Long-time behavior of solutions to a non-linear hyperbolic relaxation system

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Abstract

We study the large time behavior of the solutions of the hyperbolic relaxation system

$$\begin{cases} u_t + v_x = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ v_t + u_x = |u|^{q-1}u - v. \end{cases}$$

We write it as a non-linear damped wave equation:

$$u_{tt} + u_t - u_{xx} + (|u|^{q-1}u)_x = 0.$$

First, we prove the global existence of an unique solution with sufficiently small initial data in $[H^1 \cap L^1(\mathbb{R})] \times [L^2 \cap L^1(\mathbb{R})]$. We also show that for some large initial data, solutions blow-up in finite time.

When $q = 2$, we prove that the large time behavior of solutions is given by the fundamental solution of the viscous Burgers equation

$$u_t - u_{xx} + (|u|u)_x = 0,$$

with mass $M = \int_{\mathbb{R}}(u_0 + u_1)dx$, which is conserved along the trajectory. When $q > 2$ the convection term is too weak and the large time behavior is given by the heat kernel with the same mass.

The proofs we develop are based on the use of fine estimates of the associated linear problem in weighted spaces, a Fourier decomposition of the corresponding semigroup into the hyperbolic and parabolic component, scaling and compactness arguments. The proof of the blow-up result relies on the construction of special solutions in separated variables.

The same methods allow handling similar problems in 1-d and in several space dimensions for more general nonlinearities. We briefly indicate the corresponding results.

Contents

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1 Introduction

This paper is devoted to study the large time behavior of solutions of the following system

$$\begin{cases} u_t + \nabla \cdot \mathbf{v} = 0, \\ \mathbf{v}_t + \nabla u = \mathbf{a} f(u) - \mathbf{v}, \end{cases} \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \tag{1.1}$$

f being a nonlinear function from $\mathbb{R}$ to $\mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^N$.

System (1.1) is a typical example of hyperbolic system of conservation laws with relaxation and arises in many physical systems such as nonequilibrium gas dynamics, flood flow with friction, viscoelasticity, magnetohydrodynamics, etc. (see G.B. Whitham [19]).

The study of the behavior of solutions to hyperbolic relaxation systems has been the object of intensive work. In [14] the Cauchy problem was analyzed by T.P. Liu for the following nonhomogeneous strictly hyperbolic 2×2 system of conservation laws:

$$\begin{cases} u_t + f(u;v)_x = 0, \\ v_t + g(u;v)_x = h(u;v), \end{cases} \tag{1.2}$$

which is a generalization of the 1-d version of (1.1). Here h represents the effect of relaxation to the equilibrium state $(u; v_*(u))$ where $v_*(u)$ solves $h(u; v_*(u)) = 0$. For a finite perturbation of an equilibrium state $(u_0; v_*(u_0))$, solutions eventually relax to $(u_0; v_*(u_0))$. This process is approximated accurately by a viscous conservation law of the form

$$\begin{cases} w_t + f_*(w)_x = (b(w)w_x)_x, \\ w(x,0) = u_0, \end{cases} \tag{1.3}$$

where $f_*(w) = f(w; v_*(w))$ and

$$b(w) = \frac{\partial f_*}{\partial v}(w) \left(-v_*'(w) f_*'(w) + \frac{\partial g_*}{\partial u}(w) + \frac{\partial g_*}{\partial v}(w) v_*'(w) \right)$$

with $g_*(w) = g(w; v_*(w))$. Assuming further that the equilibrium state is (0,0), and using Chapman-Enskog expansion, it was shown that the relaxation is approximated by the diffusive wave

$$\theta_t + f_*'(0)\theta_x + \left(\frac{1}{2} f_*''(0)\theta^2 \right)_x = b(0)\theta_{xx}.$$

In I.-L. Chern [5], a rigorous proof to this approximation was given and a convergence rate in the L^p-norm for sufficiently small initial data was obtained. This smallness condition reads

$$\int u_0 dx + \left\| \int_{-\infty}^{x} u_0 dx \right\|_{H^3 \cap L^1} + \|v_0\|_{H^2 \cap L^1} \ll 1. \tag{1.4}$$

Recently, H. Liu and R. Natalini [12] studied the large time behavior of the solutions of (1.1) in the 1-d case when $f(u) = ku^2$ by means of scaling arguments. First, they obtained the needed uniform bounds on the scaled solutions by means of entropy-estimates, valid under the so-called sub-characteristic condition ($|f'(u)| < 1$) and the smallness assumptions on the initial data

$$|v_0| \leq u_0 \quad \text{and} \quad u_{0,x} \pm v_{0,x} \ll 1. \tag{1.5}$$

The main result of [12] shows that, under the assumptions above and for bounded initial data of finite mass, the first component of the system (1.1) decays towards the fundamental solution of the viscous Burgers equation in the L^p-norm, at a rate faster than $t^{(p-1)/2p}$ with $1 < p < \infty$. The main improvement in [12] with respect to the previous works was the fact of weakening the assumption (1.4) on the initial data in [5].

In this article, we consider the equation (1.1) as a nonlinear damped wave equation. Indeed, taking the time-derivative of the first equation in (1.1) and applying the divergence operator in the second one we obtain the damped wave equation

$$u_{tt} + u_t - \Delta u + \mathbf{a} \cdot \nabla f(u) = 0, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \tag{1.6}$$

with $\mathbf{a} \in \mathbb{R}^N$ and initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = -\nabla \cdot v_0(x) \qquad \text{in } \mathbb{R}^N,$$

(u_0, v_0) being the initial data of (1.1).

Equation (1.6) can be seen as a hyperbolic perturbation of the convective heat equation

$$u_t - \Delta u + \mathbf{a} \cdot \nabla f(u) = 0, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+. \tag{1.7}$$

In fact, considering the rescaling transformation

$$u_\lambda(x,t) = \lambda^N u(\lambda x, \lambda^2 t), \tag{1.8}$$

the linear equation associated with (1.6) with $f \equiv 0$ is as follows

$$\lambda^{-2} u_{\lambda,tt} + u_{\lambda,t} - \Delta u_\lambda = 0, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+.$$

Then it is obvious that, under the scaling (1.8), the term involving second order in time derivatives becomes negligible. According to this formal argument it is natural to expect the asymptotic behavior of the solutions of (1.6) to be the same as that of the solutions of the convection-diffusion equation (1.7).

On the other hand, equation (1.7) is a simple model combining both diffusive and convective effects and has been studied in [7], [8] and [20], among others, for non-linearities satisfying

$$f(u) = |u|^{q-1} u \qquad \text{and} \qquad q \geq 1 + \frac{1}{N}.$$

It is by now well known that, when $q = (N+1)/N$, the asymptotic behavior of the solutions of (1.7) with initial data L^1 is given by a uniparametric family of self-similar solutions (see [8]). When $q > (N+1)/N$, the convective term $\mathbf{a} \cdot \nabla f(u)$ is too weak and disappears as $t \to \infty$, the large time behavior being given in a first approximation by the linear heat equation. Finally, when the exponent is subcritical $q < (N+1)/N$, the solutions of (1.7) behave like the entropy solutions of the following convective and partially diffusive equation (see [7] and [9])

$$u_t - \Delta' u + \mathbf{a} \cdot \nabla f(u) = 0, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \tag{1.9}$$

where $-\Delta'$ denotes a $(N-1)$-dimensional Laplacian in the directions which are orthogonal to the convective one. In this paper we will not address the case $q < 1 + 1/N$ that can not be handled with the techniques we develop.

In our study of the solutions of (1.6), we first develop a Fourier splitting argument that allows decomposing the semigroup of the linear problem associated to (1.6) into a exponentially decaying one and a slowly decaying one. For the second one, which corresponds to the low frequency components, the decay properties are those of the semigroup generated by the heat equation.

After, we study the one dimensional case. Based on this Fourier splitting and fixed point arguments one can get global in time solutions for small initial data in

$$V \equiv H^1 \cap L^1(\mathbb{R}) \times L^2 \cap L^1(\mathbb{R}).$$

This is the unique smallness hypothesis on the initial data we impose. Thus, with respect to the previously existing results [5] and [12], we relax the sub-characteristic and the smallness assumptions (1.4) and (1.5).

We also show that when $q = 2$ large solutions may blow-up in finite time. This justifies the smallness condition on the initial data. This blow-up result is also extended for even nonlinearities of the form $f(u) = |u|^q$.

It is worth mentioning that there is an extensive literature on the global existence and blow up for equations of the form

$$u_{tt} + u_t - \Delta u = f(u) \qquad \text{in } [0,\infty) \times \mathbb{R}^N.$$

We refer to [18] and the references therein. However, as for as we know, our blow-up result is the first one for equations of the form (1.6) where the nonlinearity enters as a convective term.

Note that our blow-up result is in contrast with the behavior of solutions of convection-diffusion equations of the form (1.7). Indeed, in (1.7) all solutions with initial data in $L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ are global in time and decay in $L^p(\mathbb{R}^N)$ as $t^{-N(1-1/p)/2}$ when $t \to \infty$ and this for all nonlinear term f.

We also obtain sharp decay properties as $t \to \infty$ of the global solutions of (1.6). This yields uniform estimates on the scaled functions u_λ in (1.8). Then, scaling and compactness arguments yield the asymptotic form of solutions. In this step we need to assume that the initial data belong to the weighted space $L^2(1+|x|;\mathbb{R})$. This is a natural condition in the study of the asymptotic behavior of solutions of heat and convection-diffusion equations. Indeed, the first momentum of the initial datum being finite allows getting sharp decay properties to the asymptotic profile (see [20] and [6]).

With respect to the previously existing results, the main achievements of this paper consist in the following facts. First, we study the hyperbolic damped wave equation associated to the system (1.1) as a perturbation of a convection-diffusion equation (1.7). In particular, using a semigroup analysis we prove the global existence, uniqueness and decay estimates of the solutions of (1.6). The scaling argument allows identifying the asymptotic behavior and the rate of convergence. As a consequence we relax the assumptions on the initial data (see next section). These techniques allow us to obtain similar results in several space dimensions as well. This point will be discussed in more detail in Section 9. Moreover, we see that solutions for large data may blow-up in finite time, which is a new phenomenon with respect to the qualitative properties of convection-diffusion equations of the form (1.7).

The rest of the paper is organized as follows. In the following section present the main results of this work. In Section 3, we give some results on the asymptotic behavior in $L^p(\mathbb{R})$ of the solutions of the linear problem associated to (1.6). In particular, we develop the Fourier splitting argument. The proof of these estimates is given in Appendix A. Section 4 is devoted to prove the existence and uniqueness of the solutions of (1.6) in the one-dimensional case with the appropriate decay properties as $t \to \infty$. In Section 5, we obtain some compactness results on the family $\{u_\lambda\}$. In Section 6, the limit of the family $\{u_\lambda\}$ when $\lambda \to \infty$ is identified. In Section 7 we prove the asymptotic behavior of global solutions. Section 8 is devoted to show a blow-up result. Finally, in Section 9 we comment some possible extensions of the results of this paper to more general nonlinearities and to the multi-dimensional case.

2 Main Results

First, we study the one-dimensional case writing the system in the form (1.6).

To simplify the presentation we consider power-like nonlinearities $f(u) = |u|^{q-1}u$ and the constant a is taken to be $a = 1$. The equation under consideration reads:

$$\begin{cases} u_{tt} + u_t - u_{xx} + (|u|^{q-1}u)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x), & u_t(x,0) = u_1(x). \end{cases} \tag{2.1}$$

Recall that $(u_0, u_1) = (u_0, -v_{0,x})$, (u_0, v_0) being the initial data of (1.1).

Theorem 2.1 *Let the exponent q in (2.1) be such that $q \geq 2$ and $(u_0, u_1) \in V$ where $V = [H^1 \cap L^1(\mathbb{R})] \times [L^2 \cap L^1(\mathbb{R})]$. We assume that*

$$\|(u_0, u_1)\|_V \leq \delta, \tag{2.2}$$

with δ sufficiently small. Then, there exists an unique global solution u of (2.1) such that

$$u \in BC([0,\infty); H^1(\mathbb{R})) \cap BC^1([0,\infty); L^2(\mathbb{R})).$$

Moreover, it satisfies

$$\|u(t)\|_2 \leq c(1+t)^{-\frac{1}{4}}, \qquad t \geq 0, \tag{2.3}$$

$$\|u_x(t)\|_2 \leq c'(1+t)^{-\frac{3}{4}}, \qquad t \geq 0, \tag{2.4}$$

where the constants c, c' are proportional to the norm of the initial data in V.

The proof of this theorem is given in Section 4. It is important to note that the smallness condition on the initial data is necessary since solutions for large data blow-up in finite time.

Indeed, let us look for solutions of (2.1) with $q = 2$ such that $u(x,t) = xa(t)$. We observe that the initial data of u do not belong to V. However, truncating the support of the initial data and thanks to the finite propagation velocity, the blow-up of solutions of the form $u(x,t) = xa(t)$ leads to blow-up results for data in V too. Note that $u = xa(t)$ with $x < 0$ solves (2.1) if and only if a satisfies the differential equation:

$$a_{tt} + a_t - 2|a|a = 0, \qquad t > 0.$$

The following holds:

Lemma 2.2 *Let $(a_0, a_1) \in \mathbb{R}^2$ be the initial data of the solution of*

$$a_{tt} + a_t - \alpha |a|^{q-1} a = 0, \qquad t > 0. \tag{2.5}$$

with $q > 1$ and $\alpha > 0$. Then, if

$$|a_0| \geq \left(\frac{q+1}{8\alpha} \right)^{\frac{1}{q-1}} e^{\frac{2}{q-1}} \qquad \text{and} \qquad (\frac{1}{2}a_0 + a_1)^2 \geq \frac{1}{4}a_0^2 + \alpha \frac{2|a_0|^{q+1}}{q+1}, \tag{2.6}$$

the solution a of (2.5) blows up in finite time $t_b \leq |a_0|^{(1-q)/2} e\sqrt{2q+2}/\sqrt{\alpha}(q-1)$.

The proof of this result and the added arguments leading to the blow-up of system (2.1) will be given in Section 7. This blow-up result justifies the smallness condition in (2.2).

As a result of the previous lemma we also obtain blow up results for even nonlinearities of the form $f(u) = |u|^q$ with $q > 1$. We consider the system

$$\left\{ \begin{array}{ll} u_{tt} + u_t - u_{xx} + (|u|^q)_x = 0, & (x,t) \in \mathbb{R} \times (0, \infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \end{array} \right. \tag{2.7}$$

and get the following blow up result.

