Averaged control ⋆

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Abstract

We analyze the problem of controlling parameter-dependent systems. We introduce the notion of averaged control according to which the quantity of interest is the average of the states with respect to the parameter.

First we consider the problem of controllability for linear finite-dimensional systems and show that a necessary and sufficient condition for averaged controllability is an averaged rank condition, in the spirit of the classical rank condition for linear control systems, but involving averaged momenta of any order of the matrices generating the dynamics and representing the control action.

We also describe some open problems and directions of possible research, in particular on the average controllability of evolution partial differential equations. In this context we analyze also the averaged version of a classical optimal control problem for a parameter dependent elliptic equation and derive the corresponding optimality system.

Key words: parametrized ODE and PDE, averaged controllability, averaged observability, averaged optimal control.

1 Introduction

We analyze the problem of controlling systems submitted to parametrized perturbations, either finite or infinite dimensional ones, i.e. ordinary or partial differential equations (ODE or PDE), depending on unknown parameters in a deterministic manner. We look for a control, independent of the values of these parameters, that needs to be designed to perform well, in an averaged sense to be made precise. We do it under an averaged criterion, considering two particularly relevant cases:

- *Parameter dependent ODEs:* We introduce and analyze the problem of controlling the expected or averaged value of the systems states starting from a given and known initial datum and by means of a single control, independent of the parameters involved. First, using classical duality theory, we show that the problem is equivalent to an averaged observability inequality for the adjoint system whose distinguished feature is that all the components take the same final state independently of the value of the parameter. Then, we characterize the averaged controllability property through a suitable rank condition involving the averaged momenta of any order of the operators generating the dynamics and the control ones.
- *Parameter dependent PDEs:* We introduce the same notion of averaged control for parameter-dependent partial differential equations (PDE). By duality this leads to averaged observability problems. As we shall see, this is a challenging topic in which plenty is to be done, requiring significant further work. We also consider the problem of the optimal control for a parameter-dependent family of elliptic PDE. We provide a complete characterization of the optimal control through the corresponding optimality system.

As we shall see, the notion of averaged control addressed in this paper is weaker than the classical one of simultaneous control introduced in [20], Chapter V.

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In this paper we discuss the case in which the parameter-dependence is of deterministic nature. Similar results would hold when the operators involved vary with respect to uncertain parameters with a given probability density and one aims at controlling the expected value of the state. But, to simplify the presentation, we shall focus on the deterministic setting.

Averaged controllability is equivalent, by duality, to a property of averaged observability in which the goal is to estimate the norm of the data of the parameter-dependent adjoint system, out of partial measurements done on the averages with respect to the unknown parameters. This property is of interest on its own, when dealing with the observability of parameter-dependent systems. The actual realization of the system depending on the parameter being unknown, it is natural to address the problem based on the measurements done on averages.

The notion of averaged controllability, as formulated here, has not been analyzed until now. The nature of the results and open problems arising in its study both for finite-dimensional and infinite-dimensional systems are a good evidence of its relevance and suitability. As we shall see, when facing parameter-dependent situations, the averaged control is a natural first guess. Our results not only allow establishing whether a system is controllable in an averaged sense, but also to derive characterizations that turn out to be algorithmic and serve for computational purposes.

The problem of averaged control is also related to the issue of robust control, that has been extensibly addressed in the literature from different viewpoints. The interested reader is referred to the book [1], for instance, and to [24] where output non-anticipating feedback optimal control results are derived for linear uncertain finite-dynamical systems (see also [22] for the first results on non-anticipating control of heat-like equations). We also refer to [26] and [28] for the analysis of the problem of possible controllability of uncertain systems, to [15] and the references therein for the problem of controlling finite-dimensional systems subject to some random non-autonomous dynamics and to [29] for related notions of robust control.

The rest of this paper is organized as follows. In Section 2 we discuss the finite-dimensional averaged controllability problem. The equivalence with the property of averaged observability and with a averaged rank condition is proved. In section 3 we discuss the comparison of these problems and results with the existing ones on simultaneous control. In section 4.3 we discuss the PDE case, formulating several model problems and analyzing in detail the problem of elliptic averaged optimal control. We close this paper pointing towards some open problems and future directions of research.

2 Finite dimensional linear systems

2.1 Problem formulation and main result

Consider the finite dimensional linear control system

$$\begin{cases} x'(t) = A(\nu)x(t) + B(\nu)u(t),\ 0 < t < T, \\ x(0) = x^0. \end{cases} \tag{1}$$

In (1) the (column) vector valued function $x(t,\nu) = \big(x_1(t,\nu),\dots,x_N(t,\nu)\big) \in \mathbf{R}^N$ is the state of the system, $A(\nu)$ is a $N \times N-$matrix governing its free dynamics and $u = u(t)$ is a M-component control vector in $\mathbf{R}^M$, $M \le N$, entering and acting on the system through the control operator $B(\nu)$, a $N \times M$ parameter-dependent matrix.

The matrices A and B are assumed to depend on a parameter ν in a measurable manner, although our analysis would also be valid for the multi-parameter case. To simplify the notation we will assume that $\nu \in \mathbf{R}$, although a similar analysis can be developed when ν is a multivalued parameter or even a random one, living in a probability space. To fix ideas we will assume that the parameter ν ranges within the interval $(0,1)$. We also assume that A and B are uniformly bounded with respect to ν, so to ensure (by Lebesgue dominated convergence theorem) the integrability of the solutions of (1) (and the corresponding adjoint system) with respect to ν.

Note however that the initial datum $x^0 \in \mathbf{R}^N$ to be controlled, in principle, is independent of the parameter ν. But the state of the system itself $x(t,\nu)$ depends on ν. The case where x^0 depends on ν will be discussed as well.

The motivation of the problem we consider is the following: We address the controllability of this system whose initial datum is given, known and fully determined. However, the dynamics of the state is governed by a parametrized operator $A(\nu)$, the same as the control operator $B(\nu)$. The effective value of the parameter ν being unknown, we aim at choosing a control that would perform optimally in an averaged sense, i. e. so that, rather than controlling specific realizations of the state, the average with respect to ν is controlled. This allows building a control independent of the parameter and making a robust compromise of all the possible realizations of the system for the various possible values of the unknown parameter ν.

More precisely, the problem can be formulated as follows: *Given a control time $T > 0$ and arbitrary initial data x^0 and final target $x^1 \in \mathbf{R}^N$ we look for a control u such that the solution of (1) satisfies*

$$\int_0^1 x(T,\nu)d\nu = x^1. \tag{2}$$

Note that, contrarily to the case in which A and B are independent of ν, we can not reduce the problem to the particular case where $x^1 \equiv 0$. This will also be observed at the level of the dual observability problem. Thus, the final requirement (2) needs to be considered for all possible targets x^1.

Note also that this concept of averaged controllability differs and is weaker from that of simultaneous controllability in which one is interested on controlling all states simultaneously and not only its average.

