Approximate controllability of a semilinear heat equation in unbounded domains

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1 Introduction

In this paper, we consider the approximate controllability problem for the semilinear heat equation in an unbounded domain when the control acts on a open bounded set in the interior of the domain.

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Let Ω be an open and unbounded set of $I\!R^n$ of class C^2 uniformly, with boundary $\partial\Omega$ (see section 2.1 for a precise definition). Let ω be an open, bounded and nonempty subset of Ω. Let f be a real and globally Lipschitz function such that $f(0) = 0$. Let M be its Lipschitz constant, i.e.,

$$|f(s) - f(\sigma)| \leq M|s - \sigma| \qquad \forall s, \sigma \in I\!R.$$

We consider the following semilinear heat equation:

$$\begin{cases} y_t - \Delta y + f(y) = h\chi_\omega & \text{in } Q = \Omega \times (0, T) \\ y = 0 \quad \text{on} \quad \Sigma = \partial\Omega \times (0, T) \\ y(x, 0) = y^0(x) \quad \text{in} \quad \Omega. \end{cases} \tag{1}$$

In (1), $h = h(x, t)$ represents the control and χ_ω is the characteristic function of ω, the set where the controls are supported.

Let $1 \leq p < \infty$. We say that the system (1) is approximately controllable in $L^p(\Omega)$ at time $T > 0$ if the following holds: "For every $y^0 \in L^p(\Omega)$, the set of reachable states at time $T > 0$,

$$E(T) = \{y(x, T),\ y \ \text{ solution of (1) with } \ h \in L^p(Q)\}$$

is dense in $L^p(\Omega)$."

On the other hand (1) is approximately controllable in $C_0(\Omega)$ (the space of continuous functions vanishing at infinity and on $\partial\Omega$) at time $T > 0$ if for every $y^0 \in C_0(\Omega)$ the set

$$E(T) = \{y(x, T),\ y \ \text{ solution of (1) with } \ h \in L^\infty(Q)\}$$

is dense in $C_0(\Omega)$.

Notice that $C_0(\Omega)$ is the completion of the space $C_c(\Omega)$ of continuous functions with compact support, for the norm of the supremum.

We recall that $\lambda \in \text{sgn}(s)$ if $\lambda = s/|s|$ for $s \neq 0$ and $|\lambda| \leq 1$ for $s = 0$.

The main result of this paper is as follows:

Theorem 1 *Under the above assumptions on f and Ω, system (1) is approximately controllable in $L^p(\Omega)$ for $1 \leq p < \infty$ and in $C_0(\Omega)$ at any time $T > 0$. Moreover, we can reach a dense subset of final states by using controls of the form:*

$$h(x, t) \in \left(\int_0^T \int_\omega |\varphi(x, t)| dx dt \right) sgn(\varphi) \chi_{\omega \times (0,T)}$$

where φ is a solution of a suitable heat equation.

In the sequel, controls of this form will be refered to as "quasi bang-bang" controls.

Remark 1 In order to ensure that our control is of bang-bang form we have to prove that the zero set of φ, $\{(x,t) \in \omega \times (0,T); \varphi = 0\}$, is of zero $(n+1)$-dimensional Lebesgue measure. In space dimension $n = 1$, the results of S. Angenent [1] imply that this zero set is of zero measure. Such a result seems to be unknown in several space dimensions. ■

Remark 2 Observe that since Ω is an unbounded set we need $f(0) = 0$ to ensure that $y(x,T) \in L^p(\Omega)$. ■

When Ω is a bounded set, Fabre, Puel and Zuazua [8] proved the approximate controllability of (1) in $L^p(\Omega)$ for $1 \leq p < \infty$ and in $C_0(\Omega)$. In the proof they first show the approximate controllability of the linear system

$$\begin{cases} u_t - \Delta u + a(x,t)u = h\chi_\omega & \text{in } Q \\ u(x,t) = 0 & \text{on } \Sigma \\ u(x,0) = u^0(x) & \text{in } \Omega \end{cases} \tag{2}$$

with a potential $a(x,t) \in L^\infty(Q)$. Combining this result with a fixed point argument they obtain the approximate controllability of (1). This fixed point technique, based on Kakutani's Theorem, uses the boundness of Ω and the compactness of Sobolev's embeddings.

We prove the approximate controllability in unbounded domains by an approximation method that uses in a deep form the results in [8]. We consider the control problem in bounded domains of the form $\Omega_r = \Omega \cap B_r$, where B_r denotes the ball centered in zero of radius r. We show that the controls proposed in [8] for the problem restricted to Ω_r converge weakly in $L^p(\omega \times (0,T))$ as $r \to \infty$ to an approximate control for our problem in the whole domain Ω.

The methods of this paper do not allow to prove null-controllability results. In other words, we do not answer to the question of whether solutions may be driven exactly to zero ($y(T) = 0$). However, the methods developed by Chae, Imanulov and Moon Kim in [5] very probably may allow to answer to this question when ω is a neighborhood of the boundary $\partial\Omega$. The results in [5] are based in the use of Carleman's inequalities for solutions in the whole space and fixed point arguments. In this context the work by Lin Guo and Littman [10] is also worth mentioning. They use the non-linear Cauchy-Kovalevski theorem to show that, in a bounded one-dimensional interval, if the control acts in one extreme and the nonlinearity is analytic and in Gevrey class 2, sufficiently small (in L^∞ norm) initial data are null controllable.

Recently in [20] the second author proved that, when Ω is bounded, by means of a suitable variation of the arguments of [8] one can prove simultaneously the approximate controllability and also the exact reachability of a finite number of constraints. More precisely it was shown that for any $\psi_i \in L^q(\Omega)$, $i = 1, ..., m$ with $1/p+1/q = 1$ and $\varepsilon > 0$, initial data $y^0 \in L^p(\Omega)$ and final data $y^1 \in L^p(\Omega)$, the control h can be chosen such that the solution y of (1) satisfies both

$$\|y(T) - y^1\|_{L^p(\Omega)} \leq \varepsilon$$

and

$$\int_\Omega y^1 \psi_i dx = \int_\Omega y(T) \psi_i dx.$$

The methods developed in this paper allow to extend this type of results to unbounded domains Ω (see section 6, Theorem 3 for a precise statement).

The paper is organized as follows. In section 2 we study the existence and properties of the minima of some functionals arising in the approximate controllability of linear systems in $L^p(\Omega)$ with $1 < p < \infty$. In particular, section 2.2 is devoted to prove an uniform bound for these minima. In section 3 we prove the convergence of the solutions of system (1) in the restricted domain Q_r to the solution in the global domain. In section 4 we prove Theorem 1 when $1 < p < \infty$. In section 5 we prove Theorem 1 in $L^1(\Omega)$ and $C_0(\Omega)$ and in section 6 the exact finite reachability result mentioned above. We conclude the paper proving a compactness result used in section 2.2 (Proposition 2). This is done in section 7.

Along the paper we are going to consider the following hypotheses:

(H1) f is a globally Lipschitz function such that $f(0) = 0$;

(H2) $\Omega \subset \mathbb{R}^n$ in an unbounded, open and connected set of class C^2 uniformly;

(H3) $\omega \subset \Omega$ is an open, bounded and nonempty subset of Ω;

(H4) $h \in L^p(\mathbb{R}^n)$.

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2 Study of some functionals arising in the controllability of linear systems.

2.1 Preliminaries

For the sake of completeness, first of all, we recall the definition of domain of class C^s uniformly. We say that a domain Ω (bounded or not) is uniformly regular of class $C^s (s \geq 1)$ (see [3]), if there exists an integer $r > 0$ an sequences $\{N_j\}$ of open subsets of $I\!R^n$ and homeomorphisms $\{\psi_j\}$ of N_j on the unit ball in $I\!R^n$ such that: i) Any $(r+1)$ distinct sets N_j have empty intersection; ii) $\psi_j(N_j \cap \Omega) = \{x : |x| < 1, x_n > 0\}$, $\psi_j(N_j \cap \partial\Omega) = \{x : |x| < 1, x_n = 0\}$; iii) If $N_j' = \psi_j^{-1}(|x| < 1/2)$, $\cap_j N_j'$ contains the $(1/r)$-neigborhood of $\partial\Omega$; iv) For $y \in N_j$, $x \in \Psi_j(N_j)$ we have $|(D^\alpha \psi_j)(y)| \leq r$, $|(D^\alpha \psi_j^{-1})(x)| \leq r$, for all $|\alpha| \leq s$.

For the sake of simplicity we consider first the L^p case with $1 < p < \infty$. For $0 < t_1 < t_2$ we denote by $X^p(t_1, t_2; \Omega)$ the following Banach space:

$$X^p(t_1, t_2; \Omega) = L^p(t_1, t_2; W_0^{1,p}(\Omega)) \cap W^{1,p}(t_1, t_2; L^p(\Omega))$$

endowed with the natural norm

$$\| \ \|_{X^p(t_1,t_2;\Omega)} = \left(\| \ \|^p_{L^p(t_1,t_2;W_0^{1,p}(\Omega))} + \| \ \|^p_{W^{1,p}(t_1,t_2;L^p(\Omega))} \right)^{1/p}.$$

Let $a(x,t) \in L^\infty(Q)$. We recall (see [9], Theorem 9.1, page 341) the existence of constants $C > 0$ (depending on a, Ω and T) and $C_{t_1,t_2} > 0$ (depending on a, Ω, t_1 and t_2) such that, for every $k \in L^p(Q)$ and $w^0 \in L^p(\Omega)$, the solution w of

$$\begin{cases} w_t - \Delta w + a(x,t)w = k \ \text{ in } \ Q \\ w(x,t) = 0 \quad \text{on } \ \Sigma \\ w(x,0) = w^0(x) \quad \text{in } \ \Omega \end{cases}$$

satisfies

$$\begin{cases} \|w\|_{L^\infty(0,T;L^p(\Omega))} \le C(\|w^0\|_{L^p(\Omega)} + \|k\|_{L^p(Q)}) \\ \|w\|_{X^p(t_1,t_2;\Omega)} \le C_{t_1,t_2}(\|w^0\|_{L^p(\Omega)} + \|k\|_{L^p(Q)}). \end{cases} \tag{3}$$

For $\Omega_r = \Omega \cap B_r$, $\varphi_r^0 \in L^q(\Omega_r)$, $\frac{1}{q} + \frac{1}{p} = 1$, $a \in L^\infty(Q)$ and $y^1 \in L^p(\Omega)$ we define

$$J_r(\varphi_r^0; a(x,t), y^1) = \frac{1}{2} \left(\int_0^T \int_\omega |\varphi_r| dx\, dt \right)^2 + \alpha \|\varphi_r^0\|_{L^q(\Omega_r)} - \int_{\Omega_r} y^1 \varphi_r^0 dx \tag{4}$$

where φ_r satisfies the equation :

$$\begin{cases} -\varphi_{r,t} - \Delta\varphi_r + a(x,t)\varphi_r = 0 \ \text{ in } \ Q_r = \Omega_r \times (0,T) \\ \varphi_r = 0 \quad \text{on } \ \Sigma_r = \partial\Omega_r \times (0,T) \\ \varphi_r(T) = \varphi_r^0 \ \text{ in } \ \Omega_r. \end{cases} \tag{5}$$

We know (see [8]) that for every $\alpha > 0$, $J_r(.; a, y^1)$ is strictly convex, continuous in $L^q(\Omega_r)$ and satisfies

$$\liminf_{\|\varphi_r^0\|_{L^q(\Omega_r)} \to \infty} \frac{J_r(\varphi_r^0; a, y^1)}{\|\varphi_r^0\|_{L^q(\Omega_r)}} \ge \alpha. \tag{6}$$

Then the functional $J_r(.; a, y^1)$ achieves its minimum at a unique point $\hat{\varphi}_r^0$ in $L^q(\Omega_r)$. Furthermore, $\hat{\varphi}_r^0 = 0$ if and only if $\|y^1\chi_{\Omega_r}\|_{L^p(\Omega_r)} \leq \alpha$. ■

For $\psi^0 \in L^q(\Omega_r)$ [resp. $\psi \in L^q(Q_r)$] we denote by $\widetilde{\psi^0}$ [resp. $\widetilde{\psi}$] the extension by zero of ψ^0 [resp. ψ] to Ω [resp. to Q], i.e.

$$\widetilde{\psi^0} = \begin{cases} \psi^0 & \text{in } \Omega_r \\ 0 & \text{in } \Omega\backslash\Omega_r \end{cases}, \qquad \widetilde{\psi} = \begin{cases} \psi & \text{in } Q_r \\ 0 & \text{in } Q\backslash Q_r. \end{cases}$$

2.2 Uniform bounds on the minimizers

The following result provides a uniform bound for the minimizers of J_r with respect to r.

Proposition 1 *Let $1 < p < \infty$. Suppose that $\{a_r\}_r \subset L^\infty(Q_r)$ is such that $\{\widetilde{a_r}\}$ is a bounded sequence in $L^\infty(Q)$ and $\{y_r^1\} \subset L^p(\Omega)$ converges strongly to $y^1 \in L^p(\Omega)$ as $r \to \infty$. Then there exists a constant C independent of r, such that*

$$\|\hat{\varphi}_r^0\|_{L^q(\Omega_r)} \leq C \quad \forall r \tag{7}$$

where $\hat{\varphi}_r^0$ is the minimizer of $J_r(.; a_r, y_r^1)$.