Theorem 2.3 *Let u be the solution of (2.7). We assume that the initial data u_0 and u_1 have support in $[K_1, K_2]$ $(K_1 > 0)$. Assume also that*

$$a_i \equiv \int x u_i(x) dx > 0, \qquad i = 0, 1.$$

If

$$a_0 \geq \max \left\{ \left(\frac{q+1}{8\rho} \right)^{\frac{1}{q-1}} e^{\frac{2}{q-1}}, (2q-1) \left(\frac{e^2(2q+2)}{K_1^2 \rho (q-1)^2} \right)^{\frac{1}{q-1}} \right\}$$

and

$$(\frac{1}{2}a_0 + a_1)^2 \geq \frac{1}{4}a_0^2 + \rho \frac{2a_0^{q+1}}{q+1}$$

with

$$\rho = (K_1 + K_2)^{1-2q} \left(\frac{q-1}{2q-1} \right)^{1-q},$$

then, the function

$$a(t) = \int x u(x,t) dt. \tag{2.8}$$

blows up in finite time $t_b < K_1$.

Let us now address the problem of the asymptotic behavior for $t \to \infty$ of global solutions. We note that the smooth solutions of (2.1) with compact support (or decaying as $|x| \to \infty$) satisfy the conservation law

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} (u_t + u) dx = \int_{\mathbb{R}} \left(u_{xx} - (|u|^{q-1}u)_x \right) dx = 0.$$

That is, the mass of $u_t + u$ is conserved along time:

$$m(u_t + u) = \int_{\mathbb{R}} (u_1 + u_0) dx = M. \tag{2.9}$$

Thus, M is certainly a relevant parameter to describe the large time behavior of solutions.

Concerning the long time behavior of solutions the following holds:

Theorem 2.4 *Assume that the hypotheses of Theorem 2.1 are satisfied and that $q \geq 2$. Assume further that $u_0, u_1 \in L^2_1(\mathbb{R})$ with $L^2_1(\mathbb{R}) = \{v \in L^2(\mathbb{R}) \mid (1+|\cdot|)u \in L^2(\mathbb{R})\}$. Let u be the unique solution of (2.1). Then, there exists a self-similar function $v = t^{-1/2} f(x/\sqrt{t})$ such that*

$$\lim_{t\to\infty} t^{\frac{p-1}{2p}} \|u(\cdot,t) - v(\cdot,t)\|_p = 0, \tag{2.10}$$

for $1 \leq p \leq \infty$. When $q = 2$, v is the source type solution of the Burgers equation with the Dirac delta as initial datum

$$\begin{cases} w_t - w_{xx} + (|w|w)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ w(x,0) = M\delta_0. \end{cases} \tag{2.11}$$

When $q > 2$, v is the fundamental solution of the heat equation

$$\begin{cases} w_t - w_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ w(x,0) = M\delta_0. \end{cases} \tag{2.12}$$

In both cases $M = \int_{\mathbb{R}} (u_0 + u_1) dx$.

In Theorem 2.4 we relax the assumptions on the initial data in [12]. The initial data (u_0, u_1) in (2.1) coincide with $(u_0, -v_{0,x})$, where (u_0, v_0) are the initial data in (1.1). Thus, in addition to the sub-characteristic condition, in [12], u_0, $u_{0,x}$ and u_1 are assumed to be sufficiently small in the L^∞-norm, and of finite mass. However, in this work we do not assume the sub-characteristic condition and we only need the smallness condition in the energy space V. By the contrary we assume the initial data to be in the weighted spaces $L^2_1(\mathbb{R})$.

3 Preliminaries on the linear problem

In this section we present some results on the asymptotic behavior of the solutions of the linear problem:

$$\begin{cases} u_{tt} - \Delta u + u_t = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = \varphi^0(x), \quad u_t(x,0) = \varphi^1(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

Note that this is the linear equation involved in the nonlinear equation (1.1) we analyze.

The well-posedness of (3.1) can be easily obtained writing it as an abstract evolution equation in the energy space $H = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, with the inner product

$$< (u,v), (\widetilde{u}, \widetilde{v}) >_H = \int_{\mathbb{R}^N} u\widetilde{u}\, dx + \int_{\mathbb{R}^N} \nabla u \cdot \nabla \widetilde{u}\, dx + \int_{\mathbb{R}^N} v\widetilde{v} dx.$$

Hille-Yosida-Phillip's theorem guarantees that (3.1) generates a semigroup that we denote by $\{S(t)\}_{t\geq 0}$. Thus, for any initial data $(\varphi^0, \varphi^1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, (3.1) has an unique weak solution $u = u(x,t)$ such that $u \in C^1(\mathbb{R}^+, H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}^+, L^2(\mathbb{R}^N))$.

The following estimates on this linear semigroup are well-known (see Lemma 1 in [15]):

Lemma 3.1 *([15]) Let u be the solution of (3.1), $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}^N_+$. Then, there exist $c, c' > 0$ such that for any initial data*

$$\varphi^0 \in H^{k+|\alpha|} \cap L^a(\mathbb{R}^N), \qquad \varphi^1 \in H^{k+|\alpha|-1} \cap L^a(\mathbb{R}^N), \quad (1 \leq a \leq 2),$$

it holds

$$\|\partial_t^k D_x^\alpha u(\cdot,t)\|_2 \leq c(1+t)^{-\frac{N}{2}\left(\frac{1}{a}-\frac{1}{2}\right)-\frac{|\alpha|+2k}{2}} \left(\|\varphi^0\|_{k+|\alpha|,2} + \|\varphi^0\|_a + \|\varphi^1\|_{k+|\alpha|-1,2} + \|\varphi^1\|_a\right), \quad \forall t \geq 0. \tag{3.2}$$

Here and in the sequel $\|\cdot\|_{m,p}$ denotes the norm in $W^{m,p}(\mathbb{R}^N)$.

Now, we are going to prove estimates on the asymptotic behavior of the solutions of the linear damped wave equation (3.1) using homogeneous Sobolev spaces:

$$\dot{H}^m(\mathbb{R}^N) = \left\{ f : |\cdot|^s \widehat{f} \in L^2(\mathbb{R}^N) \right\} \quad \text{and} \quad \|f\|_{\dot{H}^m(\mathbb{R}^N)} = \||\cdot|^s \widehat{f}\|_2.$$

Lemma 3.2 *Let $k = 0, 1$ and $\alpha \in \mathbb{R}^N_+$. Then, there exist positive constants $\omega, c, c' > 0$ such that*

$$\begin{aligned} \|\partial_t^k D_x^\alpha u(\cdot,t)\|_2 &\leq ce^{-\omega t}\left(\|\varphi^0\|_{\dot{H}^{k+|\alpha|}} + \|\varphi^1\|_{\dot{H}^{k+|\alpha|-1}}\right) \\ &\quad + c'\left(\|\varphi^0\|_a + \|\varphi^1\|_a\right)(1+t)^{-\frac{N}{2}\left(\frac{1}{a}-\frac{1}{2}\right)}(1+t)^{-\frac{|\alpha|+2k}{2}}, \quad \forall t \geq 0, \end{aligned} \tag{3.3}$$

when

$$\varphi^0 \in H^{k+|\alpha|} \cap L^a(\mathbb{R}^N), \qquad \varphi^1 \in H^{k+|\alpha|-1} \cap L^a(\mathbb{R}^N), \qquad 1 \leq a \leq 2.$$

If $k + |\alpha| < 1$, $\dot{H}^{k+|\alpha|-1}$ must be replaced by $H^{k+|\alpha|-1}$ in (3.3).

The proof of this lemma is given in Appendix A.

We also need the asymptotic behavior as $t \to \infty$ of solutions of the dissipative wave equation (3.1) in the weighted space $L^2_1(\mathbb{R})$.

To do that we perform the change of variables $v(x,t) = xu(x,t)$. Given u solution of (3.1), then v is the unique solution of the following Cauchy problem:

$$\left\{ \begin{array}{l} v_{tt} + v_t - v_{xx} + 2u_x = 0 \quad \text{in } \mathbb{R} \times (0,\infty), \\ v(x,0) = x\varphi^0(x), \quad v_t(x,0) = x\varphi^1(x), \quad \text{in } \mathbb{R}. \end{array} \right. \tag{3.4}$$

Using the variation of constants formula (VCF), v can be written as follows

$$v(t) = S(t)[x\varphi^0, x\varphi^1] - \int_0^t S(t-s)[0, 2u_x(s)]\, ds.$$

Applying the decay estimates of the semigroup $S(t)$ in L^2 (Lemmas 3.1 and 3.2) in this identity, we obtain:

Lemma 3.3 *Let u be solution of (3.1). Then, there exist $\omega, c, c_1, c_2, c_3 > 0$ such that for any initial data $(\varphi^0, \varphi^1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with*

$$\varphi^0 \in L^2_1 \cap L^b_1 \cap L^a(\mathbb{R}), \qquad \varphi^0_x \in L^2_1(\mathbb{R}), \qquad \varphi^1 \in L^2_1 \cap L^b_1 \cap L^a(\mathbb{R}), \qquad a, b \in [1,2],$$

it holds

$$\begin{aligned} \|u\|_{\dot{L}^2_1} &\leq ce^{-\omega t}\left(\|x\varphi^0\|_2 + \|x\varphi^1\|_{-1,2}\right) + c_1(1+t)^{-\frac{1}{2}}(\|\varphi^0\|_2 + \|\varphi^1\|_{-1,2}) \\ &\quad + c_2(1+t)^{\frac{1}{4}-\frac{1}{2b}}\left(\|x\varphi^0\|_b + \|x\varphi^1\|_b\right) + c_3(1+t)^{\frac{3}{4}-\frac{1}{2a}}(\|\varphi^0\|_a + \|\varphi^1\|_a), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \|u_x\|_{\dot{L}^2_1} &\leq ce^{-\omega t}\left(\|x\varphi^0\|_{\dot{H}^1} + \|x\varphi^1\|_2\right) + c_1(1+t)^{-\frac{1}{4}-\frac{1}{2b}}\left(\|x\varphi^0\|_b + \|x\varphi^1\|_b\right) \\ &\quad + c_2(1+t)^{-\frac{1}{2}}(\|\varphi^0\|_{\dot{H}^1} + \|\varphi^1\|_2) + c_3(1+t)^{\frac{1}{4}-\frac{1}{2a}}(\|\varphi^0\|_a + \|\varphi^1\|_a). \end{aligned} \tag{3.6}$$

The proof of this Lemma is given in Appendix A.

4 Global existence and decay estimates

This section is devoted to prove Theorem 2.1.

Fix initial data $(u_0,u_1)\in V$. Using the semigroup notation and the variation of constants formula (VCF), we introduce the function

$$[\Phi(u)](t)=S(t)[u_0,u_1]-\int_0^t S(t-s)[f(u(s))_x]ds, \tag{4.1}$$

where we denote, to simplify the presentation, $S(t-s)[0,f(u(s))_x]$ by $S(t-s)[f(u(s))_x]$. Recall that $f(u)=|u|^{q-1}u$.

With this notation, equation (2.1) can be written as $u(t)=[\Phi(u)](t)$. In other words, the problem is reduced to the obtention of fix points of Φ. To do this we apply the Banach fix point Theorem in the Banach space

$$\begin{aligned} X \equiv \{u\in C([0,\infty);H^1(\mathbb{R}))\cap BC^1([0,\infty);L^2(\mathbb{R})) \text{ s.t. } &(1+t)^{\frac14}u\in L^\infty([0,\infty);L^2(\mathbb{R})),\\ &(1+t)^{\frac34}u_x\in L^\infty([0,\infty);L^2(\mathbb{R}))\}\end{aligned}$$

with norm

$$\|u\|_X\equiv\|(1+t)^{\frac14}u\|_{L^\infty([0,\infty);L^2(\mathbb{R}))}+\|(1+t)^{\frac34}u_x\|_{L^\infty([0,\infty);L^2(\mathbb{R}))}+\|u_t\|_{L^\infty([0,\infty);L^2(\mathbb{R}))}.$$

Define the ball $B_R=\{u\in X\quad \text{s.t. } \|u\|_X\le R\}$. We shall show that, when $\|(u_0,u_1)\|_V$ is sufficiently small, one may choose $R>0$ small enough (depending on $\|(u_0,u_1)\|_V$) such that $\Phi(B_R)\subset B_R$. This result is immediate from the following Lemma.

Lemma 4.1 *There exists $c,c'>0$ so that $\Phi(u)\in X$ and*

$$(1+t)^{\frac14}\|\Phi(u)(t)\|_2\le c'\|(u_0,u_1)\|_V+cR^q,\qquad \forall u\in B_R,\quad t\ge0, \tag{4.2}$$
$$(1+t)^{\frac34}\|\Phi(u)_x(t)\|_2\le c'\|(u_0,u_1)\|_V+cR^q,\qquad \forall u\in B_R,\quad t\ge0, \tag{4.3}$$
$$\|\Phi(u)_t(t)\|_2\le c'\|(u_0,u_1)\|_V+cR^q,\qquad \forall u\in B_R,\quad t\ge0. \tag{4.4}$$

We assume for the moment that this lemma is true. We shall return to its proof later. Then, $\Phi(u)\in X$ and choosing $R>0$ such that $c'\|(u_0,u_1)\|_V\le R/6$, we get

$$\|\Phi(u)\|_X\le\frac{R}{2}+cR^q,\qquad \forall u\in B_R.$$

Since $q>1$, choosing R sufficiently small we obtain that $\Phi(B_R)\subset B_R$. Note that the smallness condition on R imposes a smallness condition on the size of the initial data (on $\|(u_0,u_1)\|_V$) too. Thus, from now on, the initial data (u_0,u_1) are assumed to be small in V.

Now, we are going to see that Φ is a contraction. We define

$$\varphi(t) = \sup_{0\le s\le t}\{(1+s)^{\frac14}\|u(s)-v(s)\|_2\}, \tag{4.5}$$
$$\phi(t) = \sup_{0\le s\le t}\{(1+s)^{\frac34}\|u_x(s)-v_x(s)\|_2\}. \tag{4.6}$$

Lemma 4.2 *Let Φ be defined as in (4.1). Then, there exists a constant $c>0$ so that*

$$(1+t)^{\frac14}\|\Phi(u)(t)-\Phi(v)(t)\|_2\le cR^{q-1}\varphi(t),\qquad \forall u,v\in B_R,\quad t\ge0, \tag{4.7}$$
$$(1+t)^{\frac34}\|\Phi(u)_x(t)-\Phi(v)_x(t)\|_2\le cR^{q-1}[\varphi(t)+\phi(t)],\qquad \forall u,v\in B_R,\quad t\ge0, \tag{4.8}$$
$$\|\Phi(u)_t(t)-\Phi(v)_t(t)\|_2\le cR^{q-1}[\varphi(t)+\phi(t)],\qquad \forall u,v\in B_R,\quad t\ge0. \tag{4.9}$$

Using (4.7)–(4.9), we get

$$\|\Phi(u) - \Phi(v)\|_X \leq cR^{q-1}\|u - v\|_X, \qquad \forall u, v \in B_R.$$

Thus, since $q > 1$, for R sufficiently small, Φ is a strict contraction in B_R.