The particular case where A is independent of ν but $B = B(\nu)$ can be handled quite easily. Indeed, consider the system

$$\begin{cases} x'(t) = Ax(t) + B(\nu)u(t),\ 0 < t < T, \\ x(0) = x^0. \end{cases} \tag{3}$$

Obviously, the state $x = x(t, \nu)$ depends on ν and the notion of averaged controllability property (2) makes sense. But in the present case the problem is easy to solve since

$$y(t) = \int_0^1 x(t, \nu) d\nu$$

solves the system

$$\begin{cases} y'(t) = Ay(t) + \hat{B}u(t),\ 0 < t < T, \\ y(0) = x^0, \end{cases} \tag{4}$$

where $\hat{B} = \int_0^1 B(\nu)d\nu$ is the average of the control operators.

Accordingly, when $A(\nu) = A$ for all ν, the averaged controllability property holds if and only if the pair $(A, \hat{B} = \int_0^1 B(\nu)d\nu)$ satisfies the rank condition:

$$\text{rank}\Big[A^j \int_0^1 B(\nu)d\nu : 0 \le j \le N-1\Big] = N. \tag{5}$$

But this averaging principle does not apply when the operators A depend on ν. In this more general setting, the property of averaged controllability will be characterized through a rank condition in the same spirit, but involving the averages of A and all its powers, together with the control operator B, with respect to ν. However, as we shall see, in general, contrarily to the case in which A is independent of ν (see [18], [30]), this condition will involve powers of arbitrary order, and not only a finite number of powers up to order $N-1$ as in (5).

More precisely, the following holds:

Theorem 1 *System (1) fulfills the averaged controllability property (2) if and only the following rank condition is satisfied:*

$$\text{rank}\Big[\int_0^1 [A(\nu)]^j B(\nu)d\nu : j \ge 0\Big] = N. \tag{6}$$

Remark 2 *Several remarks are in order:*

- *The averaged rank condition can be interpreted and simplified when all the matrices $A(\nu), B(\nu)$ are multiples of the same constant matrices A, B: $A(\nu) = \alpha(\nu)A$, $B(\nu) = \beta(\nu)B$. In this case,*

$$\int_0^1 [A(\nu)]^k B(\nu)d\nu = \int_0^1 [\alpha(\nu)]^k \beta(\nu)d\nu A^k B,\ \forall k \ge 0$$

and

$$\Big[\int_0^1 B(\nu)d\nu, \int_0^1 A(\nu)B(\nu)d\nu, \dots\Big] = \Big[\int_0^1 \beta(\nu)d\nu B, \int_0^1 \alpha(\nu)\beta(\nu)d\nu AB, \dots\Big].$$

Thus, under the further assumption that

$$\int_0^1 [\alpha(\nu)]^k \beta(\nu)d\nu \ne 0,\ k = 1, ..., N-1, \tag{7}$$

the averaged rank condition (6) is equivalent to the classical one

$$\text{rank}\Big[B, AB, \dots, A^{N-1}B\Big] = N \tag{8}$$

involving only powers of A up to order $N-1$. Note however that, if some of the integrals in (7) vanish, then the condition differs from the classical one (8).
- *As we shall see, the property of averaged controllability of the system is equivalent to the averaged observability of the adjoint one, and both properties are equivalent to the averaged rank condition above.*

2.2 Averaged observability and proof of the main result

Our analysis is based on the classical duality principle allowing to reduce the controllability problem of a given system into an observability one for the adjoint system and to get, among all the admissible controls, the one of minimal $L^2(0, T; \mathbf{R}^M)$-norm ([25], Proposition 1.3 and [34]).

Of course, in the present situation, the adjoint system depends also on the parameter ν:

$$\begin{cases} -\varphi'(t) = A^*(\nu)\varphi(t),\ t \in (0, T) \\ \varphi(T) = \varphi^0. \end{cases} \tag{9}$$

Note that, for all values of the parameter ν, we take the same datum for φ at $t = T$. This is so because our analysis is limited to the problem of averaged controllability.

The solution of the adjoint system $\varphi = \varphi(t, \nu)$ also depends on the parameter ν, even if its datum at time $t = T$ is independent of ν, because the matrix $A^*(\nu)$ generating the dynamics does.

The *averaged observability* inequality for the adjoint system reads as follows:

$$|\varphi^0|^2 \leq C \int_0^T \left| \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu \right|^2 dt, \qquad (10)$$

for all $\varphi^0 \in \mathbf{R}^N$. In other words, the question is whether, given $T > 0$, there exists $C > 0$ such that the solutions of the adjoint system (9) fulfill the inequality (10) above for all $\varphi^0 \in \mathbf{R}^N$.

This question is also relevant on its own. Indeed, when observing a system in which the model governing the dynamics is unknown, it is natural to consider averages of observations. This problem is also relevant in the context of inverse problem theory ([3]), the issue being, in this frame, the determination of the initial datum of the system out of partial measurements. Again, when the system it is not fully known, it is natural to measure averages with respect to the parameters of dependence.

Theorem 3 *System (1) fulfills the averaged controllability property (2) if and only the adjoint system (9) satisfies the averaged observability inequality (10) and both are equivalent to the rank condition (6).*

When these properties hold, the averaged control of minimal $L^2(0,T;\mathbf{R}^M)$-norm is given by

$$u(t) = \int_0^1 B^*(\nu)\hat{\varphi}(t,\nu)d\nu, \qquad (11)$$

where $\hat{\varphi}$ is the solution of the adjoint system (9) corresponding to the datum φ^0 minimizing the functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \left| \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu \right|^2 dt$$
$$- < x^1, \varphi^0 > + < x^0, \int_0^1 \varphi(0,\nu)d\nu > \qquad (12)$$

in $\mathbf{R}^N$.

Proof of Theorem 5. Taking the scalar product (denoted by $< \cdot, \cdot >$ both in $\mathbf{R}^N$ and $\mathbf{R}^M$) of $\varphi = \varphi(t,\nu)$ with the equation satisfied by $x = x(t,\nu)$ and integrating with respect to $t \in (0,T)$ and $\nu \in (0,1)$ we get the following identity:

$$\int_0^T < u(t), \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu > dt$$
$$= \int_0^T \int_0^1 < u(t), B^*(\nu)\varphi(t,\nu) > d\nu dt$$
$$= \int_0^T \int_0^1 < B(\nu)u(t), \varphi(t,\nu) > d\nu dt$$
$$= \int_0^1 < x(T,\nu), \varphi^0 > d\nu - \int_0^1 < x^0, \varphi(0,\nu) > d\nu$$
$$=< \int_0^1 x(T,\nu)d\nu, \varphi^0 > - < x^0, \int_0^1 \varphi(0,\nu)d\nu > .$$

