c_{1,1} w_{1,1}(x) + \sum_{j=2}^{\infty} e^{(\lambda_1 - \lambda_j)(T-t)} \sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(x) = 0, \quad \forall (x,t) \in \{\gamma(t)\}_{-\infty<t\leq T}. \tag{4.23}$$

The second term on the left hand side converges to zero as $t \to -\infty$ uniformly in $x \in \Omega$. Indeed,

$$\left| \sum_{j=2}^{\infty} e^{(\lambda_1 - \lambda_j)(T-t)} \sum_{k=1}^{l(k)} c_{j,k} w_{j,k}(x) \right|^2 \leq \left\| \sum_{j=2}^{\infty} e^{(\lambda_1 - \lambda_j)(T-t)} \sum_{k=1}^{l(k)} c_{j,k} w_{j,k} \right\|^2_{L^\infty(\Omega)}$$

$$\leq \quad C \left\| \sum_{j=2}^{\infty} e^{(\lambda_1 - \lambda_j)(T-t)} \sum_{k=1}^{l(k)} c_{j,k} w_{j,k} \right\|^2_{H_1} = C \sum_{j=2}^{\infty} e^{2(\lambda_1 - \lambda_j)(T-t)} \sum_{k=1}^{l(k)} |c_{j,k}|^2$$

which converges to zero as $t \to -\infty$.

Consider now $x_0 \in P$ and $v \in D_{x_0}$. There exists a sequence $(x_n, t_n) \in \{\bar{\gamma}(t)\}_{-\infty<t\leq T}$ such that

$$x_n \to x_0, \quad \frac{x_n - x_0}{\|x_n - x_0\|} \to v \text{ and } t_n \to -\infty. \tag{4.24}$$

Passing to the limit as $(x_n, t_n) \to (x_0, -\infty)$ in (4.23) we obtain

$$G_{x_0,v,1}(0) = c_{1,1} w_{1,1}(x_0) = 0. \tag{4.25}$$

Now we complete the induction argument on r to prove (4.22). We assume that $\frac{d^r G_{x_0,v,1}}{d\mu^r}(0) = 0$ for $r \leq R-1$. Then, from the Taylor expansion of $G_{x_0,v,1}$ at $\mu = 0$ we have

$$\frac{d^R G_{x_0,v,1}}{d\mu^R}(0) = \lim_{\mu\to 0} \frac{R!\, G_{x_0,v,1}(\mu)}{\mu^R} = \lim_{\mu_n\to 0} \frac{R!\, G_{x_0,v_n,1}(\mu_n)}{\mu_n^R}, \tag{4.26}$$

where we can choose x_n and v_n as in (4.24) and $\mu_n = \|x_n - x_0\|$.

On the other hand, we have

$$\begin{aligned}
\left|\frac{G_{x_0,v_n,1}(\mu_n)}{\mu_n^R}\right|^2 &= \left|c_{1,1}\frac{w_{1,1}(x_n)}{\|x_n - x_0\|^R}\right|^2 = \left|\sum_{j=2}^{\infty} \frac{e^{(\lambda_1-\lambda_j)(T-t_n)}}{\|x_n - x_0\|^R} \sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(x_n)\right|^2 \\
&\leq \left\|\sum_{j=2}^{\infty} \frac{e^{(\lambda_1-\lambda_j)(T-t_n)}}{\|x_n - x_0\|^R} \sum_{k=1}^{l(j)} c_{j,k} w_{j,k}\right\|_{L^\infty}^2 \leq C \left\|\sum_{j=2}^{\infty} \frac{e^{(\lambda_1-\lambda_j)(T-t_n)}}{\|x_n - x_0\|^R} \sum_{k=1}^{l(j)} c_{j,k} w_{j,k}\right\|_{H_1}^2 \\
&= C \sum_{j=2}^{\infty} \frac{e^{2(\lambda_1-\lambda_j)(T-t_n)}}{\|x_n - x_0\|^{2R}} \sum_{k=1}^{l(j)} |c_{j,k}|^2 \leq C \frac{e^{-2(\lambda_2-\lambda_1)(T-t_n)}}{\|x_n - x_0\|^{2R}} \sum_{j=2}^{\infty}\sum_{k=1}^{l(j)} |c_{j,k}|^2.
\end{aligned} \tag{4.27}$$

which converges to zero as $t_n \to -\infty$ since the factor

$$\frac{e^{-2(\lambda_2-\lambda_1)(T-t_n)}}{\|x_n - x_0\|^{2R}} = \left(\frac{e^{-\frac{\lambda_2-\lambda_1}{R}(T-t_n)}}{\|x_n - x_0\|}\right)^{2R}$$

converges to zero as $t_n \to -\infty$, i.e. as $\mu \to 0$, due to the hypothesis on the accumulation directions given in (3.15).

Here we see why we only include 'non-exponential directions' in the definition of D_{x_0} in (3.15).

This concludes the proof of (4.22). From (4.22) we deduce (4.21) and then

$$c_{1,1} = 0.$$

Following an induction argument in j we easily obtain

$$\frac{d^r G_{x_0,v,j}}{d\mu^r}(0) = 0, \quad \text{for all } j \text{ and } \forall x_0 \in P,\, \forall v \in D_{x_0}, \tag{4.28}$$

where $G_{x_0,v,j}(\mu) = \sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(x_0 + \mu v)$.