Now, applying the Banach's Theorem, there exists an unique solution u of (2.1) in B_R so that, in particular, $u \in BC([0,\infty); H^1(\mathbb{R}))$ and $u \in BC^1([0,\infty); L^2(\mathbb{R}))$. Moreover, since $\Phi(B_R) \subset B_R$, we get that $\|u\|_X \leq R$ and, therefore,

$$\|u(t)\|_2 \leq R(1+t)^{-\frac{1}{4}}, \qquad \|u_x(t)\|_2 \leq R(1+t)^{-\frac{3}{4}}, \qquad t \geq 0.$$

This confirms that the constants c, c' in (2.3) and (2.4) are proportional to the norm of the initial data in V.

The same argument shows that for any initial data $u_0 \in H^1(\mathbb{R})$ and $u_1 \in L^2(\mathbb{R})$, for $\tau > 0$ sufficiently small, there exists an unique solution u of (2.1) so that

$$u \in C([0,\tau); H^1(\mathbb{R})) \cap C^1([0,\tau); L^2(\mathbb{R})).$$

This allows showing that the solution we have built in B_R is in fact the only solution of (2.4).

Now, we are going to prove Lemmas 4.1 and 4.2.

First of all, we show some estimates that we shall use in the proofs:

$$\int_0^t (1+t-s)^a(1+s)^b ds \leq \begin{cases} c_1(1+t)^{a+b+1} & \text{when } a, b > -1, \\ c_2(1+t)^a & \text{when } a > -1 \text{ and } b < -1, \\ c_3(1+t)^{\max(a,b)} & \text{when } a, b < -1. \end{cases} \tag{4.10}$$

Let us now write estimates (3.2) and (3.3) for the solution of (3.1) with initial data $\varphi_0 = 0$ and $\varphi_1 = g$. For $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}^N_+$, we have, by (3.2), that

$$\|\partial_t^k D_x^\alpha S(t)[g]\|_2 \;\; \leq \;\; c(1+t)^{-\frac{N}{2}(\frac{1}{a}-\frac{1}{2})-\frac{|\alpha|+2k}{2}}\big(\|g\|_{k+|\alpha|-1,2} + \|g\|_a\big), \quad \forall t \geq 0. \tag{4.11}$$

For $k = 0, 1$ and $\alpha \in \mathbb{R}^N_+$, by (3.3) we have that

$$\|\partial_t^k D_x^\alpha S(t)[g]\|_2 \leq ce^{-\omega t}\|g\|_{\dot{H}^{k+|\alpha|-1}} + c'\|g\|_a(1+t)^{-\frac{N}{2}(\frac{1}{a}-\frac{1}{2})-\frac{|\alpha|+2k}{2}}, \qquad \forall t \geq 0. \tag{4.12}$$

Moreover, the following interpolation inequalities hold:

$$\|u\|_\infty \;\leq\; \sqrt{2}\|u\|_2^{\frac{1}{2}}\|u_x\|_2^{\frac{1}{2}}, \qquad \forall u \in H^1(\mathbb{R}), \tag{4.13}$$

$$\|u\|_{\dot{H}^a(\mathbb{R})} \leq \|u\|_2^{1-a}\|u_x\|_2^a, \qquad \forall u \in H^1(\mathbb{R}), \quad a \in (0,1), \tag{4.14}$$

where $\dot{H}^a(\mathbb{R}^N)$ denotes the homogeneous fractional Sobolev space.

Proof of Lemma 4.1. Using the estimates (3.2) in (4.1) we get

$$\|\Phi(u)(t)\|_2 \;\leq\; c(1+t)^{-\frac{1}{4}}(\|u_0\|_2 + \|u_1\|_{-1,2} + \|u_0\|_1 + \|u_1\|_1) + \int_0^t \|S_x(t-s)[f(u(s))]\|_2 ds,$$

$$\|\Phi(u)_x(t)\|_2 \leq\; c(1+t)^{-\frac{3}{4}}(\|u_0\|_{1,2} + \|u_1\|_2 + \|u_0\|_1 + \|u_1\|_1) + \int_0^t \|S_x(t-s)[f(u(s))_x]\|_2,$$

$$\|\Phi(u)_t(t)\|_2 \leq c(1+t)^{-\frac{5}{4}}(\|u_0\|_{1,2} + \|u_1\|_2 + \|u_0\|_1 + \|u_1\|_1) + \int_0^t \|S_t(t-s)[f(u(s))_x]\|_2 ds.$$

Thus, to get (4.2)–(4.4) it is sufficient to prove the following estimates:

$$\begin{aligned}
\int_0^t \|S_x(t-s)[f(u(s))]\|_2 ds &\le c(1+t)^{-\frac{1}{4}}R^q, \qquad \forall u \in B_R, &(4.15)\\
\int_0^t \|S_x(t-s)[f(u(s))_x]\|_2 ds &\le c(1+t)^{-\frac{3}{4}}R^q, \qquad \forall u \in B_R, &(4.16)\\
\int_0^t \|S_t(t-s)[f(u(s))_x]\|_2 ds &\le cR^q, \qquad \forall u \in B_R. &(4.17)
\end{aligned}$$

Again, using the estimate (4.12) with $a=1$, $k=0$ and $\alpha=1$ in (4.15) and (4.16) and with $a=1$, $k=1$ and $\alpha=0$ in (4.17), we have

$$\begin{aligned}
\int_0^t \|S_x(t-s)[f(u(s))]\|_2 ds &\le c\int_0^t (1+t-s)^{-\frac{3}{4}}[\|f(u(s))\|_2 + \|f(u(s))\|_1]ds,\\
\int_0^t \|S_x(t-s)[f(u(s))_x]\|_2 ds &\le c\int_0^t (1+t-s)^{-\frac{3}{4}}[\|f(u(s))_x\|_2 + \||f(u(s))_x\|_1]ds,\\
\int_0^t \|S_t(t-s)[f(u(s))_x]\|_2 ds &\le c\int_0^t (1+t-s)^{-\frac{5}{4}}[\|f(u(s))_x\|_2 + \||f(u(s))_x\|_1]ds.
\end{aligned}$$

By the interpolation inequality (4.13), we get

$$\|f(u(s))\|_2 \le c\|u(s)\|_2^{\frac{q+1}{2}}\|u_x(s)\|_2^{\frac{q-1}{2}} \qquad \text{and} \qquad \|f(u(s))\|_1 \le c'\|u(s)\|_2^{\frac{2+q}{2}}\|u_x(s)\|_2^{\frac{q-2}{2}}.$$

On the other hand, thanks to the Cauchy-Schwarz inequality and (4.13),

$$\begin{aligned}
\|f(u(s))_x\|_2 &\le c\|u\|_\infty^{q-1}\|u_x\|_2 \le c\|u\|_2^{\frac{q-1}{2}}\|u_x\|_2^{\frac{q+1}{2}}, &(4.18)\\
\|f(u(s))_x\|_1 &\le c\|u\|_\infty^{q-2}\|u\|_2\|u_x\|_2 \le c\|u\|_2^{\frac{q}{2}}\|u_x\|_2^{\frac{q}{2}}. &(4.19)
\end{aligned}$$

Then, as $u \in B_R$, we obtain, for any $s \ge 0$:

$$\begin{array}{ll}
\|f(u(s))\|_2 \le cR^q(1+s)^{-\frac{2q-1}{4}}, & \|f(u(s))\|_1 \le cR^q(1+s)^{-\frac{q-1}{2}},\\
\|f(u(s))_x\|_2 \le cR^q(1+s)^{-\frac{2q+1}{4}}, & \|f(u(s))_x\|_1 \le cR^q(1+s)^{-\frac{q}{2}}.
\end{array} \qquad (4.20)$$

- Proof of (4.15) and (4.17). Using the estimates (4.20), we get

$$\begin{aligned}
\int_0^t \|S_x(t-s)[f(u(s))]\|_2 ds &\le c\,R^q\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{1}{2}}ds, \qquad \forall q \ge 2,\\
\int_0^t \|S_t(t-s)[f(u(s))_x]\|_2 ds &\le c\,R^q\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-1}ds, \qquad \forall q \ge 2.
\end{aligned}$$

  Then, (4.15) and (4.17) hold for any $q \ge 2$ thanks to (4.11).

- Proof of (4.16) with $q > 2$. Using the estimates on $[f(u(s))]_x$ in L^2 and L^1 in (4.20), we have

$$\int_0^t \|S_x(t-s)[f(u(s))_x]\|_2 ds \le cR^q\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{q}{2}}ds.$$

  Then, thanks to (4.11), we prove (4.16) for $q > 2$.

- Proof of (4.16) with $q = 2$. Now, we need to work with fractional derivatives. For $\alpha \in (0,1)$, we have

$$\int_0^t \|S_x(t-s)[f(u(s))_x]\|_2 ds = \int_0^t \|\partial_x^{2-\alpha} S(t-s)[\partial_x^\alpha (f(u(s)))]\|_2 ds.$$

Applying estimate (4.12) with $a = 2$ we get

$$\begin{aligned}&\int_0^t \|S_x(t-s)[f(u(s))_x]\|_2 ds \\ &\leq c \int_0^t e^{-\omega(t-s)} \|\partial_x^\alpha f(u(s))\|_{\dot{H}^{1-\alpha}} ds \,+\, c' \int_0^t (1+t-s)^{-\frac{2-\alpha}{2}} \|\partial_x^\alpha f(u(s))\|_2 ds. \quad (4.21)\end{aligned}$$

We note that, by (4.20), for $q = 2$,

$$\|\partial_x^\alpha f(u(s))\|_{\dot{H}^{1-\alpha}} = \|\partial_x f(u(s))\|_2 \leq cR^2 (1+s)^{-\frac{5}{4}}.$$

On the other hand, by the interpolation estimate (4.14), for $q = 2$, by (4.20) we obtain

$$\|\partial_x^\alpha f(u(s))\|_2 \leq \|\partial_x f(u(s))\|_2^\alpha \|f(u(s))\|_2^{(1-\alpha)} \leq cR^2 (1+s)^{-\frac{5\alpha}{4}} (1+s)^{-\frac{3(1-\alpha)}{4}}.$$

Therefore, coming back to (4.21), we get

$$\begin{aligned}&\int_0^t \|\partial_x^{2-\alpha} S(t-s)[\partial_x^\alpha f(u(s))]\|_2 ds \\ &\qquad \leq cR^2 \int_0^t e^{-\omega(t-s)} (1+s)^{-\frac{5}{4}} ds \,+ cR^2 \int_0^t (1+t-s)^{-\frac{2-\alpha}{2}} (1+s)^{-\frac{3+2\alpha}{4}} ds.\end{aligned}$$

Then, taking $\alpha \in (0, 1/2)$ and thanks to (4.10), we prove (4.16) for $q = 2$.

Finally, we observe that $\Phi(u)$ and $\Phi(u)_t$ are continuous in time with values in $H^1(\mathbb{R})$ and $L^2(\mathbb{R})$, respectively. Let us check the continuity of $\Phi(u)$. The term $\Phi(u)_t$ can be treated in a similar way. Indeed, given $t, t_0 > 0$ with $t \geq t_0$ (the case $t \leq t_0$ can be treated in a similar way), by the variation of constants formula (4.1) we have

$$\begin{aligned}\Phi(u)(t) - \Phi(u)(t_0) \;&=\; S(t)[u_0, u_1] - S(t_0)[u_0, u_1] - \int_{t_0}^t S(t-s)[f(u(s))_x] ds \\ &\quad - \int_0^{t_0} \left(S(t-s)[f(u(s))_x] - S(t_0 - s)[f(u(s))_x]\right) ds. \quad (4.22)\end{aligned}$$

Since $(u_0, u_1) \in V$ and taking the continuity property of the semigroup $S(t)$ into account we get:

$$\lim_{t \to t_0} \|S(t)[u_0, u_1] - S(t_0)[u_0, u_1]\|_{1,2} = 0.$$

Using (4.12), we deduce that

$$\int_{t_0}^t \|S(t-s)[f(u(s))_x]\|_{1,2} ds \leq \int_{t_0}^t [ce^{-\omega(t-s)} \|f(u(s))\|_{1,2} + c'(1+t-s)^{-\frac{3}{4}} \|f(u(s))\|_{1,1}] ds.$$

Since $u \in B_R$, $f(u(s))$ is bounded in $H^1(\mathbb{R})$ and $W^{1,1}(\mathbb{R})$. Then

$$\lim_{t \to t_0} \int_{t_0}^t \|S(t-s)[f(u(s))_x]\|_{1,2}\, ds = 0.$$

For the last term of (4.22), we use the dominated convergence theorem. First, thanks to (4.11) and taking into account that $u \in B_R$, we see that $\|S(t-s)[f(u(s))_x]\|_{1,2}$ and $\|S(t_0-s)[f(u(s))_x]\|_{1,2}$ are bounded above by a function in $L^1(0,t_0)$ depending on s which is independent of t. On the other hand, by the continuity of the semigroup $S(t)$, we get for any $s \in (0,t_0)$:

$$\lim_{t\to t_0} \|[S(t-s)-S(t_0-s)][f(u(s))_x]\|_{1,2} = 0,$$

because $f(u(s)) \in H^1(\mathbb{R})$. Then, by the dominated convergence theorem we obtain

$$\lim_{t\to t_0} \int_0^{t_0} \|[S(t-s)-S(t_0-s)][f(u(s))_x]\|_{1,2}\, ds = 0.$$

Thus, we prove that $\Phi(u)$ is continuous with respect to t with values in $H^1(\mathbb{R})$. ▪

Proof of Lemma 4.2. Given $u, v \in B_R$, let be

$$w(t) = \Phi(u)(t) - \Phi(v)(t) = \int_0^t S(t-s)[f(u(s))_x - f(v(s))_x]ds. \tag{4.23}$$