Here we have used in an essential manner that, in view of the equation satisfied by the state $x(t,\nu)$ and the adjoint one $\varphi(t,\nu)$,

$$\int_0^T \int_0^1 < B(\nu)u(t), \varphi(t,\nu) > d\nu dt$$
$$= \int_0^T \int_0^1 < [x' - A(\nu)x], \varphi > d\nu dt$$
$$= \int_0^1 \int_0^T < [x' - A(\nu)x], \varphi > dt d\nu$$
$$= \int_0^1 < x(T,\nu), \varphi^0 > d\nu - \int_0^1 < x^0, \varphi(0,\nu) > d\nu$$
$$+ \int_0^1 \int_0^T < x, [-\varphi' + A(\nu)^*]\varphi > dt d\nu$$
$$= \int_0^1 < x(T,\nu), \varphi^0 > d\nu - \int_0^1 < x^0, \varphi(0,\nu) > d\nu.$$

In other words, we have the duality identity

$$< \int_0^1 x(T,\nu)d\nu, \varphi^0 >=$$
$$= \int_0^T < u(t), \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu > dt$$
$$+ < x^0, \int_0^1 \varphi(0,\nu)d\nu > .$$

Accordingly, the controllability condition (2) can be recast as follows:

$$< x^1, \varphi^0 > = \int_0^T < u(t), \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu > dt$$
$$+ < x^0, \int_0^1 \varphi(0,\nu)d\nu >, \forall \varphi^0 \in \mathbf{R}^N. \qquad (13)$$

Following the classical theory of controllability (see [20], [34], [32]) this identity may be seen as the Euler-Lagrange equation associated to the minimization of a suitable quadratic functional over the class of solutions of the adjoint system. In the present case the functional reads:

$$J(\varphi^0) = \frac{1}{2}\int_0^T \left| \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu \right|^2 dt$$
$$- < x^1, \varphi^0 > + < x^0, \int_0^1 \varphi(0,\nu)d\nu > . \qquad (14)$$

Here and in the sequel by $| \cdot |$ we denote the Euclidean norm in $\mathbf{R}^N$ or $\mathbf{R}^M$.

The functional $J : \mathbf{R}^N \to \mathbf{R}$ is trivially continuous and convex.

Let us assume for the moment that the functional J has a minimizer $\hat{\varphi}^0$ and let $\hat{\varphi}$ be the corresponding solution of the adjoint system. Computing the first variation of J it can be seen that the control

$$u(t) = \int_0^1 B^*(\nu)\hat{\varphi}(t,\nu)d\nu, \qquad (15)$$

$\hat{\varphi}$ being the solution of the parametrized adjoint system associated to the minimizer $\hat{\varphi}^0$, ensures that the final condition (2) is satisfied, since, precisely, the identity (13) is fulfilled.

Thus, the problem is reduced to prove the existence of the minimizer of J and for this it is sufficient to prove the coercivity of the functional J or, in other words, the existence of a positive constant $C > 0$ such that the observability inequality (10) holds.

Since we are working in the finite-dimensional context, inequality (10) is equivalent to the following uniqueness property:

$$\int_0^1 B^*(\nu)\varphi(t,\nu)d\nu = 0 \quad \forall t \in [0,T] \Rightarrow \varphi^0 \equiv 0. \quad (16)$$

To analyze this uniqueness problem, using that

$$\varphi(t,\nu) = \exp[A^*(\nu)(T-t)]\varphi^0,$$

we observe that

$$\int_0^1 B^*(\nu)\varphi(t,\nu)d\nu = 0 \quad \forall t \in [0,T]$$

is equivalent to

$$\int_0^1 B^*(\nu)\exp[A^*(\nu)(t-T)]d\nu\,\varphi^0 = 0 \quad \forall t \in [0,T].$$

From the time analyticity of the matrix exponentials, and the classical argument consisting in taking consecutive derivatives at time $t = T$, this is equivalent to

$$\int_0^1 B^*(\nu)[A^*(\nu)]^k d\nu\,\varphi^0 = 0 \quad \forall k \geq 0. \quad (17)$$

Accordingly, (16) holds if and only if the averaged rank condition (6) is fulfilled.

The fact that the control obtained by minimizing the functional J is of minimal $L^2(0,T;\mathbf{R}^M)$-norm can be obtained in two different ways. One is by directly applying the Fenchel-Rockafellar duality principle. The other one is proceeding as in Proposition 1.3 of [25].

Remark 4 *Several remarks are in order:*

- *As we have seen, the adjoint state is represented by the average $\psi(t) = \int_0^1 \varphi(t,\nu)d\nu$ which does not fulfill the semigroup property. Thus the relevant adjoint state is the average of all the adjoint states, for all values of the parameter ν, whose dynamics cannot be generated by a given matrix.*
- *The control being characterized as the minimizer of the quadratic, convex and coercive functional J in $\mathbf{R}^N$, one can get effective numerical methods by implementing gradient like iterative algorithms (see [10]).*
- *Under the assumption (6) a stronger averaged controllability result can also be proved. Namely, one may consider initial data $x^0 = x^0(\nu)$ depending on ν in a measurable and bounded manner in system (1). The same averaged controllability problem makes sense in that case. The functional to be minimized has then to be slightly modified to cope with the ν-dependence of the initial data. Namely, the one to be considered is:*

$$J(\varphi^0) = \frac{1}{2}\int_0^T \left|\int_0^1 B^*(\nu)\varphi(t,\nu)d\nu\right|^2 dt$$
$$- < x^1, \varphi^0 > + \int_0^1 < x^0(\nu), \varphi(0,\nu) > d\nu. \quad (18)$$

  The condition (6) being fulfilled, the observability inequality (10) holds and, accordingly, this new functional is also coercive, ensuring the property of averaged controllability for these more general initial data too. This is the case since an estimate on $|\varphi^0|$ ensures uniform, with respect to the parameter ν, estimates on $\varphi(t,\nu)$ and this for all $0 \leq t \leq T$. Accordingly we can estimate $\int |\varphi(0,\nu)|d\nu$ too.

  In this way, we can consider initial data $x^0(\nu)$ to be controlled depending on ν. But we only control the average of solutions with respect to ν and not the full state $x(T,\nu)$ for each value of the parameter ν, as it occurs in the context of simultaneous controllability. Considering initial data that possibly depend on ν is relevant in those cases in which the initial data of the system is not completely known.

  The fact that the quantity of interest for control is the average of the state is once more reflected in the corresponding adjoint system so that its value, φ^0, at time $t = T$ is the same, independent of ν.
- *In the context of control of systems with parameter-independent matrices (A,B) the property of controllability is equivalent to the apparently weaker one of null-controllability in which the target is assumed to be $x^1 = 0$. This is so because of the linearity and backward solvability of the state equation. But this is not the case in the context of averaged control.*

  Indeed, when $x^1 = 0$ the functional to be minimized is reduced to

$$J_0(\varphi^0) = \frac{1}{2}\int_0^T \left|\int_0^1 B^*(\nu)\varphi(t,\nu)d\nu\right|^2 dt$$
$$- < x^0, \int_0^1 \varphi(0,\nu) > d\nu. \quad (19)$$

  The coercivity of the functional, the existence of its minimizer and, accordingly, the property of averaged null controllability, would then be guaranteed by the following weaker averaged observability inequality:

$$\left|\int_0^1 \varphi(0,\nu)d\nu\right|^2 \leq C\int_0^T \left|\int_0^1 B^*(\nu)\varphi(t,\nu)d\nu\right|^2 dt, \quad (20)$$

  for all $\varphi^0 \in \mathbf{R}^N$.

