Null control of a $1-d$ model of mixed hyperbolic-parabolic type

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Dedicated to Alain Bensoussan on his 60th birthday.

Abstract

In this paper we consider a simple $1-d$ model of mixed hyperbolic-parabolic type. The system consists of two intervals in which the wave and heat equations evolve respectively with transmission conditions at the interface (one single point).

We analyze the problem of controllability when the control acts on the free end of the elastic component, i.e. of the interval where the wave equation holds. We prove that the system is null controllable in a time which is twice the length of the interval where the wave equation evolves.

The proof combines sidewise energy estimates for the wave equation and Carleman inequalities for the heat equation.

1 Introduction and main result

In this article we consider the problem of null controllability for the following mixed system of hyperbolic-parabolic type:

$$
\left\{\begin{array}{lll}
y_{tt}-y_{xx}=0, & -1<x<0, & t>0\\
z_t-z_{xx}=0, & 0<x<1, & t>0\\
y=z,\, y_x=z_x, & x=0, & t>0\\
y(-1,t)=v(t), & t>0 & \\
z(1,t)=0, & t>0 & \\
y(x,0)=y_0(x),\, y_t(x,0)=y_1(x), & -1<x<0 & \\
z(x,0)=z_0(x), & 0<x<1. &
\end{array}\right. \tag{1.1}
$$

This system represents the coupling between the wave equation arising on the interval $(-1, 0)$ with state y and the heat equation that holds on the interval $(0, 1)$ with state z. At the interface, the point $x = 0$, we impose the continuity of (y, z) and (y_x, z_x). The system is complemented with boundary conditions at the free ends $x = \pm 1$ and initial conditions at time $t = 0$. The control $v = v(t)$ acts on the system through the extreme point $x = -1$.

This system might be viewed as a "toy model" of fluid-structure interaction. We refer to [9] and [12] for an analysis of the approximate controllability property for other, more complete, models in this context.

A lot of progress has been done in what concerns the controllability of heat and wave equations. In both cases, following J.L. Lions' HUM method (see [8]), the problem may be reduced to the obtention of suitable observability inequalities for the underlying uncontrolled adjoint systems. However, the techniques that have been developed to obtain such estimates differ very much from one case to the other one. In the context of the wave equation one may use multipliers (see for instance [1], [8]) or microlocal analysis ([2]) while, in the context of parabolic equations, one uses Carleman inequalities (see for instance [4], [6], [3]). Carleman inequalities have also been used to obtain observability estimates for wave equations ([14]), but, up to now, as far as we know, there is no theory describing how the Carleman inequalities for the parabolic equation may be obtained continuously from the Carleman inequalities for hyperbolic equations. This problem was addressed in [11] by viewing the heat equation $u_t - \Delta u = 0$ as limit of wave equations of the form $\varepsilon u_{tt} - \Delta u + u_t = 0$. But in [11], the Carleman inequalities were not uniform as $\varepsilon \to 0$ and therefore, Carleman inequalities were combined with a careful spectral analysis.

Summarizing, one may say that the techniques that have been developed to prove observability inequalities for wave and heat equations are difficult to combine and therefore there is, to some extent, a lack of tools to address controllability problems for systems in which both hyperbolic and parabolic components are present.

However, some examples have been addressed with succes. For instance, in [7] and [15] we considered the system of three-dimensional elasticity. There, using decoupling techniques, we were able to overcome these difficulties. However, in doing that, the fact that the hyperbolic and parabolic component of the solution of the system of thermoelasticity occupy the same domain played a crucial role.

The model we discuss here has the added difficulty that the two equations hold in two different domains and that they are only coupled through an interface where we impose transmission conditions guaranteeing the well-posedness of the initial-boundary value problem. On the contrary, our analysis is by now restricted to the $1-d$ case.