Thus, using the estimate (4.12) with $a = 1$,

$$\begin{aligned}
\|w(t)\|_2 &\leq c\int_0^t (1+t-s)^{-\frac{3}{4}}[\|f(u(s))-f(v(s))\|_2 + \|f(u(s))-f(v(s))\|_1]ds,\\
\|w_x(t)\|_2 &\leq c\int_0^t (1+t-s)^{-\frac{3}{4}}[\|f(u(s))_x-f(v(s))_x\|_2 + \|f(u(s))_x-f(v(s))_x\|_1]ds,\\
\|w_t(t)\|_2 &\leq c\int_0^t (1+t-s)^{-\frac{5}{4}}[\|f(u(s))_x-f(v(s))_x\|_2 + \|f(u(s))_x-f(v(s))_x\|_1]ds.
\end{aligned}$$

In view of the definition of f, it is immediate that there exists $c > 0$ depending on q such that

$$\begin{aligned}
\|f(u(s))-f(v(s))\|_2 &\leq c(\|u(s)\|_\infty^{q-1} + \|v(s)\|_\infty^{q-1})\|u(s)-v(s)\|_2,\\
\|f(u(s))-f(v(s))\|_1 &\leq c(\|u(s)\|_\infty^{q-2}\|u(s)\|_2 + \|v(s)\|_\infty^{q-2}\|v(s)\|_2)\|u(s)-v(s)\|_2,
\end{aligned}$$

and,

$$\begin{aligned}
\|f(u(s))_x-f(v(s))_x\|_2 &\leq c\|u(s)\|_\infty^{q-1}\|u_x(s)-v_x(s)\|_2\\
&\quad + c(\|u(s)\|_\infty^{q-2} + \|v(s)\|_\infty^{q-2})\|u(s)-v(s)\|_\infty\|v_x(s)\|_2,\\
\|f(u(s))_x-f(v(s))_x\|_1 &\leq c\|u(s)\|_\infty^{q-2}\|u\|_2\|u_x(s)-v_x(s)\|_2\\
&\quad + c(\|u(s)\|_\infty^{q-2} + \|v(s)\|_\infty^{q-2})\|u(s)-v(s)\|_2\|v_x(s)\|_2.
\end{aligned}$$

Then, using that $u, v \in B_R$, the interpolation estimate (4.13) and the functions φ and ϕ defined in (4.5) and (4.6), we get, for any $t \leq s$:

$$\begin{aligned}
&\|f(u(s))-f(v(s))\|_2 \leq cR^{q-1}(1+s)^{-\frac{2q-1}{4}}\varphi(t),\\
&\|f(u(s))-f(v(s))\|_1 \leq cR^{q-1}(1+s)^{-\frac{q-1}{2}}\varphi(t),\\
&|f(u(s))_x-f(v(s))_x\|_2 \leq cR^{q-1}(1+s)^{-\frac{2q+1}{4}}\left[\phi(t) + \varphi(t)^{\frac{1}{2}}\phi(t)^{\frac{1}{2}}\right],\\
&\|f(u(s))_x-f(v(s))_x\|_1 \leq cR^{q-1}(1+s)^{-\frac{q}{2}}\left[\phi(t)+\varphi(t)\right].
\end{aligned} \tag{4.24}$$

- Proof of (4.7) and (4.9). Thanks to the estimates (4.24), we get

$$
\begin{aligned}
\|w(t)\|_2 &\leq cR^{q-1}\varphi(t)\int_0^t (1+t-s)^{-\frac{3}{4}}[(1+s)^{-\frac{2q-1}{4}}+(1+s)^{-\frac{q-2}{2}}(1+s)^{-\frac{1}{2}}]ds,\\
\|w_t(t)\|_2 &\leq cR^{q-1}[\phi(t)+\phi(t)^{\frac{1}{2}}\varphi(t)^{\frac{1}{2}}]\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{2q+1}{4}}ds\\
&\quad +cR^{q-1}[\phi(t)+\varphi(t)]\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{q}{2}}ds,
\end{aligned}
$$

and, in view of the definition (4.23) of w, by the estimates (4.10) and the Young inequality, we prove (4.7) and (4.9).

- Proof of (4.8) with $q>2$. Again, by the estimates (4.24), we get

$$
\begin{aligned}
\|w_x(t)\|_2 &\leq c[\phi(t)+\phi(t)^{\frac{1}{2}}\varphi(t)^{\frac{1}{2}}]R^{q-1}\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{2q+1}{4}}ds\\
&\quad +c[\phi(t)+\varphi(t)]R^{q-1}\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{q}{2}}ds,
\end{aligned}
$$

and, for $q>2$, we obtain (4.8) by the Young inequality and the integral estimates (4.10).

- Proof of (4.8) with $q=2$. This case can be treated as in the proof of Lemma 4.1, using fractional derivatives. Let $\alpha\in(0,1)$ and note that, applying (4.12) with $a=2$,

$$
\|w_x(t)\|_2=\int_0^t \|\partial_x^{2-\alpha}S(t-s)[\partial_x^\alpha(f(u(s))-f(v(s)))]\|_2ds \leq
$$

$$
\leq c\int_0^t (e^{-\omega(t-s)}\|\partial_x^\alpha(f(u(s))-f(v(s)))\|_{\dot H^{1-\alpha}}+(1+t-s)^{-\frac{2-\alpha}{2}}\|\partial_x^\alpha(f(u(s))-f(v(s)))\|_2)ds.
$$

We know that, for $q=2$,

$$
\|\partial_x^\alpha(f(u(s))-f(v(s)))\|_{\dot H^{1-\alpha}}=\|\partial_x(f(u(s))-f(v(s)))\|_2\leq cR(1+s)^{-\frac{5}{4}}\left[\phi(t)+\varphi(t)^{\frac{1}{2}}\phi(t)^{\frac{1}{2}}\right].
$$

On the other hand, applying (4.14), we get, for $q=2$,

$$
\begin{aligned}
\|\partial_x^\alpha(f(u(s))-f(v(s)))\|_2 &\leq \|f(u(s))-f(v(s))\|_2^{1-\alpha}\|\partial_x(f(u(s))-f(v(s)))\|_2^\alpha\\
&\leq cR(1+s)^{-\frac{3+2\alpha}{4}}\varphi(t)^{1-\alpha}\left[\phi(t)+\varphi(t)^{\frac{1}{2}}\phi(t)^{\frac{1}{2}}\right]^\alpha,
\end{aligned}
$$

by the previous estimates on $f(u)-f(v)$. Thus, we obtain that

$$
\begin{aligned}
\|w_x(t)\|_2 &\leq cR\left[\phi(t)+\varphi(t)^{\frac{1}{2}}\phi(t)^{\frac{1}{2}}\right]\int_0^t e^{-\omega(t-s)}(1+s)^{-\frac{5}{4}}ds\\
&\quad +c'R\varphi(t)^{1-\alpha}\left[\phi(t)+\varphi(t)^{\frac{1}{2}}\phi(t)^{\frac{1}{2}}\right]^\alpha\int_0^t (1+t-s)^{-\frac{2-\alpha}{2}}(1+s)^{-\frac{3+2\alpha}{4}}ds.
\end{aligned}
$$

Taking $\alpha<1/2$, by (4.10) we get

$$
\|w_x(t)\|_2\leq cR\left[\phi(t)+\varphi(t)^{\frac{1}{2}}\phi(t)^{\frac{1}{2}}\right](1+t)^{-\frac{5}{4}}+cR\varphi(t)^{1-\alpha}[\phi(t)+\varphi(t)^{\frac{1}{2}}\phi(t)^{\frac{1}{2}}]^\alpha(1+t)^{-\frac{3}{4}}.
$$

Finally, applying Young's inequality, we prove (4.8) for $q=2$. ∎

The proof of Theorem 2.1 is now complete.

We now obtain some extra estimates on the behavior of u that will be useful in the sequel.

Proposition 4.3 *Let u be the solution of (2.1) under the hypotheses of Theorem 2.1. Then, for any $t \geq 0$, we get*

$$\|u_t(t)\|_2 \leq \begin{cases} c(1+t)^{-\frac{2q+5}{8}} & \textit{for } 2 \leq q < \dfrac{5}{2}, \\ c(1+t)^{-\frac{5}{4}} & \textit{for } q \geq \dfrac{5}{2}, \end{cases} \tag{4.25}$$

with a constant c that depends on the norm of the initial data in V and the exponent q.

Proof. Using the VCF

$$u(t) = S(t)[u_0, u_1] - \int_0^t S(t-s)[f(u(s))_x]ds,$$

and, by the estimate (3.2), we get:

$$\|u_t(t)\|_2 \leq c(1+t)^{-\frac{5}{4}}(\|u_0\|_{1,2} + \|u_0\|_1 + \|u_1\|_2 + \|u_1\|_1) + \int_0^t \|\partial_t S(t-s)[f(u(s))_x]\|_2 ds.$$

Thus, we only need to estimate the integral term to conclude the proof. To do it, we use (4.12) with $a \in [1,2]$ and we obtain

$$\int_0^t \|\partial_t S(t-s)[f(u(s))_x]\|_2 ds$$
$$\leq c\int_0^t e^{-\omega(t-s)}\|f(u(s))_x\|_2 ds \, + \, c' \int_0^t (1+t-s)^{-\frac{3}{4}-\frac{1}{2a}}\|f(u(s))_x\|_a ds. \tag{4.26}$$

Using (4.18) and estimates (2.3) and (2.4), we get for $q \geq 2$:

$$\int_0^t e^{-\omega(t-s)}\|f(u(s))_x\|_2 ds \;\; \leq \;\; C\int_0^t e^{-\omega(t-s)}(1+s)^{-\frac{2q+1}{4}}ds \leq C(1+t)^{-\frac{5}{4}},$$

where C depends on the initial data and q. Thus, we only have to study the second integral on the right hand side of (4.26).

- Case $q \geq 5/2$. Taking $a = 1$ and using (4.19) with estimates (2.3) and (2.4), we prove (4.25) for $q \geq 5/2$:

$$\int_0^t (1+t-s)^{-\frac{5}{4}}\|f(u(s))_x\|_1 ds \;\; \leq \;\; C\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{5}{4}}ds \leq C(1+t)^{-\frac{5}{4}}.$$

- Case $q \in [2, 5/2)$. By Holder's inequality

$$\|f(u(s))_x\|_a \leq c\|u_x\|_2 \left(\int_{\mathbb{R}} |u(x)|^{\frac{2a(q-1)}{2-a}} dx\right)^{\frac{2-a}{2a}}.$$

Since $a \in [1,2]$, then $2a(q-1)/(2-a) \geq 2$ and we get

$$\left(\int_{\mathbb{R}} |u(x)|^{\frac{2a(q-1)}{2-a}} dx\right)^{\frac{2-a}{2a}} \leq \|u\|_\infty^{q-\frac{2}{a}} \|u\|_2^{\frac{2-a}{a}}.$$

Therefore, thanks to (4.13), (2.3) and (2.4), we obtain:

$$\begin{aligned}\int_0^t (1+t-s)^{-\frac{3}{4}-\frac{1}{2a}}\|f(u(s))_x\|_a ds &\leq c\int_0^t (1+t-s)^{-\frac{3}{4}-\frac{1}{2a}}(1+s)^{-\frac{q+1}{2}+\frac{1}{2a}}ds \\ &\leq c[(1+t)^{\frac{1}{2a}-\frac{q+1}{2}}+(1+t)^{-\frac{3}{4}-\frac{1}{2a}}],\end{aligned}$$

by (4.10) since $(3/4+1/2a)>1$ and $(q+1)/2-1/2a>1$. We choose $a\in(1,2)$ depending on $q\in[2,5/2)$ so that the decay estimate is optimal, i.e,

$$\frac{q+1}{2}-\frac{1}{2a}=\frac{3}{4}+\frac{1}{2a} \quad \Rightarrow \quad a=\frac{4}{2q-1}\in\left(1,\frac{4}{3}\right].$$

We get

$$\int_0^t (1+t-s)^{-\frac{3}{4}-\frac{1}{2a}}\|f(u(s))_x\|_a ds \leq c\,(1+t)^{-\frac{2q+5}{8}}$$

and conclude the proof of (4.25) for $q\in[2,5/2)$. ▪

5 Compactness

The goal of this section is to obtain a compactness result in $L^2((0,T)\times\mathbb{R})$ for the family $\{u_\lambda\}_{\lambda>0}$, defined by

$$u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2 t). \tag{5.1}$$

This fact will be later used in the proof of Theorem 2.4.

For any $\lambda>0$ the function u_λ solves

$$\begin{cases} \lambda^{-2}u_{\lambda,tt}+u_{\lambda,t}-u_{\lambda,xx}+\lambda^{2-q}(|u_\lambda|^{q-1}u_\lambda)_x=0, \quad (x,t)\in\mathbb{R}\times(0,\infty), \\ u_\lambda(x,0)=u_{0,\lambda}(x)=\lambda u_0(\lambda x), \qquad u_{\lambda,t}(x,0)=u_{1,\lambda}(x)=\lambda^3 u_1(\lambda x). \end{cases} \tag{5.2}$$

On the other hand, thanks to (2.3) and (2.4), we have, for any $t\geq 0$:

$$\|u_\lambda\|_2 \leq c\,\lambda^{\frac{1}{2}}(1+\lambda^2 t)^{-\frac{1}{4}}, \tag{5.3}$$

$$\|u_{\lambda,x}\|_2 \leq c\,\lambda^{\frac{3}{2}}(1+\lambda^2 t)^{-\frac{3}{4}}. \tag{5.4}$$

By (5.3) the norm of u_λ in $L^2((0,T)\times\mathbb{R})$ is bounded when $\lambda\to\infty$. Then, by extracting subsequences (that we denote with the same subindex λ to simplify the notation), u_λ converges weakly in $L^2((0,T)\times\mathbb{R})$. Now, we proceed in several steps:

1. First, we establish that the sequence $\{u_\lambda\}$ is relatively compact in $C([t_0,T];L^p_{loc}(\mathbb{R}))$ for $1\leq p\leq\infty$ and $t_0>0$.

2. Then, for any $\varepsilon>0$ we prove the existence of $k_0>0$ such that for any $k\geq k_0$ and $\lambda\geq 1$ we get

$$\int_{|x|\geq k}|u_\lambda(x,t)|dx\leq\varepsilon, \qquad \forall t\in[t_0,T]. \tag{5.5}$$

   As a consequence, we obtain the compactness in $C([t_0,T];L^p(\mathbb{R}))$ for $t_0>0$ and any $p\in[1,\infty]$.

3. Finally, we show that for any $\varepsilon>0$, there exists $t_0>0$ sufficiently small such that, for $\lambda\geq 1$, it holds

$$\int_0^{t_0}\int_{\mathbb{R}}|u_\lambda(x,t)|^2dx\leq\varepsilon. \tag{5.6}$$

   Using the previous items, we conclude the proof of the compactness in $L^2((0,T)\times\mathbb{R})$.