  But this one does not necessarily imply the stronger coercivity inequality (10).

  Indeed, an estimate on $\int_0^1 \varphi(0,\nu)d\nu$ does not necessarily yield an estimate on φ^0. This can be easily seen, for instance, when the parameter ν takes two *single values $\nu = \nu_1, \nu_2$, and the corresponding adjoint systems are harmonic oscillators so that the corresponding solutions are $\varphi_1(t) = \varphi^0 \cos(t-T)$ and $\varphi_2(t) = \varphi^0 \cos((\pi/T+1)(t-T))$. Then their average at $t=0$ vanishes independently of what the value of φ^0 is.*

Thus, in this averaged context, null controllability does not imply the full controllability of the system, in opposition to what occurs in the purely deterministic case.

Of course such situations arise in a much more general setting in which, by symmetry considerations, because of the very dependence of the system with respect to the parameter ν, the map $\varphi^0 \to \int_0^1 \varphi(0,\nu)d\nu$ is not reversible. In those cases it makes sense to analyze the controllability of higher moments of the solutions with respect to the parameter ν. This issue will be discussed in Section 5.

- *The control we have built in (15) is an average of functions of the form $B^*\hat{\varphi}(t,\nu)$. For each value of the parameter ν this constitutes a control but not necessarily the one steering the initial datum x^0 into the final one x^1. In other words, when solving the system (1) for a given value of ν with $B^*\hat{\varphi}(t,\nu)$ as control we would get a final value that does not coincide with x^1. The control has been built so that the average with respect to ν of the reached states coincides with x^1.*
- *The weaker condition of null controllability where the only target under consideration is the null one, $x^1 = 0$, is equivalent to the weaker observability inequality (20). The later is equivalent to the following uniqueness problem: Does (17) imply $\int_0^1 \varphi(0,\nu)d\nu = 0$? Note that, since $\int_0^1 \varphi(0,\nu)d\nu = \int_0^1 \exp[-TA^*(\nu)]\varphi^0\, d\nu$, the problem is equivalent to analyzing whether the condition (17) is compatible with φ^0 being non-trivial and in the kernel of the operator $\int_0^1 \exp[-TA^*(\nu)]d\nu$. Whether this can be expressed in simpler algebraic terms is an interesting open problem.*

2.3 *Variational characterization of the controls of minimal norm*

The proof and construction of the previous section leads to the control of minimal $L^2(0,T;\mathbf{R}^M)$-norm within the class of admissible ones. They are smooth, in view of their structure (15), because of the uniform smoothness of the solutions $\hat{\varphi}(t,\nu)$ with respect to the parameter ν. We could also consider controls of minimal $L^\infty(0,T;\mathbf{R}^M)$ or $L^1(0,T;\mathbf{R}^M)$-norm and this would lead to controls of bang-bang form or of sparse nature (see [5], [34]).

In particular, the following holds:

Theorem 5 *Under the averaged rank condition (6) the averaged control of system (1) of minimal $L^2(0,T;\mathbf{R}^M)$-norm satisfying (2) is characterized as (15) where $\hat{\varphi}$ is the solution of the parametrized adjoint system (9) associated to the minimizer $\hat{\varphi}^0$ of the functional J in (14).*

Under the same conditions the control of minimal $L^\infty(0,T;\mathbf{R}^M)$-norm can be built by minimizing the functional

$$J_\infty(\varphi^0) = \frac{1}{2}\left|\int_0^T \left|\int_0^1 B^*(\nu)\varphi(t,\nu)d\nu\right| dt\right|^2 - < x^0, \int_0^1 \varphi(0,\nu)d\nu > \quad (21)$$

and takes the following bang-bang form

$$u(t) = \int_0^T \left|\int_0^1 B^*(\nu)\hat{\varphi}(t,\nu)d\nu\right| dt \,\mathrm{sgn}\left[\int_0^1 B^*(\nu)\hat{\varphi}(t,\nu)d\nu\right].$$

3 Comparison with simultaneous control

Note that the notion of averaged control we have considered in this paper is related but weaker to the one of simultaneous control of a parameter-dependent family of Ordinary Differential Equations (ODE). In the latter all the components of the system are aimed to be controlled with the same control, and not only their average. But simultaneous controllability (which is also related to the notion of ensemble controllability (see [19])) can only occur under rather restrictive assumptions on the dependence of the system with respect to the unknown parameters. By the contrary, the property of averaged controllability holds under milder and rather natural conditions.

To better emphasize this difference we consider the simplest case in which the parameter ν is discrete and takes two values, so that the system under consideration reduces to

$$\begin{cases} x_j'(t) = A_j x_j(t) + Bu(t),\ 0 < t < T, \\ x_j(0) = x_j^0, \end{cases} \quad (22)$$

with $j = 1, 2$.

To make things even simpler we consider the particular case in which the two control operators coincide: $B = B_1 = B_2$.

Note that, here, contrarily to the problem of averaged controllability discussed above, the initial data of the system also depend on j.

To fix ideas, let us assume that the target x^1 is the null one: $x^1 = 0$.

In this setting, the problem of simultaneous control is formulated as that in which one looks for a control $u =$

$u(t)$ such that the corresponding solution of (22) satisfies

$$x_1(T) = x_2(T) = 0.$$

This problem is referred to be of *simultaneous control* since the same control u is assumed to control both components of the system $x_j, j = 1, 2$.

Obviously, simultaneous control is equivalent to the superposition of the property of averaged control and that of the control of the difference of both states:

$$x_1(T) - x_2(T) = 0.$$

Accordingly the two notions can be linked by relaxing the condition on the control of the difference of both states to

$$||x_1(T) - x_2(T)|| \leq \varepsilon, \tag{23}$$

with $\varepsilon > 0$. This leads to simultaneous control when letting $\varepsilon \to 0$ whenever the corresponding controls are uniformly bounded. Of course, this limit process can only be achieved when the system under consideration fulfills the simultaneous controllability condition. By the contrary, when letting $\varepsilon \to \infty$, we recover the averaged control property.

The analysis of the simultaneous control problem requires considering the adjoint system, but with different possible data at $t = T$ for its different components $j = 1, 2$:

$$\begin{cases} -\varphi_j'(t) = A_j^*\varphi_j(t), \ t \in (0,\, T) \\ \varphi_j(T) = \varphi_j^0. \end{cases} \tag{24}$$

In other words, simultaneous control requires to consider the whole class of solutions of the adjoint system.