From (4.28) and the analyticity of the eigenfunctions $w_{j,k}$ we have

$$\sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(x_0 + \mu v) = 0, \quad \forall j,\ \forall x_0 \in P,\ \forall v \in D_{x_0} \text{ and } \forall \mu \in I\!R \text{ s.t. } [x_0, x_0 + \mu v] \in \Omega. \quad (4.29)$$

On the other hand, taking into account the fact that $\sum_{k=1}^{l(j)} c_{j,k} w_{j,k}$ is an eigenfunction and by the unique continuation hypothesis (3.16) for the eigenfunctions we obtain

$$\sum_{k=1}^{l(j)} c_{j,k} w_{j,k} \equiv 0, \quad \text{for all } j \geq 1.$$

Therefore $c_{j,k} = 0$ for all $k = 1, \dots l(j)$ because of the linear independence of $w_{j,k}$. This concludes the proof of the lemma.

■

5 Averaging of rapidly oscillating controls

In this section we prove Theorem 3.3. We introduce the variational approach of the HUM method to characterize the controls f_ε of (3.17) of minimal norm (see [L2]). Then we pass to the limit, as $\varepsilon \to 0$, in these equations to obtain the limit control. The main ingredient to establish the convergence is Lemma 5.1 below. The proof of Theorem 3.3 will be given after the proof of Lemma 5.1.

Lemma 5.1 *Let $\{\gamma(t)\}_{0\leq t\leq T}$ be an analytic $2\pi-$periodic family of $k-$dimensional manifolds. Consider a sequence $u_\varepsilon^0 \rightharpoonup u^0$ that weakly converges in H_0. Let u_ε, u be the solutions of the homogeneous system (3.1) with $f = 0$, and initial data u_ε^0, u^0 respectively. Let $\gamma_\varepsilon(t) = \gamma(t/\varepsilon)$. Then*

$$\int_0^T \int_{\gamma_\varepsilon(t)} |u_\varepsilon(x,t)|^2 \, d\sigma \, dt \to \int_0^T < u(x,t) m_\gamma(x), u(x,t) >_1 dt, \qquad \text{if } k \geq 1,$$
$$\int_0^T |u_\varepsilon(\gamma_\varepsilon(t),t)|^2 \, dt \to \int_0^T < u(x,t) m_\gamma(x), u(x,t) >_1, \qquad \text{if } k = 0, \qquad (5.1)$$

where $m_\gamma(x)$ is the weak- limit of $\delta_{\gamma(t/\varepsilon)}$ in $L^\infty(0,T;H_1')$, and $<\cdot,\cdot>_1$ denotes the duality pairing between H_1 and its dual.*

The density $m_\gamma(x)$ does not depend on the time variable t, it is supported on the nonempty open set ω, defined as the interior set, with respect to the relative topology, of the range of $\{\gamma(t)\}_{0\leq t\leq 2\pi}$, and it is characterized by (3.20).

Moreover, if $\varphi \in C_0^\infty(\Omega \times (0,T))$ *then*

$$\int_0^T \int_{\gamma_\varepsilon(t)} u_\varepsilon(x,t)\varphi(x,t)\ d\sigma\ dt \to \int_0^T < u(x,t)m_\gamma(x), \varphi(x,t) >_1 dt, \qquad \textit{if } k \geq 1,$$
$$\int_0^T u_\varepsilon(\gamma_\varepsilon(t),t)\varphi(x,t)dt \to \int_0^T < u(x,t)m_\gamma(x), \varphi(x,t) >_1 dt, \qquad \textit{if } k = 0. \tag{5.2}$$

Proof of Lemma 5.1: To simplify the presentation we consider the case $k \geq 1$. The case $k = 0$ is analogous.

The sequence $u_\varepsilon(x,t)$ of solutions of the homogeneous system (3.1) with $f = 0$ and initial data u_ε^0 can be written in the Fourier representation

$$u_\varepsilon(x,t) = \sum_{j=1}^\infty e^{-\lambda_j t} \sum_{k=1}^{l(j)} c_{j,k}^\varepsilon w_{j,k}(x).$$

We assume that $(w_{j,k})_{j,k\geq 1}$ constitute an orthonormal basis in H_0. Analogously, the solution $u(x,t)$ of the homogeneous system (2.4) with $f = 0$ and initial data u^0, is

$$u(x,t) = \sum_{j=1}^\infty e^{-\lambda_j t} \sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(x).$$

Due to the weak convergence of the initial data $u_\varepsilon^0 \rightharpoonup u^0$ in H_0 we have

$$\sum_{j,k\geq 1} |c_{j,k}^\varepsilon|^2 \leq C, \qquad \sum_{j,k\geq 1} |c_{j,k}|^2 \leq C, \tag{5.3}$$

with C independent of ε. Moreover,

$$c_{j,k}^\varepsilon \to c_{j,k}, \qquad \text{as } \varepsilon \to 0, \qquad \forall j,k \geq 1.$$

Let us prove the convergence result stated in (5.1). To avoid the singularity of the solution u_ε at $t = 0$ we divide the left hand side integral in (5.1) in two parts

$$\int_0^T \int_{\gamma_\varepsilon(t)} |u_\varepsilon(x,t)|^2\ d\sigma\ dt = \int_0^\delta \int_{\gamma_\varepsilon(t)} |u_\varepsilon(x,t)|^2\ d\sigma\ dt + \int_\delta^T \int_{\gamma_\varepsilon(t)} |u_\varepsilon(x,t)|^2\ d\sigma\ dt \tag{5.4}$$

with $\delta > 0$ to be chosen later.