The proof of Proposition 1 needs the following two results. The first one is a consequence of a classical compactness result (see [17], Theorem 5; or Theorem 4 in section 7). However since Ω is unbounded, its proof is technical and the computations are long. To make the paper easier to read we give a detailed proof of Proposition 2 in section 7 at the end of the paper. The second result is a unique continuation property, consequence of a result due to Saut and Scheurer (see [16], Theorem 1.1).

Proposition 2 *Let $1 < q < \infty$, $\{a_r\}_r \subset L^\infty(Q_r)$ be such that $\{\widetilde{a_r}\}$ is a bounded sequence in $L^\infty(Q)$, and $\psi_r^0 \in L^q(\Omega_r)$ such that $\{\widetilde{\psi_r^0}\}_r$ is bounded in $L^q(\Omega)$. Let ψ_r be the solution of*

$$\begin{cases} -\psi_{r,t} - \Delta\psi_r + a_r(x,t)\psi_r = 0 & \text{in} \quad Q_r \\ \psi_r = 0 & \text{on} \quad \Sigma_r \\ \psi_r(T) = \psi_r^0 & \text{in} \quad \Omega_r. \end{cases} \tag{8}$$

Then, there exists $\psi^0 \in L^q(\Omega)$, $\beta \in L^\infty(0,T; L^q(\Omega)) \cap C([0,T]; W_{loc}^{-1,q}(\Omega))$ and a subsequence (still denoted by the subindex r) such that

$$\widetilde{\psi_r^0} \rightharpoonup \psi^0 \quad \textit{weakly in } L^q(\Omega); \tag{9}$$

$$\widetilde{\psi_r} \rightharpoonup \beta \text{ weakly* in } L^\infty(0,T;L^q(\Omega)); \tag{10}$$

$$\widetilde{\psi_r} \to \beta \text{ strongly in } L^q(0,T-\varepsilon;L^q_{loc}(\Omega)),\ \forall 0<\varepsilon<T; \tag{11}$$

$$\widetilde{\psi_r} \to \beta \text{ strongly in } C([0,T];W^{-1,q}_{loc}(\Omega)); \tag{12}$$

$$\widetilde{\psi_r}(t) \to \beta(t) \text{ strongly in } L^q_{loc}(\Omega),\ \forall t\in[0,T) \tag{13}$$

as $r\to\infty$. Moreover, β belongs to $L^2_{loc}(0,T;H^2_{loc}(\Omega))$ and there exists $a = a(x,t)\in L^\infty(Q)$ such that $\beta=\beta(x,t)$ satisfies:

$$\begin{cases} -\beta_t - \Delta\beta + a\beta = 0 \text{ in } Q \\ \beta(T) = \psi^0 \text{ in } \Omega. \end{cases} \tag{14}$$

Remark 3 Observe that Proposition 2, allows in particular to construct a sequence such that $\widetilde{\psi_r} \to \beta$ strongly in $L^1(\omega\times(0,T))$ as $r\to\infty$ since

$$\|\widetilde{\psi_r}-\beta\|_{L^1(\omega\times(0,T))} = \int_0^{T-\varepsilon}\int_\omega |\widetilde{\psi_r}-\beta| + \int_{T-\varepsilon}^T\int_\omega |\widetilde{\psi_r}-\beta|.$$

The first term in the right hand side goes to zero (as $r\to\infty$) as a consequence of (11). The second term is upper bounded by $\varepsilon|\omega|^{(q-1)/q}\|\widetilde{\psi_r}-\beta\|_{L^\infty(0,T;L^q(\Omega))}$ and by (10) goes to zero as $\varepsilon\to 0$ uniformly on r. ■

Theorem 2 *Let $\Omega\subset I\!R^n$ be an arbitrary open and connected subset and $a(x,t)\in L^\infty(Q)$. Let ω be an open and nonempty subset of Ω and $\varphi\in L^2(0,T;H^2_{loc}(\Omega))$ such that*

$$\begin{cases} -\varphi_t - \Delta\varphi + a(x,t)\varphi = 0 \text{ in } Q \\ \varphi = 0 \text{ in } \omega\times(0,T). \end{cases} \tag{15}$$

Then $\varphi\equiv 0$ in Q.

Proof of Proposition 1 It is clear that

$$0 = J_r(0;a_r,y^1_r) \geq J_r(\hat{\varphi}^0_r;a_r,y^1_r) \text{ for every } r. \tag{16}$$

To prove (7) we argue by contradiction, i.e., suppose that there exists a sequence of minimizers $\{\hat{\varphi}^0_r\}$ such that $\|\hat{\varphi}^0_r\|_{L^q(\Omega_r)}\to\infty$ as $r\to\infty$. Let

$$\psi^0_r = \hat{\varphi}^0_r/\|\hat{\varphi}^0_r\|_{L^q(\Omega_r)}.$$

Then

$$\frac{J_r(\hat{\varphi}_r^0; a_r, y_r^1)}{\|\hat{\varphi}_r^0\|_{L^q(\Omega_r)}} = \|\hat{\varphi}_r^0\|_{L^q(\Omega_r)} \frac{1}{2}\left(\int_0^T \int_\omega |\psi_r| dxdt\right)^2 + \alpha - \int_{\Omega_r} y_r^1 \psi_r^0 dx \tag{17}$$

where ψ_r satisfies

$$\begin{cases} -\psi_{r,t} - \Delta\psi_r + a_r(x,t)\psi_r = 0 \ \text{ in } \ Q_r \\ \psi = 0 \quad \text{on} \quad \Sigma_r \\ \psi(T) = \psi_r^0 \quad \text{in } \ \Omega_r. \end{cases}$$

We have that $\|\widetilde{\psi_r^0}\|_{L^q(\Omega)} = \|\psi_r^0\|_{L^q(\Omega_r)} = 1$ for every r. In view of Proposition 2, there exists a subsequence (still denoted by the index r) such that (9)-(13) hold. For r large enough $\int_0^T \int_\omega |\psi_r| = \int_0^T \int_\omega |\widetilde{\psi_r}|$ and therefore, in view of Remark 3,

$$\lim_{r\to\infty} \int_0^T \int_\omega |\psi_r| = \int_0^T \int_\omega |\beta|.$$

Then by (16), (17) and since $\|\hat{\varphi}_r^0\|_{L^q(\Omega_r)} \to \infty$ necessarily $\int_0^T \int_\omega |\psi_r| \to 0$, and therefore $\beta \equiv 0$ in $\omega \times (0,T)$. But, by Proposition 2 we know that β belongs to $L^2(0, T-\varepsilon; H^2_{loc}(\Omega))$ for every $\varepsilon > 0$ and verifies (14). Consequently we can apply the Unique Continuation Theorem 2, obtaining $\beta \equiv 0$ in $\Omega \times (0, T-\varepsilon)$. On the other hand, $\beta \in C([0,T]; W^{-1,q}_{loc}(\Omega))$ and that implies that $\beta(T) = \psi^0 = 0$ in Ω. In view of (9), $\widetilde{\psi_r^0} \rightharpoonup 0$ in $L^q(\Omega)$, $y_r^1 \to y^1$ in $L^p(\Omega)$, $\alpha > 0$ and $\|\hat{\varphi}_r^0\|_{L^q} \to \infty$. On the other hand, $J_r(\hat{\varphi}_r^0; a_r, y_r^1) \geq \|\hat{\varphi}_r^0\|_{L^q(\Omega_r)}(\alpha - \int_\Omega y_r^1 \widetilde{\psi_r^0} dx)$ and then $J_r(\hat{\varphi}_r^0; a_r, y_r^1) \to \infty$, which contradicts (16). Therefore (7) is proved. ∎

3 Convergence of the solutions defined in the approximated domains

The main purpose of this section is to prove the following result.

Proposition 3 *Assume that* $1 \leq p < \infty$ *and that (H1)-(H4) hold. Let* $h_r \in L^p(Q_r)$ *be such that* $h_r \to h$ *weakly* in* $L^\infty(\omega \times (0,T))$ *as* $r \to \infty$. *Let* y *be the solution of*

$$\begin{cases} y_t - \Delta y + f(y) = h\chi_\omega \ \ \textit{in } Q = \Omega \times (0,T) \\ y = 0 \ \ \textit{on} \ \ \Sigma \\ y(x,0) = y^0 \ \textit{in} \ \ \Omega \end{cases} \tag{18}$$

and y_r *satisfying*

$$\left\{\begin{array}{l} y_{r,t}-\Delta y_r+f(y_r)=h_r\chi_\omega \ \ in\ Q_r \\ y_r=0 \ \ on\ \ \Sigma_r \\ y_r(x,0)=y^0 \ \ in \ \ \ \Omega_r. \end{array}\right. \tag{19}$$

Then $\|y(T)-y_r(T)\|_{L^p(\Omega_r)}\to 0,\quad as\ \ r\to\infty.$

Remark 4 In particular $\|y(T)-\widetilde{y_r}(T)\|_{L^p(\Omega)}\to 0$ as $r\to\infty$ where $\widetilde{y_r}(T)$ is the extension by zero of $y_r(T)$ to Ω. That is, given $y^1\in L^p(\Omega)$ and $\alpha>0$, if we can find controls h_r verifying the hypotheses of Proposition 3 and such that $\|y_r(T)-y^1\|_{L^p(\Omega_r)}\leq\alpha$. Then $\|y(T)-y^1\|_{L^p(\Omega)}\leq\alpha$ with the limit control h. In that case the main result would be proved. ■

In order to prove Proposition 3 we need some a priori estimates stated in the following lemmas.

Lemma 1 *Suppose that (H1)-(H4) are verified and that* $y^0\in L^p(\Omega)$ *has compact support contained in* B_ρ*, the ball centered at zero with radius* ρ*. Assume also that* $\omega\subset B_\rho$*.*

Let v *be the solution of*

$$\left\{\begin{array}{l} v_t-\Delta v+f(v)=h\chi_\omega \ \ \ in\ Q \\ v=0 \ \ \ on\ \ \Sigma \\ v(x,0)=y^0(x) \ \ \ in\ \ \Omega. \end{array}\right.$$

Then, there exists a constant $C>0$ *independent of* r*, such that*

$$\|v\|_{L^\infty(\Gamma_r)}\leq C^*\frac{e^{MT}}{(r-\rho)^n}\quad\forall r>\rho,$$

with $C^*=C\left(\|y^0\|_{L^1(I\!R^n)}+\|h\chi_\omega\|_{L^1((0,T)\times I\!R^n)}\right).$

Lemma 2 *Assume that* $g\in L^\infty(\Gamma_r)$*, with* $\Gamma_r=\partial B_r\times(0,T)$*. Let* w *satisfy:*

$$\left\{\begin{array}{l} w_t-\Delta w=0 \ \ \ in\ B_r\times(0,T) \\ w=g \ \ \ on\ \ \Gamma_r \\ w(x,0)=0 \ \ \ in \ \ \ B_r. \end{array}\right. \tag{20}$$

Then, for all $1 \leq p < \infty$, there exists $C = C(p) > 0$ independent on r and $\varepsilon > 0$ such that for every $T \geq t > 0$

$$\|w(t)\|_{L^p(B_r)} \leq C t^{\varepsilon/2} r^{\frac{n}{p}-\varepsilon} \|g\|_{L^\infty(\Gamma_r)} \ \forall r > 0.$$

Let us assume that Lemma 1 and Lemma 2 hold in order to prove Proposition 3. The proof of these lemmas will be given at the end of this section.

Proof of Proposition 3 We are going to divide the proof in two parts. In the first one we prove that the solution $y = y(h_r)$ of (18) corresponding to the control h_r instead of h satisfies $\|y(h_r;T) - y_r(T)\|_{L^p(\Omega_r)} \to 0$ as $r \to \infty$. In fact this is enough to prove the approximate controllability of the semilinear heat equation in unbounded domains. Nevertheless, since we want to extend our results to the simultaneous exact controllability of a finite number of constraints, we need to pass to the limit and to prove the statement of Proposition 3. That is done in the second part of the proof.

First Part

We divide this part of the proof in two steps. First we consider consider initial data y^0 with compact support. In the second step we will see that this assumption can be avoided.

First step: supp $y^0 \subset B_\rho$.