Local compactness

Proposition 5.1 *Let u be the unique solution of (2.1) under the hypotheses of Theorem 2.1. Let $\{u_\lambda\}$ be the family defined in (1.8) for $N = 1$. Then, for any $t_0 > 0$, $\{u_\lambda\}$ is relatively compact in $C([t_0, T]; L^p_{loc}(\mathbb{R}))$ for $1 \leq p \leq \infty$ and $t_0 > 0$.*

To prove this result, we use a variant of the Aubin-Lions compactness Lemma. The following estimates on the time derivative are needed:

Lemma 5.2 *Let $0 < t_0 < T$. Then, under the hypotheses of Proposition 5.1, $\{u_{\lambda,t}\}_{\lambda \geq 1}$ is uniformly bounded in $L^2(t_0, T; H^{-1}(\mathbb{R}))$.*

Proof of Lemma 5.2. We consider the function

$$v_\lambda(x,t) = \lambda^{-2} e^{\lambda^2 t} u_{\lambda,t}(x,t), \tag{5.7}$$

that satisfies $v_{\lambda,t} = e^{\lambda^2 t}[\lambda^{-2} u_{\lambda,tt} + u_{\lambda,t}]$. Thus, since u_λ verifies (5.2),

$$\|v_{\lambda,t}\|_{-1,2} \leq c e^{\lambda^2 t}[\|u_{\lambda,x}\|_2 + \lambda^{2-q}\|f(u_\lambda)\|_2].$$

On the one hand, we get

$$\int_{\mathbb{R}} \lambda^{4-2q}|f(u_\lambda)|^2 dx \leq c\lambda^3 \int_{\mathbb{R}} |u(y, \lambda^2 t)|^{2q} dy \leq c\lambda^3 (1 + \lambda^2 t)^{-\frac{2q-1}{2}},$$

by (4.13) and the estimates (2.3)-(2.4). With this estimate and (5.4)

$$\|v_{\lambda,t}\|_{-1,2} \leq c e^{\lambda^2 t} \lambda^{\frac{3}{2}} (1 + \lambda^2 t)^{-\frac{3}{4}},$$

for any exponent $q \geq 2$. Consequently, we get $\|v_{\lambda,t}\|_{-1,2} \leq c e^{\lambda^2 t}$, for any $t \geq t_0 > 0$. Using the Cauchy-Schwarz's inequality and the previous estimate, we have

$$\frac{\partial}{\partial t}(\|v_\lambda(t)\|^2_{-1,2}) \leq c\, e^{\lambda^2 t}\|v_\lambda(t)\|_{-1,2}, \qquad \text{for any } t \geq t_0.$$

Then,

$$\frac{\partial}{\partial t}(\|v_\lambda(t)\|_{-1,2}) \leq c\, e^{\lambda^2 t}, \qquad \text{for any } t \geq t_0.$$

Integrating between t_0 and t, we get

$$\|v_\lambda(t)\|_{-1,2} \leq \|v_\lambda(t_0)\|_{-1,2} + c\,\lambda^{-2}[e^{t\lambda^2} - e^{t_0\lambda^2}], \qquad \text{for any } t \geq t_0.$$

Since v_λ is defined by (5.7),

$$\|v_\lambda(t)\|_{-1,2} = \lambda^{-2} e^{\lambda^2 t}\|u_{\lambda,t}(t)\|_{-1,2}.$$

Thus, we get

$$\|u_{\lambda,t}(t)\|_{-1,2} \leq e^{(t_0 - t)\lambda^2}\|u_{\lambda,t}(t_0)\|_{-1,2} + 2c[1 - e^{(t_0 - t)\lambda^2}], \qquad \text{for any } t \geq t_0.$$

Thanks to (4.25), we have the following decay estimates for $u_{\lambda,t}$:

$$\|u_{\lambda,t}(t)\|_2 \leq \begin{cases} c\lambda^{\frac{5}{2}}(1+\lambda^2 t)^{-\frac{2q+5}{8}} & \text{for } 2 \leq q < \frac{5}{2}, \\ c\lambda^{\frac{5}{2}}(1+\lambda^2 t)^{-\frac{5}{4}} & \text{for } q \geq \frac{5}{2}, \end{cases} \tag{5.8}$$

and, in view of the continuity of the embedding $L^2(\mathbb{R}) \hookrightarrow H^{-1}(\mathbb{R})$, we get that, by (5.8),

$$\|u_{\lambda,t}(t_0)\|_{-1,2} \leq c\lambda^{\frac{1}{4}}$$

for any $q \geq 2$ and $t_0 > 0$. Thus, we have

$$\|u_{\lambda,t}(t)\|_{-1,2} \leq ce^{(t_0-t)\lambda^2}\lambda^{\frac{1}{4}} + 2c[1-e^{(t_0-t)\lambda^2}], \quad \text{for any } t \geq t_0,$$

and we conclude the proof. ▪

Proof of Proposition 5.1. We observe that $H^1_{loc}(\mathbb{R})$ is included in $L^p_{loc}(\mathbb{R})$ for $1 \leq p \leq \infty$ with compact embedding (see [3], p. 169). Moreover, $L^2_{loc}(\mathbb{R})$ is included in $H^{-1}_{loc}(\mathbb{R})$ and $L^p_{loc}(\mathbb{R})$ in $L^2_{loc}(\mathbb{R})$ for $2 \leq p \leq \infty$ with continuous embeddings.

On the other hand, recall that, according to Lemma 5.2, $\{u_{\lambda,t}\}_{\lambda \geq 1}$ is uniformly bounded in $L^2(t_0, T; H^{-1}(\mathbb{R}))$ for $0 < t_0 < T$. Moreover, by (5.3) and (5.4), $\{u_\lambda\}$ is uniformly bounded in $L^\infty(t_0, T; H^1(\mathbb{R}))$, for any $0 < t_0 < T$. Then, using classical compactness results (see [17], Corollary 4, p. 85), the family $\{u_\lambda\}$ is relatively compact in $C([t_0, T]; L^p(K))$ for any compact set $K \subset \mathbb{R}$ and $2 \leq p \leq \infty$. This concludes the proof. ▪

Global compactness

First, we prove (5.6). In fact, thanks to the estimate (5.3),

$$\int_0^{t_0}\int_{\mathbb{R}} |u_\lambda(x,t)|^2 dx \leq c\lambda \int_0^{t_0} (1+\lambda^2 t)^{-\frac{1}{2}} dt \leq c\lambda \int_0^{t_0} (\lambda^2 t)^{-\frac{1}{2}} dt \leq 2ct_0^{\frac{1}{2}},$$

and choosing t_0 sufficiently small, we obtain (5.6).

Now, we prove (5.5) as a consequence of the following lemma whose proof is given later.

Lemma 5.3 *Let u be solution of (2.1) given by Theorem 2.1. Moreover, assume that:*

$$u_0, u_1 \in L^2_1(\mathbb{R}).$$

Then, we have

$$\int_{\mathbb{R}} |x|^2 |u(x,t)|^2 dx \leq c(1+t)^{\frac{1}{2}}. \tag{5.9}$$

We note that

$$\int_{|x|\geq k} |u_\lambda(x,t)| dx = \int_{|y|\geq \lambda k} |u(y, \lambda^2 t)| dy.$$

Thus, by the Cauchy-Schwarz's inequality,

$$\int_{|x|\geq k} |u_\lambda(x,t)| dx \leq \sqrt{2}(\lambda k)^{-\frac{1}{2}} \|u(\lambda^2 t)\|_{\dot{L}^2_1},$$

and, thanks to (5.9), we prove

$$\int_{|x|\geq k} |u_\lambda(x,t)|dx \;\leq\; c(\lambda k)^{-\frac{1}{2}}(1+\lambda^2 t)^{\frac{1}{4}} \;\leq\; ck^{-\frac{1}{2}}(1+T)^{\frac{1}{4}}, \tag{5.10}$$

for all $0 \leq t \leq T$ and $\lambda \geq 1$.

This concludes the proof of (5.5). As a consequence, we have the following result:

Proposition 5.4 *Under the hypotheses of Lemma 5.3, the family $\{u_\lambda\}$ is relatively compact in $C([t_0,T];L^p(\mathbb{R}))$ for $0 < t_0 < T < \infty$ and any $p \in [1,\infty]$.*

Proof. First, we prove that for any $k \geq k_0$ and $\lambda \geq 1$ it holds

$$\int_{|x|\geq k} |u_\lambda(x,t)|^2 dx \leq \varepsilon, \qquad \forall t \in [t_0,T]. \tag{5.11}$$

Using (4.13) and the estimates (5.3)–(5.4), it is immediate that

$$\|u_\lambda(t)\|_\infty \leq c\lambda(1+\lambda^2 t)^{-\frac{1}{2}}. \tag{5.12}$$

Thus, using (5.10) and (5.12), we observe that

$$\int_{|x|\geq k} |u_\lambda(x,t)|^2 dx \leq \|u_\lambda(t)\|_\infty \int_{|x|\geq k} |u_\lambda(x,t)|dx \leq ck^{-\frac{1}{2}}\lambda^{\frac{1}{2}}(1+\lambda^2 t)^{-\frac{1}{4}}.$$

Consequently, we obtain (5.11) for $t \geq t_0$ choosing k sufficiently large.

Now, we consider the L^∞ norm. Let $I = \mathbb{R} - [-k,k]$. We know that

$$\|u_\lambda(t)\|_{L^\infty(I)} \leq \|u_\lambda(t)\|_{L^2(I)} \|u_{\lambda,x}(t)\|_{L^2(I)}.$$

Since $\|u_{\lambda,x}(t)\|_{L^2(I)}$ is uniformly bounded for any $\lambda \geq 1$ and $t \geq t_0$, by (5.11) we prove that for any $k \geq k_0$ and $\lambda \geq 1$

$$\|u_\lambda(t)\|_{L^\infty(I)} \leq \varepsilon, \qquad \forall t \geq t_0. \tag{5.13}$$

Thus, as a consequence of (5.11), (5.13) and Proposition 5.1, we conclude the proof. ∎

Proof of Lemma 5.3. Using the VCF, we get that:

$$\|u\|_{\dot{L}^2_1} \leq \|S(t)[u_0,u_1]\|_{\dot{L}^2_1} + \int_0^t \|\partial_x S(t-s)[f(u(s))]\|_{\dot{L}^2_1} ds,$$

with $f(u) = |u|^{q-1}u$. Applying Lemma 3.3 with $b=2$ and $a=1$ to estimate $S(t)[u_0,u_1]$ and with $b=1$ and $a=1$ to estimate $\partial_x S(t-s)[f(u(s))]$, we have

$$\begin{aligned}\|u(t)\|_{\dot{L}^2_1} \;\leq\; & c'(1+t)^{\frac{1}{4}}(\|u_0\|_{L^1} + \|u_1\|_{L^1} + \|u_0\|_{L^2_1} + \|u_1\|_{L^2_1}) \\ & + c\int_0^t e^{-\omega(t-s)}\|f(u(s))\|_{\dot{L}^2_1} ds + c_1\int_0^t (1+t-s)^{-\frac{3}{4}}\|f(u(s))\|_{\dot{L}^1_1} ds \\ & + c_2\int_0^t (1+t-s)^{-\frac{1}{2}}\|f(u(s))\|_2 ds + c_3\int_0^t (1+t-s)^{-\frac{1}{4}}\|f(u(s))\|_1 ds.\end{aligned} \tag{5.14}$$

Now, we study the integral terms. Since $f(u) = |u|^{q-1}u$, using the decay estimates (2.3) and (2.4), we obtain immediately:

$$
\begin{aligned}
\|f(u(s))\|_{\dot{L}^2_1} &\le c(1+s)^{-\frac{q-1}{2}} \|u(s)\|_{\dot{L}^2_1},\\
\|xf(u(s))\|_1 \le c\|u(s)^{q-1}\|_2 \|u(s)\|_{\dot{L}^2_1} &\le c(1+s)^{-\frac{2q-3}{4}} \|u(s)\|_{\dot{L}^2_1},\\
\|f(u(s))\|_2 \le c(1+s)^{-\frac{2q-1}{4}}, &\qquad \|f(u(s))\|_1 \le c(1+s)^{-\frac{q-1}{2}}.
\end{aligned}
$$

Therefore, applying these estimates in the integrals appearing in (5.14), we get, for any $q \ge 2$ and $t \ge 0$:

$$
\begin{aligned}
\int_0^t (1+t-s)^{-\frac12} \|f(u(s))\|_2 ds &\le c\int_0^t (1+t-s)^{-\frac12}(1+s)^{-\frac{2q-1}{4}} ds \le c(1+t)^{-\frac14},\\
\int_0^t (1+t-s)^{-\frac14} \|f(u(s))\|_1 ds &\le c\int_0^t (1+t-s)^{-\frac14}(1+s)^{-\frac{q-1}{2}} ds \le c(1+t)^{\frac14},\\
\int_0^t e^{-\omega(t-s)} \|f(u(s))\|_{\dot{L}^2_1} ds &\le c\int_0^t e^{-\omega(t-s)}(1+s)^{-\frac{q-1}{2}} \|u(s)\|_{\dot{L}^2_1} ds,\\
\int_0^t (1+t-s)^{-\frac34} \|f(u(s))\|_{\dot{L}^1_1} ds &\le c\int_0^t (1+t-s)^{-\frac34}(1+s)^{-\frac{2q-3}{4}} \|u(s)\|_{\dot{L}^2_1} ds.
\end{aligned}
$$

Then, coming back to (5.14), we deduce that, for any $t \ge 0$:

$$
\|u(t)\|_{\dot{L}^2_1} \le c(1+t)^{\frac14} + \tilde{c}\int_0^t (1+t-s)^{-\frac34}(1+s)^{-\frac{2q-3}{4}} \|u(s)\|_{\dot{L}^2_1} ds. \tag{5.15}
$$

Using Hölder's inequality for $p > 4$ in the integral in (5.15), we get

$$
\int_0^t (1+t-s)^{-\frac34}(1+s)^{-\frac{2q-3}{4}} \|u(s)\|_{\dot{L}^2_1} \le \left(\int_0^t (1+t-s)^{-\frac{3p'}{4}}(1+s)^{-p'\frac{2q-3}{4}}\right)^{\frac{1}{p'}} \left(\int_0^t \|u(s)\|^p_{\dot{L}^2_1}\right)^{\frac1p},
$$

with $1/p + 1/p' = 1$. Now, since $1 \le p' < 4/3$, we have that $(1+s)^{-p'\frac{2q-3}{4}} \le (1+s)^{-\frac{3p'}{4}}$ for $q \ge 3$. Then, by (4.10):

$$
\int_0^t (1+t-s)^{-\frac{3p'}{4}}(1+s)^{-p'\frac{2q-3}{4}} ds \le \begin{cases} c(1+t)^{-\frac{3p'}{2}+1} & \text{for } q \ge 3,\\ c(1+t)^{-\frac{qp'}{2}+1} & \text{for } q \in [2,3]. \end{cases}
$$

Thus, we have that, for $q \in [2,3]$,

$$
\|u(t)\|^p_{\dot{L}^2_1} \le c(1+t)^{\frac{p}{4}} + \tilde{c}(1+t)^{-p\frac{q-2}{2}-1} \int_0^t \|u(s)\|^p_{\dot{L}^2_1} ds, \tag{5.16}
$$

and, for $q > 3$, we obtain

$$
\|u(t)\|^p_{\dot{L}^2_1} \le c(1+t)^{\frac{p}{4}} + \tilde{c}(1+t)^{-p\frac12-1} \int_0^t \|u(s)\|^p_{\dot{L}^2_1} ds,
$$

i.e., the inequality (5.16) with $q = 3$. Thus, the proof of (5.9) for the cases $q > 3$ is reduced to the case $q = 3$.