The corresponding observability problem then reads

$$|\varphi_1^0|^2 + |\varphi_2^0|^2 \leq C \int_0^T |B^*[\varphi_1 + \varphi_2]|^2 \, dt, \tag{25}$$

for all $\varphi_1^0, \ \varphi_2^0 \in \mathbf{R}^N$. In other words, simultaneous controllability holds if and only if the observability inequality (25) is satisfied. Observe that, in this more demanding problem of simultaneous controllability, even if we observe the average of the solutions of the adjoint state through the operator B^*, we aim at recovering the norm of the data of both components of the adjoint state at the final time $t = T$. The condition of averaged controllability is weaker since it concerns only the subclass of solutions of the adjoint system in which the data at the final time $t = T$ are independent of the uncertainty parameter ν, which, in this particular example, would correspond to the system (24) with $\varphi_1^0 = \varphi_2^0$.

Note that, proceeding as in [32], it can be shown that the simultaneous observability inequality (25) (and therefore controllability) holds if and only if both the pairs (A_1, B) and (A_2, B) satisfy the rank condition under the added condition that the spectra of the two matrices A_1 and A_2 do not intersect and, in case they do at, say, a given λ, the set $[Ker(A_1 - \lambda I) \bigoplus Ker(A_2 - \lambda I)] \cap Ker(B)$ is reduced to the null state.

But, in particular, obviously, the property of simultaneous observability requires the observability of each of the systems and, therefore, the rank condition to be fulfilled by each of the systems (A_j, B), $j = 1, 2$ and this differs from the averaged rank condition in (6).

4 Partial differential equations

The problem of averaged controllability we have considered for finite dimensional systems can also be formulated in the context of PDEs. Again this leads to issues that are linked, but different, to those that are addressed in the literature devoted to simultaneous controllability of PDEs.

Problems of averaged controllability for PDEs make sense in various contexts that vary one from each other depending on the PDE model under consideration, the control objective and the kind of control action under consideration. Most of them would be extremely challenging.

4.1 The heat equation: Some open problems

Here, to motivate the issue we formulate some of them in the context of the heat equation. Similar problems could be considered also for wave equations, for instance, and this will be done in the following section.

Let Ω be a bounded domain in $\mathbf{R}^d$, $d \geq 1$, with smooth boundary and ω be an open non-empty subset of Ω. Consider the controlled heat equation:

$$\begin{cases} y_t - \operatorname{div}(a(x,\nu)\nabla y) = u(x,t)1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y^0(x) & \text{in} \quad \Omega, \end{cases} \tag{26}$$

where $Q = \Omega \times (0, T)$ stands for the space-time cylinder where the equation holds, and $\Sigma = \partial\Omega \times (0, T)$ for the lateral boundary.

The diffusivity coefficients $a(x, \nu)$, taken to be scalar to simplify the presentation, are assumed to be measurable in x, bounded above and below by positive constants, and to depend on the uncertainty parameter $\nu \in (0, 1)$ in a continuous manner.

We assume that $y^0 \in L^2(\Omega)$ and $u \in L^2(\Omega \times (0, T))$ so that (26) admits a unique solution

$$y = y(x, t; \nu) \in C\left([0, T]\,; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right),$$

for all $\nu \in (0, 1)$.

The problem of *averaged null controllability* can be formulated as follows: To find a control u so that the solution of (26) satisfies

$$\int_0^1 y(T,\nu)d\nu \equiv 0. \tag{27}$$

The same issue could be considered so to try to drive the average of the states towards a non-trivial state in the finite time $t = T$. But this is more delicate in the context of heat equations because of their intrinsic regularizing effect, as it occurs for one single equation.

Following the ideas of the finite-dimensional case, the problem can be shown to be equivalent to an averaged observability inequality for the adjoint system:

$$\begin{cases} \varphi_t + \mathrm{div}(a(x,\nu)\nabla\varphi) = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{28}$$

The control we are looking can be shown to be of the form

$$u(x,t) = \int_0^1 \hat{\varphi}(x,t,\nu)d\nu$$

where $\hat{\varphi}$ is a distinguished solution of the adjoint system determined by the datum $\hat{\varphi}^0$ minimizing the functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \left|\int_0^1 \varphi(x,t,\nu)d\nu\right|^2 dxdt + \int_\Omega y^0(x)\int_0^1 \varphi(x,0,\nu)d\nu dx. \tag{29}$$

We observe that, to prove the coercivity of the functional J, the following averaged observability inequality is needed:

$$\left|\left|\int_0^1 \varphi(x,0,\nu)d\nu\right|\right|^2_{L^2(\Omega)} \leq \leq C\int_0^T \int_\omega \left|\int_0^1 \varphi(x,t,\nu)d\nu\right|^2 dxdt.$$

Once again this inequality is weaker than the one is normally required for simultaneous controllability. As explained in Section 3, in the context of simultaneous controllability, the data of each component of the adjoint system (represented by the various values of the parameter ν in the present setting) are different. Note however that the existing literature on simultaneous controllability only addresses systems constituted by a finite number of heat equations while here the system may depend on a continuous parameter (or even several ones) (see [2]).

The main existing tools to prove observability inequalities for heat-like equations are the so-called Carleman inequalities (see Fursikov and Imanuvilov [13]). But adapting these tools to deal with these averaged observability inequalities seems to be a challenging issue. Even the case where the control set ω is the whole domain Ω does not seem to be an easy one. For a single equation, the observability inequality holds trivially in that case. But the issue is much more delicate in the context of averaged controllability.

The approach by Lebeau and Robbiano [17], based on Carleman estimates for the eigenfunctions of the underlying elliptic operator, could be adapted to the simplest case where

$$a(x,\nu) = \sigma(\nu)a(x),$$

since, then, all the operators involved share the same basis of eigenfunctions and then the average with respect to the parameter ν would be governed by a parabolic-like dynamics that could be described in Fourier series under the additional condition that

$$\int_0^1 \sigma(\nu)d\nu > 0. \tag{30}$$

But even the complete understanding of this very particular case requires of significant further developments.

When dealing with heat equations and, in general with PDEs, we are working with infinite-dimensional dynamical systems. Thus, the property of approximate controllability makes sense as well (see [7]). It refers to the density of the range of solutions one can achieve at the final time $t = T$ by making the control vary. In other words, approximate controllability consists in relaxing the final condition so to require the state to get arbitrarily close to the final target but not necessarily exactly to it.

The problem of average approximate controllability can then be formulated as follows: Given an initial condition $y^0 \in L^2(\Omega)$, a final target $y^1 \in L^2(\Omega)$ and $\varepsilon > 0$, to find a control function $u \in L^2(\omega \times (0,T))$ such that the solution of (28) satisfies

$$\left|\left|\int_0^1 y(T,\nu)d\nu - y^1\right|\right|_{L^2(\Omega)} \leq \varepsilon. \tag{31}$$

In particular, when the target is the trivial one, $y^1 = 0$, the problem consists in showing that, for all $\varepsilon > 0$, there exists a control u_ε such that the corresponding solution satisfies

$$\left|\left|\int_0^1 y(T,\nu)d\nu\right|\right|_{L^2(\Omega)} \leq \varepsilon. \tag{32}$$

Obviously, this is a relaxed version of the averaged null controllability property above.