Suppose that the data y^0 has compact support. We fix $\rho > 0$ such that supp $y^0 \subset B_\rho$. Let $z = y - y_r$. Then z is the solution of

$$\begin{cases} z_t - \Delta z + f(y) - f(y_r) = 0 \text{ in } Q_r \\ z = y \ \text{ on } \ \Sigma_r \\ z(x,0) = 0 \ \text{ in } \ \Omega_r. \end{cases}$$

We write $z = \hat{z} + \bar{z}$ where $\hat{z}$ and $\bar{z}$ satisfy, respectively,

$$\begin{cases} \hat{z}_t - \Delta \hat{z} = -f(y) + f(y_r) \text{ in } Q_r \\ \hat{z} = 0 \text{ on } \ \Sigma_r \\ \hat{z}(x,0) = 0 \ \text{ in } \ \Omega_r \end{cases} \tag{21}$$

$$\begin{cases} \bar{z}_t - \Delta \bar{z} = 0 \text{ in } Q_r \\ \bar{z} = y \ \text{ on } \ \Sigma_r \\ \bar{z}(x,0) = 0 \ \text{ in } \ \Omega_r. \end{cases} \tag{22}$$

We first estimate the norm of $\bar{z}$ in $L^p(Q_r)$. Let w be the solution of

$$\begin{cases} w_t - \Delta w = 0 \ \text{ in } B_r \times (0,T) \\ w = k = \begin{cases} |y| \ \text{ on } \ (\partial B_r \cap \partial\Omega_r) \times (0,T) = \Sigma_r' \\ 0 \ \text{ in } \ \Gamma_r \backslash \Sigma_r' \end{cases} \\ w(x,0) = 0 \ \text{ in } \ B_r \end{cases}$$

where Γ_r denotes $\partial B_r \times (0,T)$. Since $y = 0$ on $\Sigma_r \backslash \Sigma_r'$, by the maximum principle, for every $t \in [0,T]$, $\|\bar{z}(t)\|_{L^p(\Omega_r)} \leq \|w(t)\|_{L^p(B_r)}$, but, in view of Lemma 2, for $1 \leq p < \infty$,

$$\|w(t)\|_{L^p(B_r)} \leq Ct^{\varepsilon/2} r^{\frac{n}{p}-\varepsilon} \|k\|_{L^\infty(\Gamma_r)} \ = Ct^{\varepsilon/2} r^{\frac{n}{p}-\varepsilon} \|y\|_{L^\infty(\Sigma_r)} \tag{23}$$

and then

$$\|\bar{z}\|_{L^p(Q_r)} \leq Cr^{\frac{n}{p}-\varepsilon} \|y\|_{L^\infty(\Sigma_r)}$$

So, in view of Lemma 1,

$$\|\bar{z}\|_{L^p(Q_r)} \leq C\frac{r^{\frac{n}{p}-\varepsilon}}{(r-\rho)^n} C \left(\|y^0\|_{L^1(I\!R^n)} + \|h_r \chi_\omega\|_{L^1((0,T)\times I\!R^n)} \right) \text{ for every } r > R.$$

In the same way, it is easy to estimate $\|\bar{z}(T)\|_{L^p(\Omega_r)}$, obtaining

$$\|z(\bar{T})\|_{L^p(\Omega_r)} \leq C\frac{r^{\frac{n}{p}-\varepsilon}}{(r-\rho)^n} C \left(\|y^0\|_{L^1(I\!R^n)} + \|h_r \chi_\omega\|_{L^1((0,T)\times I\!R^n)} \right) \text{ for every } r > R.$$

The weak* convergence of h_r in $L^\infty(\omega \times (0,T))$ implies the uniform bound of the sequence in $L^1(\omega \times (0,T))$ and then

$$\|\bar{z}(T)\|_{L^p(\Omega_r)} \to 0 \text{ as } r \to \infty.$$

We need to show that $\|\hat{z}(T)\|_{L^p(\Omega_r)} \to 0$ as $r \to \infty$. We multiply (21) by $|\hat{z}|^{p-2}\hat{z}$ (by $|\hat{z} + \varepsilon|^{p-2}\hat{z}$ if $1 \leq p < 2$ and then let $\varepsilon \to 0$) and integrate by parts in Ω_r. Then

$$\frac{1}{p}\frac{d}{dt}\int_{\Omega_r} |\hat{z}|^p + \int_{\Omega_r} |\nabla |\hat{z}|^{p/2}|^2 = \int_{\Omega_r} (f(y_r) - f(y))|\hat{z}|^{p-2}\hat{z}.$$

Therefore, since f is globally Lipschitz,

$$\frac{1}{p}\frac{d}{dt}\int_{\Omega_r} |\hat{z}|^p \leq M \int_{\Omega_r} |\bar{z} + \hat{z}||\hat{z}|^{p-1}$$

with M the Lipschitz constant of f. Therefore, by Holder's, Schwartz's and Gronwall's inequalities, there exists a constant $C > 0$ independent of r such that

$$\int_{\Omega_r} |\hat{z}(T)|^p dx \leq C \int_0^T \int_{\Omega_r} |\bar{z}|^p dxdt$$

So, in view of Lemma 1 and the assumptions on h_r, we obtain

$$\|\hat{z}(T)\|_{L^p(\Omega_r)} \to 0 \text{ as } r \to \infty.$$

This implies that $\|y(T) - y_r(T)\|_{L^p(\Omega_r)} \to 0$ as $r \to \infty$.

Second step: Arbitrary data

Suppose now that the data y^0 doesn't have compact support. We decompose the solution y_r of (19) in the following way: $y_r = z_r + w_r$ with

$$\begin{cases} w_{r,t} - \Delta w_r + f(w_r) = h_r \text{ in } Q_r \\ w_r = 0 \text{ on } \Sigma_r \\ w_r(0) = y^0 \chi_{B_\rho} \text{ in } \Omega_r \end{cases}$$

with $\rho > 0$ and B_ρ the ball of radius ρ to be chosen later and

$$\begin{cases} z_{r,t} - \Delta z_r + f(w_r + z_r) = f(w_r) \text{ in } Q_r \\ z_r = 0 \text{ on } \Sigma_r \\ z_r(0) = y^0(I - \chi_{B_\rho}) \text{ in } \Omega_r. \end{cases}$$

Notice that z_r and w_r depend also on ρ. However, to simplify the notation we will not make this dependence explict by means of another index.

In the same way we write y solution of (18) as $y = w + z$ with

$$\begin{cases} w_t - \Delta w + f(w) = h_r \text{ in } Q \\ w = 0 \text{ on } \Sigma \\ w(0) = y^0 \chi_{B_\rho} \text{ in } \Omega \end{cases} ; \quad \begin{cases} z_t - \Delta z + f(w + z) = f(w) \text{ in } Q \\ z = 0 \text{ on } \Sigma \\ z(0) = y^0(I - \chi_{B_\rho}) \text{ in } \Omega. \end{cases}$$

We observe that w and w_r have initial data with compact support and, as a consequence of the first part of the proof, $w_r(T) \to w(T)$ strongly in $L^p(\Omega_r)$ as $r \to \infty$. So, given $\varepsilon > 0$ we can chose $r_1 = r(\rho)$ such that if $r > r_1$ then

$$\|w_r(T) - w(T)\|_{L^p(\Omega_r)} \leq \varepsilon/2.$$

Observe that

$$\|y^0(I - \chi_{B_\rho})\|_{L^p(\Omega)} \to 0 \text{ as } \rho \to \infty.$$

Therefore we can expect that for ρ large enough

$$\|z_r(T)\|_{L^p(\Omega_r)} + \|z(T)\|_{L^p(\Omega)} \leq \varepsilon/2. \tag{24}$$

Let us prove it. As in the first step of the proof we consider, to simplify, $2 \leq p \leq \infty$. We multiply (18) by $|z|^{p-2}z$. Then

$$\frac{1}{p}\frac{d}{dt}\int_\Omega |z|^p + \int_\Omega |\nabla |z|^{p/2}|^2 + \int_\Omega (f(w+z) - f(w))|z|^{p-2}z = 0.$$

Since f is globally Lipschitz

$$|f(w+z) - f(w)| \leq M|z|.$$

By Gronwall's inequality we obtain

$$\int_\Omega |z(t)|^p \leq \int_\Omega |z(0)|^p e^{pMt}.$$

In particular

$$\|z(T)\|_{L^p(\Omega)} \leq e^{MT}\|z(0)\|_{L^p(\Omega)} = e^{MT}\|y^0(I - \chi_{B_\rho})\|_{L^p(\Omega)}$$

and therefore

$$\|z(T)\|_{L^p(\Omega)} \to 0 \text{ as } \rho \to \infty.$$

We can apply the same argument to z_r to obtain (24). ■

Second Part: Passing to the limit In view of the first part of the proof, it is enough to prove that if v_r solves

$$\begin{cases} v_{r,t} - \Delta v_r + f(v_r) = h_r \chi_\omega \text{ in } Q \\ v_r = 0 \text{ on } \Sigma \\ v_r(0) = y^0 \text{ in } \Omega, \end{cases}$$

then $\|y(T) - v_r(T)\|_{L^p(\Omega)} \to 0$ as $r \to \infty$ where y is the solution of (18) corresponding to the limit control h. Observe that $\gamma_r = y - v_r$ solves

$$\begin{cases} \gamma_{r,t} - \Delta\gamma_r + f(y) - f(v_r) = (h - h_r)\chi_\omega \text{ in } Q \\ \gamma_r = 0 \text{ on } \Sigma \\ \gamma_r(0) = 0 \text{ in } \Omega \end{cases} \tag{25}$$

For simplicity, let us consider $\infty > p \geq 2$. We multiply (25) by $|\gamma_r|^{p-2}\gamma_r$ and integrate by parts. Since f is globally Lipschitz and by Gronwall's inequality it is not difficult to see that there exists a constant $C > 0$ independent of r such that

$$\|\gamma_r(t)\|_{L^p(\Omega)} \leq C \quad \forall t \in [0,T]. \tag{26}$$

In particular

$$\|\gamma_r(T)\|_{L^p(\Omega)} \leq C \int_0^T \int_\Omega (h - h_r)\chi_\omega |\gamma_r|^{p-2}\gamma_r \quad \forall t \in [0,T] \tag{27}$$

and for a subsequence,

$$\gamma_r(T) \rightharpoonup \gamma(T) \text{ weakly in } L^p(\Omega). \tag{28}$$

We need to prove that this convergence is strong and that $\gamma(T) = 0$. First of all observe that (27) allows to prove that $\gamma(T) = 0$ if we can prove for every $\varepsilon > 0$ that

$$\gamma_r \to \gamma \text{ strongly in } L^p(\varepsilon, T; L^p(\omega)). \tag{29}$$

In fact if (29) holds, proceeding as in Remark 3, it is clear that (27) implies

$$\|\gamma(T)\|_{L^p(\Omega)} \leq \lim_{r\to\infty} \int_\varepsilon^T \int_\Omega (h - h_r)\chi_\omega |\gamma_r|^{p-2}\gamma_r + \varepsilon C$$

where C is independent of r. Therefore since (for a subsequence) $h - h_r \rightharpoonup 0$ weakly in $L^p(\omega \times (\varepsilon, T))$ and $\gamma_r \to \gamma$ strongly in $L^p(\omega \times (\varepsilon, T))$ we obtain that $\gamma(T) = 0$.

That means that the weak limit in (28) vanishes, i.e. $\gamma(T) = 0$. Observe that this fact allows to prove the strong convergence of $\gamma_r(T)$ proving only that

$$\gamma_r(T) \to \gamma(T) \text{ strongly in } L^p_{loc}(\Omega). \tag{30}$$

Indeed, if (30) holds we can write $\gamma_r = \beta_r + \eta_r$ with η_r satisfying:

$$\begin{cases} \eta_{r,t} - \Delta\eta_r + f(y) - f(v_r) = 0 \text{ in } \Omega\backslash\Omega_\rho \times (0,T) \\ \eta_r = \gamma_r \text{ on } \Sigma_\rho \\ \eta_r(0) = 0 \text{ in } \Omega\backslash\Omega_\rho \end{cases} \tag{31}$$

with $\rho > 0$ to be chosen later. By Lemma 1, if y^0 has compact support contained in $B_{\tilde{\rho}}$, it is easy to see that there exists a constant $C > 0$ such that for every $r > 0$

$$\|\gamma_r\|_{L^\infty(\Gamma_\rho)} \leq C(\|y^0\|_{L^1(\mathbb{R}^n)} + H)\frac{e^{MT}}{(\rho - \tilde{\rho})} \quad \forall \rho > \tilde{\rho}$$

where $H > \|h\chi_\omega\|_{L^1(Q)} + \|h_r\chi_\omega\|_{L^1(Q)}$ for every $r > 0$. Proceeding as in the first part of the proof, it is not difficult to obtain that

$$\|\eta_r(T)\|_{L^p(\Omega\setminus\Omega_\rho)} \to 0 \text{ as } \rho \to \infty \text{ uniformly in } r. \tag{32}$$

Observe that $\eta_r = \gamma_r|_{\Omega\setminus\Omega_\rho}$ and $\beta_r = \gamma_r|_{\Omega_\rho}$. Then, combining (32) and (30) we obtain the strong convergence of $\gamma_r(T)$.

So, the proof is reduced to the proof of (29) and (30).

