

DISPERSION FOR 1-D SCHRÖDINGER AND WAVE EQUATIONS WITH BV COEFFICIENTS

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ABSTRACT. In this paper we analyze the dispersion for one dimensional wave and Schrödinger equations with BV coefficients. In the case of the wave equation we give a complete answer in terms of the variation of the logarithm of the coefficient showing that dispersion occurs if this variation is small enough but it may fail when the variation goes beyond a sharp threshold. For the Schrödinger equation we prove that the dispersion holds under the same smallness assumption on the variation of the coefficient. But, whether dispersion may fail for larger coefficients is unknown for the Schrödinger equation.

1. INTRODUCTION

In this paper we consider the following two equations with variable coefficients: The one-dimensional wave equation

$$(1.1) \quad \begin{cases} v_{tt}(t, x) - \partial_x(a(x)\partial_x v_x)(t, x) = 0, & (t, x) \in \mathbb{R}^2, \\ v(0, x) = v_0(x), \quad v_t(0, x) = 0, & x \in \mathbb{R}, \end{cases}$$

and the Schrödinger equation

$$(1.2) \quad \begin{cases} iu_t(t, x) + \partial_x(a(x)\partial_x u)(t, x) = 0, & (t, x) \in \mathbb{R}^2, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

Along the paper we will consider nonnegative functions a with bounded variation and satisfying the following lower and upper bounds

$$(1.3) \quad 0 < m \leq a(x) \leq M, \quad x \in \mathbb{R}.$$

The main positive results of this paper are as follows.

Theorem 1.1. *For any $a \in BV(\mathbb{R})$ satisfying (1.3) and $\text{Var}(\log(a)) < 2\pi$ there exists a positive constant $C(\text{Var}(a), m, M)$ such that*

$$(1.4) \quad \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |v(t, x)| dt \leq C(\text{Var}(a), m, M) \|v_0\|_{L^1(\mathbb{R})}.$$

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Theorem 1.2. *For any $a \in BV(\mathbb{R})$ satisfying (1.3) and $\text{Var}(\log(a)) < 2\pi$ there exists a positive constant $C(\text{Var}(a), m, M)$ such that*

$$(1.5) \quad \|u(t)\|_{L^\infty(\mathbb{R})} \leq \frac{C(\text{Var}(a), m, M)}{\sqrt{t}} \|u_0\|_{L^1(\mathbb{R})}.$$

In the case of the wave equation a counterexample can also be established when the total variation of the logarithm of the coefficient is large, showing that our dispersion result above is sharp.

Theorem 1.3. *Let be $0 < m < M$ and $\alpha \geq 2\pi$. For any positive number N large enough there exists a piecewise constant function, $m \leq a \leq M$, with $\text{Var}(\log(a)) = \alpha$ such that for some $v_0 \in L^1(\mathbb{R})$ the solution v of problem (1.1) satisfies*

$$(1.6) \quad \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |v(t, x)| dt \geq N \|v_0\|_{L^1(\mathbb{R})}.$$

Such a counterexample is not available for the Schrödinger equation. Thus, whether the above dispersion result is sharp for this model is an open problem.

Our results are given in terms of the total variation of function $\log(a)$. However under the boundedness assumption above (1.3), $\text{Var}(a)$ and $\text{Var}(\log a)$ are comparable.

The main ideas of the proofs of the above results come from the analysis of wave propagation in multi-layer structures [5, Ch. 3] and [1]. The proof follows mainly the ideas in [1] but with finer resolvent estimates.

We recall that, once the dispersion is established for the solutions of the linear Schrödinger equation, more general space-time estimates can be obtained, namely, the so-called Strichartz estimates

$$\|u\|_{L^q(\mathbb{R}, L^r(\mathbb{R}))} \leq C(q, r) \|\varphi\|_{L^2(\mathbb{R})},$$

for some admissible pairs (q, r) . Strichartz estimates for BV coefficients in 1-d without smallness conditions have been established in [2] without making use of the dispersion property. This paper is devoted to investigate under which assumptions the dispersion property still holds.

Estimates similar to these in Theorem 1.1 but integrating on the space variable x instead of time, have been obtained in [4] under a smallness assumption on the BV-norm of $\log(a)$. The methods developed in this paper could very likely be useful to further analyze the problems addressed in [4]. But this is still to be done.

The paper is organized as follows. In section 2 we present some preliminary results from [1] and state two technical lemmas that allow us to improve the results in [1]. In section 3 we prove the main results stated in the introduction. We point out that the proof of Theorem 1.2 uses previous results from the proof of Theorem 1.1. Section 4 contains the proofs of the two technical lemmas. We will obtain estimates on some almost periodic functions by using some tools from analytical number theory.

2. RESOLVENT ESTIMATES ON A LAMINAR MEDIA

In this section we consider a laminar media and collect some previous results from [1], keeping the same notations.

Let us consider a partition of the real axis

$$(2.7) \quad -\infty = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = \infty$$

and a step function

$$(2.8) \quad a(x) = b_k^{-2}, \quad x \in I_k = (x_{k-1}, x_k), \quad k = 1, \dots, n,$$

where $1/M^2 \leq b_k \leq 1/m^2$.

Let us now consider the self-adjoint operator $-\partial_x a(x) \partial_x$ defined from $\{h \in H^1(\mathbb{R}), a \partial_x h \in L^2(\mathbb{R})\}$ to $L^2(\mathbb{R})$. For $\omega \geq 0$ let us consider R_ω its resolvent:

$$R_\omega g = (-\partial_x a(x) \partial_x + \omega^2 I)^{-1} g.$$

It follows that for $x \in I_k = (x_{k-1}, x_k)$, $k = 1, \dots, n$, we have

$$(2.9) \quad R_\omega g(x) = c_{2k-1}(\omega) e^{\omega b_k x} + c_{2k}(\omega) e^{-\omega b_k x} + b_k \int_{I_k} \frac{g(y)}{2\omega} e^{-\omega b_k |x-y|} dy,$$

where $c_2 = c_{2n-1} = 0$ and the other coefficients are determined by solving the system obtained from the continuity of $R_\omega g$ and $a(x) \partial_x R_\omega g$ at the points x_k , $k = 1, \dots, n-1$. It follows that

$$D_n(\omega) C = T$$

where $C = [c_1, c_3, c_4, \dots, c_{2n-3}, c_{2n-2}, c_{2n}]^T$, $T = (t_1, \dots, t_{n-1})^T$, $t_k = (t_{k1}, t_{k2})^T$,

$$t_k = \begin{pmatrix} -b_k \int_{I_k} \frac{g(y)}{2\omega} e^{-\omega b_k (x_k - y)} dy + b_{k+1} \int_{I_{k+1}} \frac{g(y)}{2\omega} e^{-\omega b_{k+1} (y - x_k)} dy \\ b_k b_{k+1} \left(\int_{I_k} \frac{g(y)}{2\omega} e^{-\omega b_k (x_k - y)} dy + \int_{I_{k+1}} \frac{g(y)}{2\omega} e^{-\omega b_{k+1} (y - x_k)} dy \right) \end{pmatrix}, \quad k = 1, \dots, n-1,$$

and

$$D_n = \begin{pmatrix} \mathbf{a}_1 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & B_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & B_3 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & A_{n-3} & B_{n-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{n-2} & B_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{n-1} & \mathbf{b}_{n-1} \end{pmatrix}$$

with

$$\mathbf{a}_1 = \begin{pmatrix} e^{\omega b_1 x_1} \\ b_2 e^{\omega b_1 x_1} \end{pmatrix}, \quad \mathbf{b}_{n-1} = \begin{pmatrix} -e^{-\omega b_n x_{n-1}} \\ b_{n-1} e^{-\omega b_n x_{n-1}} \end{pmatrix},$$

$$A_k = \begin{pmatrix} e^{\omega b_k x_k} & e^{-\omega b_k x_k} \\ b_{k+1} e^{\omega b_k x_k} & -b_{k+1} e^{-\omega b_k x_k} \end{pmatrix}, \quad B_k = \begin{pmatrix} -e^{\omega b_{k+1} x_k} & -e^{-\omega b_{k+1} x_k} \\ -b_k e^{\omega b_{k+1} x_k} & b_k e^{-\omega b_{k+1} x_k} \end{pmatrix}.$$

For technical reasons we introduce the matrix \tilde{D}_n which has the same structure as D_n but replacing vector \mathbf{b}_{n-1} with

$$\mathbf{b}_{n-1} = \begin{pmatrix} -e^{\omega b_n x_{n-1}} \\ -b_{n-1} e^{\omega b_n x_{n-1}} \end{pmatrix}.$$

We point out that the vectors \mathbf{b}_{n-1} appearing in D_n and \tilde{D}_n are given by the second and respectively first column of B_{n-1} .

Let us introduce now the reflection coefficients

$$(2.10) \quad d_{k-1} = \frac{b_{k-1} - b_k}{b_{k-1} + b_k}, \quad k = 2, \dots, n$$

and the functions Q_k , $k = 1, \dots, n$, defined as follows: $Q_1(\omega) \equiv 0$ and

$$(2.11) \quad Q_k(\omega) = \begin{cases} e^{-2\omega b_k(x_k - x_{k-1})} \frac{-d_{k-1} + Q_{k-1}(\omega)}{1 - d_{k-1} Q_{k-1}(\omega)}, & k = 2, \dots, n-1, \\ e^{2\omega b_k x_{k-1}} \frac{-d_{k-1} + Q_{k-1}(\omega)}{1 - d_{k-1} Q_{k-1}(\omega)}, & k = n. \end{cases}$$

It follows that for $2 \leq k \leq n$

$$Q_k(\omega) = \begin{cases} e^{-2\omega b_k x_k} \frac{\det \tilde{D}_k(\omega)}{\det D_k(\omega)}, & k = 2, \dots, n-1, \\ \frac{\det \tilde{D}_k(\omega)}{\det D_k(\omega)}, & k = n, \end{cases}$$

and for any $2 \leq k \leq n$

$$(2.12) \quad \det D_k(\omega) = \prod_{j=1}^{k-1} (b_j + b_{j+1}) e^{\omega(b_j - b_{j+1})x_j} (1 - d_j Q_j(\omega)).$$

It has been proved in [1] that there exists a $\delta > 0$ such that for any $\omega \in \mathbb{C}$ with $\Re(\omega) > -\delta$ we have $|Q_k(\omega)| < 1$, $k = 2, \dots, n$. It implies that $(\det D_n(\omega))^{-1}$ is uniformly bounded in the same region of the complex plane and moreover $\omega R_\omega u_0$ can be analytically continued. Thus the spectral calculus gives us the following representation of the solutions of equations (1.1) and (1.2).