Now, we define

$$
\varphi_q(t) = (1+t)^{p\frac{q-2}{2}+1} \|u(t)\|^p_{\dot{L}^2_1}, \qquad \forall t \ge 0. \tag{5.17}
$$

Thus, for $q \in [2,3]$, (5.16) can be written as

$$\varphi_q(t) \le c(1+t)^{p\frac{2q-3}{4}+1} + \widetilde{c}\int_0^t (1+s)^{-p\frac{q-2}{2}-1}\varphi_q(s)ds. \tag{5.18}$$

We distinguish two cases:
Case $2 < q \le 3$. Applying the Gronwall's inequality (see [10], pp. 4) in (5.18), we get

$$\begin{aligned}\varphi_q(t) &\le c(1+t)^{p\frac{2q-3}{4}+1} + c'\int_0^t (1+s)^{\frac{p}{4}} \exp[\widetilde{c}\int_s^t (1+\rho)^{-p\frac{q-2}{2}-1}d\rho]ds \\ &\le c(1+t)^{p\frac{2q-3}{4}+1} + c'\exp[-2\widetilde{c}\frac{(1+t)^{-p\frac{q-2}{2}}}{p(q-2)}]\int_0^t (1+s)^{\frac{p}{4}} \exp[2\widetilde{c}\frac{(1+s)^{-p\frac{q-2}{2}}}{p(q-2)}]ds.\end{aligned}$$

Since $q \in (2,3]$,

$$\begin{aligned}\exp[\frac{2\widetilde{c}}{p(q-2)}(1+s)^{-p\frac{q-2}{2}}] &\le \exp[\frac{2\widetilde{c}}{p(q-2)}], \qquad \forall s \ge 0, \\ \int_0^t (1+s)^{\frac{p}{4}}ds &\le c(1+t)^{\frac{p}{4}+1} \le c(1+t)^{p\frac{2q-3}{4}+1}, \qquad \forall t \ge 0.\end{aligned}$$

Therefore,

$$\varphi_q(t) \le C(1+t)^{p\frac{2q-3}{4}+1}, \qquad \forall t \ge 0, \quad q \in (2,3].$$

Then, in view of the definition (5.17) of φ_q, we get (5.9) for any $q \ge 2$.
Case $q = 2$. Applying again Gronwall's inequality (see [10], pp. 4) in (5.18) with $q = 2$, we get

$$\varphi_2(t) \le c(1+t)^{\frac{p}{4}+1} + c'(1+t)^{\widetilde{c}}\int_0^t (1+s)^{\frac{p}{4}-\widetilde{c}}ds.$$

We now analyze carefully the last integral. The constant $\widetilde{c} > 0$ is proportional to the norm of the initial data in V. Therefore, as the norm of (u_0, u_1) is small in V, we can assume that $(p/4 - \widetilde{c}) > -1$. Thanks to this fact, we have the following integral estimate

$$\int_0^t (1+s)^{\frac{p}{4}-\widetilde{c}}ds \le c(1+t)^{\frac{p}{4}-\widetilde{c}+1}$$

and, consequently, we get

$$\varphi_2(t) \le C(1+t)^{\frac{p}{4}+1}, \qquad \forall t \ge 0.$$

In view of the definition (5.17) of φ_2, this implies (5.9) for $q = 2$. ▪

6 Identification of the limit

In this section we are going to identify the limit of the sequence $\{u_\lambda\}_{\lambda>0}$. As $\lambda \to \infty$, the equation (5.2) formally reduces to the heat equation for $q > 2$ and to the Burgers equation for $q = 2$. In particular, we prove the following proposition:

Proposition 6.1 *Under the hypotheses of Theorem 2.1 and Lemma 5.3, the sequence* $\{u_\lambda\}_{\lambda>0}$ *converges in* $L^2((0,T)\times\mathbb{R})$ *to* v *where:*

- *If the exponent* $q > 2$*,* v *is the solution of the heat equation (2.12).*

- *If the exponent $q = 2$, v is the solution of the Burgers equation (2.11).*

Proof. To do it, we use the weak formulation of solutions. We consider the following test functions

$$\mathcal{D}(T) = \{\varphi \in C([0,T]; H^2(\mathbb{R})) \cap C^1([0,T]; L^2(\mathbb{R})) \ : \ \varphi(T,x) = 0\}.$$

Thus, the solutions $\{u_\lambda\}$ of the rescaled problem (5.2) satisfy:

$$\int_0^T \int_{\mathbb{R}} u_\lambda(\varphi_t + \varphi_{xx}) dxdt \ + \ \lambda^{2-q} \int_0^T \int_{\mathbb{R}} |u_\lambda|^{q-1} u_\lambda \varphi_x dxdt + \\ + \lambda \int_{\mathbb{R}} [u_0(\lambda x) + u_1(\lambda x)]\varphi(x,0)dx + \lambda^{-2} \int_0^T \int_{\mathbb{R}} u_{\lambda,t}\varphi_t dxdt = 0, \qquad \varphi \in \mathcal{D}(T). \quad (6.1)$$

Letting $\lambda \to \infty$, we are going to see that the limit $v \in L^2([0,T] \times \mathbb{R})$ of the sequence $\{u_\lambda\}_{\lambda>0}$ satisfies, when $q = 2$, the weak formulation of the Burgers equation (2.11)

$$\int_0^T \int_{\mathbb{R}} v(\varphi_t + \varphi_{xx}) dxdt + \int_0^T \int_{\mathbb{R}} |v| v \varphi_x dxdt + M\varphi(0,0) = 0, \qquad \forall \varphi \in \mathcal{D}(T), \qquad (6.2)$$

and, in the case $q > 2$, the weak formulation of the heat equation (2.12)

$$\int_0^T \int_{\mathbb{R}} v(\varphi_t + \varphi_{xx}) dxdt + M\varphi(0,0) = 0, \qquad \forall \varphi \in \mathcal{D}(T). \qquad (6.3)$$

Since u_λ is relatively compact in $L^2((0,T) \times \mathbb{R})$, there exists a subsequence (denoted with the same subindex λ) u_λ converging in $L^2((0,T) \times \mathbb{R})$ to some function v. Then, for any $\varphi \in \mathcal{D}(T)$,

$$\lim_{\lambda \to \infty} \int_0^T \int_{\mathbb{R}} u_\lambda(\varphi_t + \varphi_{xx}) = \int_0^T \int_{\mathbb{R}} v(\varphi_t + \varphi_{xx}). \qquad (6.4)$$

By the change of variable $y = \lambda x$,

$$\lambda \int_{\mathbb{R}} [u_0(\lambda x) + u_1(\lambda x)]\varphi(x,0)dx = \int_{\mathbb{R}} [u_0(y) + u_1(y)]\varphi(y/\lambda, 0)dy,$$

and, applying the Dominated Convergence Theorem, we get

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}} [u_0(y) + u_1(y)]\varphi(y/\lambda, 0)dy = M\varphi(0,0), \qquad \forall \varphi \in \mathcal{D}(T). \qquad (6.5)$$

Thanks to (5.8), since $\min(5/4, 2q + 5/8) > 1$ for $q \geq 2$, we get

$$\int_0^T \|u_{\lambda,t}(s)\|_2 ds \leq c\lambda^{\frac{1}{2}},$$

and therefore

$$\lim_{\lambda \to \infty} \lambda^{-2} \int_0^T \int_{\mathbb{R}} u_{\lambda,t}\varphi_t = 0. \qquad (6.6)$$

Now, we study the integral in (6.1) involving the term $|u_\lambda|^{q-1}u_\lambda$. We consider two cases:

- Case $q > 2$. Using (4.13) and the estimates (5.3) and (5.4), we obtain

$$\int_{\mathbb{R}} |u_\lambda|^q dx \le \|u_\lambda\|_\infty^{q-2} \|u_\lambda\|_2^2 \le \lambda^{q-1}(1+\lambda^2 t)^{-\frac{q-1}{2}},$$

and, for $T > 0$, we conclude the existence of $c > 0$, depending on q, such that

$$\int_0^T (1+\lambda^2 t)^{-\frac{q-1}{2}} dt \le \begin{cases} c\lambda^{-2}, & \text{with } q > 3, \\ c\lambda^{-2}\ln(1+\lambda^2 T) & \text{with } q = 3, \\ c\lambda^{-2}(1+\lambda^2 T)^{\frac{3-q}{2}} & \text{with } q \in [2,3). \end{cases}$$

Then, for $0 < T < \infty$:

$$\lim_{\lambda\to\infty} \lambda^{2-q} \int_0^T \int_{\mathbb{R}} ||u_\lambda|^{q-1} u_\lambda \varphi_x| dx dt \le \lim_{\lambda\to\infty} c\lambda \int_0^T (1+\lambda^2 t)^{-\frac{q-1}{2}} dt = 0, \tag{6.7}$$

provided $q > 2$. Therefore, (6.4)–(6.7) guarantee that the limit v of u_λ in $L^2((0,T)\times\mathbb{R})$ satisfies the weak formulation of the heat equation (6.3) when the exponent $q > 2$.

- Case $q = 2$. Now, since the sequence $\{u_\lambda\}$ converges to v in $L^2((0,T)\times\mathbb{R})$, we get, thanks to $\varphi_x \in C((0,T)\times L^\infty(\mathbb{R}))$,

$$\lim_{\lambda\to\infty} \int_0^T \int_{\mathbb{R}} |u_\lambda| u_\lambda \varphi_x dx dt = \int_0^T \int_{\mathbb{R}} |v| v \varphi_x dx dt. \tag{6.8}$$

Thus, thanks to (6.4)–(6.6) and (6.8), the limit v of the sequence u_λ satisfies the weak formulation of the Burgers equation (6.2) when the exponent $q = 2$.

As a final point, we note that the uniqueness of the weak solutions of (2.11) and (2.12) in $L^2((0,T)\times\mathbb{R})$ is well known thanks to the classical transposition method (see [11]). This fact guarantees that the limit v is unique and that it is actually the limit of the whole family $\{u_\lambda\}$. This concludes the proof of Proposition 6.1. ▪

7 Asymptotic behavior

In this section we derive the asymptotic behavior of the solutions of (2.1) (proof of Theorem 2.4).

Proof of Theorem 2.4. We know by Proposition 5.4 that, when $\lambda \to \infty$, there exists v such that the sequence $\{u_\lambda\}$ converges to v in $L^p(\mathbb{R})$ $(1 \le p \le \infty)$ for $t = 1$:

$$\lim_{\lambda\to\infty} \|u_\lambda(\cdot,1) - v(\cdot,1)\|_p = 0. \tag{7.1}$$

We note that, by Proposition 6.1, this convergence holds for the whole sequence $\{u_\lambda\}$. Depending on the value of q, one can get the explicit form of the solution v.

- **Case** $q > 2$. v is the solution of (2.12) defined by $v(x,t) = MG(x,t)$ $(q > 2)$, where G is the heat kernel

$$G(x,t) = (4\pi t)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4t}\right).$$

Thus, v is invariant under the rescaling transformation (5.1), i.e. $v_\lambda \equiv v$. Then,

$$\int_{\mathbb{R}} |u_\lambda(x,1) - v(x,1)|^p dx \quad = \quad \lambda^{p-1} \int_{\mathbb{R}} |u(y,\lambda^2) - v(y,\lambda^2)|^p dy.$$

Now, we choose $\lambda^2 = t$ and thanks to (7.1), we obtain the convergence result (2.10) for $q > 2$.

- **Case** $q = 2$. In this case, by Proposition 6.1, the limit v is the solution of (2.11) whose explicit formula is

$$v(x,t) = t^{-\frac{1}{2}} \exp\left(-\frac{x^2}{t}\right) \left\{ C_M + \int_{-\infty}^{x/\sqrt{t}} \exp\left(-\frac{s^2}{4}\right) ds \right\},$$

where $C_M \in \mathbb{R}$ is a constant so that

$$\int_{\mathbb{R}} v(x,t) dx = M, \qquad \forall t > 0.$$

This solution is also self-similar: $v(x,t) = t^{-1/2} f_M(x/\sqrt{t})$, for a suitable profile f_M. Again, v is invariant under the rescaling transformation. Then, we conclude the proof as in the case $q > 2$. ▪

8 Blow-up

In this section we show the results of blow-up in finite time (Lemma 2.2 and Theorem 2.3).

As we mentioned in Section 2, the blow-up result for the ODE (2.5) with exponent $q = 2$ implies the existence of blowing-up solutions of the form $u(x,t) = xa(t)$ for the PDE (2.1) with $q = 2$. The corresponding initial data are of the form $u_0(x) = xa_0$, $u_1(x) = xa_1$. Obviously they do not belong to the space V. Note however that, due to the finite speed of propagation (=1 in model (2.1)) one can modify the solution so that it blows-up and has compact support in x. Indeed, let a be solution of (2.5) blowing up in time T. Let $\varphi \in C_c^\infty(\mathbb{R})$ be such that $\varphi(x) = x$ for all $x \in [-3T, 0]$. Let the initial data for (2.1) be $u_0(x) = a_0\varphi(x)$, $u_1(x) = a_1\varphi(x)$. Then, the solution of (2.1) in $[-3T + t, -t]$ is of the form $u \equiv xa(t)$. Consequently it blows up in time T for all $x \in [-2T, -T]$.