In fact, this is an adaptation to the proof of Proposition 2 (see section 7). For this reason we only give an sketch of the proof. Observe that since Ω is of class C^2 uniformly, the operator $-\Delta$ with Dirichlet boundary conditions generates an analytic semigroup $S(t)$ in $L^p(\Omega)$ for $1 < p < \infty$ (see [18] p. 88). We know that for $v \in L^p(\Omega)$, $S(t)v \in W^{2,p}(\Omega)$ for every $t > 0$ with $\|S(t)v\|_{W^{2,p}(\Omega)} \leq Ct^{-1}\|v\|_{L^p(\Omega)}$ and $\|S(t)v\|_{L^p(\Omega)} \leq C\|v\|_{L^p(\Omega)}$ (see [13] p.74). Interpolating for $0 < s < 2$ (see [12]) we obtain

$$\|S(t)v\|_{W^{s,p}(\Omega)} \leq C(s)t^{-s/2}\|v\|_{L^p(\Omega)}, \qquad \forall v \in L^p(\Omega). \tag{33}$$

On the other hand, we can write $y(t), v_r(t)$ by the variation of constants formula, and we obtain that $\gamma_r(t)$ can be written as:

$$\gamma_r(t) = \int_0^t S(t-\sigma)(h(\sigma) - h_r(\sigma))d\sigma + \int_0^t S(t-\sigma)[f(y(\sigma)) - f(v_r(\sigma))]d\sigma$$

Since f is globally Lipschitz, by (33) and (26), we obtain that

$$\|\gamma_r(t)\|_{W^{1,p}(\Omega)} \leq C.$$

Observe that since Ω is unbounded we do not have the compactness of the Sobolev's embeddings. Nevertheless the last estimate is still valid in $W^{1,p}(K)$ with K compact. This allows to prove (30), that is

$$\gamma_r(T) \to \gamma(T) \text{ strongly in } L^p_{loc}(\Omega).$$

Proceeding as in the proof of Proposition 2, third and fourth steps in section 7, it is not difficult to see that

$$\gamma_r \to \gamma \text{ strongly in } L^p(\varepsilon, T; L^p_{loc}(\Omega)).$$

■

We proceed now to prove Lemma 1. In this aim we use the following estimates for the solution of the system in $I\!R^n$.

Lemma 3 *Suppose that (H1)-(H4) are verified and that $y^0 \in L^p(\Omega)$ has compact support contained in B_ρ, the ball centered at zero with radius ρ. Assume also that $\omega \subset B_\rho$. Let Y be the solution of*

$$\left\{ \begin{array}{l} Y_t - \Delta Y + f(Y) = |h|\chi_\omega \quad \textit{in } Q^* = I\!R^n \times (0,T) \\ Y(x,0) = |\widetilde{y^0}(x)| \quad \textit{in } \; I\!R^n \end{array} \right. \tag{34}$$

where $\widetilde{y^0}$ is the extension by zero of y^0 to $I\!R^n$. Then, there exists a constant $C > 0$ such that, for every $r > \rho$

$$\|Y\|_{L^\infty(\Gamma_r)} \leq C^* \frac{e^{MT}}{(r-\rho)^n}$$

where $C^* = C\left\{\|y^0\|_{L^1(I\!R^n)} + \|h\|_{L^1(I\!R^n \times (0,T))}\right\}$, $M = \|f'\|_\infty$ and $\Gamma_r = \partial B_r \times (0,T)$.

Remark 5 Observe that the upper bound includes the L^1-norm of y^0. Since we are in an unbounded set and $y^0 \in L^p(\Omega)$, this norm can be estimated because the data y^0 is of compact support. ■

Proof of Lemma 3

Since $Y(0) = |\widetilde{y^0}| \geq 0$, by the maximum principle Y is positive. Moreover $|f(Y)| < MY$ and then Y is a subsolution of the problem

$$\left\{ \begin{array}{l} u_t - \Delta u - Mu = |h|\chi_\omega \quad \text{in } Q^* \\ u(x,0) = |\widetilde{y^0}(x)| \quad \text{in } \; I\!R^n. \end{array} \right. \tag{35}$$

Let $v = e^{-Mt}u$. Then v verifies

$$\left\{ \begin{array}{l} v_t - \Delta v = e^{-Mt}|h|\chi_\omega \quad \text{in } Q^* \\ v(x,0) = |y^0(x)| \quad \text{in } \; I\!R^n. \end{array} \right. \tag{36}$$

We can express v by the variation of constants formula:

$$\begin{aligned} v(x,t) &= (4\pi t)^{-n/2}\int_{I\!R^n} e^{\frac{-|x-z|^2}{4t}}|y^0(z)|dz \\ &+ \int_0^t\int_{|z|<\rho}(4\pi(t-\sigma))^{-n/2}e^{\frac{-|x-z|^2}{4(t-\sigma)}}e^{-M\sigma}|h(\sigma,z)|\chi_\omega d\sigma dz. \end{aligned}$$

Since $\operatorname{supp} y^0$ and $\omega\subset B_\rho$ we have, for every x such that $|x|>\rho$,

$$\begin{aligned} v(x,t) &\le (4\pi t)^{-n/2}\int_{|z|<\rho} e^{\frac{-||x|-\rho|^2}{4t}}|y^0(z)|dz \\ &+ \int_0^t\int_{|z|<\rho}(4\pi(t-\sigma))^{-n/2}e^{\frac{-||x|-\rho|^2}{4(t-\sigma)}}|h(\sigma,z)|\chi_\omega d\sigma dz \end{aligned}$$

and then

$$\begin{aligned} v(x,t) &\le (4\pi t)^{-n/2}e^{\frac{-||x|-\rho|^2}{4t}}|y^0|_{L^1(B_\rho)} \\ &+ \int_0^t\int_{|z|<\rho}(4\pi(t-\sigma))^{-n/2}e^{\frac{-||x|-\rho|^2}{4(t-\sigma)}}|h(\sigma,z)|\chi_\omega d\sigma dz. \end{aligned} \tag{37}$$

Let $g(s)=s^{-\alpha}e^{-\beta/s}$. By elementary calculus we obtain that g reaches its maximum at $s=\beta/\alpha$. Then

$$s^{-\alpha}e^{-\beta/s}\le\left(\frac{e}{\alpha}\right)^{-\alpha}\beta^{-\alpha}.$$

Substituting this bound in (37) with $\alpha=n/2$ and $\beta=||x|-\rho|^2$, we obtain

$$\|v\|_{L^\infty(\Gamma_r)}\le\left(\frac{2\pi e}{n}\right)^{-n/2}\frac{1}{(r-\rho)^n}\left(\|y^0\|_{L^1(I\!R^n)}+\|h\chi_\omega\|_{L^1((0,T)\times I\!R^n)}\right).$$

We set

$$C^*=\left(\frac{2\pi e}{n}\right)^{-n/2}\left(\|y^0\|_{L^1(I\!R^n)}+\|h\chi_\omega\|_{L^1((0,T)\times I\!R^n)}\right).$$

Since $u(x,t)=e^{Mt}v\le e^{MT}v$ we have that

$$\|u\|_{L^\infty(\Gamma_r)}\le C^*\frac{e^{MT}}{(r-\rho)^n}\qquad\forall r>\rho$$

and since Y is a non negative subsolution of (35):

$$\|Y\|_{L^\infty(\Gamma_r)} \leq C^* \frac{e^{MT}}{(r-\rho)^n} \quad \forall r > \rho.$$

■

Remark 6 Following the same procedure it is easy to see that if $\tilde{y}$ satisfies

$$\begin{cases} \tilde{y}_t - \Delta \tilde{y} + f(\tilde{y}) = -|h|\chi_\omega \text{ in } Q^* \\ \tilde{y}(x,0) = -|\widetilde{y^0(x)}| \text{ in } I\!R^n \end{cases}$$

then, for every $r > \rho$,

$$\tilde{y}(x,t) \geq -C^* \frac{e^{MT}}{(r-\rho)^n} \text{ in } \Gamma_r.$$

Since $\tilde{y}$ is non positive that implies that $\|\tilde{y}\|_{L^\infty(\Gamma_r)} \leq C^* e^{MT}/(r-\rho)^n$ for all $r > \rho$. ■

Proof of Lemma 1 Let $g = Y - v$ with Y verifying (34). Then g verifies

$$\begin{cases} g_t - \Delta g + f(Y) - f(v) = 2h^-\chi_\omega \text{ in } Q \\ g = Y \text{ on } \Sigma \\ g(x,0) = |y^0(x)| - y^0(x) \text{ in } \Omega. \end{cases} \tag{38}$$

Since f is globally Lipschitz we can apply the maximum principle to see that $g \geq 0$. By Lemma 1 we have, for every $r > \rho$ and for almost every x such that $|x| = r$,

$$C^* \frac{e^{MT}}{(r-\rho)^n} \geq Y(x,t) \geq v(x,t) \quad \forall t \in (0,T).$$

In a similar way, but now with $v - \tilde{y}$, we obtain, for every $r > \rho$ and for almost every x such that $|x| = r$,

$$-C^* \frac{e^{MT}}{(r-\rho)^n} \leq \tilde{y}(x,t) \leq v(x,t) \quad \forall r > \rho, \quad \forall t \in (0,T).$$

■

Proof of Lemma 2 Consider first the case where $r = 1$ and suppose that w solves (20) with $r = 1$. Since $g \in L^\infty(\Gamma_1)$, by the maximum principle we know that

$$w(t) \in L^p(B_1) \;\text{ for every } t \in (0,T), \text{ for all } 1 \leq p \leq \infty. \tag{39}$$

For $1 < q < \infty$ such that $1/p + 1/q = 1$ we consider the following transposed problem: for $\eta^0 \in L^q(B_1)$ and $t \in (0,T)$, let η be the solution of the equation

$$\begin{cases} -\eta_t - \Delta\eta = 0 \;\text{ in } B_1 \times (0,t) \\ \eta = 0 \;\text{ on }\; \Gamma_1 \\ \eta(x,t) = \eta^0 \;\text{ in }\; B_1. \end{cases}$$

Since $\eta_0 \in L^q(B_1)$ we have that $\|\eta(t)\|_{W^{s,q}(B_1)} \leq t^{-s/2}\|\phi^0\|_{L^q(B_1)}$ for $0 < s < 2$ (For the proof see Lions-Magenes [12] or (72) in section 7). Now, if $\frac{1+q}{q} < s < 2$, we have that $\frac{\partial\eta}{\partial\nu}(t)$ belongs to $L^1(\partial B_1)$ [see Triebel [19], p. 330], and

$$\left\|\frac{\partial\eta}{d\nu}(t)\right\|_{L^1(\partial B_1)} \leq Ct^{-s/2}\|\eta^0\|_{L^q(B_1)}$$

and then

$$\left\|\frac{\partial\eta}{d\nu}\right\|_{L^1(\partial B_1\times(0,t))} \leq C(B_1,s)t^{(1-s/2)}\|\eta^0\|_{L^q(B_1)} \;\; \forall t \in (0,T). \tag{40}$$

Multiplying (20) by η we obtain:

$$\int_{B_1} w(t)\eta^0 = \int_0^t \int_{\partial B_1} \frac{\partial\eta}{\partial\nu} g$$

and in view of (40) we have

$$|\int_{B_1} w(t)\eta^0| \leq \|\frac{\partial\eta}{\partial\nu}\|_{L^1(\Gamma_1)}\|q\|_{L^\infty(\Gamma_1)} \leq C(B_1,\varepsilon)t^{\varepsilon/2}\|\eta^0\|_{L^q(B_1)}\|g\|_{L^\infty(\Gamma_1)}.$$

Therefore

$$\|w(t)\|_{L^p(B_1)} \leq C(B_1,\varepsilon)t^{\varepsilon/2}\|g\|_{L^\infty(\Gamma_1)} \tag{41}$$

where $\frac{1}{p} > \varepsilon = 2 - s$, $2 > s > \frac{1+q}{q}$ and $C(B_1,\varepsilon)$ is independent of t. Let now w be the solution of (20), $\bar{w}(x,t) = w(rx, r^2t)$ and $\bar{g}(x,t) = g(rx, r^2t)$. Then $\bar{w}$ satisfies

$$\begin{cases} \bar{w}_t - \Delta\bar{w} = 0 \;\text{ in } B_1 \times (0, T/r^2) \\ \bar{w} = \bar{g} \;\text{ on }\; \partial B_1 \times (0,T/r^2) = \bar{\Gamma}_1 \\ \bar{w}(x,0) = 0 \;\text{ in }\; B_1. \end{cases}$$

In view of (41)

$$\left\|\bar{w}\left(\frac{t}{r^2}\right)\right\|_{L^p(B_1)} \leq C\left(\frac{t}{r^2}\right)^{\varepsilon/2} \|\bar{g}\|_{L^\infty(\bar{\Gamma}_1)}$$

and then

$$\left(\int_{B_r} w^p(x,t)dx\right)^{1/p} \leq r^{n/p} C\left(\frac{t}{r^2}\right)^{\varepsilon/2} \|g\|_{L^\infty(\Gamma_r)}.$$

That is,

$$\|w(t)\|_{L^p(B_r)} \leq Ct^{\varepsilon/2} r^{\frac{n}{p}-\varepsilon} \|g\|_{L^\infty(\Gamma_r)} \tag{42}$$

where C is independent on r. ■

In the next section we prove that the controls given in [8] for the approximate controllability of (18) on bounded domains verify the hypotheses of Proposition 3.