When considering a single equation, independent of the unknown parameter ν, approximate controllability is motivated by the fact that the range of the semigroup is dense. This density property is natural in the context of parabolic problems because the very smoothing effect. Indeed, as a consequence of the regularizing properties of the semigroup, its range at the final time $t = T$ can not cover the whole space $L^2(\Omega)$. But this is compatible with it being dense. This property was analyzed in [11] even for semi-linear heat equations with globally Lipschitz nonlinearities showing that it is actually equivalent to the property of backward unique continuation for parabolic problems.

The density of the averages of the solutions generated by the different semigroups at the final time is much less clear in the context of parametric parabolic equations we are considering here. This problem is equivalent to that of the averaged version of the backward unique continuation property, which is also worth being investigated. In other words: *Under which conditions is it true that* $\int \varphi(x,0,\nu)d\nu = 0$ *implies that* $\varphi^0 = 0$*?*

In the context of approximate controllability, the functional to be minimized is:

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \left| \int_0^1 \varphi(x,t,\nu)d\nu \right|^2 dxdt + \varepsilon ||\varphi^0||_{L^2(\Omega)} + \int_\Omega y^0(x) \int_0^1 \varphi(x,0,\nu)d\nu dx. \quad (33)$$

Its coercivity is then equivalent to the following averaged unique continuation property:

$$\int_0^1 \varphi(x,t,\nu)d\nu \equiv 0 \text{ in } \omega \times (0,T) \implies \varphi^0 \equiv 0. \quad (34)$$

Using the time-analyticity of solutions this problem is easier to handle, and leads to interesting problems of unique continuation for averages of eigenfunctions. In particular the following issue would emerge: Assume that $\mu > 0$ is a common eigenvalue of the operators $-\text{div}(a(x,\nu)\nabla\cdot)$ for the values of the parameters ν in a given set I_μ so that the corresponding eigenfunctions are $w(x,\nu)$. Is it true that $\int_{I_\mu} w(x,\nu)d\nu = 0$ in ω implies that $\int_{I_\mu} w(x,\nu)d\nu = 0$ everywhere in Ω? This issue does not seem easy to handle based on the existing tools relying in Carleman inequalities.

Note however that this proof of approximate controllability, based purely on unique continuation, does not yield any estimate on how the control depends on the parameter ε. As shown in [12], actually, in parabolic problems, often, the approximate control depends in a very unstable manner on ε, blowing up exponentially as $\varepsilon \to 0$ if the final target is not exactly reachable.

The same questions arise in the context of boundary controllability too. In that case the control system reads as follows:

$$\begin{cases} y_t - \text{div}(a(x,\nu)\nabla y) = 0 & \text{in} \quad Q \\ y = u & \text{on} \quad \Sigma \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \quad (35)$$

Here the control $u = u(x,t)$, which is independent of ν as well, acts on the boundary of the domain where the equation holds. Similar questions can be formulated when the control only acts on a subset of the boundary.

The corresponding averaged observability inequality would now read:

$$\left|\left| \int_0^1 \varphi(x,0,\nu)d\nu \right|\right|^2_{L^2(\Omega)} \leq C \int_0^T \int_{\partial\Omega} \left| \int_0^1 \frac{\partial \varphi(x,t,\nu)}{\partial n} d\nu \right|^2 d\sigma(x)dt,$$

where $\partial \cdot /\partial n$ stands for the normal derivative on the boundary and $d\sigma(x)$ for the boundary measure.

This issue will very likely be more complex than the previous one on internal control and is completely open.

In the $1-d$ case these problems could be handled using Fourier expansions. We refer to [21] and [23] for related problems on the control and homogenization of the heat equation and the simultaneous control of a system of a finite number of heat equations, respectively. But the problem of averaged control is still to be analyzed.

Of course, in this setting of boundary control, the problem of approximate controllability also makes sense and can be reduced to an unique continuation one.

4.2 Wave equations: Some open problems

The same problems can be formulated in the context of wave equations.

Simple $1-d$ could be treated with the techniques developed in [8] based on non-harmonic Fourier series. This would allow considering systems on a finite number of equations, i. e. with the case where the parameter ν ranges over a finite-dimensional set.

The multi-dimensional problem is much more complex. As described in [16] the existing tools based on the propagation of singularities and microlocal defect measures can be used to obtain averaged controllability results when the number of equations involved is finite.

But the issue would be much more complex, and it is essentially open, when the parameter ν ranges over an infinite set.

As observed in [16], once the problem is solved for the wave equation, transmutation techniques allow to handle the case of parabolic too, provided the coefficients involved in the systems under consideration are time-independent.

4.3 Optimal averaged control of elliptic equations

In this section, rather than considering the averaged controllability problems above, we focus on the steady-state version and consider the averaged optimal in the context a quadratic minimization problem for an elliptic equation depending upon a parameter. There is an extensive literature on optimal control for PDEs (see, for instance, [31]) but the problems of averaged control we address here are new. As we shall see, the key, once more, is the identification of the corresponding adjoint state as the average of the parameter dependent adjoint states.

Let Ω be a bounded domain in $\mathbf{R}^d$, $d \geq 1$, with smooth boundary and ω be an open non-empty subset of Ω. Consider the controlled elliptic equation:

$$\begin{cases} -\text{div}(a(x,\nu)\nabla y) = u(x)1_\omega & \text{in} \quad \Omega \\ y = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{36}$$

The diffusivity coefficients $a(x,\nu)$, are taken to be scalar to simplify the presentation, and are assumed to be measurable in x, bounded above and below by positive constants, and to depend on the uncertainty parameter $\nu \in (0,1)$ in a measurable manner. Of course, under these conditions, given $u \in L^2(\Omega)$, for each value of ν there is a unique solution $y = y(x,\nu) \in H_0^1(\Omega)$.

We are interested in the control of the averaged state

$$z(x) = \int_0^1 y(x,\nu)d\nu \in H_0^1(\Omega).$$

Given a target $z_d \in L^2(\Omega)$, consider the quadratic optimal control problem consisting on minimizing the functional

$$J(u) = \frac{1}{2}\left[||z - z_d||^2_{L^2(\Omega)} + ||u||^2_{L^2(\omega)}\right]. \tag{37}$$

It is easy to see, by the Direct Method of the Calculus of Variations (DMCV) (see [4]) that the minimizer of J in $L^2(\omega)$ exists and it is unique. This is due to the fact that J is continuous, convex and coercive in a Hilbert space. Uniqueness is due to strict convexity.