Lemma 2.1. *The solution of the wave equation (1.1) verifies*

$$(2.13) \quad v(t, x) \mathbf{1}_{\{t>0\}} = \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{it\omega} \omega (R_{i\omega} v_0)(x) d\omega.$$

Lemma 2.2. *The solution of the Schrödinger equation (1.2) verifies*

$$(2.14) \quad u(t, x) = \frac{1}{i\pi} \int_{-\infty}^{\infty} e^{-it\omega^2} \omega (R_{i\omega} u_0)(x) d\omega.$$

For completeness we prove these lemmas.

Proof of Lemma 2.1. Set $v_1(t) = v(t)\mathbf{1}_{\{t>0\}}$. It follows that v_1 satisfies

$$\partial_{tt}v_1 - \partial_x(a(x)\partial_x v_1) = \partial_t \delta_0 f.$$

Since $v_1(t)$ is supported on $(0, \infty)$ it follows that the Fourier transform in time variable of v_1 is holomorphic in the domain $\{\Im z < 0\}$ and verifies the equation

$$(-z^2 - \partial_x(a(x)\partial_x)\hat{v}_1(z, \cdot) = izf(\cdot), \quad \Im(z) < 0.$$

Taking $z = \omega - i\varepsilon$, $\omega \in \mathbb{R}$, $\varepsilon > 0$ small enough we obtain that

$$\hat{v}_1(\omega - i\varepsilon) = i(\omega - i\varepsilon)R_{i(\omega - i\varepsilon)}f.$$

Using the inverse Fourier transform we get

$$v_1(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\omega} i(\omega - i0)R_{i(\omega - i\varepsilon)}f d\omega.$$

Since ωR_ω can be analytically continued on $\{\Re z > -\delta\}$ we obtain the desired result.

A similar argument shows that

$$v(t)\mathbf{1}_{\{t<0\}} = -\frac{1}{2\pi} \int_{\mathbb{R}} e^{it\omega} i(\omega + i0)R_{i(\omega + i\varepsilon)}f d\omega.$$

The proof is now complete. □

Proof of Lemma 2.2. Using the identity

$$\begin{aligned} h(\lambda) &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathbb{R}} h(s) \left(\frac{1}{\lambda - (s - i\varepsilon)} - \frac{1}{\lambda - (s + i\varepsilon)} \right) ds \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} h(s) \left(\frac{1}{\lambda - (s - i0)} - \frac{1}{\lambda - (s + i0)} \right) ds, \end{aligned}$$

classical spectral calculus gives us that

$$u(t, x) = e^{-itA} = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-its} \left((A - (s - i0))^{-1} - (A - (s + i0))^{-1} \right) u_0 ds.$$

Since $\sigma(A) = (0, \infty)$ we have that $(A + z)^{-1}$ is analytic on $\mathbb{C} \setminus (-\infty, 0)$ and that

$$(A + (-\tau^2 + i0))^{-1} = R_{i|\tau|+0}, \quad (A + (-\tau^2 - i0))^{-1} = R_{-i|\tau|+0}.$$

Then

$$\begin{aligned} u(t, x) &= \frac{1}{2\pi i} \int_0^\infty e^{-its} \left((A - (s - i0))^{-1} - (A - (s + i0))^{-1} \right) u_0 ds \\ &= \frac{1}{\pi i} \int_0^\infty e^{-it\tau^2} \tau \left((A - (\tau^2 - i0))^{-1} - (A - (\tau^2 + i0))^{-1} \right) u_0 d\tau \\ &= \frac{1}{\pi i} \int_0^\infty e^{-it\tau^2} \tau \left(R_{i\tau+0} u_0 - R_{-i\tau+0} u_0 \right) d\tau \\ &= \frac{1}{\pi i} \int_{-\infty}^\infty e^{-it\tau^2} \tau R_{i\tau+0} u_0 d\tau. \end{aligned}$$

Using now that ωR_ω can be analytically continued on $\{\Re z > -\delta\}$ we obtain the desired result. \square

The proof of the main results of this paper requires the theory of almost periodic functions. A function $f : \mathbb{R} \rightarrow \mathbb{C}$ is said to be almost-periodic if it can be represented as

$$f(t) = \sum_n c_n e^{i\lambda_n t},$$

and the following norm satisfies

$$\|f\|_{AP} = \sum_n |c_n| < \infty.$$

It is easy to see that the space of almost periodic functions is an algebra. For more details on the properties of these functions we refer to [3].

As observed in [1], the function $\det D_n(i\omega)$ is an almost periodic function. The same property is satisfied by $1/\det D_n(i\omega)$ even if this property is not trivial (see [1], section 2.2).

Here, in addition to the results in [1], we will compute exactly the coefficients c_k in terms of vector T and sequence $\{Q_k\}_{k=1}^n$ by solving the system $D_n(\omega)C = T$ (see Section 3 below). The argument in [1] only uses the fact that, since C is a solution of the above system, then its components are finite sums of the terms in the vector T . Also, instead of using the results in [1, Section 2.2] we control in a finer way the sequence $\{Q_k\}_{k=1}^n$ introduced in (2.11) and prove the following two key lemmas.

Lemma 2.3. *Let us consider two sequences of real numbers $(c_n)_{n \geq 1}$ and $(d_n)_{n \geq 1}$ with $|d_n| \leq d < 1$ satisfying*

$$(2.15) \quad \sum_{n \geq 1} \operatorname{arctanh}(|d_n|) < \frac{\pi}{2}.$$

We also consider the following sequence of functions $Q_1(i\omega) \equiv 0$ and

$$(2.16) \quad Q_k(i\omega) = e^{ic_{k-1}\omega} \frac{-d_{k-1} + Q_{k-1}(i\omega)}{1 - d_{k-1}Q_{k-1}(i\omega)}, \omega \in \mathbb{R}, k \geq 2.$$

For any $n \geq 2$ the following holds

$$(2.17) \quad \|Q_n\|_{AP} \leq \tan \left(\sum_{k \geq 1} \operatorname{arctanh}(|d_k|) \right).$$

Lemma 2.4. *Let $\alpha \geq \pi/2$ and $0 < d < 1$. For any $N > 0$ there exist a positive integer n and a sequence $0 < \tilde{d}_k \leq d$, $k = 1, \dots, n-1$, with*

$$\sum_{k=1}^{n-1} \operatorname{arctanh}(\tilde{d}_k) = \alpha$$

such that for any sequence $\{c_k\}_{k=1}^{n-1}$ of numbers linearly independent over the rationals and any choice of $\varepsilon_k \in \{\pm 1\}$, $k = 1, \dots, n-1$ the sequence $\{Q_k\}_{k=1}^n$ defined in (2.16) with $\{c_k\}_{k=1}^{n-1}$ above and $d_k = \varepsilon_k \tilde{d}_k$, $k = 1, \dots, n-1$, satisfies

$$(2.18) \quad \|Q_n\|_{AP} \geq N.$$

Remark 2.1. Let us remark that the reflection coefficients d_k in (2.10) can be rewritten as

$$(2.19) \quad d_k = \tanh\left(\frac{\log(b_{k-1}) - \log(b_k)}{2}\right).$$

Since $|\tanh(x)| = \tanh(|x|)$ we have

$$|d_k| = \tanh\left|\frac{\log(b_{k-1}) - \log(b_k)}{2}\right|.$$

As a consequence:

$$(2.20) \quad \sum_{k \geq 1} \operatorname{arctanh}|d_k| = \sum_{k \geq 1} \left|\frac{\log(b_{k-1}) - \log(b_k)}{2}\right| = \frac{\operatorname{Var}(\log(a))}{4}.$$

Also, since a satisfies (1.3) we also have

$$(2.21) \quad \frac{\operatorname{Var}(a)}{M} \leq \operatorname{Var}(\log a) \leq \frac{\operatorname{Var}(a)}{m}.$$

With these two lemmas we can prove Theorem 1.1 and Theorem 1.3. Theorem 1.2 will be a consequence of Theorem 1.1.

Let us now comment on how these lemmas apply to obtain the main results in this paper. The key point in the proof of Theorem 1.1 is, as we will see in Section 3, that for a step function a as in (2.7) and (2.8) the following holds

$$(2.22) \quad \sup_{v_0 \in L^1(\mathbb{R})} \frac{\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |v(t, x)| dt}{\int_{\mathbb{R}} |v_0| dx} \simeq \sup_{v_0 \in L^1(\mathbb{R})} \frac{\|(Q_n(i\omega)\widehat{v}_0)^\vee\|_{L^1(\mathbb{R})}}{\|v_0\|_{L^1(\mathbb{R})}} = \|Q_n\|_{AP}.$$

Thus the results given by Lemma 2.3 and Lemma 2.4 on the AP -norm of Q_n provide results for the behavior of the solutions of the wave equation (1.1).

In the case of the Schrödinger equation (1.2), using the same arguments as in the case of the wave equation, we have that

$$(2.23) \quad \sup_{u_0 \in L^1(\mathbb{R})} \frac{t^{1/2} \|u(t)\|_{L^\infty(\mathbb{R})}}{\|u_0\|_{L^1(\mathbb{R})}} \simeq \sup_{u_0 \in L^1(\mathbb{R})} \frac{t^{1/2} \|(e^{it\omega^2} Q_n(i\omega)\widehat{u}_0)^\vee\|_{L^\infty(\mathbb{R})}}{\|u_0\|_{L^1(\mathbb{R})}}.$$

Applying Young's inequality it is immediate that for all $t > 0$ the following holds

$$\sup_{u_0 \in L^1(\mathbb{R})} \frac{t^{1/2} \|(e^{it\omega^2} Q_n(i\omega)\widehat{u}_0)^\vee\|_{L^\infty(\mathbb{R})}}{\|u_0\|_{L^1(\mathbb{R})}} \leq \sup_{v_0 \in L^1(\mathbb{R})} \frac{\|(Q_n(i\omega)\widehat{v}_0)^\vee\|_{L^1(\mathbb{R})}}{\|v_0\|_{L^1(\mathbb{R})}} \leq \|Q_n\|_{AP}.$$

However, we cannot say that the right hand side in (2.23) is comparable with $\|Q_n\|_{AP}$. This is why we have only a positive result when the BV -norm of the coefficient a is small. The optimality of the result in Theorem 1.2 is still an open problem.