4 Proof of Theorem 1

4.1 Construction of the controls in the truncated domain

First of all we are going to recall the construction of [8] for the controls in the truncated domains Ω_r. Assume first that $f \in C^1(I\!R)$. Our hypotheses on f imply that the function g defined by

$$g(s) = \begin{cases} f(s)/s & \text{if} \quad s \neq 0 \\ f'(0) & \text{if} \quad s = 0 \end{cases}$$

is continuous and belongs to $L^\infty(I\!R)$. Notice that in [8], $g(s) = (f(s)-f(0))/s$ if $s \neq 0$, but, in our case, we are supposing $f(0) = 0$.

Let $\alpha > 0$, $y^0, y^1 \in L^p(\Omega_r)$ be fixed . For $\eta_r \in L^p(Q_r)$ we write $y_r = V_r + u_r$ where u_r is solution of

$$\begin{cases} u_{r,t} - \Delta u_r + g(\eta_r)u_r = 0 \ \text{ in } Q_r \\ u_r = 0 \ \text{ on } \ \Sigma_r \\ u_r(x,0) = y_r^0(x) \ \text{ in } \ \Omega_r. \end{cases} \tag{43}$$

In [8] it is proved that there exists $\varphi_r \in L^p(Q_r)$ and $\lambda_r \in \mathrm{sgn}(\varphi_r)$ such that if V_r is the solution of

$$\begin{cases} V_{r,t} - \Delta V_r + g(\eta_r)V_r = \|\varphi_r\|_{L^1(\omega\times(0,T))}\lambda_r\chi_\omega \ \text{ in } Q_r \\ V_r = 0 \ \text{ on } \ \Sigma_r \\ V_r(x,0) = 0 \ \text{ on } \ \Omega_r \end{cases} \tag{44}$$

then $\|V_r(T) - y^1 + u_r(T)\|_{L^p(\Omega_r)} \leq \alpha$, that is $\|y_r(T) - y^1\|_{L^p(\Omega_r)} \leq \alpha$. Moreover, φ_r solves the adjoint system

$$\begin{cases} -\varphi_{r,t} - \Delta\varphi_r + g(\eta_r)\varphi_r = 0 \ \text{ in } Q_r \\ \varphi_r = 0 \ \text{ on } \ \Sigma_r \\ \varphi_r(T,x) = \varphi_r^0 \ \text{ in } \ \Omega, \end{cases}$$

with φ_r^0 the minimizer of

$$J_r(\varphi_r^0, g(\eta_r), y^1 - u_r(T)) = \frac{1}{2}(\int_{\omega\times(0,T)} |\varphi_r|)^2 + \alpha\|\varphi_r^0\|_{L^q(\Omega_r)} - \int_{\Omega_r}(y^1 - u_r(T))\varphi_r^0.$$

Let us consider the set-valued mapping:

$$\Lambda_r : L^p(Q_r) \to \mathcal{P}(L^p(Q_r))$$

with

$$\Lambda_r(\eta_r) = \ \{y_r : \lambda_r \in \mathrm{sgn}(\varphi_r(\eta_r, y^0, y^1)), \\ \|y_r(T) - y^1\|_{L^p(\Omega_r)} < \alpha, \varphi_r^0 \text{ minimizing } J_r\Big\}$$

where y_r is the solution of (19) corresponding to η_r and $h_r = \|\varphi_r\|_{L^1(\omega\times(0,T))}\lambda_r$. In [8] it was proved that Λ_r has a fixed point such that

$$\begin{cases} -\varphi_{r,t} - \Delta\varphi_r + g(y_r)\varphi_r = 0 \text{ in } \ Q_r \\ \varphi_r = 0 \ \text{ on } \ \Sigma_r \\ \varphi_r(T,x) = \varphi_r^0 \text{ in } \ \Omega \\ y_{r,t} - \Delta y_r + f(y_r) = \|\varphi_r\|_{L^1(\omega\times(0,T))}\lambda_r\chi_\omega \ \text{ in } Q_r \\ y_r = 0 \ \text{ on } \ \Sigma_r \\ y_r(x,0) = y^0 \ \text{ in } \ \Omega_r \\ \|y_r(T) - y^1\|_{L^p(\Omega_r)} \leq \alpha. \end{cases} \tag{45}$$

Observe that Proposition 3 reduces the proof of Theorem 1 to proving that a subsequence of these controls $\|\varphi_r\|_{L^1(q)}\lambda_r$ converges weakly in $L^\infty(\omega\times(0,T))$ as $r \to \infty$. The limit will then be the control of the limit problem in Ω. That is done in the next section.

4.2 Convergence of the controls

Since $\lambda_r \in \text{sgn}\varphi_r$ the controls $h_r = |\varphi_r|_{L^1(\omega\times(0,T))}\lambda_r$ belong to $L^\infty(\omega \times (0,T))$ and $|h_r(x,t)| \leq |\varphi_r|_{L^1(\omega\times(0,T))}$ almost everywhere in $\omega \times (0,T)$. Moreover, since ω is a bounded set, h_r belongs to $L^p(\omega \times (0,T))$ for every $1 \leq p \leq \infty$. On the other hand, by Proposition 2 and Remark 3 we know that a uniform bound of the minimizers implies convergence of the corresponding solutions of (8) in $L^1(\omega \times (0,T))$ (and for a subsequence a.e. in $\omega \times (0,T)$). In that case, since $\lambda_r \in \text{sgn}\varphi_r$ we will obtain the weak* convergence of the controls. That is, we are going to prove that there exits $C > 0$ such that

$$\|\varphi_r^0\|_{L^q(\Omega_r)} \leq C \quad \text{for every } r \tag{46}$$

with φ_r^0 the minimizer of $J_r(\cdot; g(y_r), y_r^1 - u_r(T))$.

As a matter of facts, in view of Proposition 1, that is a direct consequence of the following result:

Proposition 4 *Let u_r be the solution of (43) corresponding to $g(y_r)$, with y_r a fixed point of the set valued mapping Λ_r, and initial data $y_r^0 = y^0\chi_{\Omega_r}$ with $y^0 \in L^p(\Omega)$. Then, $\{\widetilde{u_r}(T)\}$ is relatively compact in $L^p(\Omega)$, where $\widetilde{u_r}(T)$ is the extension by zero of $u_r(T)$ to Ω.*

Proof As before, we can use the maximum principle to verify that the solution U_r of

$$\begin{cases} U_{r,t} - \Delta U_r - CU_r = 0 \text{ in } Q_r \\ U_r = 0 \text{ on } \Sigma_r \\ U_r(x,0) = |y_r^0(x)| \text{ on } \Omega_r \end{cases}$$

with $C = \|g\|_\infty$ satisfies $U_r \geq u_r$ in $\Omega_r \times (0,T)$ for every r.

Let $w_r = e^{-Ct}U_r$. Then by the maximum principle $z \geq w_r$ in Q_r for every r where z satisfies

$$\begin{cases} z_t - \Delta z = 0 \text{ in } Q \\ z = 0 \text{ on } \Sigma \\ z(x,0) = |y^0(x)| \text{ in } \Omega \end{cases} \tag{47}$$

and then $e^{Ct}z \geq u_r$ in $\Omega_r \times (0,T)$ for every r.

In a similar way we can see that $u_r > -e^{Ct}z$ in $\Omega_r \times (0,T)$ and then

$$|u_r(x,t)| \leq e^{Ct}z(x,t) \text{ in } \Omega_r \times (0,T) \tag{48}$$

for every $r > 0$. Since $\widetilde{u_r}(T)$ is bounded in $L^p(\Omega)$ there exists $u \in L^p(\Omega)$ and a subsequence (still denoted by r) such that

$$\widetilde{u_r}(T) \rightharpoonup u \quad \text{weakly in} \quad L^p(\Omega), \text{ as } r \to \infty.$$

Observe that if we put $t' = T - t$ then $u_r(t')$ satisfies (8). We can apply Proposition 2 to the sequence $\widetilde{u_r}$. Then there exists $\gamma \in L^p(\Omega \times (0,T)) \cap C([0,T]; W^{-1,p}(\Omega))$ such that (for a subsequence)

$$\widetilde{u_r} \to \gamma \text{ strongly in } C([0,T]; W^{-1,p}_{loc}(\Omega)), \text{ as } r \to \infty,$$

$$\widetilde{u_r} \to \gamma \text{ strongly in } L^p(\varepsilon, T; L^p_{loc}(\Omega)), \ \forall 0 < \varepsilon < T, \text{ as } r \to \infty$$

and

$$\widetilde{u_r}(t) \to \gamma(t) \text{ strongly in } L^p_{loc}(\Omega), \ \forall t \in (0,T).$$

Therefore $\gamma(T) = u$ and $\widetilde{u_r}(T) \to \gamma(T)$ strongly in $L^p_{loc}(\Omega)$. On the other hand, in view of (48), for every $\varepsilon > 0$ there exists $R > 0$, such that, for every $r > R$:

$$\|\widetilde{u_r}(T)\|_{L^p(\Omega'_R)} \leq e^{CT} \|z(T)\|_{L^p(\Omega'_R)} < \varepsilon/3 \tag{49}$$

and for R large enough $\|\gamma(T)\|_{L^p(\Omega'_R)} \leq \varepsilon/3$. Then, in view of (49), for every $r > R$ we have

$$\|\widetilde{u_r}(T) - \gamma(T)\|_{L^p(\Omega)} \leq \|u_r(T) - \gamma(T)\|_{L^p(\Omega_R)} + e^{CT}\|z(T)\|_{L^p(\Omega'_R)} + \varepsilon/3.$$

It is then easy to conclude that $\widetilde{u_r}(T) \to \gamma(T)$ strongly in $L^p(\Omega)$, $r \to \infty$.
■

This convergence can be generalized to the case in which f is a globally Lipschitz function. Indeed, from [8] we have:

Proposition 5 *Let f be a globally Lipschitz function. There exist $A > 0$ and a sequence $(f_n)_n$ in $C^1(I\!R)$ such that $f_n(0) = 0$ and*

$$\forall n \in I\!N, \quad \forall s \in I\!R, \qquad \left|\frac{f_n(s)}{s}\right| \leq A,$$
$$\lim_{n\to\infty} f_n = f \ \textit{in} \qquad C_c(I\!R).$$

Moreover, denoting by φ_{nr}, $\lambda_{nr} \in \operatorname{sgn}(\varphi_{nr})$ and v_{nr} the solutions of (45) associated to f_n (i.e. with $g(s) = g_n(s) = f_n(s)$), there exists $G_r \in L^\infty(Q_r)$

such that φ^0_{nr} *converges strongly in* $L^q(\Omega_r)$ *to a minimizer* φ^0_r *of* $J_r(.;G_r, y^1 - u_r(T))$ *and* (φ_{nr}, v_{nr}) *converges strongly in* $L^q(Q_r) \times L^p(Q_r)$ *to the solution of*

$$\begin{cases} -\varphi_{r,t} - \Delta\varphi_r + G_r\varphi_r = 0 \quad in \ Q_r \\ \varphi_r = 0 \quad on \ \ \Sigma_r \\ \varphi_r(T,x) = \varphi^0_r \quad in \quad \Omega \\ y_{r,t} - \Delta y_r + f(y_r) = |\varphi_r|_{L^1(\omega\times(0,T))}\lambda_r\chi_\omega \quad in \ Q_r \\ y_r = 0 \quad on \ \ \Sigma_r \\ y_r(x,0) = y^0 \quad in \quad \Omega \\ |y_r(T) - y^1|_{L^p(\Omega_r)} \leq \alpha \end{cases}$$

where $\lambda_r \in \operatorname{sgn}(\varphi_r)\chi_q$*. Furthermore,* $G_r(x,t) = g(y_r(x,t))$ *on the set* $[y_r(x,t) \neq 0]$.

5 $L^1(\Omega)$ and $C_0(\Omega)$ cases

5.1 The $L^1(\Omega)$ case

The results obtained in section 2 concerning the uniform bound of the minimizers of the functional J_r in $L^q(\Omega)$ for $1 < q < \infty$ can be adapted without major changes to the case $q = \infty$, $p = 1$.