We first estimate the first integral in (5.4). By classical estimates on the heat kernel (see [CH], p. 44) we know that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{n}{2q}} \left\|u_\varepsilon^0\right\|_{L^q(\Omega)}$$

with C a constant that does not depend on ε. Therefore the first integral in (5.4) can be estimated by

$$\int_0^\delta \int_{\gamma_\varepsilon(t)} |u_\varepsilon(x,t)|^2 \, d\sigma \, dt \le C \left\| u_\varepsilon^0 \right\|^2_{L^q(\Omega)} \int_0^\delta t^{-\frac{n}{q}} \int_{\gamma_\varepsilon(t)} d\sigma \, dt$$
$$\le C \left\| u_\varepsilon^0 \right\|^2_{L^q(\Omega)} \max_{t\in[0,\delta]} (\text{ meas } (\gamma(t/\varepsilon))) \int_0^\delta t^{-\frac{n}{q}} \, dt = C \left\| u_\varepsilon^0 \right\|^2_{L^q(\Omega)} \frac{\delta^{1-\frac{n}{q}}}{1-\frac{n}{q}} \max_{t\in[0,2\pi]} \text{ meas } (\gamma(t)).$$

We set $q = 2$ if $n = 1$ and $q = 4$ if $n = 2, 3$. Then, by the continuous Sobolev embeddings $H_0^1(\Omega) \to L^\infty(\Omega)$ if $n = 1$ and $H_0^1(\Omega) \to L^4(\Omega)$ for $n = 2, 3$ we easily deduce the estimate

$$\int_0^\delta \int_{\gamma_\varepsilon(t)} |u_\varepsilon(x,t)|^2 \, d\sigma \, dt \le C(\delta, n, \gamma) \left\| u_\varepsilon^0 \right\|^2_{H_0} \tag{5.5}$$

with $C(\delta, n, \gamma)$ independent of ε and such that

$$C(\delta, n, \gamma) \to 0, \qquad \text{as } \delta \to 0.$$

Note that the estimate above holds since the manifold γ is time-periodic and consequently

$$\max_{t\in[0,\delta]} (\text{ meas } \gamma(t/\varepsilon)) = \max_{t\in[0,2\pi]} (\text{ meas } \gamma(t/\varepsilon)) < \infty$$

as $\varepsilon \to 0$.

Thus, to prove the convergence result stated in (5.1), it suffices to show that the second integral in (5.4), for $\delta > 0$ fixed, tends to

$$\int_\delta^T < u(x,t) m_\gamma(x,t), u(x,t) >_1 \, dt$$

as $\varepsilon \to 0$. Indeed, once this is proved, (5.1) is obtained passing to the limit, as $\delta \to 0$, in (5.4).

We have

$$\begin{aligned} \int_\delta^T \int_{\gamma_\varepsilon(t)} |u_\varepsilon(x,t)|^2 \, d\sigma \, dt &= \int_\delta^T \int_{\gamma_\varepsilon(t)} \sum_{j,i=1}^\infty \sum_{k=1}^{l(j)} \sum_{m=1}^{l(i)} e^{-(\lambda_i+\lambda_j)t} c^\varepsilon_{j,k} c^\varepsilon_{i,m} w_{j,k}(x) w_{i,m}(x) \, d\sigma \, dt \\ &= \sum_{j,i=1}^\infty \sum_{k=1}^{l(j)} \sum_{m=1}^{l(i)} \int_\delta^T \int_{\gamma_\varepsilon(t)} e^{-(\lambda_i+\lambda_j)t} c^\varepsilon_{j,k} c^\varepsilon_{i,m} w_{j,k}(x) w_{i,m}(x) \, d\sigma \, dt. \end{aligned}$$

Now we take the limit as $\varepsilon \to 0$,

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_\delta^T \int_{\gamma_\varepsilon(t)} |u_\varepsilon(x,t)|^2 \, d\sigma \, dt \\ &= \lim_{\varepsilon\to 0} \sum_{j,i=1}^\infty \sum_{k=1}^{l(j)} \sum_{m=1}^{l(i)} \int_\delta^T \int_{\gamma_\varepsilon(t)} e^{-(\lambda_i+\lambda_j)t} c^\varepsilon_{j,k} c^\varepsilon_{i,m} w_{j,k}(x) w_{i,m}(x) \, d\sigma \, dt \\ &= \sum_{j,i=1}^\infty \sum_{k=1}^{l(j)} \sum_{m=1}^{l(i)} c_{j,k} c_{i,m} \lim_{\varepsilon\to 0} \int_\delta^T \int_{\gamma_\varepsilon(t)} e^{-(\lambda_i+\lambda_j)t} w_{j,k}(x) w_{i,m}(x) \, d\sigma \, dt. \end{aligned} \tag{5.6}$$

Interchanging the sum and the limit is justified because of the dominated convergence theorem. Indeed, each term of the series can be bounded above as follows

$$\begin{aligned}
&\left| c^{\varepsilon}_{j,k} c^{\varepsilon}_{i,m} \int_{\delta}^{T} \int_{\gamma_\varepsilon(t)} e^{-(\lambda_i+\lambda_j)t} w_{j,k}(x) w_{i,m}(x)\ d\sigma\ dt \right| \\
\leq\ & |c^{\varepsilon}_{j,k} c^{\varepsilon}_{i,m}| \max_{t\in[\delta,T]} \left| \int_{\gamma_\varepsilon(t)} w_{j,k}(x) w_{i,m}(x)\ d\sigma \right| \int_{\delta}^{T} e^{-(\lambda_i+\lambda_j)t} dt \\
\leq\ & \left(\sum_{i=1}^{\infty} \sum_{m=1}^{l(i)} |c^{\varepsilon}_{i,m}|^2 \right) \max_{t\in[0,2\pi]} \left| \int_{\gamma(t)} w_{j,k}(x) w_{i,m}(x)\ d\sigma \right| \frac{e^{-(\lambda_i+\lambda_j)\delta} - e^{-(\lambda_i+\lambda_j)T}}{\lambda_i+\lambda_j} \\
\leq\ & \|u^{\varepsilon}_0\|^2_{H_0} \max_{t\in[0,2\pi]} \left| \int_{\gamma(t)} w_{j,k}(x) w_{i,m}(x)\ d\sigma \right| \frac{e^{-(\lambda_i+\lambda_j)\delta}}{\lambda_i+\lambda_j} \\
\leq\ & C \max_{t\in[0,2\pi]} \left| \int_{\gamma(t)} w_{j,k}(x) w_{i,m}(x)\ d\sigma \right| \frac{e^{-(\lambda_i+\lambda_j)\delta}}{\lambda_i+\lambda_j},
\end{aligned} \tag{5.7}$$