3. PROOF OF THE MAIN RESULTS

The aim of this section is to prove the main results of this paper.

We first concentrate on the case of the wave equation (1.1). We will prove that the solution of equation (1.1) satisfies

$$(3.1) \quad \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |v(t, x)| dt \leq C(\text{Var}(a), m, M) \|v_0\|_{L^1(\mathbb{R})}.$$

Let us consider a laminar medium as in Section 2. The key point in our proof is that the above estimate is equivalent with the fact that $Q_n(i\omega)$ is an $L^1(\mathbb{R})$ -Fourier multiplier and the norm of Q_n as a $L^1(\mathbb{R})$ -multiplier can be estimated in terms of the variation of $\log(a)$. Moreover, we point out that it is sufficient to consider v_0 to be supported in one of the intervals I_k , $k = 1, \dots, n$ since by linearity the result extends to any function $v_0 \in L^1(\mathbb{R})$.

We denote by \hat{f} and f^\vee the Fourier and the inverse Fourier transform of the function f :

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx, \quad f^\vee(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\xi) e^{i\xi x} d\xi.$$

Proof of Theorem 1.1. Using the spectral formula (2.13) and the representation of the resolvent $R_{i\omega}$ obtained in the previous section, for any $x \in I_k$, the solution v of equation (1.1) can be written as

$$\begin{aligned} & v(t, x) \mathbf{1}_{\{t > 0\}} \\ &= \int_{\mathbb{R}} \left(i\omega c_{2k-1}(i\omega) e^{i\omega(t+b_k x)} + i\omega c_{2k}(i\omega) e^{i\omega(t-b_k x)} \right) \frac{d\omega}{2\pi} + \frac{b_k}{4\pi} \int_{\mathbb{R}} e^{it\omega} \int_{I_k} v_0(y) e^{-i\omega b_k |x-y|} dy d\omega \\ &= (i\omega c_{2k-1}(i\omega))^\vee(t + b_k x) + (i\omega c_{2k}(i\omega))^\vee(t - b_k x) + \tilde{v}(t, x) \end{aligned}$$

where

$$\begin{aligned} \tilde{v}(t, x) &= \frac{b_k}{4\pi} \int_{\mathbb{R}} e^{it\omega} \int_{I_k} (v_0 \mathbf{1}_{\{x_{k-1} < y < x\}})(y) e^{-i\omega b_k (x-y)} dy d\omega \\ &\quad + \frac{b_k}{4\pi} \int_{\mathbb{R}} e^{it\omega} \int_{I_k} (v_0 \mathbf{1}_{\{x < y < x_k\}})(y) e^{-i\omega b_k (y-x)} dy d\omega \\ &= \frac{1}{2} \int_{\mathbb{R}} e^{i\omega(t/b_k - x)} (v_0 \mathbf{1}_{\{x_{k-1} < y < x\}})^\vee(\omega) d\omega + \frac{1}{2} \int_{\mathbb{R}} e^{i\omega(t/b_k + x)} (v_0 \mathbf{1}_{\{x < y < x_k\}})^\wedge(\omega) d\omega \\ &= \frac{1}{2} (v_0 \mathbf{1}_{\{x_{k-1} < y < x\}}) \left(x - \frac{t}{b_k} \right) + \frac{1}{2} (v_0 \mathbf{1}_{\{x < y < x_k\}}) \left(x + \frac{t}{b_k} \right). \end{aligned}$$

It easy to see that

$$(3.2) \quad \int_{\mathbb{R}} |\tilde{v}(t, x)| dt \leq b_k \|v_0\|_{L^1(\mathbb{R})} \leq m^{-2} \|v_0\|_{L^1(\mathbb{R})}.$$

Since for $t < 0$, $v(t, x) = v(-t, x)$ we have

$$\int_{\mathbb{R}} |v(t, x)| dt = 2 \int_0^{\infty} |v(t, x)| dt.$$

Hence, in order to prove estimate (3.1), it remains to show that for any $j = 1, \dots, n$ and for any $x \in I_j$ the following holds:

$$(3.3) \quad \int_{\mathbb{R}} |(i\omega c_{2j-1}(i\omega))^\vee(t + b_j x) + (i\omega c_{2j}(i\omega))^\vee(t - b_j x)| dt \leq C(\text{Var}(a), m, M) \|v_0\|_{L^1(\mathbb{R})}.$$

In fact we will prove a stronger estimate: for any $1 \leq j \leq n - 1$ the following holds:

$$(3.4) \quad \int_{\mathbb{R}} \left(|(i\omega c_{2j-1}(i\omega))^\vee(x)| + |(i\omega c_{2j}(i\omega))^\vee(x)| \right) dx \leq C(\text{Var}(a), m, M) \|v_0\|_{L^1(\mathbb{R})}.$$

We also remark that since $c_2 = c_{2n-1} = 0$, when $j \in \{1, n - 1\}$, the two estimates (3.3) and (3.4) are the same. Estimate (3.4) is the key not only in the proof of Theorem 1.1 but also in the one of Theorem 1.2.

Since there is a strong connexion between the c 's and t 's we observe that for $k = 1, \dots, n - 1$,

$$(3.5) \quad i\omega t_{k,1}(i\omega) = -\frac{b_k}{2} e^{-i\omega b_k x_k} \widehat{v_0 \mathbf{1}_{I_k}}(-b_k \omega) + \frac{b_{k+1}}{2} e^{-i\omega b_{k+1} x_k} \widehat{v_0 \mathbf{1}_{I_{k+1}}}(b_{k+1} \omega)$$

and

$$(3.6) \quad i\omega t_{k,2}(i\omega) = \frac{b_k b_{k+1}}{2} \left(e^{-i\omega b_k x_k} \widehat{v_0 \mathbf{1}_{I_k}}(-b_k \omega) + e^{-i\omega b_{k+1} x_k} \widehat{v_0 \mathbf{1}_{I_{k+1}}}(b_{k+1} \omega) \right).$$

An immediate consequence is that

$$(3.7) \quad \int_{\mathbb{R}} |(i\omega t_{k,1}(i\omega))^\vee(x)| dx \leq \frac{|b_k| + |b_{k+1}|}{2} \int_{\mathbb{R}} |v_0| \leq c(m) \int_{\mathbb{R}} |v_0|$$

and

$$(3.8) \quad \int_{\mathbb{R}} |(i\omega t_{k,2}(i\omega))^\vee(x)| dx \leq |b_k| |b_{k+1}| \int_{\mathbb{R}} |v_0| \leq c(m) \int_{\mathbb{R}} |v_0|.$$

The main steps in the proof of (3.4) are the following:

- Prove (3.4) for $j = n$ and $\text{supp } v_0 \subset I_n$.
- Prove (3.4) for $j = n$ and $\text{supp } v_0 \subset I_1$.
By symmetry the same holds for $j \in \{1, n\}$ and $\text{supp } v_0 \subset I_1 \cup I_n$.
- Prove (3.4) for $j = n$ and $\text{supp } v_0 \subset I_k$, $2 \leq k \leq n - 1$.
- Prove (3.4) for $j \in \{k, \dots, n\}$ and $\text{supp } v_0 \subset I_k$ with $2 \leq k \leq n - 1$ with $k \leq j \leq n$.
By symmetry the same holds for $x \in I_j$ and $\text{supp } v_0 \subset I_k$, $2 \leq k \leq n - 1$ with $1 \leq j \leq k$.

Case 1. Computing c_{2n} when $\text{supp } v_0 \subset I_n$. In this case all t_k , $k = 1, \dots, n-2$, vanish. Moreover $t_{n-1,2} = b_{n-1}t_{n-1,1}$. It follows that

$$c_{2n}(i\omega) = (\det D_n(i\omega))^{-1} \begin{vmatrix} \mathbf{a}_1 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & B_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & B_3 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & A_{n-3} & B_{n-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{n-2} & B_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{n-1} & t_{n-1} \end{vmatrix}.$$

Developing over the last two lines the above determinant we obtain

$$\begin{aligned} c_{2n}(i\omega) &= - \begin{vmatrix} e^{i\omega b_{n-1}x_{n-1}} & t_{n-1,1} \\ b_n e^{i\omega b_{n-1}x_{n-1}} & t_{n-1,2} \end{vmatrix} \frac{\det D_{n-1}(i\omega)}{\det D_n(i\omega)} + \begin{vmatrix} e^{-i\omega b_{n-1}x_{n-1}} & t_{n-1,1} \\ -b_n e^{-i\omega b_{n-1}x_{n-1}} & t_{n-1,2} \end{vmatrix} \frac{\det \tilde{D}_{n-1}(i\omega)}{\det D_n(i\omega)} \\ &= -e^{i\omega b_{n-1}x_{n-1}} \begin{vmatrix} 1 & 1 \\ b_n & b_{n-1} \end{vmatrix} t_{n-1,1} \frac{\det D_{n-1}(i\omega)}{\det D_n(i\omega)} \\ &\quad + e^{-i\omega b_{n-1}x_{n-1}} \begin{vmatrix} 1 & 1 \\ -b_n & b_{n-1} \end{vmatrix} t_{n-1,1} \frac{\det \tilde{D}_{n-1}(i\omega)}{\det D_n(i\omega)} \\ &= t_{n-1,1} \left((b_{n-1} + b_n) e^{-i\omega b_{n-1}x_{n-1}} \frac{\det \tilde{D}_{n-1}(i\omega)}{\det D_n(i\omega)} + (b_n - b_{n-1}) e^{i\omega b_{n-1}x_{n-1}} \frac{\det D_{n-1}(i\omega)}{\det D_n(i\omega)} \right) \\ &= -t_{n-1,1} e^{-i\omega b_n x_{n-1}} \frac{\det \tilde{D}_n(i\omega)}{\det D_n(i\omega)}, \end{aligned}$$

where the last identity involving $\det \tilde{D}_n$, $\det \tilde{D}_{n-1}$ and $\det D_{n-1}$ has been proved in [1, p. 871].

From (3.5) we have that

$$i\omega t_{n-1,1}(i\omega) = \frac{b_n}{2} e^{-i\omega b_n x_{n-1}} \widehat{v_0 I_n}(b_n \omega)$$

and then we obtain the exact formula of $c_{2n}(i\omega)$ in terms of $Q_n(i\omega)$:

$$(3.9) \quad i\omega c_{2n}(i\omega) = -\frac{b_n}{2} e^{-2i\omega b_n x_{n-1}} \frac{\det \tilde{D}_n(i\omega)}{\det D_n(i\omega)} = -\frac{b_n}{2} e^{-2i\omega b_n x_{n-1}} Q_n(i\omega) \widehat{v_0 I_n}(b_n \omega).$$

This identity is the key point in proving not only Theorem 1.1 but also Theorem 1.3.