We consider the solution φ_r of (5) with final data $\varphi^0_r \in L^\infty(\Omega_r)$. Then $\varphi_r \in L^\infty(\Omega_r \times (0,T))$ and $\varphi_r \in C([0,T]; L^q_{loc}(\Omega_r))$ for every $1 \leq q < \infty$. Given $y^1 \in L^1(\Omega)$ and $\alpha > 0$ we define

$$J_r(\varphi^0_r; a, y^1) = \frac{1}{2}\left(\int_0^T \int_\omega |\varphi_r|\right)^2 - \alpha\|\varphi^0_r\|_{L^\infty(\Omega_r)} - \int_{\Omega_r} y^1\varphi^0_r dx.$$

The main difference with the case $1 < q < \infty$ is that J_r achieves its minimum at some $\hat{\varphi}^0_r \in L^\infty(\Omega_r)$ but this minimizer is not necessarily unique. Nevertheless it is not difficult to show under the hypotheses of Proposition 1 (with $q = \infty$ and $p = 1$) that there exists a constant $C > 0$ such that

$$\|\hat{\varphi}^0_r\|_{L^\infty(\Omega_r)} \leq C \ \forall r$$

and for every minimizer $\hat{\varphi}^0_r$ of J_r. In this aim Proposition 2 has to be adapted to the present case. That is, for convergences (9)-(10) we have to make use of the weak* convergence in $L^\infty(\Omega)$ and convergences (11)-(13) must be changed to the following:

$$\widetilde{\psi_r} \to \beta \ \text{ strongly in } L^q(0, T-\varepsilon; L^q_{loc}(\Omega)),\ \forall 0 < \varepsilon < T; \tag{50}$$

$$\widetilde{\psi_r} \to \beta \ \text{ strongly in } C([0,T]; W^{-1,q}_{loc}(\Omega)); \tag{51}$$

$$\widetilde{\psi_r}(t) \to \beta(t) \ \text{ strongly in } L^q_{loc}(\Omega),\ \forall t \in [0,T) \tag{52}$$

as $r \to \infty$ for every $1 < q < \infty$.

Since Proposition 1 is still valid the only point that needs an adaptation is the convergence of the controls. That is, we need to prove Proposition 4 in the $L^1(\Omega)$ case. The main difference in the proof lies in the fact that the initial data lies in $L^1(\Omega)$ (the analogue of (3) is no longer valid). Nevertheless (48) is still valid with $z(x,t) \in L^\infty(0,T; L^1(\Omega))$. However, due to the smoothing effects of the heat equation it is not difficult to prove that

$$\widetilde{u_r} \to \gamma \ \text{ strongly in } L^p(\delta, T; L^p_{loc}(\Omega)), \tag{53}$$

$$\widetilde{u_r} \to \gamma \text{ strongly in } C([\delta,T]; W^{-1,p}_{loc}(\Omega)) \tag{54}$$

and

$$\widetilde{u_r}(t) \to \gamma(t) \text{ strongly in } L^p(\Omega)\ \forall t \in (\delta, T] \tag{55}$$

for some $p > 1$ and for every $\delta > 0$.

Therefore, for any R large enough and in view of the continuity of the inclusion $L^p(\Omega_R) \subset L^1(\Omega_R)$ it is straightforward to prove that

$$\|u_r(T) - \gamma(T)\|_{L^1(\Omega_R)} \to 0 \ \text{ as } r \to \infty$$

and in view of (48) to obtain that $\widetilde{u_r}(T) \to \gamma(T)$ strongly in $L^1(\Omega)$ as $r \to \infty$.
■

5.2 The $C_0(\Omega)$ case

Let us introduce the dual space of $C_0(\Omega)$, $M(\Omega)$ which is a space of bounded Radon measures on Ω. The norm in $M(\Omega)$ is defined as follows:

$$\|\mu\|_{M(\Omega)} = \sup_{\vartheta \in C_0(\Omega)} \frac{|<\mu, \vartheta>|}{\|\vartheta\|_{L^\infty(\Omega)}}$$

where $< \cdot, \cdot >$ denotes the duality product $M(\Omega)$-$C_0(\Omega)$. Observe that for $\mu \in M(\Omega)$ the Riesz Representation Theorem allows to write $< \mu, \vartheta >= \int_\Omega \vartheta d\mu$ for every $\vartheta \in C_0(\Omega)$ (see e.g. [15] th. 2.3.1).

As in the L^p case we consider first the linear equation with potential (5). For every $\varphi_r^0 \in M(\Omega_r)$ system (5) has a unique solution $\varphi_r \in L^\infty(0,T;L^1(\Omega_r)) \cap C([0,T);L^1(\Omega_r))$ that takes the final data in the following sense: For every $\vartheta \in C_0(\Omega_r)$

$$\int_{\Omega_r} \varphi_r(t)\vartheta dx \to \int_{\Omega_r} \vartheta d\varphi_r^0 \text{ as } t \to T.$$

For $y^1 \in C_0(\Omega)$ and $\alpha > 0$ fixed we define

$$J_r(\varphi_r^0; a, y^1) = \frac{1}{2}\left(\int_0^T \int_\omega |\varphi_r|\right)^2 + \alpha\|\varphi_r^0\|_{M(\Omega_r)} - \int_{\Omega_r} y^1 d\varphi_r^0.$$

We know (see [8]) that $J_r(\cdot; a, y^1)$ achieves its minimum at a unique point $\hat{\varphi}_r^0 \in M(\Omega_r)$.

By Hahn-Banach Theorem any measure φ_r^0 in $M(\Omega_r)$ can be extended to an element $\widetilde{\varphi_r^0}$ of $M(\Omega)$ such that $||\widetilde{\varphi_r^0}||_{M(\Omega)} = ||\varphi_r^0||_{M(\Omega_r)}$.

The analogue of Proposition 2 is straightforward having in mind the regularizing effects of the heat equation. Indeed, for every $T > \varepsilon > 0$ we have that $\varphi_r^\varepsilon = \varphi_r(T-\varepsilon) \in L^2(\Omega_r)$ with φ_r^ε bounded in $L^2(\Omega_r)$. Under the hypotheses of Proposition 2 with $q = \infty$ and $\varphi_r^0 \in M(\Omega_r)$, it is not difficult to prove that there exists a subsequence of solutions of (5), $\varphi^0 \in M(\Omega)$ and $\beta \in L^\infty(0,T;L^1(\Omega)) \cap L^2_{loc}(0,T;H^2_{loc}(\Omega))$ such that

$$\widetilde{\varphi_r^0} \rightharpoonup \varphi^0 \text{ weakly* in } M(\Omega)$$

$$\widetilde{\varphi_r} \rightharpoonup \beta \text{ weakly* in } L^\infty(0,T;L^1(\Omega))$$

$$\widetilde{\varphi_r} \to \beta \text{ strongly in } L^2(0,T-\varepsilon;L^2_{loc}(\Omega)),\ \forall 0 < \varepsilon < T$$

$$\widetilde{\varphi_r}(t) \to \beta(t) \text{ strongly in } L^1_{loc}(\Omega), \quad \forall t \in (0,T)$$

as $r \to \infty$. Let us show that the limit β verifies (14) with final data $\beta(T) = \varphi^0$. Observe first that for every $\vartheta \in C_c(\Omega) \cap C^2(\bar{\Omega})$ and for r large enough, $\widetilde{\varphi_r}$ satisfies

$$\int_\Omega \vartheta\widetilde{\varphi_r}(t)dx - \int_\Omega \vartheta d\widetilde{\varphi_r^0} - \int_t^T \int_\Omega \widetilde{\varphi_r}\Delta\vartheta dx + \int_t^T \int_\Omega \widetilde{\varphi_r} a_r \vartheta dx = 0$$

so that, when $r \to \infty$ we obtain

$$| \int_\Omega \beta(t)\vartheta dx - \int_\Omega \vartheta d\varphi^0| \leq C(T-t), \ \forall \vartheta \in C^2(\bar{\Omega}) \cap C_c(\Omega)$$

where C is a constant that only depends on the uniform $L^\infty(0,T;L^1(\Omega))$-bound of φ_r and the $L^\infty(Q)-$bound of a_r. As $\beta \in L^\infty(0,T;L^1(\Omega))$ and taking into account that $C^2(\bar{\Omega}) \cap C_c(\Omega)$ is dense in $C_0(\Omega)$, this implies that

$$\int_\Omega \beta(t)\vartheta dx \to \int_\Omega \vartheta d\varphi^0 \ \text{ as } t \to T^-, \ \forall \vartheta \in C_0(\Omega).$$

The regularity of the limit $\beta \in L^2_{loc}(0,T;H^2_{loc}(\Omega))$ allows to prove Proposition 1 as in the L^p cases.

Since the fixed point argument used in [8] to obtain the controllability result for the semilinear heat equation in the case C_0 is similar to the argument used in the L^p case we conclude adapting Proposition 4 to the present case. That is, we need to prove that $\{\widetilde{u_r(T)}\}$ is relatively compact in $C_0(\Omega)$ with $\widetilde{u_r(T)}$ the extension by zero to Ω of $u_r(T)$ the solution of (30) corresponding to $g(y_r)$, with y_r a fixed point of the set valued mapping Λ_r and initial data $y^0_r = y^0\chi_{\Omega_r}$ with $y^0 \in C_0(\Omega)$.

First of all we observe (see (48)) that there exists $z \in L^\infty(0,T;C_0(\Omega))$ such that

$$|u_r(x,t)| \leq Cz(x,t) \text{ in } \Omega_r \times (0,T)$$

for every $r > 0$. Observe that if we put $t' = T - t$ then $u_r(t')$ satisfies (8). We are then in the hypotheses of Proposition 2 with $q = \infty$ (see section 5.1 above) and then we can prove that

$$\widetilde{u_r} \to \gamma \ \text{ strongly in } L^p(\delta,T;L^p_{loc}(\Omega)), \ \forall 0 < \delta < T; \tag{56}$$

$$\widetilde{u_r} \to \gamma \text{ strongly in } C([0,T];W^{-1,p}_{loc}(\Omega)); \tag{57}$$

$$\widetilde{u_r}(t) \to \gamma(t) \ \text{ strongly in } L^p_{loc}(\Omega), \ \forall t \in (0,T] \tag{58}$$

as $r \to \infty$ for every $1 < p < \infty$. On the other hand it can be proved (see Section 7, third step) that for every compact subset $K \subset \Omega$

$$\|\widetilde{u_r}(T)\|_{W^{1,p}(K)} \leq C$$

for p large enough such that $W^{1,p}(K) \subset C(K)$ with compact inclusion. Therefore we obtain also that $\widetilde{u_r}(T)|_K \to \gamma(T)|_K$ strongly in $C(K)$ for every compact set $K \subset \Omega$. This, with (48) concludes the proof. ■

6 Exact finite controllability

For Ω an open and bounded domain Zuazua [20] recently proved the simultaneous approximate and finite controllability of the semilinear heat equation. That is, let $\psi_i \in L^q(\Omega)$, $i = 1, ..., m$ be fixed, $\alpha > 0$ and $y^1 \in L^p(\Omega)$, then there exists a control h such that the corresponding solution of (1) simultaneously satisfies

$$\|y(T) - y^1\|_{L^p(\Omega)} \le \alpha \quad \text{and} \int_\Omega y(T)\psi^i = \int_\Omega y^1\psi^i, \; i = 1, ..., m. \tag{59}$$

The techniques introduced in this paper allow to extend that result to unbounded domains Ω. In this aim we introduce the functional $\hat{J}_r(\varphi_r^0; a_r(x,t), y^1)$ defined as

$$\hat{J}_r(\varphi_r^0) = (\int_0^T \int_\omega |\varphi_r|)^2 + \alpha\|(I - \Pi_{E_r})\varphi_r^0\|_{L^q(\Omega_r)} - \int_{\Omega_r} y^1\varphi_r^0 + \int_{\Omega_r} y^0\varphi_r(0)$$

where E_r is the finite dimensional subspace of $L^q(\Omega_r)$ generated by $\psi^i|_{\Omega_r}$ and Π_{E_r} a projector over that subspace (see [6], p. 7). Observe that, as proved in [20] for each $r > 0$ fixed, $\hat{J}_r$ is continuous, convex and coercive, that is

$$\liminf_{\|\phi_r^0\|_{L^q(\Omega_r)} \to \infty} \frac{\hat{J}_r(\phi_r^0)}{\|\phi_r^0\|_{L^q(\Omega_r)}} \ge \alpha.$$

Thus, for every $r > 0$ the minimizer of $\hat{J}_r$ exists and this provides the control in the truncated domain Ω_r, with the potential a_r and such that (59) holds in Ω_r. We can prove the uniform bound of the minimizers like we did in section 4. Remember that once the uniform bound of the minimizers is proved, Proposition 2 allows to construct a control h as limit of the controls h_r in the restricted domains Ω_r. In particular given $y^0 \in L^p(\Omega)$, from Lemma 1, we obtain that $y_r(T)$, the solution of (19) with control h_r and initial data $y^0\chi_{\Omega_r}$ converges in $L^p(\Omega)$ to $y(T)$ the solution of (18) corresponding to h and initial data y^0. Therefore $\int_\Omega y_r(T)\psi^i\chi_{\Omega_r} \to \int_\Omega y(T)\psi^i$. Since $\int_\Omega y_r(T)\psi^i\chi_{\Omega_r} = \int_\Omega y^1\psi^i\chi_{\Omega_r}$ we have

$$\int_\Omega y(T)\psi^i = \int_\Omega y^1\psi^i.$$

Thus we obtain the following finite-approximately result:

Theorem 3 *Let f be a globally Lipschitz function such that $f(0)=0$ and Ω an open and unbounded set of $\mathbb{R}^n$ of class C^2 uniformly. Let $1 \le p \le \infty$, $1/q+1/p=1$, then for every $\alpha>0$, $y^0, y^1 \in L^p(\Omega)$ [$\in C_0(\Omega)$ if $p=\infty$] and $\psi_i \in L^q$, $i=1,...,m$ there exists a control $h \in L^p(\Omega)$ such that the corresponding solution y of (1) simultaneously satisfies :*

$$\|y(T)-y^1\|_{L^p(\Omega)} \le \alpha \text{ and } \int_\Omega y(T)\psi_i = \int_\Omega y^1\psi_i,\ \forall i=1,...,m.$$

7 Proof of Proposition 2

In this section we prove Proposition 2. To do this we need the following classical compactness result (see Simon [17], Theorem 5, p. 84):

Theorem 4 *Let X, B, Y be Banach spaces such that $X \subset B \subset Y$ with continuous embeddings, the embedding $X \subset B$ being compact. Let $1 \le p \le \infty$. If $\mathcal{F}$ is a bounded subset of $L^p(0,T;X)$ and*

$$\|\tau_h f - f\|_{L^p(0,T-h;Y)} \to 0 \text{ as } h \to 0 \text{ uniformly for } f \in \mathcal{F}$$

where $\tau_h f(t) = f(t+h)$, then $\mathcal{F}$ is relatively compact in $L^p(0,T;B)$ [in $C([0,T];B)$ if $p=\infty$.]