In the absence of control, i.e. when $v \equiv 0$, the energy

$$E(t) = \frac{1}{2}\int_{-1}^{0}\left[\mid y_x(x,t)\mid^2 + \mid y_t(x,t)\mid^2\right]dx + \frac{1}{2}\int_0^1 |z_x(x,t)|^2 dx \tag{1.2}$$

is decreasing. More precisely,

$$\frac{dE}{dt}(t) = -\int_0^1 |z_{xx}(x,t)|^2 dx. \tag{1.3}$$

Therefore, when $v \equiv 0$, for initial data

$$(y_0, y_1) \in H^1(-1,0) \times L^2(-1,0),\ z_0 \in H^1(0,1) \tag{1.4}$$

with

$$y_0(-1) = 0, y_0(0) = z_0(0), z_0(1) = 0, \tag{1.5}$$

system (1.1) admits an unique solution

$$\begin{cases} y \in C\left([0,\infty);\ H^1(-1,0)\right) \cap C^1\left([0,\infty);\ L^2(-1,0)\right) \\ z \in C\left([0,\infty);\ H^1(0,1)\right) \cap L^2(0,T; H^2(0,1)). \end{cases} \tag{1.6}$$

Note that, when (1.4) hold, y_0 and z_0 are simply the restriction of a function of $H_0^1(-1,1)$ to the left and right intervals $(-1,0)$ and $(0,1)$, respectively. Thus, as a consequence of (1.6), and abusing of notation, we may write that

$$(y,z) \in C\left([0,\infty);\ H_0^1(-1,1)\right). \tag{1.7}$$

The same existence and uniqueness result holds when $v \not\equiv 0$ but it is smooth enough.

Here we are interested on the problem of null-controllability. More precisely, given $T > 0$ and initial data $\{(y_0, y_1),\ z_0\}$ as above, we look for a control $v = v(t)$ (say, in $L^2(0,T)$), such that the solution of (1.1) is at rest at time $t = T$.

Here, being at rest at time $t = T$ means fulfilling the conditions

$$y(x,T) \equiv y_t(x,T) \equiv 0,\ -1 < x < 0;\ z(x,T) \equiv 0,\ 0 < x < 1. \tag{1.8}$$

As we mentioned above, there is a large literature in the subject in what concerns wave and heat equations, but much less is known when both components are coupled. We refer to the survey articles [17] and [18] for a description of the state of the art in this field.

If we relax the controllabillity condition (1.8) to a weaker one requiring the distance of the solution at time T to the target to be less than an arbitrarily small ε, i. e. to the so called approximate controllability property, the main difficulties disappear. Indeed, as a consequence of Holmgren's Uniqueness Theorem, this property turns out to hold even in several space dimensions. But, as we shall see, when doing this, the main difficulty arising when analyzing the null-control problem, i. e. the so called observability inequality, is avoided.

2 Observability of the adjoint system

As usual, when studying controllability problems, the key point is the obtention of suitable observability estimates for the adjoint system. Once this is done the null control may be easily obtained minimizing a suitable quadratic functional on a Hilbert space (see, for instance, [8]).

Let us therefore consider the adjoint system

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = f & \text{in} \quad (-1,0)\times(0,T) \\ -\psi_t - \psi_{xx} = g & \text{in} \quad (0,1)\times(0,T) \\ \varphi(0,t) = \psi(0,t) & \text{for} \quad t\in(0,T) \\ \varphi_x(0,t) = \psi_x(0,t) & \text{for} \quad t\in(0,T) \\ \varphi(-1,t) = \psi(1,t) = 0 & \text{for} \quad t\in(0,T) \\ \varphi(x,T) = \varphi_0(x),\ \varphi_t(x,T) = \varphi_1(x) & \text{in} \quad (-1,0) \\ \psi(x,T) = \psi_0(x) & \text{in} \quad (0,1). \end{cases} \tag{2.1}$$

Multiplying in (2.1) formally by (y,z) and integrating by parts it follows that

$$\begin{aligned} &\int_{-1}^{0}\int_{0}^{T} fydxdt + \int_{0}^{1}\int_{0}^{T} gzdxdt \\ &= \int_{0}^{T} \varphi_x(-1,t)v(t)dt - \int_{0}^{1} \left[\psi_0(x)z(x,T) - \psi(x,0)z_0(x)\right] dx \\ &+ \int_{-1}^{0} \left[\varphi_1(x)y(x,T) - \varphi_0(x)y_t(x,T) - \varphi_t(x,0)y_0(x) + \varphi(x,0)y_1(x)\right] dx. \end{aligned} \tag{2.2}$$

Obviously, in the obtention of (2.2) the transmission conditions in (1.1) and (2.1) have played a crucial role to cancel the terms appearing at the interface $x=0$ when integrating by parts.