Proof of Lemma 2.2. First, let $b(t) = e^{t/2}a(t)$. Then (2.5) becomes:

$$b_{tt} - \frac{b}{4} = \alpha e^{-\frac{q-1}{2}t} |b|^{q-1} b, \qquad t > 0, \tag{8.1}$$

with initial data $b(0) = a_0$ and $b'(0) = a_0/2 + a_1$. Now, we multiply (8.1) by b_t and we have

$$\frac{1}{2} b_t^2 = \frac{1}{2}(\frac{1}{2}a_0 + a_1)^2 - \frac{1}{8}a_0^2 + \frac{1}{8}b^2 + \alpha \int_0^t e^{-\frac{q-1}{2}s} \frac{(|b|^{q+1})_t(s)}{q+1} ds.$$

Integrating by parts, we get

$$\int_0^t e^{-\frac{q-1}{2}s} \frac{(|b|^{q+1})_t(s)}{q+1} ds = e^{-\frac{q-1}{2}t} \frac{|b(t)|^{q+1}}{q+1} - \frac{|a_0|^{q+1}}{q+1} + \frac{q-1}{2} \int_0^t e^{-\frac{q-1}{2}t} \frac{|b(s)|^{q+1}}{q+1} ds$$

and, then, we obtain

$$\frac{1}{2}b_t^2 \geq \frac{1}{2}(\frac{1}{2}a_0 + a_1)^2 - \frac{1}{8}a_0^2 + \alpha e^{-\frac{q-1}{2}t}\frac{|b(t)|^{q+1}}{q+1} - \alpha\frac{|a_0|^{q+1}}{q+1}.$$

Given an arbitrary a_0 we assume that a_1 is sufficiently large such that

$$\frac{1}{2}(\frac{1}{2}a_0 + a_1)^2 - \frac{1}{8}a_0^2 - \alpha\frac{|a_0|^{q+1}}{q+1} \geq 0, \tag{8.2}$$

so that,

$$\frac{1}{2}b_t^2 \geq \alpha e^{-\frac{q-1}{2}t}\frac{|b(t)|^{q+1}}{q+1}.$$

Given $\tau > 0$ arbitrary, we have

$$b_t^2 \geq c^2|b|^{q+1},$$

for $0 \leq t \leq \tau$ with $c^2 = 2\alpha e^{-(q-1)\tau/2}/(q+1)$. Then $(|b|)_t \geq c|b|^{(q+1)/2}$ and after integration we obtain that

$$|b(t)| \geq \left(|a_0|^{\frac{1-q}{2}} - c\frac{q-1}{2}t\right)^{-\frac{2}{q-1}},$$

and, thus, when $q > 1$ the solution b blows up in time $t_b \leq 2|a_0|^{(1-q)/2}/c(q-1)$ and, $t_b \leq \tau$, if

$$|a_0| \geq \left(\frac{2(q+1)}{\alpha(q-1)^2}\right)^{\frac{1}{q-1}}\tau^{-\frac{2}{q-1}}e^{\frac{\tau}{2}}, \qquad \tau > 0. \tag{8.3}$$

Now, we consider the function $h(\tau) = \tau^{-2/(q-1)}e^{\tau/2}$ in the right side of (8.3). The minimum critical point of this function in $(0,\infty)$ is $\tau = 4/(q-1)$. Taking this value in (8.3), we get

$$|a_0| \geq \left(\frac{q+1}{8\alpha}\right)^{\frac{1}{q-1}}e^{\frac{2}{q-1}}. \tag{8.4}$$

So that, if a_0 and a_1 satisfy (2.6), a blows up in finite time $t_b \leq |a_0|^{(1-q)/2}e\sqrt{2q+2}/\sqrt{\alpha}(q-1)$. This concludes the proof of Lemma 2.2. ▪

Now, we prove the blow up result for even nonlinearities.

Proof of Theorem 2.3. We first observe that, due to the finite speed of propagation, if the initial data (u_0, u_1) of u have its support in the interval $[K_1, K_2]$, then the support of u is contained in $[K_1 - t, K_2 + t]$ for all $t > 0$. Thus, to reduce the difficulty, we study the function $a(t)$ defined in (2.8) for $t \leq K_1$.

Now, applying Hölder inequality, we get for any $t \leq K_1$

$$\begin{aligned} a(t)^q &\leq \left(\frac{(K_2+t)^{\frac{2q-1}{q-1}} - (K_1-t)^{\frac{2q-1}{q-1}}}{(2q-1)/(q-1)}\right)^{q-1}\int |u|^q dx \\ &\leq (K_1+K_2)^{2q-1}\left(\frac{q-1}{2q-1}\right)^{q-1}\int |u|^q dx. \end{aligned} \tag{8.5}$$

Now, we multiply the equation (2.1) by x and integrate. Thus, we get

$$a_{tt} + a_t = \int |u|^q dx$$

Then, thanks to the inequality (8.5), a satisfies

$$a_{tt} + a_t \geq \rho a^q, \qquad t > 0, \tag{8.6}$$

with $\rho = (K_1 + K_2)^{1-2q}\left((q-1)/(2q-1)\right)^{1-q}$.

It is clear that, if the solution of

$$a_{tt} + a_t = \rho a^q, \qquad t > 0, \tag{8.7}$$

blows up in finite time t_b, also the solution of the inequality (8.6) blows up in finite time $\leq t_b$. Now, we show a blow-up result of the solution of (8.7) as a consequence of Lemma 2.2.

Under the hypotheses of Theorem 2.3, the initial data a_0 and a_1 of (8.7) are positive. With these initial data the solution of (8.7) is positive and also satisfies (2.5) replacing α with ρ. Then, applying the Lemma 2.2, the functions (2.8) satisfying (8.6) blow up in finite time t_b,

$$t_b \leq a_0^{\frac{1-q}{2}} \frac{e\sqrt{2q+2}}{\sqrt{\rho}(q-1)},$$

if the constants (a_0, a_1) satisfy

$$a_0 \geq \left(\frac{q+1}{8\rho}\right)^{\frac{1}{q-1}} e^{\frac{2}{q-1}} \qquad \text{and} \qquad (\frac{1}{2}a_0 + a_1)^2 \geq \frac{1}{4}a_0^2 + \rho\frac{2a_0^{q+1}}{q+1}. \tag{8.8}$$

By the definition of ρ, $t_b \leq K_1$ if

$$a_0 \geq (2q-1)\left(\frac{e^2(2q+2)}{K_1^2\rho(q-1)^2}\right)^{\frac{1}{q-1}}. \tag{8.9}$$

Then, if a_0 and a_1 satisfy (8.8) and (8.9), we conclude the proof of the Theorem 2.3. ▪

9 Extensions to other convective nonlinearities and to the multi-dimensional case

More general nonlinearities. One of the most natural extensions of the previous results is based on the observation that, for large values of time, small solutions tend to zero for a very large class of nonlinearities. Hence only the behavior near zero of the convective nonlinearity should matter when describing the asymptotic behavior of solutions. Consider the equation

$$u_{tt} + u_t - u_{xx} + (f(u))_x = 0$$

where f and f' are locally Lipschitz real functions with $f(0) = 0$ and for which there exists $q \geq 2$ such that

$$\lim_{u\to 0} \frac{f(u)}{|u|^{q-1}u} = a, \qquad a \in \mathbb{R} - \{0\}, \tag{9.1}$$

or,

$$\lim_{u\to 0} \frac{f(u)}{|u|^q} = a, \qquad a \in \mathbb{R} - \{0\}. \tag{9.2}$$

The same results on the global existence, decay and asymptotic behavior of small solutions hold for this more general class of nonlinearities. In particular, under these conditions, the asymptotic behavior of solutions of (2.1) is the same as those of the convection-diffusion

$$u_t - u_{xx} + (f(u))_x = 0$$

which, in turn, is the same as that of the simplified equation

$$u_t - u_{xx} + a(|u|^{q-1}u)_x = 0 \tag{9.3}$$

or

$$u_t - u_{xx} + a(|u|^q)_x = 0. \tag{9.4}$$

Note also that the asymptotic behavior of both equations (9.3) and (9.4) is the same since the fundamental solution of both of them has constant sign.

It is also interesting to observe that when $c = 0$ in (9.1) or (9.2) the asymptotic behavior of small solutions is given by the fundamental solution of the linear heat equation.

The multi-dimensional case. The results of this paper can also be easily extended to the multidimensional case.

Consider for instance the following dissipative wave equation with a non-linear convective term

$$\begin{cases} u_{tt} + u_t - \Delta u + \mathbf{a}\cdot\nabla f(u) = 0, \qquad (x,t)\in\mathbb{R}^N\times\mathbb{R}^+, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \end{cases} \tag{9.5}$$

where $\mathbf{a}\in\mathbb{R}$ is a fixed vector and $f:\mathbb{R}\to\mathbb{R}$ is a C^2 function.

When analyzing (9.5) the first difficulty to overcome is related with the local existence and uniqueness of solutions. The classical energy method requires the non-linearity arising in the wave equation to be locally Lipschitz from $H^1(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$ and this holds in space dimension $N = 1$ whenever f is of class C^2. But this is not true in higher dimensions because of the lack of embedding from $H^1(\mathbb{R}^N)$ into $L^\infty(\mathbb{R})$. Consequently we rather look for solutions in the class

$$C([0,\tau]; H^k(\mathbb{R}^N)) \cap C^1([0,\tau]; H^{k-1}(\mathbb{R}^N)) \tag{9.6}$$

with $k > 0$ such that $2k > N$ so that $H^k(\mathbb{R}^N)$ becomes an algebra because of the continuous embedding $H^k(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$.

Assuming that f is of class C^{k+1} and its derivatives up to order $k+1$ are uniformly bounded, i.e. $f \in BC^{k+1}(\mathbb{R}^N)$, then classical methods yield local existence and uniqueness of solutions in the class (9.6) for every pair of initial data $(u_0, u_1) \in H^k(\mathbb{R}^N)\times H^{k-1}(\mathbb{R}^N)$.

The method of this paper allows showing that for sufficiently small initial data such that, moreover, $[(u_0,u_1)\in L^1(\mathbb{R}^N)]^2$, solutions are global in time and decay as $t\to\infty$ with the same velocity as the solutions of the heat equation with initial data in $L^1(\mathbb{R}^N)$, i.e., with the rate

$$\|u(t)\|_{L^p(\mathbb{R}^N)} \le c(1+t)^{-\frac{N}{2}\left(1-\frac{1}{p}\right)}, \qquad \forall t > 0 \tag{9.7}$$

for all $1 \le p \le \infty$.

The scaling and compactness arguments we have employed allow also describing the first term of the asymptotic development of solutions. Indeed, assuming that the initial data belong to $L^2(1+|x|^{(N+1)/2};\mathbb{R}^N)$ (to obtain the global compactness) and (9.1) is satisfied with $c = 1$ and $q \ge (N+1)/N$ it then follows that

$$(1+t)^{-\frac{N}{2}\left(1-\frac{1}{p}\right)}\|u(t)-v(t)\|_{L^p(\mathbb{R}^N)} \to 0, \qquad \text{as } t\to\infty$$

for all $1\le p\le\infty$ where v is the solution of

$$\begin{cases} v_t - \Delta v + \mathbf{a}\cdot\nabla(|v|^{q-1}v) = 0, \qquad (x,t)\in\mathbb{R}^N\times\mathbb{R}^+, \\ v(x,0) = \displaystyle\int_{\mathbb{R}^N}(u_0+u_1)dx\,\delta_0, \qquad \text{in } \mathbb{R}^N \end{cases} \tag{9.8}$$

when $q = (N+1)/N$ and of

$$\begin{cases} v_t - \Delta v = 0, & (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \\ v(x,0) = \displaystyle\int_{\mathbb{R}^N} (u_0 + u_1) dx\, \delta_0, & \text{in } \mathbb{R}^N \end{cases} \tag{9.9}$$

when $q > (N+1)/N$.

The same techniques allow obtaining similar results for the hyperbolic version of the Navier-Stokes equations analyzed in [2]:

$$\begin{cases} u_{tt} - u_t - \Delta u + u \cdot \nabla u = \nabla p, & (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\ \operatorname{div}(u) = 0. \end{cases} \tag{9.10}$$

Note that the quadratic nonlinearity in (9.10) corresponds to $q = 2$ in our previous analysis. When $N = 2$, $q = 2 > (N+1)/N = 3/2$. Consequently, small solutions of (9.10) behave as $t \to \infty$ as the solutions of the linear Stokes equation which are given by the fundamental solution of the heat equation. This fact was already observed for the Navier-Stokes in dimension $N = 2$ in [4].

Some open problems. The methods we have developed in this paper do not allow to describe the asymptotic behavior of solutions when $1 < q < (N+1)/N$. To fix ideas, consider the case $N = 1$ so that the new range of exponents becomes $1 < q < 2$.

When $1 < q < 2$, in [7] it was proved that the solutions of for the convection-diffusion equation (9.3) with integrable initial data behave as $t \to \infty$ as the entropy solution of

$$\begin{cases} v_t + a(|v|^{q-1}v)_x = 0, & \text{in } \mathbb{R} \times (0,\infty) \\ v(x,0) = \displaystyle\int_{\mathbb{R}} u_0 dx\, \delta_0, & \text{in } \mathbb{R}. \end{cases} \tag{9.11}$$

The first difficulty encountered in [7] for proving this result was to show that the solutions of the convection-diffusion equation (9.3) decay as fast as the solutions of the hyperbolic conservation law (9.11), namely, that

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq c(1+t)^{-\frac{1}{q}}, \qquad \forall t > 0. \tag{9.12}$$

Note that the decay rate in (9.12) is faster than that of the solutions of (9.3) in the range $1 < q < 2$.

When dealing with the damped wave equation in this range of nonlinearities (i.e. $1 < q < 2$) the methods of this paper do not yield a better decay than that in Theorem 2.1. This decay rate is insufficient to be able to apply the scaling of the hyperbolic conservation law (9.11) ($u_\lambda(x,t) = \lambda u(\lambda x, \lambda^q t)$) and to obtain a uniformly bounded family of scaled solutions. On the other hand, the entropy inequalities used in [7] when dealing with (9.3) do not seem to apply either. Consequently obtaining the decay rate (9.12) for solutions of (9.3) with $N = 1$ and $1 < q < 2$ and small initial data is an interesting open problem. The same can be said about the multidimensional version in the range $1 < q < (N+1)/N$. Once this sharp decay rate were obtained it would not be difficult to apply the scaling arguments in [7] and [9] together with the techniques developed in this article to show that solutions behave as the entropy solutions of the hyperbolic conservation law in 1-d or of the convection-diffusion equation with partial diffusivity of the form (1.7) in several space dimensions.