since, by hypothesis, the sequence of initial data u^0_ε is uniformly bounded in H_0 (see (5.3)). Now, by Hölder inequality and the trace theorem,

$$\left| \int_{\gamma(t)} w_{j,k}(x) w_{i,m}(x)\ d\sigma \right| \leq \|w_{j,k}\|_{L^2(\gamma(t))} \|w_{i,m}\|_{L^2(\gamma(t))} \leq C(\gamma(t)) \|w_{j,k}\|_{H_1} \|w_{i,m}\|_{H_1}, \tag{5.8}$$

where $C(\gamma(t))$ is a constant that only depends on the measure of $\gamma(t)$. Note that, in view of the hypothesis on $\gamma(t)$, this constant can be chosen independently of time, i.e. there exists $C(\gamma)$ independent of t such that $C(\gamma(t)) \leq C(\gamma)$ for all $t \in [0, 2\pi]$. Moreover, taking into account the normalization of the eigenfunctions in H_0, we have

$$\|w_{j,k}\|_{H_1} \leq C\sqrt{\lambda_j}\, \|w_{j,k}\|_{H_0} = C\sqrt{\lambda_j},$$

and the terms in (5.7) can be bounded above by

$$C(\gamma) \frac{\sqrt{\lambda_j}\sqrt{\lambda_i}}{\lambda_j+\lambda_i} e^{-(\lambda_j+\lambda_i)\delta} \leq C(\gamma) e^{-(\lambda_j+\lambda_i)\delta} \tag{5.9}$$

with $C(\gamma)$ a constant that does not depend on ε, i, j, k, m. The sum of all these terms in i, j, k, m is finite due to the well-known asymptotic behavior of the eigenvalues of the Laplace operator. Indeed,

$$\sum_{i,j\geq 1} \sum_{m=1}^{l(i)} \sum_{k=1}^{l(j)} e^{-(\lambda_i+\lambda_j)\delta} = \left(\sum_{i\geq 1} \sum_{m=1}^{l(i)} e^{-\lambda_i \delta} \right)^2 .$$

The last sum is finite. Recall that the number of eigenvalues less than a constant λ, including multiplicity, is asymptotically equal to $\lambda|\Omega|/4\pi$ if $n = 2$, and $\lambda^{3/2}|\Omega|/6\pi^2$ if $n = 3$ (see [CoH], pag. 442). Then, for example, in the case $n=3$ we have

$$\sum_{i\geq 1}\sum_{m=1}^{l(i)} e^{-\lambda_i\delta} = \sum_{k=1}^{\infty}\sum_{k-1\leq\lambda_i\leq k} l(i)e^{-\lambda_i\delta} \leq C\sum_{k=1}^{\infty} k^{3/2}e^{-(k-1)\delta} < \infty.$$

Once we have checked (5.6), we pass to the limit in each one of the terms in the right hand side of (5.6). Then,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\delta^T\int_{\gamma_\varepsilon(t)} e^{-(\lambda_i+\lambda_j)t}w_{j,k}(x)w_{i,m}(x)\, d\sigma\, dt = \lim_{\varepsilon\to 0}\int_\delta^T <\delta_{\gamma_\varepsilon(t)}, e^{-(\lambda_i+\lambda_j)t}w_{j,k}(x)w_{i,m}(x)>_1 \\
&= \int_\delta^T < m_\gamma(x,t), e^{-(\lambda_j+\lambda_i)t}w_{j,k}(x)w_{i,m}(x)>_1\ dt \\
&= \int_\delta^T < e^{-\lambda_i t}w_{i,m}(x)m_\gamma(x,t), e^{-\lambda_j t}w_{j,k}(x)>_1\ dt,
\end{aligned}$$

where $m_\gamma(x,t)$ is the weak-* limit of $\delta_{\gamma_\varepsilon(t)}$ in $L^\infty(0,T;H_1)$ and $<\cdot,\cdot>_1$ is the duality pairing between H_1 and its dual space.

Thus,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\delta^T\int_{\gamma_\varepsilon(t)} |u_\varepsilon(x,t)|^2 d\sigma\ dt \\
=&\ \sum_{j,i=1}^{\infty}\sum_{k=1}^{l(j)}\sum_{m=1}^{l(i)} c_{j,k}c_{i,m}\int_\delta^T < e^{-\lambda_i t}w_{i,m}(x)m_\gamma(x,t), e^{-\lambda_j t}w_{j,k}(x)>_1\ dt \\
=&\ \int_\delta^T < u(x,t)m_\gamma(x,t), u(x,t)>_1 dt,
\end{aligned}$$

for all $\delta>0$.