Using classical energy estimates it can be shown that, when $f \in L^1\left(0,T;L^2(-1,0)\right)$, $g \in L^2\left(0,T;L^2(0,1)\right)$, $(\varphi_0,\psi_0) \in H_0^1(-1,1)$ and $\varphi_1 \in L^2(-1,0)$, system (2.1) admits an unique solution

$$\begin{cases} (\varphi,\psi) \in C\left([0,T];\ H_0^1(-1,1)\right);\ \varphi_t \in C^1\left([0,T];\ L^2(-1,0)\right) \\ \psi \in L^2\left(0,T;\ H^2(0,1)\right). \end{cases} \tag{2.3}$$

It is then easy to see using the classical results on the "hidden regularity" of solutions of the wave equation that

$$\varphi_x(-1,t) \in L^2(0,T) \tag{2.4}$$

as well, since this property holds locally around the boundary for finite energy solutions of the wave equation (see [8], Tome 1). Thus, in the present case, the presence of the heat component to the right of $x = 0$ is not an obstacle for this property of regularity of the trace of the normal derivative of the wave component to hold.

By transposition, we deduce that, whenever $v \in L^2(0,T)$, $y_0 \in L^2(-1,0)$ and $(y_1, z_0) \in H^{-1}(-1,1)$, system (1.1) admits an unique solution

$$y \in C\left([0,T];\, L^2(-1,0)\right),\ (y_t, z) \in C\left([0,T];\, H^{-1}(-1,1)\right). \tag{2.5}$$

Our goal is to prove the null-controllability of system (1.1) in this functional setting.

For this we need the following observability property for the solutions of the adjoint system:

Proposition 2.1 *Assume that* $f \equiv g \equiv 0$.

Let $T > 2$. *Then, there exists* $C > 0$ *such that*

$$\| (\varphi(x,0),\, \psi(x,0)) \|^2_{H_0^1(-1,1)} + \| \varphi_t(x,0) \|^2_{L^2(-1,0)} \leq C \| \varphi_x(-1,t) \|^2_{L^2(0,T)} \tag{2.6}$$

for every solution of (2.1) *with* $f \equiv g \equiv 0$.

Proof. We proceed in three steps.

Step 1. Sidewise energy estimates for the wave equation.

Arguing as in [16] and using the fact that φ satisfies the homogeneous wave equation on the left space interval $x \in (-1,0)$ (since $f \equiv 0$) we deduce that

$$\int_{1+x}^{T-(1+x)} \left[|\varphi_t(x,t)|^2 + |\varphi_x(x,t)|^2 \right] dt \leq \int_0^T |\varphi_x(-1,t)|^2\, dt, \quad \forall x \in [-1,0]. \tag{2.7}$$

In particular, integrating with respect to $x \in (-1,0)$:

$$\int_{-1}^{0} \int_{1+x}^{T-(1+x)} \left(\varphi_t^2 + \varphi_x^2\right) dx dt \leq \int_0^T |\varphi_x(-1,t)|^2\, dt \tag{2.8}$$

and, at $x = 0$,

$$\int_{1}^{T-1} \left[|\varphi_t(0,t)|^2 + |\varphi_x(0,t)|^2 \right] dt \leq \int_0^T |\varphi_x(-1,t)|^2\, dt. \tag{2.9}$$

Using the fact that $\varphi = 0$ at $x = -1$ and Poincaré inequality we also deduce that

$$\int_{1}^{T-1} |\varphi(0,t)|^2\, dt \leq \int_{-1}^{0} \int_{1+x}^{T-(1+x)} \left(\varphi_t^2 + \varphi_x^2\right) dx dt \tag{2.10}$$

.

This inequality, combined with (2.8) yields

$$\int_1^{T-1} |\varphi(0,t)|^2\, dt \le C \int_0^T |\varphi_x(-1,t)|^2\, dt \tag{2.11}$$

for some $C > 0$, independent of φ.

Step 2. Estimates for the heat equation.