Proof of Proposition 2 We divide the proof in six steps. The first four steps correspond to convergence results, the fifth step is devoted to prove that β satisfies equation (14) and the sixth that β belongs to $L^2_{loc}(0,T;H^2_{loc}(\Omega))$. On the other hand at some parts of the proof we are going to distinguish the following two cases: i) $2 \le q < \infty$, ii) $1<q<2$.

First of all, for $1<q<\infty$, we observe that since $\psi_r^0 \in L^q(\Omega_r)$ and $\psi_r \in L^q(0,T;W_0^{1,q}(\Omega_r))$, then $\nabla\widetilde{\psi_r} = \widetilde{\nabla\psi_r}$ and we have that $\widetilde{\psi_r}$ satisfies

$$\begin{cases} -\widetilde{\psi_{r,t}} - \Delta\widetilde{\psi_r} + \widetilde{a_r}(x,t)\widetilde{\psi_r} = \frac{\partial\psi_r}{\partial\nu}\delta_{\partial\Omega_r} & \text{in } Q \\ \widetilde{\psi_r} = 0 \quad \text{on } \partial\Omega\times(0,T) \\ \widetilde{\psi_r}(T) = \widetilde{\psi_r^0} \text{ in } \Omega \end{cases} \tag{60}$$

where $\delta_{\partial\Omega_r}$ is the Dirac measure of the set $\partial\Omega_r$. That is, for $\phi \in C^0(\overline{\Omega})$, $< \frac{\partial\psi_r(t)}{\partial\nu}\delta_{\partial\Omega_r}, \phi > = \int_{\partial\Omega_r} \frac{\partial\psi_r(t)}{\partial\nu}\phi d\sigma$. In view of (60) we have that for every compact subset $K \subset \Omega$, $R>0$ such that $K \subset B_R$ and for every $r>R$,

$$-\widetilde{\psi_{r,t}} = \Delta\widetilde{\psi_r} - \widetilde{a_r}(x,t)\widetilde{\psi_r} \text{ in } K\times(0,T). \tag{61}$$

First step: $\widetilde{\psi_r^0} \rightharpoonup \psi^0$ weakly in $L^q(\Omega)$

Since

$$\|\widetilde{\psi_r^0}\|_{L^q(\Omega)} = \|\psi_r^0\|_{L^q(\Omega_r)} \leq C \quad \text{for every } r \tag{62}$$

there exists a subsequence verifying

$$\widetilde{\psi_r^0} \rightharpoonup \psi^0 \ \text{ weakly in } \ L^q(\Omega). \tag{63}$$

We denote this subsequence with the index r to simplify the notation. The result stated in Proposition 2 holds precisely for that subsequence, as we shall see.

Second step: $\widetilde{\psi_r} \rightharpoonup \beta$ weakly* in $L^\infty(0,T;L^q(\Omega))$

At this point we distinguish between case i) and ii).

Case i): $2 \leq q < \infty$. We multiply equation (8) by $|\psi_r|^{q-2}\psi_r$ and integrate over Ω_r:

$$-\frac{1}{q}\frac{d}{dt}\int_{\Omega_r} |\psi_r(t)|^q + \frac{4(q-1)}{q^2}\int_{\Omega_r} |\nabla|\psi_r(t)|^{q/2}|^2 + \int_{\Omega_r} a_r(x,t)|\psi_r(t)|^q = 0.$$

Therefore, since $A \geq \|a_r\|_{L^\infty(Q)}$ for every r, we can apply Gronwall's inequality to obtain

$$\int_{\Omega_r} |\psi_r(t)|^q + \frac{4}{p}\int_t^T e^{Aq(t-s)}\int_{\Omega_r} |\nabla|\psi_r(s)|^{q/2}|^2 dxds \leq e^{qAT}\int_{\Omega_r} |\psi_r(T)|^q \leq C.$$

Therefore, for every r,

$$\int_{\Omega_r} |\psi_r(t)|^q \ \leq \ C \quad \forall t \in (0,T) \tag{64}$$

$$\int_t^T \int_{\Omega_r} |\nabla|\psi_r|^{q/2}|^2 dxds \ \leq \ C \quad \forall t \in (0,T). \tag{65}$$

In view of (64) we can extract a subsequence (from that we have chosen in (63)) verifying

$$\widetilde{\psi_r} \rightharpoonup \beta \quad \text{weakly* in } \ L^\infty(0,T;L^q(\Omega)), \tag{66}$$

and in particular,

$$\widetilde{\psi_r} \rightharpoonup \beta \quad \text{weakly in } \ L^{q+1}(0,T;L^q(\Omega)). \tag{67}$$

Case ii): $1 < q < 2$. For $\varepsilon > 0$ we multiply (8) by $(\varepsilon + |\psi_r|^2)^{q/2-1}\psi_r$ and we obtain:

$$-\frac{1}{q}\frac{d}{dt}\int_{\Omega_r}(\varepsilon+|\psi_r|^2)^{q/2}+\int_{\Omega_r}\nabla\psi_r\nabla[(\varepsilon+|\psi_r|^2)^{\frac{q-2}{2}}\psi_r] \leq A\int_{\Omega_r}(\varepsilon+|\psi_r|^2)^{\frac{q-2}{2}}|\psi_r(t)|^2,$$

where $A \geq \|\widetilde{a_r}\|_{L^\infty(Q)}$ for every $r > 0$. We have that

$$\begin{aligned}\int_{\Omega_r}\nabla\psi_r\nabla[(\varepsilon+|\psi_r|^2)^{\frac{q-2}{2}}\psi_r] = & \int_{\Omega_r}|\nabla\psi_r|^2(\varepsilon+|\psi_r|^2)^{\frac{q-2}{2}} \\ & +(q-2)\int_{\Omega_r}|\nabla\psi_r|^2(\varepsilon+|\psi_r|^2)^{\frac{q-4}{2}}\psi.\end{aligned}$$

Since $q - 2 < 0$, we deduce that

$$(q-2)\int_{\Omega_r}|\nabla\psi_r|^2(\varepsilon+|\psi_r|^2)^{\frac{q-2}{2}} \leq (q-2)\int_{\Omega_r}|\nabla\psi_r|^2(\varepsilon+|\psi_r|^2)^{\frac{q-4}{2}}\psi_r^2$$

and therefore

$$-\frac{1}{q}\frac{d}{dt}\int_{\Omega_r}(\varepsilon+|\psi_r|^2)^{q/2} \leq A\int_{\Omega_r}(\varepsilon+|\psi_r|^2)^{q/2}.$$

That implies that for every $t \in [0, T]$, for every r and for every $\varepsilon > 0$ we have

$$\int_{\Omega_r}(\varepsilon+|\psi_r(t)|^2)^{q/2} \leq C\int_{\Omega_r}(\varepsilon+|\psi_r(T)|^2)^{q/2}.$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$\int_{\Omega_r}|\psi_r(t)|^q \leq C\int_{\Omega_r}|\psi_r(T)|^q \leq C \quad \text{for every } t \in [0, T]. \tag{68}$$

Therefore $\|\widetilde{\psi_r}\|_{L^\infty(0,T;L^q(\Omega))} \leq C$ for every r. Proceeding as in case i), in view of (68), (61), (62) and extracting subsequences, we obtain

$$\widetilde{\psi_r^0} \rightharpoonup \psi^0 \text{ weakly in } L^q(\Omega); \widetilde{\psi_r} \rightharpoonup \beta \text{ weakly* in } L^\infty(0,T;L^q(\Omega));$$

$$\widetilde{\psi_r} \rightharpoonup \beta \text{ weakly in } L^q(Q).$$

Third step: $\widetilde{\psi_r(t)} \to \beta(t)$ strongly in $L^q_{loc}(\Omega)$ for every t.

For $K \subset \Omega$ compact, from (61) and (64) we have

$$\|\widetilde{\psi_{r,t}}\|_{L^\infty(0,T;W^{-2,q}(K))} \leq C.$$

In view of Theorem 4, taking $B = W^{-1,q}(K)$, $X = L^q(K)$, $Y = W^{-2,q}(K)$, $p = \infty$ and $F = \{\widetilde{\psi_r}\}_r$, there exists $\rho(K)$ and a subsequence $\widetilde{\psi_r}$ (from the previous one) such that $\widetilde{\psi_r} \to \rho(K)$ strongly in $C([0,T]; W^{-1,q}(K))$.

By uniqueness of the weak limit and extracting diagonal subsequences we observe that for every compact subset K, $\rho(K) = \beta|_K$, i.e.

$$\widetilde{\psi_r} \to \beta \quad \text{strongly in } C([0,T]; W^{-1,q}_{loc}(\Omega)) \tag{69}$$

and $\psi^0|_K = \beta(T)|_K$.

At this point we observe that we can reduce the case $1 < q < 2$ to the case $q = 2$. Indeed, due to the smoothing effect of the heat equation, for $T > \varepsilon > 0$ we have

$$\begin{aligned} \psi_r^\varepsilon &= \psi_r(T - \varepsilon) \in L^2(\Omega_r) \\ \|\psi_r^\varepsilon\|_{L^2(\Omega_r)} &\leq (4\pi(T-\varepsilon))^{\frac{-n}{2}(\frac{1}{q}-\frac{1}{2})} \|\psi_r^0(T)\|_{L^q(\Omega_r)}, \end{aligned}$$

(see [4], p. 46). Therefore, ψ_r satisfies

$$\begin{cases} -\psi_{r,t} - \Delta\psi_r + a_r(x,t)\psi_r = 0 \ \text{ in } \ \ \Omega_r \times (0, T-\varepsilon) \\ \psi_r = 0 \quad \text{on } \ \partial\Omega_r \times (0, T-\varepsilon) \\ \psi_r(T-\varepsilon) = \psi_r^\varepsilon \quad \text{ in } \ \Omega_r \end{cases}$$

with $\{\psi_r^\varepsilon\}$ bounded in $L^2(\Omega_r)$ and we are in the conditions of case $q = 2$.

In order to get compactness in L^q, $2 \leq q < \infty$, we need some estimates on the gradients. More precisely, we need to prove the following: For every $K \subset \Omega$ compact, there exist R large enough and a constant $C = C(K) > 0$ independent of r such that for every $r > R$,

$$||\nabla\psi_r\,(t)||_{L^q(K)} \leq C([T-t]^{-1/2} + 1) \tag{70}$$

for every $0 < t < T$. For $\delta > 0$ given, we choose R and $K_R \subset\subset \Omega_R$ an open set of class C^2 with compact closure (observe that $\Omega_R = \Omega \cap B_R$ has not necessarily this regularity) such that $\text{dist}(\partial K, \partial K_R) > 3\delta > 0$. Let K_δ be an open set of class C^2 with compact closure such that $K \subset K_\delta \subset K_R$ and $\text{dist}(\partial K_\delta, \partial K_R) > \delta$. We construct $\phi \in C^\infty(\Omega)$ such that $\phi = 1$ in K_δ and $\phi = 0$ in $\Omega \backslash K_R \cup \partial K_R$. For every $r > R$ we define

$$U_r = \psi_r \phi. \tag{71}$$

Then U_r satisfies

$$\begin{cases} -U_{r,t} - \Delta U_r + a_r(x,t)U_r = \psi_r \Delta\phi - 2\mathrm{div}(\psi_r \nabla\phi) \quad \text{in} \quad K_R \times (0,T) \\ U_r = 0 \quad \text{on } \partial K_R \times (0,T) \\ U_r(T) = \psi_r^0 \phi \quad \text{in} \quad K_R. \end{cases}$$

Let $S(t)$ be the semigroup generated by the operator $-\Delta$ in $L^q(K_R)$ with Dirichlet boundary conditions. By the variation of constants formula we observe that for every $0 \le t < T$,

$$\begin{aligned} U_r(t) &= S(T-t)U_r(T) + \int_t^T S(\sigma - t)\psi_r(\sigma)\Delta\phi d\sigma \\ &\quad -2\int_t^T S(\sigma-t)\mathrm{div}(\psi_r(\sigma)\nabla\phi)d\sigma - \int_t^T S(\sigma-t)a_r(\sigma)U_r(\sigma)d\sigma. \end{aligned}$$

Since $S(\cdot)$ is an analytic semigroup, we have (see [13], p.74)

$$\|S(t)v\|_{L^q(K_R)} \le \|v\|_{L^q(K_R)};\ \|S(t)v\|_{W^{2,q}(K_R)} \le \frac{C(K_R)}{t}\|v\|_{L^q(K_R)},\ \forall v \in L^q(K_R).$$

Interpolating for $0 < s < 2$ (see Lions-Magenes [12]) we obtain

$$\|S(t)v\|_{W^{s,q}(K_R)} \le C(K_R, s, q)t^{-s/2}\|v\|_{L^q(K_R)},\ \forall v \in L^q(K_R). \tag{72}$$

Therefore,

$$\begin{aligned} \|U_r(t)\|_{W^{1/2,q}(K_R)} &\le C(K_R)(T-t)^{-1/4}\|U_r(T)\|_{L^q(K_R)} \\ &\quad + C(K_R,\phi)\|\psi_r\|_{L^\infty(0,T;L^q(\Omega_r))}\int_t^T(\sigma-t)^{-1/4}d\sigma \\ &\quad + C(K_R,\phi)\|\psi_r\|_{L^\infty(0,T;L^q(\Omega_r))}\int_t^T(\sigma-t)^{-1/4-1/2}d\sigma \\ &\quad + C(K_R,A)\|U_r\|_{L^\infty(0,T;L^q(K_R))}\int_t^T(\sigma-t)^{-1/4}d\sigma. \end{aligned}$$

In view of (62), (64) and (71) we obtain

$$\|U_r(t)\|_{W^{1/2,q}(K_R)} \le C[(T-t)^{-1/4} + 1]$$

where the constant C depends on ϕ, K_R, T and $A \ge |a_r|_\infty$.