Let us denote by $u^* \in L^2(\omega)$ the minimizer of J . We now focus on the identification of u^* through an optimality system. The following holds:

Theorem 6 *The unique optimal control u^* for the averaged optimal control problem consisting in the minimization of the functional J in (37) is given by*

$$u^* = -\psi^* \ \ in \ \omega \tag{38}$$

where

$$\psi^*(x) = \int_0^1 \varphi^*(x,\nu)d\nu \tag{39}$$

and

$$\begin{cases} -\text{div}(a(x,\nu)\nabla\varphi^*) = z^* - z_d & in \quad \Omega \\ \varphi^* = 0 & on \quad \partial\Omega, \end{cases} \tag{40}$$

(φ^*, y^*) being the unique solution of the optimality system

$$\begin{cases} -\text{div}(a(x,\nu)\nabla y^*) = -\int_0^1 \varphi^*(x,\nu)d\nu 1_\omega \\ -\text{div}(a(x,\nu)\nabla\varphi^*) = \int_0^1 y^*(x,\nu)d\nu - z_d \\ y^*|_{\partial\Omega} = \varphi^*|_{\partial\Omega} = 0. \end{cases} \tag{41}$$

Sketch of the proof. The Euler-Lagrange equations characterizing the property that u^* minimizes J, using the Gateaux derivative of J at u^* in the direction v, leads to

$$< DJ(u^*), \delta u >$$
$$= \int_\Omega (z^* - z_d)\delta z dx + \int_\omega u^* \delta u dx = 0, \tag{42}$$

where δz is the derivative of z with respect to u, in the direction of δu. It is characterized by the average

$$\delta z(x) = \int_0^1 \delta y(x,\nu)d\nu \tag{43}$$

and $\delta y(x,\nu)$ is the derivative of the state $y(x,\nu)$ with respect to u, which is the solution of the system

$$\begin{cases} -\text{div}(a(x,\nu)\nabla(\delta y)) = \delta u(x)1_\omega & \text{in} \quad \Omega \\ \delta y = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{44}$$

Here and in the sequel we denote by the super index * the various quantities associated to the minimizer u^*.

Let us now introduce the ν-dependent adjoint state $\varphi^*(x,\nu)$ solution of

$$\begin{cases} -(a(x,\nu)\nabla\varphi^*) = z^* - z_d & \text{in} \quad \Omega \\ \varphi^* = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{45}$$

and the corresponding average

$$\psi^*(x) = \int_0^1 \varphi^*(x,\nu)d\nu. \tag{46}$$

The key computation is the following

$$\begin{aligned} \int_\Omega (z^* - z_d)\delta z dx &= \int_\Omega (z^* - z_d)\int_0^1 \delta y d\nu dx \\ &= \int_0^1 \int_\Omega (z^* - z_d)\delta y dx d\nu \\ &= \int_0^1 \int_\Omega -\text{div}(a(x,\nu)\nabla\varphi^*)\delta y dx d\nu \\ &= \int_0^1 \int_\Omega -\text{div}(a(x,\nu)\nabla(\delta y))\varphi^* dx d\nu \\ &= \int_0^1 \int_\omega \delta u \varphi^* dx d\nu = \int_\omega \delta u \psi^* dx. \end{aligned} \tag{47}$$

Equation (42) then reads

$$\int_\omega \delta u \psi^* dx + \int_\omega u^* \delta u dx = 0, \tag{48}$$

which shows that the optimal control u^* is as in (38).

The optimal control is thus given by (38), where ψ^* is given by (46) where (y^*, φ^*) are the optimal state and adjoint solutions of the optimality system 41. This completes the proof.

We can also consider the case where there is some uncertainty in the control operator as well. The system then reads

$$\begin{cases} -\mathrm{div}(a(x,\nu)\nabla y) = b(x,\nu)u(x) & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{49}$$

As above, given a target $z_d \in L^2(\Omega)$, we consider the quadratic optimal control problem consisting on minimizing the functional

$$J(u) = \frac{1}{2}\left[||z - z_d||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}\right]. \tag{50}$$

In this case the optimality system reads

$$\begin{cases} -\mathrm{div}(a(x,\nu)\nabla y^*) & \\ \quad = -b(x,\nu)\int_0^1 b(x,\nu)\varphi^*(x,\nu)d\nu & \text{in } \Omega \\ -\mathrm{div}(a(x,\nu)\nabla \varphi^*) = z^* - z_d & \text{in } \Omega \\ y^* = \varphi^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{51}$$

5 Perspectives

In this paper we have introduced the notion of averaged control and controllability and underlined how it leads to new problems of observability for the corresponding adjoint systems. We have also shown that these problems are related but different to the problem simultaneous control and observation.

This topic raises plenty of interesting and important problems, most of them, possibly, rather complex and requiring further important developments.

We mention here briefly some of them:

- **Weighted averages.** The same methods can be applied to control other averages of the states, with respect to different weights. Let us consider a different weight function $\rho = \rho(\nu)$. And let us consider the following problem of weighted averaged control:

$$\int_0^1 \rho(\nu)x(T,\nu)d\nu = x^1. \tag{52}$$

Then, the adjoint system is the same, but the observability inequality needed to ensure this weighted property is:

$$|\varphi^0|^2 \le C\int_0^T \left|B^*(\nu)\int_0^1 [\rho(\nu)\varphi(t,\nu)]d\nu\right|^2 dt.$$

Again this can be reduced to an averaged rank condition. The problem can be dealt with even more directly with the change of variables

$$y = y(t,\nu) = \rho(\nu)x(t,\nu)$$

since, then, y solves

$$\begin{cases} y'(t) = A(\nu)y(t) + \rho(\nu)B(\nu)u(t),\ 0 < t < T, \\ y(0) = \rho(\nu)x^0. \end{cases}$$

The results in Section 2 then apply directly.

- **Higher order moments.** In this paper we have addressed the problem of averaged controllability which consists simply on controlling the average of the ν-dependent states $x = x(t,\nu)$. But we could consider stronger controllability conditions in which, in addition to controlling the average, we also require to control higher order moments. The problem then would be that of searching for a control $u = u(t)$ such that the solution $x = x(t,\nu)$ satisfies both

$$\int_0^1 x(T,\nu)d\nu = x^1 \tag{53}$$

and

$$\int_0^1 \rho(\nu)x(T,\nu)d\nu = y^1 \tag{54}$$

for an extra given weight function $\rho = \rho(\nu)$.

This issue can be viewed as that of simultaneous averaged control of a system of two equations depending on the parameter ν. Indeed, in addition to the state $x = x(t,\nu)$ let us also introduce a new state corresponding to the added moment:

$$y = y(t,\nu) = \rho(\nu)x(t,\nu).$$

Then, y solves

$$\begin{cases} y'(t) = A(\nu)y(t) + \rho(\nu)B(\nu)u(t),\ 0 < t < T, \\ y(0) = \rho(\nu)x^0. \end{cases}$$

The new state $z = (x,y)$ combining the two states x and y solves a system of the form

$$\begin{cases} z'(t) = \hat{A}(\nu)z(t) + \hat{B}(\nu)u(t),\ 0 < t < T, \\ z(0) = z^0, \end{cases} \tag{55}$$

where $z^0(\nu) = (x^0, \rho(\nu)x^0)$, $\hat{A}(\nu)$ is the diagonal matrix involving twice $A(\nu)$ in its diagonal terms and $\hat{B}(\nu)$ is the vector control operator $\hat{B}(\nu) = (B(\nu), \rho(\nu)B(\nu))$.