In view of (2.9)-(2.10) and using the transmission conditions at $x = 0$ we deduce that

$$\int_1^{T-1} \left[|\ \psi(0,t)\ |^2 + |\ \psi_t(0,t)\ |^2 + |\ \psi_x(0,t)\ |^2 \right] dt \le C \int_0^T |\varphi_x(-1,t)|^2\, dt. \tag{2.12}$$

Our goal in this second step is to determine how much of the energy of ψ we can estimate in terms of the left hand side of (2.12). Note that (2.12) provides estimates on the Cauchy data of ψ at $x = 0$ in the time interval $(1, T-1)$, which is non empty because of the assumption $T > 2$. In order to simplify the notation, in this step we translate the interval $(1, T-1)$ into $(0, T')$ with $T' = T - 2$. This can be done because the system under consideration is time independent. On the other hand, taking into account that the inequalities for the heat equation we shall use hold in any interval of time, we can replace T' by T to simplify the notation.

We have to use the fact that ψ satisfies

$$\begin{cases} \psi_t + \psi_{xx} = 0, & \text{in} \quad (0,1) \times (0,T) \\ \psi(1,t) = 0, & \text{for} \quad t \in (0,T). \end{cases} \tag{2.13}$$

Note that the boundary condition of ψ at $x = 0$ is unknown, although, according to (2.12), we have an estimate on its $H^1(0,T)$ norm.

We decompose ψ as follows:

$$\psi = \theta + \eta \tag{2.14}$$

with θ solution of

$$\begin{cases} \theta_t + \theta_{xx} = 0 & \text{in} \quad (0,1) \times (0,T) \\ \theta(x,T) = 0 & \text{in} \quad (0,1) \\ \theta(0,t) = \psi(0,t) & \text{for} \quad t \in (0,T) \\ \theta(1,t) = 0 & \text{for} \quad t \in (0,T), \end{cases} \tag{2.15}$$

and η solving

$$\begin{cases} \eta_t + \eta_{xx} = 0 & \text{in} \quad (0,1) \times (0,T) \\ \eta(x,T) = \psi(x,T) & \text{in} \quad (0,1) \\ \eta(0,t) = \eta(1,t) = 0 & \text{for} \quad t \in (0,T). \end{cases} \tag{2.16}$$

Analyzing the regularity of solutions of (2.15) one can deduce that

$$\|\ \theta\ \|_{L^2\left(0,T;\, H^{5/2-\delta}(0,1)\right)} + \|\ \theta_t\ \|_{L^2\left(0,T;\, H^{1/2-\delta}(0,1)\right)} \le C_\delta \ \|\ \psi(0,t)\ \|_{H^1(0,1)} \tag{2.17}$$

for all $\delta > 0$.

In particular

$$\| \theta_x(0,t) \|_{L^2(0,T)} \leq C \| \psi(0,t) \|_{H^1(0,T)} . \tag{2.18}$$

Combining (2.12) and (2.18) we deduce that

$$\begin{aligned} \| \eta_x(0,t) \|^2_{L^2(0,T)} &\leq C \left[\| \psi(0,t) \|^2_{H^1(0,T)} + \| \psi_x(0,t) \|^2_{L^2(0,T)} \right] \\ &\leq C \| \varphi_x(-1,t) \|^2_{L^2(0,T)} . \end{aligned} \tag{2.19}$$

Now, using the classical observability estimates (see [10] and [13]) for the solutions η of the heat equation (2.16) with homogeneous Dirichlet boundary conditions we deduce that

$$\| \eta \|_{L^2(0,T-s;\, H^\sigma(0,1))} \leq C_{s,\sigma} \| \eta_x(0,t) \|_{L^2(0,T)} \tag{2.20}$$

for all $s \in (0,T)$ and for all $\sigma > 0$, with $C_{s,\sigma}$ independent of η, which, combined with (2.19), yields

$$\| \eta \|_{L^2(0,T-s;\, H^\sigma(0,1))} \leq C_{s,\sigma} \| \varphi_x(-1,t) \|_{L^2(0,T)} \tag{2.21}$$

Combining (2.17) and (2.21) and going back to the time interval $(1, T-1)$ we deduce that

$$\| \psi \|_{L^2(1,T-1-\delta,\, H^1(0,1))} \leq C_\delta \| \varphi_x(-1,t) \|_{L^2(0,T)} \tag{2.22}$$

for all $\delta \in (0, T-2)$.