It remains to prove that the limit density $m_\gamma(x,t)$ does not depend on the time variable t and that it is supported on the region ω, constituted by the range of $\gamma(t)$ over the period $t\in[0,2\pi]$. In fact, the limit density m_γ is characterized by

$$\int_0^T < m_\gamma,\varphi>_1\ dt = \lim_{\varepsilon\to 0}\int_0^T\int_{\gamma(t/\varepsilon)}\varphi(x,t)d\sigma\ dt, \qquad \forall\varphi(x,t)\in L^1(0,T;H_1).$$

Taking into account that $C_0^\infty(\Omega)\times C_0^\infty(0,T)$ is sequentially dense in $C_0^\infty(\Omega\times(0,T))$, which is dense in $L^1(0,T;H_1)$, m_γ is also characterized by

$$\int_0^T < m_\gamma,\varphi>_1\ dt = \lim_{\varepsilon\to 0}\int_0^T\int_{\gamma(t/\varepsilon)}\varphi(x)\psi(t)d\sigma\ dt, \qquad \forall\varphi(x)\in H_1, \quad \psi(t)\in L^1(0,T).$$

Note that $F(s)=\int_{\gamma(t/\varepsilon)}\varphi(x)d\sigma$ is a 2π-periodic function and then $F(s/\varepsilon)$ converges weakly to its average in L^2_{loc} as $\varepsilon\to 0$. Therefore,

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\gamma_\varepsilon(t)}\varphi(x)\psi(t)\ d\sigma\ dt = \frac{1}{2\pi}\int_0^{2\pi}\int_{\gamma(s)}\varphi(x)\ d\sigma\ ds\int_0^T\psi(t)\ dt. \tag{5.10}$$

This last integral can be written as an integral over the region $\omega \times (0,T)$ where ω is the interior set of the range of $\{\gamma(t)\}_{t\in[0,2\pi]}$. The control is weighted by a suitable density function m_γ, supported on ω, and independent of the time variable t. Moreover, this limit $m_\gamma(x)$ is characterized by

$$\int_\omega \varphi(x)m_\gamma(x)\ d\sigma = \frac{1}{2\pi}\int_0^{2\pi}\int_{\gamma(s)}\varphi(x)\ d\sigma\ ds, \qquad \forall\varphi\in H_1. \tag{5.11}$$

Note that m_γ depends, roughly, on the time derivative of $\gamma(t)$ over a period $[0,2\pi]$.

The proof of (5.2) is similar. We only have to take into account that $C_0^\infty(\Omega)\times C_0^\infty(0,T)$ is sequentially dense in $C_0^\infty(\Omega\times(0,T))$ and then it suffices to check (5.2) for test functions in separated variables. This concludes the proof of the lemma.

■

Proof of Theorem 3.3: We first restrict ourselves, without loss of generality, to the case where $u^0=0$ and $\|u^1\|_{H_0}\geq\alpha$.

Given $\varepsilon>0$, system (3.1) is approximately controllable. Indeed according to Lemma 4.2 and, in view of the assumptions on the curve $\gamma(t)$, the unique continuation property (4.2) holds for $\gamma_\varepsilon(t)=\gamma(t/\varepsilon)$ for all $\varepsilon>0$. Then the control that makes (3.10) to hold is given by $f_\varepsilon=\overline{\varphi}_\varepsilon(\gamma_\varepsilon(t),t)$, where $\overline{\varphi}_\varepsilon$ solves (4.1), the initial data $\overline{\varphi}_\varepsilon^0$ being the minimizer of the functional

$$J_\varepsilon(\varphi^0)=\frac{1}{2}\int_0^T\int_{\gamma_\varepsilon(t)}|\varphi(x,t)|^2\ d\sigma dt+\alpha\left\|\varphi^0\right\|_{H_0}-<u^1,\varphi^0>_{H_0',H_0} \tag{5.12}$$

over H_0. Note, in particular, that the coercivity of this functional is guaranteed by the unique continuation property (4.2) (see [FPZ]).

The adjoint system associated to the limit system (3.18) is also given by (4.1) and the corresponding functional associated to (3.18) is given by

$$J(\varphi^0)=\frac{1}{2}\int_0^T\int_\omega|\varphi(x,t)|^2m_\gamma(x)d\sigma dt+\alpha\left\|\varphi^0\right\|_{H_0}-<u^1,\varphi^0>_{H_0',H_0}, \tag{5.13}$$

where φ is the solution of (4.1) with final data φ^0. Recall that Theorem 3.3 is stated for the particular case $k=n-1$ to simplify the presentation. In the general case ($0\leq k\leq n-1$), the first term in (5.13) would be

$$\frac{1}{2}\int_0^T<\varphi(x,t)m_\gamma(x),\varphi(x,t)>_1\ dt, \tag{5.14}$$

and the rest of this proof could be adapted in a straightforward manner. Note that, when $k=n-1$ the weak limit of $\delta_{\gamma_\varepsilon(t)}$ is supported in an open subset $\omega\subset\Omega$ and then we can write (5.14) as in (5.13).

We set

$$M_\varepsilon=\inf_{\varphi^0\in H_0}J_\varepsilon(\varphi^0). \tag{5.15}$$

For each $\varepsilon > 0$ the functional J_ε achieves its minimum M_ε in H_0. This is a consequence of the unique continuation property (4.2) which allows us to prove the coercivity of J_ε for each $\varepsilon > 0$. This unique continuation property is obtained applying the result of Lemma 4.2 to the curve $\gamma_\varepsilon(t) = \gamma(t/\varepsilon)$, which satisfies the hypotheses of Lemma 4.2.

Lemma 5.2 below establishes that the coercivity of J_ε is in fact uniform in ε. Moreover, if $f(t) = \overline{\varphi}_\varepsilon(\gamma(t/\varepsilon), t)$ where $\overline{\varphi}_\varepsilon$ solves (4.1) with data $\overline{\varphi}^0_\varepsilon$, the solution of (2.4) satisfies (3.10).