It follows that

$$(3.10) \quad \int_{\mathbb{R}} |(i\omega c_{2n}(i\omega))^\vee(x)| dx \leq \frac{b_n}{2} \|Q_n(i\omega)\|_{AP} \|v_0\|_{L^1(\mathbb{R})}$$

and in view of Lemma 2.3 we have

$$\int_{\mathbb{R}} |(i\omega c_{2n}(i\omega))^\vee(x)| dx \leq C \left(M, m, \tan \left(\sum_{k=1}^n \arctanh |d_n| \right) \right).$$

Thus by (2.20) and (2.21) the proof of the theorem in this case is finished.

Case 2. Computing c_{2n} when $\text{supp } v_0 \subset I_1$. Here all the terms t_k , $k = 2, \dots, n-1$ vanish and $t_{1,2} = -b_2 t_{1,1}$. Hence

$$c_{2n}(i\omega) = (\det D_n(i\omega))^{-1} \begin{vmatrix} \mathbf{a}_1 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 \\ 0 & A_2 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & B_3 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & A_{n-3} & B_{n-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{n-2} & B_{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{n-1} & 0 & 0 \end{vmatrix}.$$

Developing the above determinant over blocks of two lines we obtain that

$$c_{2n}(i\omega) = \frac{\det A_2 \cdots \det A_{n-1}}{\det D_n(i\omega)} \begin{vmatrix} e^{i\omega b_1 x_1} & t_{1,1} \\ b_2 e^{i\omega b_1 x_1} & t_{1,2} \end{vmatrix} = \frac{\det A_1 \cdots \det A_{n-1}}{\det D_n(i\omega)} e^{i\omega b_1 x_1} t_{1,1}(i\omega).$$

Using estimate (3.7) on $t_{1,1}$ we get

$$(3.11) \quad \int_{\mathbb{R}} |(i\omega c_{2n}(i\omega))^\vee(x)| dx \leq \left\| \frac{\det A_1 \cdots \det A_{n-1}}{\det D_n(i\omega)} \right\|_{AP} \|v_0\|_{L^1(\mathbb{R})}.$$

Since the proof of estimate (3.4) is the same as in Case 3 below (choose $k = 1$ in (3.12)) we will skip it here.

Case 3. Computing c_{2n} when $\text{supp } v_0 \subset I_k$, $2 \leq k \leq n-1$. Here $t_{k-1,2} = b_{k-1} t_{k-1,1}$, $t_{k,2} = -b_{k+1} t_{k,1}$, all the other terms in vector T vanishing. Let us now compute c_{2n} . It is given by

$$c_{2n}(i\omega)(\det D_n(i\omega)) = \begin{vmatrix} a_1 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & B_3 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ - & - & - & A_{k-1} & B_{k-1} & - & - & - & t_{k-1} \\ 0 & 0 & 0 & 0 & A_k & B_k & 0 & 0 & t_k \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & A_{n-2} & B_{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{n-1} & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & B_3 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ - & - & - & A_{k-1} & B_{k-1} & - & - & - & t_{k-1} \\ 0 & 0 & 0 & 0 & A_k & B_k & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & A_{n-2} & B_{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{n-1} & 0 & 0 \end{vmatrix} + \begin{vmatrix} a_1 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & B_3 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ - & - & - & A_{k-1} & B_{k-1} & - & - & - & 0 \\ 0 & 0 & 0 & 0 & A_k & B_k & 0 & 0 & t_k \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & A_{n-2} & B_{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{n-1} & 0 & 0 \end{vmatrix}.$$

Developing the determinants over blocks of two lines we find that

$$c_{2n}(i\omega)(\det D_n(i\omega)) = \det A_k \cdots \det A_{n-1} \begin{vmatrix} a_1 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & B_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & B_3 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & A_{k-3} & B_{k-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{k-2} & B_{k-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{k-1} & t_{k-1} \end{vmatrix} \\ + \det A_{k+1} \cdots \det A_{n-1} \begin{vmatrix} a_1 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & B_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & B_3 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & A_{k-2} & B_{k-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{k-1} & B_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_k & t_k \end{vmatrix}.$$

Since the components of t_{k-1} satisfy $t_{k-1,2} = b_{k-1}t_{k-1,1}$ we can use the same argument as in Case 1 and we obtain that the first determinant equals

$$-t_{k-1,1}e^{-i\omega b_k x_{k-1}} \det \tilde{D}_k(i\omega).$$

The second one could be computed in a similar way by expanding the determinant over the last two lines and using that $t_{k,2} = -b_{k+1}t_{k,1}$

$$\begin{vmatrix} a_1 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & B_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & B_3 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & A_{k-2} & B_{k-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{k-1} & B_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_k & t_k \end{vmatrix} \\ = - \begin{vmatrix} e^{i\omega b_k x_k} & t_{k,1} \\ b_{k+1}e^{i\omega b_k x_k} & -b_{k+1}t_{k,1} \end{vmatrix} \det D_k(i\omega) + \begin{vmatrix} e^{-i\omega b_k x_k} & t_{k,1} \\ -b_{k+1}e^{-i\omega b_k x_k} & -b_{k+1}t_{k,1} \end{vmatrix} \det \tilde{D}_k(i\omega) \\ = 2b_{k+1}t_{k,1} \det D_k(i\omega)e^{i\omega b_k x_k} = -\det(A_k)t_{k,1} \det D_k(i\omega)e^{i\omega b_k x_k}.$$

This gives us that

$$c_{2n}(i\omega) = -\frac{\det A_k \cdots \det A_{n-1}}{\det D_n(i\omega)} \left(t_{k-1,1}e^{-i\omega b_k x_{k-1}} \det \tilde{D}_k(i\omega) + t_{k,1}e^{i\omega b_k x_k} \det D_k(i\omega) \right) \\ = -\frac{\det A_k \cdots \det A_{n-1} \det D_k(i\omega)}{\det D_n(i\omega)} \left(t_{k-1,1}e^{i\omega b_k (x_k - x_{k-1})} Q_k(i\omega) + t_{k,1} \right) e^{i\omega b_k x_k}.$$

Applying estimate (3.7) for $t_{k,1}$ and $t_{k-1,1}$ we obtain that

$$(3.12) \quad \int_{\mathbb{R}} |(i\omega c_{2n}(i\omega))^\vee(t)| dt \leq C(m) (\|Q_k\|_{AP} + 1) \left\| \frac{\det A_k \cdots \det A_{n-1} \det D_k(i\omega)}{\det D_n(i\omega)} \right\|_{AP} \|v_0\|_{L^1(\mathbb{R})}.$$

Using now formula (2.12) that gives us the explicit expression of $\det D_n$ and $\det D_k$ we find that

$$(3.13) \quad \frac{\det A_k \cdots \det A_{n-1} \det D_k(i\omega)}{\det D_n(i\omega)} = \prod_{j=k}^{n-1} \frac{\det A_j}{b_j + b_{j+1}} \frac{e^{-i\omega(b_j - b_{j+1})x_j}}{(1 - d_j Q_j(i\omega))} \\ = (-1)^{n-k} \prod_{j=k}^{n-1} \frac{(1 - d_j)}{(1 - d_j Q_j(i\omega))} e^{-i\omega(b_j - b_{j+1})x_j}.$$

Observe now that

$$\frac{(1 - d_j)}{(1 - d_j Q_j(i\omega))} = \begin{cases} \frac{1}{1+d_j} (1 + d_j Q_{j+1}(i\omega) e^{2i\omega b_{j+1}(x_{j+1} - x_j)}), & j = 1, \dots, n-2, \\ \frac{1}{1+d_j} (1 + d_j Q_{j+1}(i\omega) e^{-2i\omega b_{j+1}x_j}), & j = n-1. \end{cases}$$

It gives us that for any $1 \leq j \leq n-1$ we have

$$\left\| \frac{(1 - d_j)}{(1 - d_j Q_j(i\omega))} \right\|_{AP} \leq \frac{1 + |d_j| \|Q_{j+1}\|_{AP}}{1 + d_j} \leq \frac{\exp(|d_j| \|Q_{j+1}\|_{AP})}{1 + d_j}$$

and then

$$(3.14) \quad \left\| \prod_{j=k}^{n-1} \frac{(1 - d_j)}{(1 - d_j Q_j(i\omega))} \right\|_{AP} \leq \exp \left(\max_{j=k, \dots, n-1} \|Q_{j+1}\|_{AP} \sum_{j=k}^{n-1} |d_j| \right) \cdot \prod_{j=k}^{n-1} \frac{1}{1 + d_j} \\ \leq \exp \left(c(m, M) \text{Var}(a) \max_{j=k+1, \dots, n} \|Q_j\|_{AP} \right) \cdot \prod_{j=k}^{n-1} \frac{1}{1 + d_j} \\ \leq \exp \left(c(m, M) \text{Var}(a) \tan(\text{Var}(\log(a))/4) \right) \cdot \prod_{j=k}^{n-1} \frac{1}{1 + d_j}.$$

The last term also satisfies

$$(3.15) \quad \prod_{j=k}^{n-1} \frac{1}{1 + d_j} = \prod_{j=k}^{n-1} \frac{b_j + b_{j+1}}{2b_j} = \prod_{j=k}^{n-1} \left(1 + \frac{|b_{j+1} - b_j|}{2b_j} \right) \leq \exp \left(C(m, M) \text{Var}(a) \right).$$

Putting now together estimates (3.12), (3.13), (3.14) and (3.15) we obtain that estimate (3.4) also holds in the case considered here.

Case 4. Prove (3.4) when $\text{supp } v_0 \subset I_k$, $2 \leq k \leq n-1$ and $k \leq j \leq n-1$. The previous cases prove (3.4) for $j = n$. Let us now prove that it holds for any $k \leq j \leq n-1$. We point out that once estimate (3.4) will be proved then it also holds for $1 \leq j \leq k-1$.