Step 3. Conclusion.

Combining (2.8) and (2.22) we have that

$$\begin{aligned} &\int_1^{T-1-\delta} \int_{-1}^0 \left[|\varphi_t(x,t)|^2 + |\varphi_x(x,t)|^2 \right] dxdt + \int_1^{T-1-\delta} \int_0^1 |\psi_x(x,t)|^2 \, dxdt \\ &\leq C_\delta \int_0^T |\varphi_x(-1,t)|^2 \, dt, \end{aligned} \tag{2.23}$$

for all $\delta > 0$ with $T - 2 - \delta > 0$.

Taking into account that the energy

$$E(t) = \frac{1}{2} \int_{-1}^0 \left[|\varphi_t(x,t)|^2 + |\varphi_x(x,t)|^2 \right] dx + \frac{1}{2} \int_0^1 |\psi_x(x,t)|^2 \, dx$$

is a non decreasing function of time when (φ, ψ) solve (2.1) with $f \equiv g \equiv 0$, inequality (2.6) holds.

■

3 Null-controllability

As a consequence of Proposition 2.1 the following null-controllability property of system (1.1) may be deduced:

Theorem 3.1 *Assume that* $T > 2$. *Then, for every* $y_0 \in L^2(-1,0)$, $(y_1, z_0) \in H^{-1}(-1,1)$ *there exists a control* $v \in L^2(0,T)$ *such that the solution* (y,z) *of* (1.1) *satisfies* (1.8).

The proof of Theorem 3.1 may be done as in [3]. Using the variational approach to approximate controllability (see [5]), for any $\varepsilon > 0$, one can easily find a control v_ε such that

$$\| y(T) \|_{L^2(-1,0)} + \| (y_t(T), z(T)) \|_{H^{-1}(-1,1)} \leq \varepsilon.$$

Moreover, according to (2.6) one can show that v_ε remains bounded in $L^2(0,T)$ as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$ one gets the desired null-control.

4 Further comments

- The tools we have developed can be easily extended to treat similar systems with variable coefficients. One can also handle the case in which the space interval is divided in three pieces so that the heat equation arises in the middle one and the wave equation holds in the other two. Controlling on both extremes of the interval through the two wave equations allows to control to zero the whole process.

- The same techniques allow to treat other boundary and transmission conditions. For instance, in the context of fluid-structure interaction it is more natural to consider transmission conditions of the form:

  $$y_t = z; \quad y_x = z_x \quad \text{at } x = 0, \text{ for all } t > 0. \tag{4.1}$$

  When doing this, y_t represents the velocity in the displacement of the structure and z the velocity of the fluid and the energy of the system is then:

  $$E(t) = \frac{1}{2}\int_{-1}^{0} \left[| y_x(x,t) |^2 + | y_t(x,t) |^2 \right] dx + \frac{1}{2}\int_0^1 |z(x,t)|^2 dx \tag{4.2}$$

  The method of proof of the observability inequality developed in section 2 applies in this case too.

However, many interesting questions are completely open. For instance:

- A similar result is true when the control acts on the right extreme point $x = 1$ through the parabolic component?

  In what concerns observability, this problem is equivalent to replacing $\| \varphi_x(-1,t) \|_{L^2(0,T)}$ by $\| \psi_x(1,t) \|_{L^2(0,T)}$ in (2.6). The proof given above does not apply readily in this case because of the lack of sidewise energy estimates for the heat equation.

  The same question arises for the boundary conditions (4.1).

- Extending the result of this paper to the case of several space dimensions is also a challenging open problem. Given a domain Ω and an open subset $\omega \subset\subset \Omega$ we consider the wave equation in the outer region $\Omega \backslash \bar{\omega}$ and the heat equation in the inner one ω, coupled by suitable transmission conditions in the interface $\partial\omega$ as in (1.1) or (4.2). Can we control the whole process acting on the outer boundary $\partial\Omega$ on the wave component during a large enough time?

  The techniques developed in the literature up to now to deal with multi-dimensional controllability problems seem to be insufficient to address this question.

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