Lemma 5.2 *We have*

$$\liminf_{\substack{\|\varphi^0\|_{H_0}\to\infty \\ \varepsilon\to 0}} \frac{J_\varepsilon(\varphi^0)}{\|\varphi^0\|_{H_0}} \geq \alpha. \tag{5.16}$$

Furthermore, the minimizers $\{\overline{\varphi^0_\varepsilon}\}_{\varepsilon\geq 0}$ are uniformly bounded in H_0.

Proof of Lemma 5.2. Let us consider sequences $\varepsilon_j \to 0$ and $\varphi^0_{\varepsilon_j} \in H_0$ such that $\left\|\varphi^0_{\varepsilon_j}\right\|_{H_0} \to \infty$ as $j \to \infty$.

Let us introduce the normalized data

$$\eta^0_{\varepsilon_j} = \frac{\varphi^0_{\varepsilon_j}}{\left\|\varphi^0_{\varepsilon_j}\right\|_{H_0}}$$

and the corresponding solutions of (4.1):

$$\eta_{\varepsilon_j} = \frac{\varphi_{\varepsilon_j}}{\left\|\varphi^0_{\varepsilon_j}\right\|_{H_0}}.$$

We have

$$I_j = \frac{J_{\varepsilon_j}(\varphi^0_{\varepsilon_j})}{\left\|\varphi^0_{\varepsilon_j}\right\|_{H_0}} = \frac{1}{2}\left\|\varphi^0_{\varepsilon_j}\right\|_{H_0} \int_0^T \int_{\gamma_{\epsilon_j}(t)} |\eta_{\varepsilon_j}(x,t)|^2 d\sigma dt + \alpha - < u^1, \psi^0_{\varepsilon_j} >_{H_0',H_0} .$$

We distinguish the following two cases:

Case 1: $\liminf_{j\to\infty} \int_0^T \int_{\gamma_{\epsilon_j}(t)} |\eta_{\varepsilon_j}(x,t)|^2 d\gamma_\varepsilon dt > 0$. In this case, we have clearly $\liminf_{j\to\infty} I_j = \infty$.

Case 2: $\liminf_{j\to\infty} \int_0^T \int_{\gamma_{\epsilon_j}(t)} |\eta_{\varepsilon_j}(x,t)|^2 d\gamma_\varepsilon dt = 0$. In this case we argue by contradiction. Assume that there exists a subsequence, still denoted by the index j, such that

$$\int_0^T \int_{\gamma_{\epsilon_j}(t)} |\eta_{\varepsilon_j}(x,t)|^2 d\sigma dt \to 0 \tag{5.17}$$

and

$$\liminf_{j\to\infty} I_j < \alpha. \tag{5.18}$$

By extracting a subsequence, still denoted by the index j, we have

$$\eta^0_{\varepsilon_j} \rightharpoonup \eta^0 \text{ weakly in } H_0,$$

and therefore

$$\eta_{\varepsilon_j} \rightharpoonup \eta \text{ weakly-* in } L^\infty(0,T;H_0)$$

where η is the solution of (4.1) with initial data η^0. By Lemma 5.1 we have

$$\eta = 0 \text{ in } \gamma_\varepsilon(t) \times (0,T).$$

Now, recall that, by hypothesis, γ_ε is a strategic curve and then Lemma 4.2 establishes that $\eta^0 = 0$. Thus

$$\eta^0_{\varepsilon_j} \rightharpoonup 0 \text{ weakly in } H_0$$

and therefore

$$\liminf_{j\to\infty} I_j \geq \lim \inf_{j\to\infty} (\alpha- < u^1, \eta^0_{\varepsilon_j} >_{H_0', H_0}) = \alpha$$

since u^1_j converges strongly in H_0. This is in contradiction with (5.18) and concludes the proof of (5.16).

On the other hand, it is obvious that $I_\varepsilon \leq 0$ for all $\varepsilon > 0$. Thus, (5.16) implies the uniform boundedness of the minimizers in H_0.

Concerning the convergence of the minimizers we have the following lemma:

Lemma 5.3 *The minimizers $\overline{\varphi}^0_\varepsilon$ of J_ε converge strongly in H_0 as $\varepsilon \to 0$ to the minimizer $\overline{\varphi}^0$ of J in (5.13) and M_ε converges to*

$$M = \inf_{\overline{\varphi}^0 \in H_0} J(\overline{\varphi}^0). \tag{5.19}$$

Moreover, the corresponding solutions $\overline{\varphi}_\varepsilon$ of (4.1) converge in $C([0,T];H_0)$ to the solution $\overline{\varphi}$ as $\varepsilon \to 0$.

Proof of Lemma 5.3. We adapt a classical argument in $\Gamma-$convergence (see [DM]). By extracting a subsequence, that we still denote by ε, we have

$$\overline{\varphi}^0_\varepsilon \rightharpoonup \eta^0 \text{ weakly in } H_0$$

as $\varepsilon \to 0$. It is sufficient to check that $\overline{\varphi}^0 = \eta^0$ or, equivalently, that η^0 is the minimizer of J, i.e.

$$J(\eta^0) \leq J(\varphi^0) \text{ for all } \varphi^0 \in H_0. \tag{5.20}$$

We know that

$$\overline{\varphi}_\varepsilon \rightharpoonup \eta \text{ weakly-* in } L^\infty(0,T;H_0)$$

where η is the solution of (4.1) with initial data ψ^0. By Lemma 5.1 we deduce that

$$J(\eta^0) = \lim_{\varepsilon\to 0} J_\varepsilon(\overline{\varphi}^0_\varepsilon). \tag{5.21}$$

On the other hand, for each $\varphi^0 \in H_0$ we have

$$\lim_{\varepsilon \to 0} J_\varepsilon(\overline{\varphi}_\varepsilon^0) \leq \lim_{\varepsilon \to 0} J_\varepsilon(\varphi^0). \tag{5.22}$$

Observe also that for $\varphi^0 \in H_0$ fixed, Lemma 5.1 ensures that

$$\lim_{\varepsilon \to 0} J_\varepsilon(\varphi^0) = J(\varphi^0). \tag{5.23}$$

Combining (5.21)-(5.23) it is easy to see that (5.20) holds.