Note that, in this case, the initial datum z^0 depends on the parameter ν. But the results of section 2 apply in this case too as explained in Remark 2. Thus the property of averaged controllability of the state and its momentum can be characterized in terms of a rank condition involving the ν-dependent operators $\hat{A}(\nu)$ and $\hat{B}(\nu)$, namely:

$$\mathrm{rank}\Big[\int_0^1 [\hat{A}(\nu)]^j\hat{B}(\nu)d\nu : j \ge 0\Big] = 2N. \tag{56}$$

The control can be computed by minimizing a suitable functional associated to the corresponding adjoint system consisting on two copies of the same adjoint system:

$$\begin{cases} -\varphi'(t) = A^*(\nu)\varphi(t),\ t \in (0,T) \\ \varphi(T) = \varphi^0, \end{cases} \tag{57}$$

$$\begin{cases} -\psi'(t) = A^*(\nu)\psi(t),\ t \in (0,T) \\ \psi(T) = \psi^0. \end{cases} \tag{58}$$

The functional to be minimized then reads

$$J(\varphi^0,\psi^0) = \\ = \frac{1}{2}\int_0^T \left| \int_0^1 B^*(\nu)[\varphi(t,\nu) + \rho(\nu)\psi(t,\nu)]d\nu \right|^2 dt \\ - < x^1, \varphi^0 > - < y^1, \psi^0 > + \int_0^1 < x^0, \varphi(0,\nu)d\nu > \\ + \int_0^1 < x^0, \rho(\nu)\psi(0,\nu)d\nu > . \tag{59}$$

Let us assume that the functional achieves its minimum at $(\tilde{\varphi}^0, \tilde{\psi}^0)$. Writing the Euler-Lagrange equations associated to the minimization of J, both in the directional variations corresponding to φ^0 and ψ^0, with the control

$$u(t) = \int_0^1 B^*(\nu)[\tilde{\varphi}(t,\nu) + \rho(\nu)\tilde{\psi}(t,\nu)]d\nu,$$

we get

$$0 = \int_0^T B(\nu)u(t)\int_0^1 \varphi(t,\nu)d\nu dt - < x^1, \varphi^0 > \\ + < x^0, \int_0^1 \varphi(0,\nu)d\nu >$$

and

$$0 = \int_0^T B(\nu)u(t)\int_0^1 \rho(\nu)\psi(t,\nu)d\nu dt - < y^1, \psi^0 > \\ + \int_0^1 < x^0, \rho(\nu)\psi(0,\nu)d\nu >$$

respectively, which are equivalent to the control requirements (53) and (54).

In order to ensure the existence of the minimizer of J the following observability inequality is needed:

$$|\varphi^0|^2 + |\psi^0|^2 \\ \leq C\int_0^T \left| \int_0^1 B^*(\nu)[\varphi(t,\nu) + \rho(\nu)\psi(t,\nu)]d\nu \right|^2 dt.$$

Of course, this inequality is stronger than the one needed for averaged controllability to hold.

Similar results can be obtained when the averaged controllability condition is enriched by adding an arbitrary number of averaged constraints for different momenta of the state.

A systematic analysis of this issue would be of interest.

- **Control of variances.** Inspired in the theory of probability it is also natural to consider the problem in which, in addition to controlling the averages,

$$\int_0^1 x(T,\nu)d\nu = x^1, \tag{60}$$

one also addresses the problem of analyzing the effectiveness of the control for each realization of the parameter ν.

This can be done, for instance, under an assumption of the Lipschitz character of the dependence of $A(\nu)$ with respect to ν. Indeed, let us assume that

$$|A(\nu_1) - A(\nu_2)| \leq L|\nu_1 - \nu_2|. \tag{61}$$

In this case it is easy to see that $x(T,\nu)$ depends on ν in a Lipschitz manner with respect to the parameter ν, with a Lipschitz constant that depends on L in (61) and on the norm of x^0 and the control $u = u(t)$.

Taking into account that the average $\int x(T,\nu)d\nu$ coincides with x^1, and that this holds componentwise, this means that each component of $x(T,\nu) - x^1$ is a Lipschitz function of zero mean. All this leads to rough estimates of the variance $\int |x(T,\nu) - x^1|^2 d\nu$.

Note that here we have estimated the variance of the states obtained by the control u that was chosen so to control the average, being of minimal $L^2(0,T)$-norm. Of course, different choices of the control would also be possible. For instance, one could look for the optimal control minimizing this variance, and then estimate the obtained variance.

Of course, this problem is also related to the one above on the control of higher order moments since, by a suitable choice of the basis with respect to the ν variable, the variance, by Parseval's identity, can be written as a discrete ℓ^2-norm of all the moments on that basis.

A further analysis of this issue would be desirable.

- **Averaged optimal control of elliptic PDEs.** In the previous section we have considered the simplest possible problem of optimal control for a parameter dependent elliptic system. There is an extensive literature of optimal control for PDEs of various kinds (elliptic, parabolic, ...) and of various forms: Internal/boundary controls, constraints on states and controls, L^2 versus sparse controls, etc. One of the goals of the existing literature being the characterization of optimal controls by means of suitable optimality conditions, it would be of interest to extend the analysis to deal with the notion of averaged optimal control.

- **Nonlinear problems.** There is an extensive literature on the control of nonlinear ODE and PDE models ([6]). Of course all the issues presented here can also be considered for nonlinear problems.

But, as pointed to us by E. Trélat, the problem above can also be seen as a particular instance of a more general class of nonlinear control problems. This point of view emerges naturally when considering ν as part of the state, governed by the trivial dynamics:

$$\frac{d}{dt}(\nu) = 0.$$

The state is then the vector (x, ν) and we look for the control so that its average with respect to the initial data of ν is under control.

Under this point of view the control system is nonlinear in the state. More generally speaking the problem can be then recast in the context of non-linear control systems

$$x'(t) = f(x(t), u(t)),$$

$x = x(t)$ being the state and $u = u(t)$ the control so that, if the state is split into two subsets of components $x = (x_1, x_2)$, each of them taking initial values (x_1^0, x_2^0), then the condition of averaged controllability concerns the average of $x_1(T)$ with respect to x_2^0.

This leads to an interesting class of problems worth to be analyzed but much beyond the scope of this paper.

- **Numerical approximation.** There is also an extensive literature on the numerical approximation of control problems (see for instance [14], [27], [31], [33]). This question also arises in the context of averaged control.
- **Random dependence.** In all the problems we have considered here the matrices and operators were assumed to depend on an uncertain real valued parameter. The same analysis could be developed when the number of parameters under consideration is finite or even infinite. The case where the parameter is a random variable is also of interest. The corresponding notion of averaged controllability should then be formulated in terms of the expected value of the randomly dependent state.

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