We now use that

$$(3.16) \quad A_{n-1} \begin{pmatrix} c_{2n-3} \\ c_{2n-2} \end{pmatrix} + \mathbf{b}_{n-1} c_{2n} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and for $j \leq n-2$,

$$(3.17) \quad A_j \begin{pmatrix} c_{2j-1} \\ c_{2j} \end{pmatrix} + B_j \begin{pmatrix} c_{2j+1} \\ c_{2j+2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From identity (3.16) and the results of Case 3, we obtain that (3.4) holds for $j = n-1$:

$$\begin{aligned} \int_{\mathbb{R}} \left(|(i\omega c_{2n-2}(i\omega))^\vee(x)| + |(i\omega c_{2n-3}(i\omega))^\vee(x)| \right) dx &\leq C(m, M) \int_{\mathbb{R}} |(i\omega c_{2n}(i\omega))^\vee(x)| dx \\ &\leq C(m, M, \text{Var}(a)) \int_{\mathbb{R}} |v_0|. \end{aligned}$$

Applying identity (3.17) we obtain

$$\begin{aligned} \begin{pmatrix} c_{2j-1} \\ c_{2j} \end{pmatrix} &= -A_j^{-1} B_j \begin{pmatrix} c_{2j+1} \\ c_{2j+2} \end{pmatrix} \\ &= \frac{1}{2b_{j+1}} \begin{pmatrix} (b_{j+1} + b_j) e^{i\omega(b_{j+1}-b_j)x_j} & (b_{j+1} - b_j) e^{-i\omega(b_{j+1}+b_j)x_j} \\ (b_{j+1} - b_j) e^{i\omega(b_{j+1}+b_j)x_j} & (b_{j+1} + b_j) e^{-i\omega(b_{j+1}-b_j)x_j} \end{pmatrix} \begin{pmatrix} c_{2j+1} \\ c_{2j+2} \end{pmatrix}. \end{aligned}$$

It implies that for $j \leq n-2$ we have

$$\begin{aligned} \int_{\mathbb{R}} |(i\omega c_{2j-1}(i\omega))^\vee(t)| + |(i\omega c_{2j}(i\omega))^\vee(t)| dt \\ \leq \frac{|b_{j+1} - b_j| + b_j + b_{j+1}}{2b_{j+1}} \int_{\mathbb{R}} |(i\omega c_{2j+1}(i\omega))^\vee(t)| + |(i\omega c_{2j+2}(i\omega))^\vee(t)| dt \\ \leq \left(1 + \frac{|b_{j+1} - b_j|}{b_{j+1}} \right) \int_{\mathbb{R}} |(i\omega c_{2j+1}(i\omega))^\vee(t)| + |(i\omega c_{2j+2}(i\omega))^\vee(t)| dt. \end{aligned}$$

Using the same argument as in (3.15) we obtain that for any $k \leq j \leq n-1$

$$(3.18) \quad \int_{\mathbb{R}} |(i\omega c_{2j-1}(i\omega))^\vee(t)| + |(i\omega c_{2j}(i\omega))^\vee(t)| dt \leq C(m, M, \text{Var}(a)) \int_{\mathbb{R}} |v_0|.$$

The proof of Theorem 1.1 is now complete. \square

Proof of Theorem 1.3. In this section we describe the manner in which the coefficient a satisfying the conditions in Theorem 1.3 can be constructed. Without restricting the generality, we will construct a coefficient a with $1/2 \leq a \leq 2$.

Let fix $d = (\log 2)/2$. Since $\alpha \geq \pi/2$, by Lemma 2.4, for any $N > 0$ there exists a sequence $\{\tilde{d}_k\}_{k=1}^{n-1}$ such that

$$\sum_{k=1}^{n-1} \operatorname{arctanh}(|\tilde{d}_k|) = \alpha, \quad \operatorname{arctanh}|d_k| \leq d, \quad \forall k = 1, \dots, n-1,$$

and for any sequence of rationally independent numbers $\{c_k\}_{k=1}^n$ and any $\varepsilon_k \in \{\pm 1\}$, $k = 1, \dots, n-1$, the sequence $\{Q_k\}_{k=1}^n$ associated to $\{c_k\}_{k=1}^{n-1}$ and $\{\varepsilon_k \tilde{d}_k\}_{k=1}^{n-1}$ satisfies $\|Q_n(i\omega)\|_{AP} \geq 2N$.

Let us set $d_k = \varepsilon_k \tilde{d}_k = \varepsilon_k |d_k|$ where $\varepsilon_k \in \{\pm 1\}$ will be chosen later. We now choose a sequence $\{b_k\}_{k=1}^n$ such that

$$d_k = \varepsilon_k |d_k| = \frac{b_k - b_{k+1}}{b_k + b_{k+1}} < 0, \quad k = 1, \dots, n-1.$$

This is possible since we can define recursively

$$\frac{b_k}{b_{k+1}} = \exp(2\operatorname{arctanh}(d_k)), \quad k = 1, \dots, n-1, \quad b_1 = 1.$$

Let a be the piecewise constant function defined by

$$a(x) = b_k^{-2}, \quad x \in (x_{k-1}, x_k), \quad k = 1, \dots, n,$$

where $x_0 = -\infty$, $x_n = \infty$ and the points $\{x_k\}_{k=1}^{n-1}$ are chosen such that the numbers

$$b_k(x_k - x_{k-1}), \quad k = 2, \dots, n-1$$

are linearly independent over \mathbb{Q} . Using (2.20) we get that

$$\operatorname{Var}(\log(a)) = 4 \sum_{k=1}^{n-1} \operatorname{arctanh}(|d_k|) = 4\alpha.$$

We now show how to choose the sequence $\{\varepsilon_k\}_{k=1}^{n-1}$ such that $a \in [1/2, 2]$. Observe that for $x \in (x_{k-1}, x_k)$, the coefficient a is given by

$$a(x) = b_k^{-2} = \exp\left(2 \sum_{j=1}^{k-1} \operatorname{arctanh}(d_j)\right) = \exp\left(2 \sum_{j=1}^{k-1} \varepsilon_j \operatorname{arctanh}(|d_j|)\right).$$

Since $\operatorname{arctanh}(|d_k|) \leq d = (\log 2)/2$ we always can choose $\varepsilon_k \in \{\pm 1\}$ such that

$$\left| \sum_{j=1}^{k-1} \varepsilon_j \operatorname{arctanh}(|d_j|) \right| \leq \frac{\log 2}{2}, \quad \forall k = 2, \dots, n-1.$$

Indeed, for $k = 2$ choose $\varepsilon_1 = 1$. Assume that the above inequality is true for some k . If $\sum_{j=1}^{k-1} \varepsilon_j \operatorname{arctanh}(|d_j|)$ belongs to $(0, \log(2)/2)$ then choose $\varepsilon_k = -1$, otherwise $\varepsilon_k = 1$. It follows that the above inequality will also hold for $k+1$. With this choice of $\{\varepsilon_k\}_{k=1}^{n-1}$ we obtain that coefficient a satisfies $a \in (1/2, 2)$.

Denoting $m = 1/2$, in view of estimate (3.2) we have

$$\frac{\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |v(t, x)| dt}{\int_{\mathbb{R}} |v_0| dx} \geq \frac{\int_{\mathbb{R}} |i\omega c_{2n}(i\omega)^\vee(t)| dt}{\int_{\mathbb{R}} |v_0| dx} - m^{-2} \gtrsim \frac{\|(Q_n(i\omega)\widehat{v}_0)^\vee\|_{L^1(\mathbb{R})}}{\|v_0\|_{L^1(\mathbb{R})}} - m^{-2}.$$

Choosing now a sequence v_{0k} such that

$$\frac{\|(Q_n(i\omega)\widehat{v}_{0k})^\vee\|_{L^1(\mathbb{R})}}{\|v_{0k}\|_{L^1(\mathbb{R})}} \rightarrow \|Q_n\|_{AP}$$

we get for any N large enough that

$$\sup_{v_0 \in L^1(\mathbb{R})} \frac{\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |v(t, x)| dt}{\int_{\mathbb{R}} |v_0| dx} \gtrsim \|Q_n\|_{AP} - m^{-2} \geq 2N - 4 \geq N.$$

For the function a chosen above we obtain that the left hand side of the above inequality can be arbitrarily large. The proof is now finished. \square

Proof of Theorem 1.2. We use formula (2.9) for the resolvent to find that for $x \in (x_{k-1}, x_k)$ solution u of system (1.2) is given by

$$\begin{aligned} u(t, x) &= \frac{1}{i\pi} \int_{\mathbb{R}} e^{-it\omega^2} \omega c_{2k-1}(i\omega) e^{i\omega b_k x} d\omega + \frac{1}{i\pi} \int_{\mathbb{R}} e^{-it\omega^2} \omega c_{2k}(i\omega) e^{-i\omega b_k x} d\omega \\ &\quad + \frac{b_k}{2i\pi} \int_{\mathbb{R}} e^{-it\omega^2} \omega \int_{I_k} u_0(y) e^{-i\omega_k |x-y|} dy d\omega. \end{aligned}$$

It follows that

$$|u(t, x)| \leq \frac{C}{\sqrt{t}} \int_{\mathbb{R}} (|(i\omega c_{2k-1}(i\omega))^\vee(x)| + |(i\omega c_{2k}(i\omega))^\vee(x)|) dx + \frac{C}{\sqrt{t}} \|u_0\|_{L^1(\mathbb{R})}.$$

Using now estimate (3.4) that has already been proved in the proof of Theorem 1.1 we obtain the desired estimate and the proof is finished. \square

4. PROOF OF THE TWO TECHNICAL LEMMAS

In this section we prove Lemma 2.3 and Lemma 2.4 using a fine analysis of the sequence $\{Q_n\}_{n \geq 1}$ defined by (2.16) by means of multi-variable series.