This concludes the proof of the weak convergence of the minimizers and it also shows that

$$\liminf_{\varepsilon \to 0} M_\varepsilon \geq M = J(\bar{\varphi}^0) = \limsup_{\varepsilon \to 0} J_\varepsilon(\overline{\varphi}^0) \geq \limsup_{\varepsilon \to 0} J_\varepsilon(\overline{\varphi}_\varepsilon^0) = \limsup_{\varepsilon \to 0} M_\varepsilon. \tag{5.24}$$

Therefore we deduce that $M_\varepsilon \to M$.

Observe that (5.19), combined with the weak convergence of $\overline{\varphi}_\varepsilon^0$ in H_0, implies that

$$\lim_{\varepsilon \to 0} \left(\frac{1}{2} \int_0^T \int_{\gamma_\varepsilon(t)} |\overline{\varphi}_\varepsilon(x,t)|^2 d\sigma \; dt + \alpha \left\| \overline{\varphi}_\varepsilon^0 \right\|_{H_0} \right) = \frac{1}{2} \int_0^T \int_\omega |\overline{\varphi}|^2 m_\gamma(x) d\omega dt + \alpha \left\| \overline{\varphi}^0 \right\|_{H_0},$$

since the last term in $J_\varepsilon(\bar{\varphi}_\varepsilon^0)$, which is linear in $\bar{\varphi}_\varepsilon^0$, passes trivially to the limit.

This identity, combined with the weak convergence of $\overline{\varphi}_\varepsilon^0$ to $\overline{\varphi}^0$ in H_0 and Lemma 5.1 implies that

$$\overline{\varphi}_\varepsilon^0 \to \overline{\varphi}^0 \text{ strongly in } H_0. \tag{5.25}$$

Therefore, we have

$$\overline{\varphi}_\varepsilon \to \overline{\varphi} \text{ strongly in } C([0,T]; H_0).$$

This concludes the proof of Theorem 3.3 when $u^0 = 0$ and $\|u^1\|_{L^2(\Omega)} \geq \alpha$.

Let us consider now the case where u^0 is non-zero. We set $v^1 = v(T)$ where v is the solution of (2.4) with $f = 0$. Now observe that the solution u of (2.4) can be written as $u = v + w$ where w is the solution of (2.4) with zero initial data that satisfies $w(T) = u(T) - v^1$. In this way, the controllability problem for u can be reduced to a controllability problem for w with zero initial data $w^0 = 0$. This is the problem we solved. The proof is now complete.

■

Remark 5.1 *The proof guarantees that the coercivity property (5.16) is also true for the limit functional J. This fact could also be proved arguing directly on J. In this case we would use the corresponding unique continuation property for the solutions of the adjoint limit system.*

6 Further results and open problems

Let us describe briefly some generalizations and open problems related with the results in this paper:

1. In this paper we have restricted ourselves to space dimensions $n \leq 3$. However, only the results for space dimension $n = 1$, that was treated separately, are based in arguments that cannot be generalized to higher dimensions. The proofs given for the cases $n = 2, 3$ do not depend on the dimension and can be easily generalized to any space dimension n. For, it is sufficient to make an appropriate choice of the Sobolev spaces (3.3) where system (3.1) is well-posed.

2. As we mentioned in the introduction, when the control acts on an open non-empty subset of the domain Ω for all $0 \leq t \leq T$ the heat equation is null-controllable. The problem of null-controllability for system with a singular control concentrated on the curve γ is completely open. It is well known that the null controllability of system (3.1) is equivalent to the following observability inequality for the solutions of the adjoint system (4.1):

$$\int_\Omega |\varphi(x,0)|^2 dx \leq C \int_0^T \int_{\gamma(s)} |\varphi(x,t)|^2 d\sigma \, ds, \qquad \forall \varphi \text{ solution of (4.1).} \tag{6.1}$$

This observability inequality is much stronger than the unique continuation property (4.2). The arguments and techniques developed in this paper do not allow to obtain (6.1). In recent years, Carleman estimates has been used systematically as the most efficient tool to obtain observability inequalities (see [FI] and [FZ]) but they seem not to be sufficient to obtain quantitative results as (6.1) when $\gamma(t)$ is of dimension $k \leq n-1$.

3. The results of this paper (Theorem 3.3) show that increasing the time oscillations of the control region improves the controllability properties of the heat equation. It would be interesting to further analyze this property in the context of the cost of controlling the system. In other words, the initial and final data u^0, u^1 being fixed, as well as $\alpha > 0$, it would be interesting to analyze the size of the minimal control allowing to achieve (3.10) and its behavior as $\varepsilon \to 0$. This was done in [FZ] in the case of controls acting on open subsets of Ω independent of the time. But the techniques in [FZ], based on the use of Carleman inequalities, do not seem to apply in the present situation.

4. The results in Theorem 3.3 are also valid when considering controls that simultaneously guarantee the approximate control condition (2.2) and the exact control a finite-dimensional projection (see [Z2] for details). In this case, the proof is very similar. We only have to replace the term $\alpha \left\|\varphi^0\right\|_{H_0}$ by $\alpha \left\|(I-\pi_E)\varphi^0\right\|_{H_0}$ in both the functionals J_ε and J, given in (5.12) and (5.13) respectively. Here, I represents the identity and π_E the orthogonal projection of H_0 on a finite dimensional subspace E. It is not difficult to see that the arguments in the proof of Theorem 3.3 can be easily adapted to this situation.