Since $K_\delta \subset K_R$ and $U_r(t) = \psi_r(t)$ in K_δ for every t, we have proved that for every $r > R$ and for every $T > t \geq 0$,

$$\|\psi_r(t)\|_{W^{1/2,q}(K_\delta)} \leq C[(T-t)^{-1/4} + 1]. \tag{73}$$

Let $\vartheta \in C^\infty(\Omega)$ be such that $\vartheta = 1$ in K and $\vartheta = 0$ in $\Omega \backslash K_\delta \cup \partial K_\delta$. For every $r > 0$ we define

$$V_r = \psi_r \vartheta. \tag{74}$$

Then,

$$\begin{cases} -V_{r,t} - \Delta V_r + a_r(x,t)V_r = \psi_r \Delta\vartheta - 2\mathrm{div}(\psi_r \nabla\vartheta) \quad \text{in} \quad K_\delta \times (0,T) \\ V_r = 0 \quad \text{on} \quad \partial K_\delta \times (0,T) \\ V_r(T) = \psi_r^0 \vartheta \quad \text{in} \quad K_\delta. \end{cases}$$

Let $\widetilde{S}(t)$ be the semigroup generated by the operator $-\Delta$ in $L^q(K_\delta)$ with Dirichlet boundary conditions. Using again the variation of constants formula, we obtain

$$\begin{aligned} \|\nabla V_r(t)\|_{L^q(K_\delta)} \quad & \leq C\left(K_\delta\right)(T-t)^{-1/2}\|V_r(T)\|_{L^q(K_\delta)} \\ & + C\left(K_\delta, \vartheta\right)\|\psi_r\|_{L^\infty(0,T;L^q(K_\delta))} \int_t^T (\sigma - t)^{-1/2}\, d\sigma \\ & + C\left(K_\delta, A\right)\|V_r\|_{L^\infty(0,T;L^q(K_\delta))} \int_t^T (\sigma - t)^{-1/2}\, d\sigma \\ & + C\left(K_\delta, \vartheta\right) \int_t^T (\sigma - t)^{-1/2-1/4} \|\psi_r(\sigma)\|_{W^{1/2,q}(K_\delta)}\, d\sigma. \end{aligned}$$

In view of (62), (64), (73) and (74) we have that

$$\begin{aligned} \|\nabla V_r(t)\|_{L^q(K_\delta)} \quad & \leq \quad C(T-t)^{-1/2} + C\int_t^T (\sigma - t)^{-3/4}[(T-\sigma)^{-1/4}] + 1] \\ & \leq \quad C[(T-t)^{-1/2} + 1]. \end{aligned}$$

Since $V_r(t) = \psi_r(t)$ in K we obtain (70).

We observe that (70) implies $\nabla|\psi_r|^{q/2} \in L^\infty(0, T-\varepsilon; L^2(K))$ for every $\varepsilon > 0$ and

$$\begin{aligned} \int_K |\nabla|\psi_r(s)|^{q/2}|^2 \quad & \leq \quad C\int_K |\nabla\psi_r(s)|^2|\psi_r(s)|^{q-2} \\ & \leq \quad C\|\psi_r\|^{q-2}_{L^\infty(0,T-\varepsilon;L^q(\Omega_r))}\|\nabla\psi_r\|^2_{L^\infty(0,T-\varepsilon;L^q(K))}, \end{aligned}$$

for every $s \in (0, T-\varepsilon)$.

Indeed, from (70) and (64) we have

$$\begin{aligned}\int_K |\nabla |\psi_r(s)|^{q/2}|^2 &\leq C(\frac{1}{(T-s)}+1)\|\psi_r(T)\|^{q-2}_{L^q(\Omega_r)} \\ &= C(\frac{1}{(T-s)}+1), \ \forall s \in (0, T-\varepsilon).\end{aligned}$$

In other words, we have proved that $|\widetilde{\psi_r}|^{q/2-1}\widetilde{\psi_r} \in L^\infty(0, T-\varepsilon; H^1(K))$ and

$$\||\widetilde{\psi_r}|^{q/2-1}\widetilde{\psi_r}\|_{L^\infty(0,T-\varepsilon;H^1(K))} \leq C(\varepsilon), \tag{75}$$

for every $r > R$. In view of (75) for every $t \in (0, T-\varepsilon)$ there exists $r_i = r_i(t)$ [r_i subsequence of (69)] and $\gamma(t, K)$ such that $|\widetilde{\psi_{r_i}}(t)|^{q/2-1}\widetilde{\psi_{r_i}}(t) \to \gamma(t, K)$ strongly in $L^2(K)$ and

$$|\widetilde{\psi_{r_i}}(t)|^{q/2-1}\widetilde{\psi_{r_i}}(t)(x) \to \gamma(t, K)(x) \text{ for almost every } x \in K.$$

Moreover, there exists $g_{K,t} \in L^2(K)$ such that $|\widetilde{\psi_{r_i}}(x,t)|^{q/2} \leq g_{K,t}$, for all r_i and for almost every $x \in K$. Let $G(s) = |s|^{q/2-1}s$. Then $\widetilde{\psi_{r_i}}(x,t) \to G^{-1}(\gamma(t, K)(x))$ for almost every $x \in K$. As a consequence of Lebesgue's Dominated Convergence Theorem we have that $\widetilde{\psi_{r_i}}(x,t) \to G^{-1}(\gamma(t,K)(x))$ strongly in $L^q(K)$. In view of (69), $G^{-1}(\gamma(t,K)(x)) = \beta(t)|_K$ for $t \in (0, T-\varepsilon)$ and therefore $\widetilde{\psi_{r_i}}(t) \to \beta(t)|_K$ strongly in $L^q(K)$.

The subsequence r_i depends (in principle) on t. However, since the limit has been identified in a unique way as $\beta(t)|_K$ we deduce that the whole sequence converges, i.e. for every compact $K \subset \Omega$ and $0 < t < T$, $\widetilde{\psi_r}(t) \to \beta(t)$ in $L^q(K)$.

We observe that in case $1 < q < 2$ we have obtained

$$\widetilde{\psi_r}(t) \to \beta|_K(t) \text{ strongly in } L^2(K), \quad \forall t \in (0, T-\varepsilon)$$

and this convergence implies the convergence in $L^q(K)$ by the continuity of the embedding $L^2(K) \subset L^q(K)$. ■

Fourth step: $\widetilde{\psi_r} \to \beta$ **strongly in** $L^q(0, T-\varepsilon; L^q_{loc}(\Omega))$.

Let $0 \leq \varepsilon < T$ and $h_r(t) = |\widetilde{\psi_r}(t) - \beta(t)|_{L^q(K)}$. From (67) we know that there exists a constant $C > 0$ such that

$$\left(\int_0^{T-\varepsilon} |h_r(t)|^{q+1}dt\right)^{1/(q+1)} \leq C.$$

Since $\widetilde{\psi_r}(t) \to \beta|_K(t)$ strongly in $L^q(K)$ and by Egorov's Theorem, for any $\delta > 0$ there exists a set $B_\delta \subset [0, T-\varepsilon]$ such that $A_\delta = [0, T-\varepsilon]\backslash B_\delta$ with measure $|A_\delta| < \delta^q/C^q$ and such that $h_r \to 0$ uniformly in B_δ. Let R be such that for every $r > R$, $|h_r| < |\frac{\delta}{|A|}|^{1/q}$ in B_δ. Then, for every $r > R$ we have

$$\int_0^{T-\varepsilon} |h_r(t)|^q \leq \int_{A_\delta} |h_r(t)|^q + \int_{B_\delta} |h_r(t)|^q \leq C\frac{\delta}{C} + |B_\delta|\frac{\delta}{|B_\delta|} = 2\delta.$$

Therefore $\widetilde{\psi_r} \to \beta|_K$ strongly in $L^q(0, T-\varepsilon; L^q(K))$. Clearly, the limit β does not depend on the compact set $K \subset \Omega$. Thus

$$\widetilde{\psi_r} \to \beta \text{ strongly in } L^q(0, T-\varepsilon; L^q_{loc}(\Omega)).$$

■

Fifth step: Equation verified by β

We will see now that the limit β verifies (14).

Let $v \in \mathcal{D}(Q)$ and $R > 0$ be such that $\operatorname{supp} v \subset [B_R \cap \Omega] \times (0, T-\varepsilon)$. Then, for every $r > R$

$$\int_0^{T-\varepsilon}\int_\Omega \widetilde{\psi_r} v_t - \int_0^{T-\varepsilon}\int_\Omega \widetilde{\psi_r}\Delta v + \int_0^{T-\varepsilon}\int_\Omega \widetilde{a_r}(x,t)\widetilde{\psi_r} v = 0. \qquad (76)$$

Since $|\widetilde{a_r}|_{L^\infty(Q)} \leq A$, there exists a subsequence (still denoted by r) and $a \in L^\infty(Q)$ such that $\widetilde{a_r} \rightharpoonup a$ weakly* in $L^\infty(Q)$. Passing to the limit in (76) along that subsequence we obtain

$$\int_0^{T-\varepsilon}\int_\Omega \beta v_t - \int_0^{T-\varepsilon}\int_\Omega \beta\Delta v + \int_0^{T-\varepsilon}\int_\Omega a(x,t)\beta v = 0$$

and therefore β verifies the equation

$$-\beta_t - \Delta\beta + a\beta = 0 \text{ in } \mathcal{D}'(\Omega \times (0,T)). \qquad (77)$$

■

Sixth step: β belongs to $L^2_{loc}(0,T; H^2_{loc}(\Omega))$

As we have seen in the third step, when $1 < q < 2$ the smoothing effect of the heat equation allows to treat the equation verified by ψ_r as in the case $q = 2$ provided we work in a time interval $t \in (0, T-\varepsilon)$ excluding a neighborhood of the time T. On the other hand, when $2 < q < \infty$ for any compact set $K \subset \Omega$ the data $\psi_r^0|_K$ belong to and are uniformly bounded in $L^2(K)$ with respect to r. Therefore all cases $1 < q < \infty$ can be reduced to $q = 2$.

In order to estimate the norm of the solution in $L^2(0, T-\varepsilon; H^2(K))$ we proceed as in the third step. That is, we choose R and $K_R \subset\subset \Omega_R$ an open set of class C^2 such that $\text{dist}(\partial K, \partial K_R) > 3\delta > 0$. Let K_δ be an open set of class C^2 such that $K \subset\subset K_\delta \subset K_R$ and $\text{dist}(\partial K_\delta, \partial K_R) > \delta$. We construct $\phi \in C^\infty(\Omega)$ such that $\phi = 1$ in K_δ and $\phi = 0$ in $\Omega \backslash K_R \cup \partial K_R$. For every $r > R$ we define $U_r = \psi_r \phi$ and by the variation of constant formula we estimate first the norm of $\psi_r(t)$ in $H^{3/2}(K_\delta)$. By a bootstrap argument we obtain that

$$\int_0^{T-\varepsilon} \|\psi_r(t)\|_{H^2(K)} \leq C$$

and therefore (for a subsequence) $\widetilde{\psi_r}|_K \rightharpoonup \beta|_K$ weakly in $L^2(0, T-\varepsilon; H^2(K))$. This concludes the proof of Proposition 2. ■

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