Another open problem to be studied is the behavior of solutions for intermediate data for which neither the global existence nor the blow up results in this paper apply. This region, so-called as threshold, has been studied in [13] for a Burgers equation with a convolution term, describing the motion of radiating gas in therm-nonequilibrium.

A Appendix: Linear estimates

In this appendix we prove Lemmas 3.2 and 3.3. The proofs can be carried out by means of a careful analysis of the Fourier transform of solutions.

Using the Fourier transform, the equation (3.1) can be written as the following Cauchy problem:

$$\begin{cases} \widehat{u}_{tt} + |\xi|^2\widehat{u} + \widehat{u}_t = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ \widehat{u}(\xi,0) = \widehat{\varphi^0}(\xi), & \widehat{u}_t(\xi,0) = \widehat{\varphi^1}(\xi). \end{cases} \tag{A.1}$$

Define $\widehat{v}(\xi,t) = \widehat{u}(\xi,t)\chi_{|\xi|\le 1/4}$ and $\widehat{w}(\xi,t) = \widehat{u}(\xi,t)\chi_{|\xi|>1/4}$ where χ stands for the characteristic function. Using the inverse Fourier transform one obtains a decomposition of the solution u of (3.1) itself $u = v + w$, where v and w denote respectively the inverse Fourier transform of $\widehat{u}$ and $\widehat{v}$.

Lemma A.1 *Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}^N_+$. Then,*

$$\|\partial_x^\alpha \partial_t^k w(\cdot,t)\|_2 \le ce^{-\omega t}\left[\|\varphi^0\|_{\dot{H}^{k+|\alpha|}} + \|\varphi^1\|_{\dot{H}^{k+|\alpha|-1}}\right], \qquad \forall t \ge 0. \tag{A.2}$$

Proof. In order to prove (A.2), we consider the Lyapunov function $L(\xi,t) = E(\xi,t) + \varepsilon F(\xi,t)$, where

$$E(\xi,t) = \frac{1}{2}\left(|\partial_t \widehat{w}(\xi,t)|^2 + |\xi|^2|\widehat{w}(\xi,t)|^2\right), \qquad F(\xi,t) = \partial_t\widehat{w}(\xi,t)\overline{\widehat{w}(\xi,t)} + \frac{1}{2}|\widehat{w}(\xi,t)|^2.$$

The value ε will be chosen later such that we obtain $\partial_t L(\xi,t) \le -cL(\xi,t)$ for some positive constant $c > 0$. Since $|\xi| > 1/4$, we get

$$\begin{aligned} |L(\xi,t) - E(\xi,t)| &\le \varepsilon\left[\frac{|\partial_t\widehat{w}(\xi,t)|^2}{2|\xi|^2} + \frac{1}{2}|\xi|^2|\widehat{w}(\xi,t)|^2 + \frac{1}{2}|\widehat{w}(\xi,t)|^2\right] \\ &\le \varepsilon\left[8|\partial_t\widehat{w}(\xi,t)|^2 + \frac{1}{2}|\xi|^2|\widehat{w}(\xi,t)|^2 + 8|\xi|^2|\widehat{w}(\xi,t)|^2\right] \le 17\varepsilon\, E(\xi,t). \end{aligned}$$

Thus, considering $\varepsilon < 1/17$, we have

$$(1 - 17\varepsilon)E(\xi,t) \le L(\xi,t) \le (1 + 17\varepsilon)E(\xi,t). \tag{A.3}$$

Since $w(\xi,t)$ satisfies (A.1), multiplying by $\overline{\partial_t\widehat{w}(\xi,t)}$, we have

$$\partial_t E(\xi,t) = -|\partial_t\widehat{w}(\xi,t)|^2.$$

Multiplying by $\overline{\widehat{w}(\xi,t)}$, we obtain $\partial_t F(\xi,t) = -|\xi|^2|\widehat{w}(\xi,t)|^2 + |\partial_t\widehat{w}(\xi,t)|^2$. Adding the previous equalities and thanks to the fact that $\varepsilon < 1/17$, we get

$$\partial_t L(\xi,t) \le -2\varepsilon E(\xi,t).$$

Then, using (A.3) we prove $\partial_t L(\xi,t) \le -cL(\xi,t)$ with $c = 2\varepsilon/(1+17\varepsilon)$. Moreover, by (A.3), we get $E(\xi,t) \le c'e^{-ct}E(\xi,0)$, and prove (A.2) when $(|\alpha|,k) = (0,0)$, $(0,1)$ and $(1,0)$. We conclude the proof of (A.2) thanks to the classical property of the Fourier transform guaranteeing that $\|\partial_x^\alpha w\|_2 = \||\cdot|^{|\alpha|}\widehat{w}\|_2$. ▪

Now, we can decompose $\widehat{v}$ as $\widehat{v}(\xi,t) = \widehat{p}(\xi,t) + \widehat{q}(\xi,t)$, where, for $|\xi| \le 1/4$,

$$\begin{aligned} \widehat{p}(\xi,t) &= \left[\widehat{\varphi}^0(\xi)\big(\frac{1}{2} + r(\xi)\big) + \widehat{\varphi}^1(\xi)\right]\frac{e^{-\frac{t}{2}}}{2r(\xi)}e^{r(\xi)t}, \\ \widehat{q}(\xi,t) &= -\left[\widehat{\varphi}^0(\xi)\big(\frac{1}{2} - r(\xi)\big) + \widehat{\varphi}^1(\xi)\right]\frac{e^{-\frac{t}{2}}}{2r(\xi)}e^{-r(\xi)t}, \end{aligned}$$

with

$$r(\xi) = \sqrt{\frac{1}{4} - |\xi|^2}, \quad \text{for } |\xi| \leq \frac{1}{4}.$$

Let p and q be the inverse Fourier transforms of $\widehat{p}$ and $\widehat{q}$, respectively. We have:

Lemma A.2 *Let $k = 0, 1$ and $\alpha \in \mathbb{R}^N_+$. Then,*

$$\|\partial_x^\alpha \partial_t^k q(\cdot, t)\|_2 \leq c e^{-\frac{t}{2}} \left[\|\varphi^0\|_{\dot{H}^{|\alpha|+k}} + \|\varphi^1\|_{\dot{H}^{|\alpha|-1+k}}\right], \qquad \forall t \geq 0, \tag{A.4}$$

and moreover, when $k + |\alpha| < 1$, in (A.4) the homogeneous space $\dot{H}^{k+|\alpha|-1}$ must be replaced by the Sobolev space $H^{k+|\alpha|-1}$.

Proof. We observe that for $|\xi| \leq 1/4$, $r(\xi)$ is positive, and

$$|\partial_t^k \widehat{q}(\xi, t)| \leq \left|\widehat{\varphi}^0(\xi)\big(\frac{1}{2} - r(\xi)\big) + \widehat{\varphi}^1(\xi)\right| \frac{1}{2r(\xi)}\big(\frac{1}{2} + r(\xi)\big)^k e^{-\frac{t}{2}}.$$

Thus, q decays exponentially. Since $|\xi| \leq 1/4$, we get

$$2 \leq \frac{1}{r(\xi)} \leq \frac{4}{\sqrt{3}} \quad \text{and} \quad \frac{3}{4} \leq \frac{1}{2} + r(\xi) \leq 1.$$

Moreover, since

$$\frac{3}{4}\big(\frac{1}{2} - r(\xi)\big) \leq \big(\frac{1}{2} - r(\xi)\big)\big(\frac{1}{2} + r(\xi)\big) = |\xi|^2,$$

then

$$\big(\frac{1}{2} - r(\xi)\big) \leq \frac{4}{3}|\xi|^2 \qquad \forall |\xi| \leq \frac{1}{4}.$$

Using these inequalities and Parseval's identity, the decay estimate (A.4) is obtained immediately. ∎

Finally, we prove the following estimate:

Lemma A.3 *Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}^N_+$. Then,*

$$\|\partial_t^k \partial_x^\alpha p(\cdot, t)\|_2 \leq c(1+t)^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{2})}(1+t)^{-\frac{2k+|\alpha|}{2}} \left[\|\varphi^0\|_a + \|\varphi^1\|_a\right], \qquad \forall t \geq 0. \tag{A.5}$$

Combining this estimate, (A.2) and (A.4), we conclude the proof of Lemma 3.2.

Proof of Lemma A.3. We observe that

$$|\partial_t^k \widehat{p}(\xi, t)| \leq c\left|\widehat{\varphi}^0(\xi)\big(\frac{1}{2} + r(\xi)\big) + \widehat{\varphi}^1(\xi)\right| \frac{1}{r(\xi)}\big(-\frac{1}{2} + r(\xi)\big)^k e^{[-\frac{1}{2} + r(\xi)]t}.$$

Since $|\xi| \leq 1/4$, we know that $-3|\xi|^2/4 \leq r(\xi) - 1/2 \leq -|\xi|^2$, and therefore,

$$|\partial_t^k \widehat{p}(\xi, t)| \leq c\left(|\widehat{\varphi}^0(\xi)| + |\widehat{\varphi}^1(\xi)|\right)|\xi|^{2k} e^{-|\xi|^2 t}.$$

Using Parseval's identity, we obtain

$$\|\partial_x^\alpha \partial_t^k p(\cdot, t)\|_2^2 \leq c \int_{|\xi| \leq \frac{1}{4}} \left(|\widehat{\varphi}^0(\xi)|^2 + |\widehat{\varphi}^1(\xi)|^2\right)|\xi|^{2|\alpha| + 4k} e^{-2|\xi|^2 t} d\xi.$$

Now, by Hölder's inequality,

$$\|\partial_x^\alpha \partial_t^k p(\cdot,t)\|_2 \le c(\|\widehat{\varphi}^0\|_{2p} + \|\widehat{\varphi}^1\|_{2p}) \left(\int_{\mathbb{R}^N} |\xi|^{2|\alpha|p'+4kp'} e^{-2p'|\xi|^2 t} d\xi \right)^{\frac{1}{2p'}} \quad \text{with} \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

On the other hand, it is easy to check (see [1], p. 263)

$$\int_{|\xi|\le\delta} e^{-t\omega|\xi|^2} |\xi|^k d\xi \ \le\ c(\omega,\delta,k)(1+t)^{-\frac{k+N}{2}}.$$

Combining these two inequalities, we get

$$\|\partial_x^\alpha \partial_t^k p(\cdot,t)\|_2 \quad\le\quad c(\|\widehat{\varphi}^0\|_{2p} + \|\widehat{\varphi}^1\|_{2p})\,(1+t)^{-\frac{N}{4p'}}(1+t)^{-\frac{2k+|\alpha|}{2}}, \qquad \forall t \ge 0.$$

Finally, choosing $1 \le a \le 2$ such that $1/a + 1/(2p) = 1$, taking into account that $1/2p' = 1/a - 1/2$, and using Hausdorff-Young's inequality (see [16], pp. 328), we prove (A.5). ∎

Finally, we prove Lemma 3.3.

Proof of the Lemma 3.3. By Lemma 3.2, we have

$$\begin{aligned} \|v\|_2 \quad &\le \quad ce^{-\omega t}\left(\|x\varphi^0\|_2 + \|x\varphi^1\|_{-1,2}\right) + c'(1+t)^{\frac{1}{4}-\frac{1}{2b}}\left(\|x\varphi^0\|_b + \|x\varphi^1\|_b\right) + \\ &\qquad +2\int_0^t \|S_x(t-s)[0,u(s)]\|_2 ds. \end{aligned} \tag{A.6}$$

By the estimate (3.2) with ($\varphi^0 = 0, \varphi^1 = u(s)$) and ($a = 2, k = 0, \alpha = 1$), we have

$$\int_0^t \|S_x(t-s)[0,u(s)]\|_2 ds \le \int_0^t (1+t-s)^{-\frac{1}{2}} \|u(s)\|_2 ds.$$

Applying estimate (3.3) for ($\alpha = 0, k = 0$), we obtain

$$\begin{aligned} \int_0^t \|S_x(t-s)[0,u(s)]\|_2 ds \ &\le c(\|\varphi^0\|_2 + \|\varphi^1\|_{-1,2}) \int_0^t (1+t-s)^{-\frac{1}{2}} e^{-\omega s} ds \\ &\quad + c'\left(\|\varphi^0\|_a + \|\varphi^1\|_a\right) \int_0^t (1+t-s)^{-\frac{1}{2}} (1+s)^{\frac{1}{4}-\frac{1}{2a}} ds \\ &\le c(\|\varphi^0\|_2 + \|\varphi^1\|_{-1,2})(1+t)^{-\frac{1}{2}} + c'(\|\varphi^0\|_a + \|\varphi^1\|_a)(1+t)^{\frac{3}{4}-\frac{1}{2a}}, \end{aligned}$$

thanks to (2.11). Then, returning to (A.6), we prove (3.5).

Note that $xu_x = v_x - u$. Since the decay of $\|u(t)\|_2$ is known by (3.2), we only need to obtain the behavior of v_x to get (3.6). Using the variation of constants formula and the estimate (3.3), we get

$$\begin{aligned} \|v_x\|_2 \quad &\le \quad ce^{-\omega t}\left(\|x\varphi^0\|_{\dot{H}^1} + \|x\varphi^1\|_2\right) + c'(1+t)^{-\frac{1}{4}-\frac{1}{2b}}\left(\|x\varphi^0\|_b + \|x\varphi^1\|_b\right) \\ &\qquad +2\int_0^t \|S_x(t-s)[0,u_x(s)]\|_2 ds. \end{aligned} \tag{A.7}$$

By (3.2) with ($a = 2, k = 0, \alpha = 1$), we have

$$\int_0^t \|S_x(t-s)[0,u_x(s)]\|_2 ds \le c\int_0^t (1+t-s)^{-\frac{1}{2}} \|u_x(s)\|_2 ds.$$

Applying estimate (3.3) with $k = 0$ and $\alpha = 1$, we get

$$\begin{aligned}\int_0^t \|S_x(t-s)[0,u_x(s)]\|_2 ds &\leq c(\|\varphi^0\|_{\dot{H}^1} + \|\varphi^1\|_2)\int_0^t (1+t-s)^{-\frac{1}{2}} e^{-\omega s} ds \\ &\quad + c'\left(\|\varphi^0\|_a + \|\varphi^1\|_a\right)\int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{4}-\frac{1}{2a}} ds \\ &\leq c(\|\varphi^0\|_{\dot{H}^1} + \|\varphi^1\|_2)(1+t)^{-\frac{1}{2}} + c'(\|\varphi^0\|_a + \|\varphi^1\|_a)(1+t)^{\frac{1}{4}-\frac{1}{2a}},\end{aligned}$$

by (2.11). Then, coming back to (A.7) we obtain (3.6). ▪

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