4.1. The functions $\alpha(q)$, $\beta(d)$, $\gamma(t)$, $E(d_1, \dots, d_n; q_1, \dots, q_{n-1})$ and $R(t_1, \dots, t_n; q_1, \dots, q_{n-1})$. We introduce the notations that will be used in this section. We denote by $\alpha(q)$ and $\beta(\sigma)$ the mappings:

$$\alpha(q) : z \mapsto qz, \quad \beta(\sigma) : z \mapsto \frac{z + \sigma}{1 + \sigma z}.$$

Note that $q \mapsto \alpha(q)$ is a multiplicative mapping, i.e. $\alpha(q)\alpha(q') = \alpha(qq')$, and β satisfies

$$\beta(\sigma)\beta(\sigma') = \beta\left(\frac{\sigma + \sigma'}{1 + \sigma\sigma'}\right).$$

From the formula

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

we deduce that mapping $t \mapsto \gamma(t) := \beta(\tanh t)$ has the property

$$\gamma(a)\gamma(b) = \gamma(a+b), \quad a, b \in \mathbb{R}.$$

If $d = (d_1, \dots, d_n)$, with $0 \leq d_i < 1$ and $q = (q_1, \dots, q_{n-1})$ is a multivariable then we define the multivariable series

$$E(d; q) := (\beta(d_n) \alpha(q_{n-1}) \beta(d_{n-1}) \cdots \alpha(q_1) \beta(d_1)) (0).$$

With the convention that for $j = (j_1, \dots, j_{n-1})$ we write $q^j := q_1^{j_1} \cdots q_{n-1}^{j_{n-1}}$. We introduce the norm $\|\cdot\|$ of a multivariable series by

$$\left\| \sum_{j \in \mathbb{N}^{n-1}} a_j q^j \right\| = \sum_{j \in \mathbb{N}^{n-1}} |a_j|.$$

It is easy to see that for any two series $\|ab\| \leq \|a\| \|b\|$.

Lemma 4.1. *Let be $\{Q_k\}_{k \geq 1}$ defined by (2.16). Then for any $n \geq 1$*

$$(4.19) \quad \|Q_{n+1}\|_{AP} \leq \|E(|d_1|, \dots, |d_n|; (q_1, \dots, q_{n-1}))\|$$

with equality when c_1, \dots, c_{n-1} are linearly independent over \mathbb{Q} .

Proof. We write $d_k = \varepsilon_k |d_k|$ with $\varepsilon_k = \pm 1$. The sequence $\{Q_k\}_{k \geq 1}$ can be written as

$$\begin{aligned} Q_k(i\omega) &= -\varepsilon_{k-1} e^{i\omega c_{k-1}} \frac{-\varepsilon_{k-1} Q_{k-1}(i\omega) + |d_{k-1}|}{1 - \varepsilon_{k-1} |d_{k-1}| Q_{k-1}(i\omega)} \\ &= (\alpha(-\varepsilon_{k-1} e^{i\omega c_{k-1}}) \beta(|d_{k-1}|) \alpha(-\varepsilon_{k-1})) (Q_{k-1}(i\omega)), k \geq 2. \end{aligned}$$

It follows that

$$Q_{n+1}(i\omega) = \left(\alpha(-\varepsilon_n e^{i\omega c_n}) \beta(|d_n|) \alpha(\varepsilon_{n-1} \varepsilon_n e^{i\omega c_{n-1}}) \beta(|d_{n-1}|) \cdots \alpha(\varepsilon_1 \varepsilon_2 e^{i\omega c_1}) \beta(|d_1|) \right) (0).$$

With the above notations it follows that

$$Q_{n+1}(i\omega) = -\varepsilon_n e^{i\omega c_n} E(|d_1|, \dots, |d_n|; \varepsilon_1 \varepsilon_2 e^{i\omega c_1}, \dots, \varepsilon_{n-1} \varepsilon_n e^{i\omega c_{n-1}}).$$

Since $|\varepsilon_n| = 1$ and $|\varepsilon_k \varepsilon_{k+1}| = 1$ for $1 \leq k \leq n-1$ we have

$$\|Q_{n+1}\|_{AP} \leq \|E(|d_1|, \dots, |d_n|; q_1, \dots, q_{n-1})\|,$$

with equality when c_1, \dots, c_{n-1} are linearly independent over \mathbb{Q} . □

In the following we will estimate in an clever way the norm of $E(|d_1|, \dots, |d_n|; (q_1, \dots, q_{n-1}))$. For any $t = (t_1, \dots, t_n)$ we define the series $R(t; q)$ in the multivariable $q = (q_1, \dots, q_{n-1})$ by

$$(4.20) \quad \begin{aligned} R(t; q) &= (\gamma(t_n) \alpha(q_{n-1}) \gamma(t_{n-1}) \cdots \alpha(q_1) \gamma(t_1)) (0) \\ &= E((\tanh(t_1), \dots, \tanh(t_n)); q). \end{aligned}$$

When there is no risk of confusion we will write only $R(t)$.

Using that the map $t \rightarrow \tanh(t)$ is one to one from $[0, \infty)$ to $[0, 1)$ we will estimate the norm of $R((t_1, \dots, t_n); (q_1, \dots, q_{n-1}))$ by taking the advantage of the multiplicative property of function γ .

4.2. Estimates on partitions. For any positive number x we consider the set of all partitions of x :

$$A_x = \left\{ t = (t_1, \dots, t_n) \mid t_j > 0, \sum_{j=1}^n t_j = x \right\}.$$

If $t = (t_1, \dots, t_n) \in A_x$ the sequence of the partial sums $x_k = \sum_{j=1}^k t_j$, $k = 1, \dots, n$ is a partition of the interval $[0, x]$, i.e. we have $0 = x_0 < x_1 < \dots < x_n = x$. We get a one-to-one correspondence between A_x and the partitions of $[0, x]$.

If $t = (t_1, \dots, t_n)$ and $s = (s_1, \dots, s_m)$ are two elements in A_x we say that s is finer than t and we write $t \prec s$ if the sequence of partial sums of s contains the sequence of partial sums of t , i.e. there is a sequence $1 \leq k_1 < \dots < k_n = m$ such that

$$\sum_{j=1}^l t_j = \sum_{j=1}^{k_l} s_j, \quad \forall 1 \leq l \leq n.$$

It follows that (A_x, \prec) is a directed set.

Lemma 4.2. *Let be $0 \leq d_j \leq 1$, $1 \leq j \leq n$ and $P = \prod_{j=1}^n (1 + d_j)$, $p = \prod_{j=1}^n (1 - d_j)$. Then the map*

$$\beta(d_n)\alpha(q_{n-1}) \dots \beta(d_1)\alpha(q_0)$$

is given by the map $z \mapsto (az+b)/(cz+d)$, where $a, b, c, d \in \mathbb{R}[q_0, \dots, q_{n-1}]$ have non-negative coefficients,

$$\|a\| = \|c\| = \frac{1}{2}(P + p), \quad \|b\| = \|d\| = \frac{1}{2}(P - p),$$

the coefficient of the first order powers $q_0 \dots q_{n-1}$ in polynomial a and the one of zero order term in polynomial d is 1.

Proof. Note that

$$\frac{1}{2}(P + p) = \sum_{k \text{ even}} \sum_{1 \leq i_1 < \dots < i_k \leq n} d_{i_1} \dots d_{i_k}$$

and

$$\frac{1}{2}(P - p) = \sum_{k \text{ odd}} \sum_{1 \leq i_1 < \dots < i_k \leq n} d_{i_1} \dots d_{i_k}.$$

Let $M(q_0, \dots, q_{n-1})$ be the matrix

$$M(q_0, \dots, q_{n-1}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & d_n \\ d_n & 1 \end{pmatrix} \begin{pmatrix} q_{n-1} & 0 \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & d_1 \\ d_1 & 1 \end{pmatrix} \begin{pmatrix} q_0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the map $\beta(d_n)\alpha(q_{n-1}) \dots \beta(d_1)\alpha(q_0)$ is given by

$$z \mapsto \frac{az + b}{cz + d}.$$

Since d_j , $j \geq 1$ are nonnegative, $a, b, c, d \in R[q_0, \dots, q_{n-1}]$ are polynomials with non-negative coefficients. Therefore $\|a\| = a(1, \dots, 1)$ and similarly for b, c, d .

Denoting

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

we have that

$$(4.21) \quad M(q_0, \dots, q_{n-1}) = (I + d_n A)(B_0 + q_{n-1} B_1) \cdots (I + d_1 A)(B_0 + q_0 B_1).$$

Since $B_0 + B_1 = I$ and $A^2 = I$, it follows that

$$\begin{aligned} M(1, \dots, 1) &= (I + d_n A) \cdots (I + d_1 A) = \sum_{k=0}^n \sum_{1 \leq i_1 < \dots < i_k \leq n} d_{i_1} \cdots d_{i_k} A^k \\ &= \left(\sum_{k \text{ even}} \sum_{1 \leq i_1 < \dots < i_k \leq n} d_{i_1} \cdots d_{i_k} \right) I + \left(\sum_{k \text{ odd}} \sum_{1 \leq i_1 < \dots < i_k \leq n} d_{i_1} \cdots d_{i_k} \right) A \\ &= \frac{1}{2}(P + p)I + \frac{1}{2}(P - p)A = \begin{pmatrix} \frac{1}{2}(P + p) & \frac{1}{2}(P - p) \\ \frac{1}{2}(P - p) & \frac{1}{2}(P + p) \end{pmatrix}. \end{aligned}$$

It implies that

$$a(1, \dots, 1) = c(1, \dots, 1) = \frac{1}{2}(P + p)$$

and

$$b(1, \dots, 1) = d(1, \dots, 1) = \frac{1}{2}(P - p).$$

Then the first part of Lemma 4.2 is proved.

Let us now compute the coefficient of the first order powers $q_0 \dots q_{n-1}$ in polynomial a and the one of zero order term in polynomial d . In view of (4.21) we have

$$M(q_0, \dots, q_{n-1}) = \sum_{(k_0, \dots, k_{n-1}) \in \{0,1\}^n} q_0^{k_0} \cdots q_{n-1}^{k_{n-1}} M_{k_0, \dots, k_{n-1}},$$

where

$$M_{k_0, \dots, k_{n-1}} = (I + d_n A) B_{k_{n-1}} \cdots (I + d_1 A) B_{k_0}.$$

To determine the coefficient of $q_0 \cdots q_{n-1}$ and the free term of d one has to calculate $M_{1, \dots, 1}$ and $M_{0, \dots, 0}$, respectively.

Note that $B_0^2 = B_0$, $B_1^2 = B_1$ and $B_0 A B_0 = B_1 A B_1 = 0$ so $B_0(I + d_j A)B_0 = B_0$ and $B_1(I + d_j A)B_1 = B_1$. Hence

$$M_{0, \dots, 0} = (I + d_n A) B_0 \cdots (I + d_1 A) B_0 = (I + d_n A) B_0 = \begin{pmatrix} 0 & d_n \\ 0 & 1 \end{pmatrix}$$

so the constant term of d is one. Also

$$M_{1, \dots, 1} = (I + d_n A) B_1 \cdots (I + d_1 A) B_1 = (I + d_n A) B_1 = \begin{pmatrix} 1 & 0 \\ d_n & 0 \end{pmatrix}$$

so the coefficient of $q_1 \cdots q_{n-1}$ in a is one. □

Corollary 4.1. *Let $(t_1, \dots, t_n) \in A_x$. The map*

$$\gamma(t_n)\alpha(q_{n-1}) \cdots \gamma(t_1)\alpha(q_0)$$

is given by $z \mapsto (az + b)/(cz + d)$, where $a, b, c, d \in \mathbb{R}[q_0, \dots, q_{n-1}]$ have non-negative coefficients,

$$\|a\| = \|d\| < \cosh x, \quad \|b\| = \|c\| < \sinh x,$$

the coefficient of $q_0 \cdots q_{n-1}$ in a and the constant term of d is 1.