5. The main ingredients to prove the approximate controllability results of this paper, and in particular Theorem 3.2, are the Fourier decomposition of solutions and the time-analyticity of the underlying semigroup. Therefore, we can easily extend the results to more general equations where these two properties hold. This is the case, for instance, in equations of the form

$$\rho(x)u_t - \text{ div } (A(x)\nabla u) = 0,$$

with Dirichlet, Neuman or mixed boundary conditions.

The situation is more complex when considering time-dependent coefficients. Then, in general, it is not possible to write the solutions in Fourier series. However, there are some particular cases where a certain decomposition is still possible. For example, consider the heat equation

$$\rho(t)u_t - \Delta u = 0$$

with $0 < \rho_m \leq \rho(t) \leq \rho_M < \infty$ and some boundary conditions. With the change of variable

$$s(t) = \int_0^t \rho^{-1}(r)dr,$$

the equation is transformed into the constant coefficients heat equation

$$u_s - \Delta u = 0,$$

for which a Fourier decomposition is known. Coming back to the original variable t we obtain a decomposition of the solutions that allows to adapt the results of this paper.

But the problem is completely open for general linear equations with coefficients depending both in space and time.

6. The techniques used in this paper cannot be adapted to the semi-linear case. Indeed, the approximate controllability of the semi-linear heat equation is usually derived from the approximate controllability of the linearized equation with a potential. As this potential depends on both the space and time variables our techniques do not apply, as pointed out above.

7. The techniques used in this paper cannot be adapted to wave or plate equations since the time-analyticity of the solutions fails. For the wave equation there are some partial controllability results, for the 1-d case, when the control acts on a point that follows particular time-dependent trajectories (see [K2]).

References

[B] M. Berggrem, *Optimal Control of Time Evolution Systems: Controllability Investigations and Numerical Algorithms*, PhD Thesis, Houston, 1995.

[CH] T. Cazenave and A. Haraux, *Introduction to Semi-Linear Evolution Equations*, Oxford University Press, 1999.

[C] C. Chapman, *Real mathematical analysis*, Springer Verlag, 2002.

[CoH] R. Courant and D. Hilbert. *Methods of mathematical physics*, volume 1. Wiley: New York, 1989.

[DM] G. Dal Maso, *Introduction to* Γ*-convergence*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 8, Birkhauser, 1992.

[DN] P. Donato and A. Nabil, *Approximate controllability of linear parabolic equations in perforated domains,* ESAIM Control, Optimization and Calculus of Variations, **6** (2001), 21-38.

[DF] H. Donnelly and C, Fefferman, *Nodal sets of eigenfunctions in Riemmanian manifolds*, Invent. Math., 93 (1988), 161-183.

[FPZ] C. Fabre, J.-P. Puel and E. Zuazua, *Approximate controllability of the semilinear heat equation*, Proc. Royal Soc. Edinbourgh, **125 A**, (1995), 31-61.

[FZ] E. Fernández-Cara and E. Zuazua, *The cost of approximate controllability for heat equations: The linear case*, Advances Diff. Eqs., **5** (4-6) (2000), 465-514.

[FI] A.V. Fursikov and O.Yu. Imanuvilov, *Controllability of evolution equations*, Lecture Note Series 34, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, 1996.

[H] M. Hervé, *Les fonctions analytiques*, Presses Universitaires de France, 1982.

[JL] D. Jerison and G. Lebeau, *Nodal sets of sums of eigenfunctions, in Harmonic analysis and partial differential equations*, (Chicago, Il, 1996), 223-239, Chicago Lectures in Mathematics, Univ. Chicago Press, Chicago, IL, 1999.

[K] A. Khapalov, *Controllability of the wave equation with a moving point control*, Appl. Math. Optim., **31**, (1995) 155-175.

[K2] A. Khapalov, *Mobile point controls versus locally distributed ones for the controllability of the semilinear parabolic equation*, SIAM J. Cont. Optim., **40** (1), (2001) 231-252.

[LR] G. Lebeau and L. Robiano, *Contrôle exact de l'équation de la chaleur*, Comm. P.D.E., **20**, (1995), 335-356.

[L] F. H. Lin, *Nodal sets of solutions of elliptic and parabolic equations*, Comm Pure Appl. Math. 44 (1991) 287-308.

[L1] J.-L. Lions, *Contrôlabilité exacte, stabilisation et perturbations de systèmes distribués.* Tomes 1 & 2. Masson, RMA **8** & **9**, Paris, 1988.

[L2] J.-L. Lions, *Some methods in the mathematical analysis of systems and their control*, Gordon and Breach, 1981.

[L3] J.-L. Lions, *Pointwise control for distributed systems*, in "Control and estimation in distributed parameter systems", edited by H.T. Banks, SIAM, 1992, 1-39.

[LM] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Springer Verlag, 1973.

[N] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson, Paris, 1967.

[P] A. Pazy, *Semigroups of linear operators and applications to Partial Differential Equations*, Applied Mathematics Sciences **44**, Springer-Verlag, 1993

[Z1] E. Zuazua, *Approximate controllability for linear parabolic equations with rapidly oscillating coefficients*, Control and Cybernetics, **23** (1994), 793-801.

[Z2] E. Zuazua, *Some problems and results on the controllability of Partial Differential Equations*, Second European Congress of Mathematics, Budapest, July 1996. Progress in Mathematics, Vol. 169, 1998, Birkhäuser Verlag, Basel/Switzerland, 276-311.

[Z3] E. Zuazua, *Controllability of Partial Differential Equations*, Discrete and Continuous Dynamical Systems, 8 (2) (2002), 469-513.