Proof. Since $\gamma(t) = \beta(\tanh t)$ we can use Lemma 4.2 with $d_j = \tanh t_j$. Using the notations of Lemma 4.2, we must prove that

$$(4.22) \quad \frac{P+p}{2} < \cosh x \quad \text{and} \quad \frac{P-p}{2} < \sinh x.$$

Since

$$1 + d_j = 1 + \tanh t_j = \frac{e^{t_j}}{\cosh t_j}, \quad 1 - d_j = 1 - \tanh t_j = \frac{e^{-t_j}}{\cosh t_j}$$

we get

$$P = e^x \left(\prod_j \cosh t_j \right)^{-1}, \quad p = e^{-x} \left(\prod_j \cosh t_j \right)^{-1}.$$

Thus (4.22) immediately follows. \square

Lemma 4.3. *For any two partitions $s, t \in A_x$ with $t \prec s$ the following holds for any positive integer r*

$$\|R(t)^r\| \leq \|R(s)^r\|.$$

Proof. Let $t = (t_1, \dots, t_n) \prec s = (s_1, \dots, s_m)$ be two elements in A_x . In order to prove this result we relate the two series $R(t)$ and $R(s)$. Observe that $R(t)$ and $R(s)$ depend on $n-1$ and $m-1$ variables respectively, $m \geq n$. Since $t \prec s$ there are $0 = k_0 < k_1 < \dots < k_n = m$ such that

$$t_l = \sum_{j=k_{l-1}+1}^{k_l} s_j, \quad \forall 1 \leq l \leq n.$$

Using that $\gamma(a+b) = \gamma(a)\gamma(b)$ it follows that

$$\gamma(t_l) = \prod_{j=k_{l-1}+1}^{k_l} \gamma(s_j).$$

Since $\alpha(1) = 1$ we also have

$$\gamma(t_l) = \gamma(s_{k_{l-1}+1})\alpha(1)\gamma(s_{k_{l-1}+2})\alpha(1) \cdots \alpha(1)\gamma(s_{k_l}).$$

Replacing $\gamma(t_l)$ by the above formula in the definition of $R(t)$, (4.20), one gets

$$\begin{aligned} R(t; (q_1, \dots, q_{n-1})) &= (\gamma(t_n)\alpha(q_{n-1})\gamma(t_{n-1}) \cdots \alpha(q_1)\gamma(t_1)) (0) \\ &= R(s; (1, \dots, 1, q_1, 1, \dots, 1, q_2, \dots, q_{n-1}, 1, \dots, 1)), \end{aligned}$$

where the blocks of one above have lengths $k_n - k_{n-1} - 1, \dots, k_2 - k_1 - 1, k_1 - k_0 - 1$. Thus $\|R(t)\| \leq \|R(s)\|$. For $r \geq 2$ the argument is similar since

$$R(t; (q_1, \dots, q_{n-1}))^r = R(s; (1, \dots, 1, q_1, \dots, q_{n-1}, 1, \dots, 1))^r.$$

The proof is finished. \square

4.3. Upper bounds for $R(t)$. We now obtain some properties of the multivariable series $R(t)$ introduced above. For any integer $r \geq 0$ we define the function $f_r : (0, \infty) \rightarrow (0, \infty)$ by

$$f_r(x) = \sup_{t \in A_x} \|R(t)^r\|.$$

In particular, $f_0 \equiv 1$. We note that $f_{r_1+r_2}(x) \leq f_{r_1}(x)f_{r_2}(x)$. In particular for any integer $r \geq 1$, $f_r(x) \leq f_1(x)^r$.

The first estimate for f_1 is given in the following lemma.

Lemma 4.4. *For any $x \in (0, \log(2 + \sqrt{3}))$ the following holds*

$$(4.23) \quad f_1(x) \leq \frac{\sinh(x)}{2 - \cosh(x)}.$$

Proof. Let us choose $t = (t_1, \dots, t_n) \in A_x$ and denote $q = (q_1, \dots, q_{n-1})$. We will show that

$$\|R(t; q)\| \leq \frac{\sinh(x)}{2 - \cosh(x)}.$$

By definition

$$R(t; q) = \left(\gamma(t_n)\alpha(q_{n-1})\gamma(t_{n-1}) \cdots \alpha(q_1)\gamma(t_1) \right) (0).$$

From Corollary 4.1 it follows that

$$(\gamma(t_n)\alpha(q_{n-1})\gamma(t_{n-1}) \cdots \alpha(q_1)\gamma(t_1)\alpha(q_0))(z) = \frac{a(q_0, \dots, q_{n-1})z + b(q_0, \dots, q_{n-1})}{c(q_0, \dots, q_{n-1})z + d(q_0, \dots, q_{n-1})}.$$

Hence choosing $q_0 = 1$ and $z = 0$ we get

$$R(t; q) = \frac{b(1, q)}{d(1, q)}.$$

Using Corollary 4.1, we have that

$$\|b\| < \sinh(x) \text{ and } \|d(1, q) - 1\| = \|d\| - 1 < \cosh(x) - 1.$$

Then, for any x satisfying $\cosh(x) < 2$ the following holds

$$\|R(t; q)\| \leq \frac{\|b(1, q)\|}{1 - \|1 - d(1, q)\|} \leq \frac{\sinh(x)}{2 - \cosh(x)}.$$

Since the partition $t \in A_x$ has been arbitrarily chosen we obtain that estimate (4.23) holds for all $x < \operatorname{arccosh}(2) = \log(2 + \sqrt{3})$. \square

Lemma 4.5. *For any positive integer r , function f_r is increasing.*

Proof. Let us observe that for any $d \in [0, 1)$ we have $\beta(d)\alpha(-1)\beta(d)\alpha(-1) = 1$. Thus, for any nonnegative t , $\gamma(t)$ satisfies $\gamma(t)\alpha(-1)\gamma(t)\alpha(-1) = 1$.

Let us now choose $0 < x < y$, where $y = x + z$ with $z > 0$. Let $t = (t_1, \dots, t_n) \in A_x$. Then $s := (t_1, \dots, t_n, z/2, z/2) \in A_y$. Since $\gamma(z/2)\alpha(-1)\gamma(z/2)\alpha(-1) = 1$ we have:

$$\begin{aligned} R(t; (q_1, \dots, q_{n-1})) &= \left(\gamma\left(\frac{z}{2}\right)\alpha(-1)\gamma\left(\frac{z}{2}\right)\alpha(-1)\gamma(t_n)\alpha(q_{n-1}) \cdots \alpha(q_1)\gamma(q_1) \right) (0) \\ &= R(s; (q_1, \dots, q_{n-1}, -1, -1)) \end{aligned}$$

Hence for any integer $r \geq 1$,

$$R(t; q_1, \dots, q_{n-1})^r = R(s; q_1, \dots, q_{n-1}, -1, -1)^r,$$

which implies that $\|R(t)^r\| \leq \|R(s)^r\|$. This implies that for any $t \in A_x$ there is some $s \in A_y$ with $\|R(t)^r\| \leq \|R(s)^r\|$. Therefore f_r is an increasing function and the proof finishes. \square

We denote $I = \{x \in (0, \infty) \mid f_1(x) < \infty\}$. In view of Lemma 4.4 and Lemma 4.5, the set I is an interval that includes $(0, \log(2 + \sqrt{3}))$. Moreover, all the functions f_r , $r \geq 1$, are finite on interval I since $f_r(x) \leq f_1^r(x)$. Now we prove that f_r are differentiable.

Lemma 4.6. *The set I is an open interval. For any integer $r \geq 1$, function f_r is differentiable on I and satisfies $f'_r = r(f_{r-1} + f_{r+1})$.*

Proof. Let ε and x be positive numbers. For any partition $s \in A_{x+\varepsilon}$ there is a finer partition, \tilde{s} , of the form $\tilde{s} = (t, t') \in A_x \times A_\varepsilon$, (because for any partition of $[0, x + \varepsilon]$ there is a finer one containing x). Then

$$\|R(s)\| \leq \|R(\tilde{s})\| \leq f_1(x + \varepsilon)$$

and so

$$f_1(x + \varepsilon) = \sup_{(t, t') \in A_x \times A_\varepsilon} \|R(t, t')\|.$$

Let us consider $(t, t') \in A_x \times A_x$, $t = (t_1, \dots, t_n)$, $t' = (t_{n+1}, \dots, t_m)$. Denoting $q = (q_1, \dots, q_{n-1})$ and $q' = (q_n, \dots, q_{m-1})$ we obtain that

$$R((t, t'); (q, q')) = (\gamma(t_m)\alpha(q_{m-1}) \cdots \gamma(t_{n+1})\alpha(q_n)) (R(t; q)).$$

Since $t \in A_\varepsilon$ by Corollary 4.1 we have that

$$\gamma(t_m)\alpha(q_{m-1}) \cdots \gamma(t_{n+1})\alpha(q_n)$$

is given by $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}[q_n, \dots, q_{m-1}]$ have non-negative coefficients, $\|a\| = \|d\| < \cosh \varepsilon$, $\|b\| = \|c\| < \sinh \varepsilon$, the coefficient of $q_n \cdots q_{m-1}$ in a and the constant term of d are 1. Then $\|a\| = 1 + \|a - q_n \cdots q_{m-1}\|$ and $\|d\| = 1 + \|d - 1\|$ so

$$\|a - q_n \cdots q_{m-1}\| = \|a\| - 1 = \|d\| - 1 = \|d - 1\| < \cosh \varepsilon - 1.$$

Then

$$R((t, t')) = \frac{aR(t) + b}{cR(t) + d} = \frac{aR(t) + b}{1 + (cR(t) + d - 1)}.$$

Since $t \in A_x$ we have $\|R(t)\| \leq f_1(x) < \infty$. It follows that

$$aR(t) + b = q_n \cdots q_{m-1}R(t) + (a - q_n \cdots q_{m-1})R(t) + b = q_n \cdots q_{m-1}R(t) + b + O(\varepsilon^2)$$

and

$$cR(t) + d = 1 + cR(t) + (d - 1) = 1 + cR(t) + O(\varepsilon^2).$$

Then for $\varepsilon < \varepsilon_0$, ε_0 small enough depending on $f_1(x)$, the following holds

$$\begin{aligned} R((t, t')) &= (q_n \cdots q_{m-1}R(t) + b + O(\varepsilon^2))(1 - cR(t) + O(\varepsilon^2)) \\ &= q_n \cdots q_{m-1}R(t) + b - q_n \cdots q_{m-1}cR(t)^2 + O(\varepsilon^2). \end{aligned}$$

Since $b - q_n \cdots q_{m-1}cR(t) = O(\varepsilon)$ this implies that for any $r \geq 1$ we have

$$\begin{aligned} R((t, t'))^r &= (q_n \cdots q_{m-1})^r R(t)^r + r(q_n \cdots q_{m-1})^{r-1} R(t)^{r-1} (b - q_n \cdots q_{m-1}cR(t)^2) + O(\varepsilon^2) \\ &= (q_n \cdots q_{m-1})^r R(t)^r + r(q_n \cdots q_{m-1})^{r-1} (bR(t)^{r-1} - q_n \cdots q_{m-1}cR(t)^{r+1}) + O(\varepsilon^2). \end{aligned}$$

Using that $\sinh \varepsilon = \varepsilon + O(\varepsilon^2)$ it follows that

$$\begin{aligned} \|R(t, t')^r\| &\leq \|R(t)^r\| + r\|R(t)^{r-1}\| \cdot \|b\| + r\|R(t)^{r+1}\| \cdot \|c\| + O(\varepsilon^2) \\ &\leq f_r(x) + r f_{r-1}(x) \sinh \varepsilon + r f_{r+1}(x) \sinh \varepsilon + O(\varepsilon^2) \\ &= f_r(x) + r(f_{r-1}(x) + f_{r+1}(x))\varepsilon + O(\varepsilon^2). \end{aligned}$$

Thus, for ε small enough we have

$$(4.24) \quad f_r(x + \varepsilon) \leq f_r(x) + r(f_{r-1}(x) + f_{r+1}(x))\varepsilon + O(\varepsilon^2).$$

This implies that I is an open interval.

For the reverse inequality we will take $m = n + 1$ so t' has dimension one, $t' = (\varepsilon)$, and $q' = q_n$. Then $(t, \varepsilon) \in A_{x+\varepsilon}$ so $f_r(x + \varepsilon) \geq \|R(t, \varepsilon)^r\|$. We have $R((t, \varepsilon); (q, q_n)) = \gamma(\varepsilon)\alpha(q_n)(R(t, q))$, i.e.

$$R((t, \varepsilon)) = \frac{q_n R(t) + \tanh \varepsilon}{q_n \tanh \varepsilon R(t) + 1}.$$

The same argument as above shows that

$$R(t, \varepsilon)^r = q_n^r R(t)^r + r\varepsilon q_n^{r-1} (R(t)^{r-1} - q_n^2 R(t)^{r+1}) + O(\varepsilon^2).$$

Since $R(t)^{r-1}, R(t)^r, R(t)^{r+1} \in \mathbb{R}[[q_0, \dots, q_{n-1}]]$ it follows that

$$\|q_n^r R(t)^r + r\varepsilon(q_n^{r-1} R(t)^{r-1} - q_n^{r+1} R(t)^{r+1})\| = \|R(t)^r\| + r\varepsilon(\|R(t)^{r-1}\| + r\varepsilon\|R(t)^{r+1}\|).$$

Hence for any $t \in A_x$ and ε small enough

$$f_r(x + \varepsilon) \geq \|R(t, \varepsilon)^r\| \geq \|R(t)^r\| + r\varepsilon(\|R(t)^{r-1}\| + \|R(t)^{r+1}\|) + O(\varepsilon^2)$$

and then

$$(4.25) \quad f_r(x + \varepsilon) \geq f_r(x) + \varepsilon r(f_{r-1}(x) + f_{r+1}(x)) + O(\varepsilon^2).$$

Using (4.24) and (4.25) we obtain that f_r satisfies

$$f_r(x + \varepsilon) = f_r(x) + \varepsilon r(f_{r-1}(x) + f_{r+1}(x)) + O(\varepsilon^2).$$

Thus function f_r is right differentiable and satisfies

$$\lim_{\varepsilon \searrow 0} \frac{f_r(x + \varepsilon) - f_r(x)}{\varepsilon} = r(f_{r-1}(x) + f_{r+1}(x)).$$

Moreover, since $f_k(x - \varepsilon) \leq f_k(x) \leq f_1^k(x)$, $k \in \{r-1, r+1\}$, by applying the same argument as in the proof of (4.24) and (4.25) to $x' = x - \varepsilon$ we obtain that

$$f_r(x) = f_r(x - \varepsilon) + \varepsilon r(f_{r-1}(x - \varepsilon) + f_{r+1}(x - \varepsilon)) + O(\varepsilon^2) = f_r(x - \varepsilon) + O(\varepsilon)$$

which proves that f_r is also left continuous.

For the left derivative of f_r at x we apply the previous analysis to the point $x' = x - \varepsilon$

$$\frac{f_r(x) - f_r(x - \varepsilon)}{\varepsilon} = r(f_{r-1}(x - \varepsilon) + f_{r+1}(x - \varepsilon)) + O(\varepsilon).$$

Since f_{r-1}, f_{r+1} are continuous we obtain

$$\lim_{\varepsilon \searrow 0} \frac{f_r(x) - f_r(x - \varepsilon)}{\varepsilon} = r(f_{r-1}(x) + f_{r+1}(x)).$$

The proof of Lemma 4.6 is now finished. \square

Theorem 4.1. *We have $I = (0, \pi/2)$ and for any $r \geq 1$ function f_r is given by*

$$f_r(x) = \tan^r x, \quad x \in I.$$

Proof. We first show that $[0, \pi/2) \subseteq I$ and for any $r \geq 1$ the following holds

$$f_r(x) \leq \tan^r x, \quad \forall x \in (0, \frac{\pi}{2}).$$

For any $x \in I$ we have $f_1'(x) = f_0(x) + f_2(x) \leq 1 + f_1(x)^2$. Lemma 4.4 gives us that $\lim_{x \rightarrow 0} f_1(x) = 0$ and then by integrating the last inequality we obtain that $\arctan f_1(x) \leq x$. Thus $f_1(x) \leq \tan x$. For $r \geq 2$ similar estimates hold since $f_r(x) \leq (f_1(x))^r$.

Let $S = \{(x, y) \in (0, \pi/2) \times \mathbb{R} \mid |y \tan x| < 1\}$. Using the properties of functions f_r above we get that function g defined by

$$g(x, y) = f_1(x) + f_2(x)y + f_3(x)y^2 + \dots$$

is well defined and differentiable on S . Explicit computations show that g satisfies the following first order equation

$$(y^2 + 1)g_y(x, y) - g_x(x, y) = -2yg(x, y) - 1, \quad \forall (x, y) \in S.$$

In order to solve it we need some boundary conditions. Observe that since $f_r(x) \leq \tan^r x$ for all $r \geq 1$, function g satisfies $\lim_{x \rightarrow 0} g(x, y) = 0$ for all $y \in \mathbb{R}$. Solving the above equation by the method of characteristics we obtain that

$$g(x, y) = \frac{\tan x}{1 - y \tan x}.$$

Developing in y power series we get that $f_r(x) = \tan^r x$, as claimed. In particular, $f_1(x) = \tan x$. Since f_1 is increasing we also obtain that $I = (0, \pi/2)$ and the proof is finished. \square

4.4. Proofs of the two Lemmas. We are now able to prove Lemma 2.3 and Lemma 2.4.

Proof of Lemma 2.3. By definition of E and Theorem 4.1 we have that

$$\begin{aligned} \|Q_{n+1}\|_{AP} &= \|E((|d_1|, \dots, |d_n|))\| = \|R((\operatorname{arctanh}|d_1|, \dots, \operatorname{arctanh}|d_n|))\| \\ &\leq f_1\left(\sum_{k=1}^n \operatorname{arctanh}|d_k|\right) = \tan\left(\sum_{k=1}^n \operatorname{arctanh}|d_k|\right). \end{aligned}$$

Then estimate (2.17) follows. \square

Proof of Lemma 2.4. Let us choose $\alpha \geq \pi/2$. By Theorem 4.1, $f_1(\pi/2) = \infty$. This means that for any positive $N > 0$ there exists a partition $t = (t_1, \dots, t_{n-1}) \in A_\alpha$ such that $\|R(t)\| \geq N$. Choosing eventually finer partitions we can assume that $t_k < \operatorname{arctanh}(d)$, $k = 1, \dots, n-1$.

Choosing $\tilde{d}_k = \tanh(t_k)$, $k = 1, \dots, n-1$, we have $0 < \tilde{d}_k \leq d$,

$$\|E((\tilde{d}_1, \dots, \tilde{d}_{n-1}))\| = \|R(t)\| \geq N.$$

We finally choose the $\{c_k\}_{k=1}^{n-1}$ such that they are linearly independent over rationals. Using Lemma 4.1 it implies that $\{Q_k\}_{k=1}^n$ defined by (2.16) satisfies

$$\|Q_n\|_{AP} = \|E((|d_1|, \dots, |d_{n-1}|))\| = \|E((\tilde{d}_1, \dots, \tilde{d}_{n-1}))\| \geq N$$

and the proof is finished. \square

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REFERENCES

- [1] V. Banica. Dispersion and Strichartz inequalities for Schrödinger equations with singular coefficients. *SIAM J. Math. Anal.*, 35(4):868–883 (electronic), 2003.
- [2] N. Burq and F. Planchon. Smoothing and dispersive estimates for 1D Schrödinger equations with BV coefficients and applications. *J. Funct. Anal.*, 236(1):265–298, 2006.
- [3] C. Corduneanu. *Almost periodic oscillations and waves*. Springer, New York, 2009.
- [4] L. Demanet and G. Peyré. Compressive wave computation. *Found. Comput. Math.*, 11(3):257–303, 2011.
- [5] J-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna. *Wave propagation and time reversal in randomly layered media*, volume 56 of *Stochastic Modelling and Applied Probability*. Springer, New York, 2007.

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