Large time behavior in $\mathbb{R}^N$ for linear parabolic equations with periodic coefficients

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Abstract

In this paper we study the asymptotic behavior of solutions of linear parabolic equations in $\mathbb{R}^N$ with periodic coefficients and L^1 initial data, as $t \to \infty$. It was already known that, in a first approximation, solutions behave as the fundamental solution of the homogenized system. We use the Bloch waves decomposition to obtain a complete expansion of the solutions as $t \to \infty$.

1 Introduction

This work is devoted to the analysis of the asymptotic behavior as $t \to +\infty$ of the solutions of the linear parabolic equation:

$$\begin{cases} u_t - \dfrac{\partial}{\partial x_j}\left(a_{jk}\dfrac{\partial u}{\partial x_k}\right) = 0, & \text{in } \mathbb{R}^N \times (0,+\infty) \\ u(\cdot,0) = \varphi, & \text{in } \mathbb{R}^N. \end{cases} \tag{1}$$

The coefficients $a = (a_{jk})_{1\le j,k\le N}$ are assumed to be bounded, symmetric, coercive and periodic. More precisely we set $Y = [0,2\pi)^N$ and we denote by $L^\infty_\#(Y)$ the subspace of $L^\infty(\mathbb{R}^N)$ of functions which are $Y-$periodic:

$$L^\infty_\#(Y) = \left\{\phi \in L^\infty(\mathbb{R}^N) : \phi(x+2\pi p) = \phi(x), \forall x \in \mathbb{R}^N, \forall \mathrm{p} \in \mathbb{Z}^N\right\}.$$

We assume that

$$a_{jk} \in L^\infty_\#(Y), \quad \forall\, 1 \le j,k \le N. \tag{2}$$

Moreover, we assume that a is symmetric:

$$a_{jk} = a_{kj}, \quad \forall\, 1 \le j,k \le N. \tag{3}$$

Finally we assume that there exists $\alpha > 0$ such that

$$\sum_{j,k=1}^{N} a_{jk}\xi_j\xi_k \geq \alpha \left|\xi\right|^2, \quad \forall x \in \mathbb{R}^N, \xi \in \mathbb{R}^N. \tag{4}$$

We are interested in the asymptotic behavior of solutions of (1) with initial data in $L^1\left(\mathbb{R}^N\right)$.

It is well known that for any $\varphi \in L^1\left(\mathbb{R}^N\right)$ system (1) admits a unique solution such that:

$$u \in C\left([0,\infty); L^1\left(\mathbb{R}^N\right)\right) \cap L^\infty_{loc}\left(0,\infty; L^p\left(\mathbb{R}^N\right)\right), \quad \text{for any } 1 \leq p \leq \infty; \tag{5}$$

$$\int_{\mathbb{R}^N} u\left(x,t\right) dx = \int_{\mathbb{R}^N} \varphi\left(x\right) dx, \forall t > 0 \quad \text{(conservation of mass)}; \tag{6}$$

$$\left\|u\left(t\right)\right\|_p \leq C_p t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)} \left\|\varphi\right\|_1, \ \forall t > 0, \ \forall 1 \leq p \leq \infty. \tag{7}$$

When the initial datum φ belongs to $L^q\left(\mathbb{R}^N\right)$ the following $L^p - L^q$ estimate holds:

$$\left\|u\left(t\right)\right\|_p \leq C(p,q) t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \left\|\varphi\right\|_q, \ \forall t > 0, \ \forall q \leq p \leq \infty. \tag{8}$$

Here and in the sequel by $\left\|\cdot\right\|_p$ we denote the norm in $L^p\left(\mathbb{R}^N\right)$.

In this paper we derive a complete asymptotic expansion of solutions of (1) as $t \to \infty$. The first term in the asymptotic expansion was obtained in [10] and [11]. It was shown that

$$t^{\left(\frac{N}{2}\right)\left(1-\frac{1}{p}\right)} \left\|u\left(t\right) - MG_h\left(t\right)\right\|_p \to 0, \quad \text{as } t \to \infty, \tag{9}$$

for all $1 \leq p \leq \infty$ with $M = \int_{\mathbb{R}^N} \varphi\left(x\right) dx$, G_h being the fundamental solution of the heat equation with homogenized coefficients $q = \left(q_{jk}\right)_{1\leq j,k\leq N}$:

$$\begin{cases} u_t - q_{kj}\dfrac{\partial^2 u}{\partial x_j \partial x_k} & = 0, \quad \text{in } \mathbb{R}^N \times (0,+\infty) \\ u\left(\cdot,0\right) & = \delta_0, \quad \text{in } \mathbb{R}^N, \end{cases} \tag{10}$$

δ_0 being the Dirac delta at the origin.

Let us recall that the homogenized coefficients associated to the periodic matrix a are given by (see [2, 18])

$$q_{jk} = \frac{1}{|Y|}\int_Y a_{jk} dy + \frac{1}{|Y|}\int_Y a_{jm}\frac{\partial \chi_j}{\partial y_m} dy, \quad \forall j,k = 1,\ldots,N, \tag{11}$$

χ_k being the solution of the Y-periodic elliptic problem

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(a_{jl}\dfrac{\partial \chi_k}{\partial x_l}\right) = \dfrac{\partial a_{kl}}{\partial y_l} \\ \\ \chi_k \text{ is } Y\text{-periodic}, \forall k = 1,\ldots,N. \end{cases} \tag{12}$$

Note that the solution χ_k of (12) is uniquely determined up to an additive constant. However, the value of this constant does not affect the definition of q_{jk} in (11).

Let us briefly recall how homogenization theory enters in the description of the asymptotic behavior of (1).

In view of (6) and (7) it is natural to introduce the following scaling of solutions u of (1):

$$u_\lambda = \lambda^N u\left(\lambda x, \lambda^2 t\right), \quad \lambda > 0. \tag{13}$$

Indeed, $u_\lambda(t)$ is uniformly bounded in $L^p(\mathrm{I\!R}^N)$ as $\lambda \to \infty$ for all $1 \le p \le \infty$ and $t > 0$. Moreover u_λ solves

$$\begin{cases} u_{\lambda,t} - \dfrac{\partial}{\partial x_j}\left(a_{\lambda,jk}(x)\dfrac{\partial u_\lambda}{\partial x_k}\right) = 0, & \text{in } \mathrm{I\!R}^N \times (0,+\infty) \\ u_\lambda(\cdot,0) = \lambda^N \varphi(\lambda x), & \text{in } \mathrm{I\!R}^N, \end{cases} \tag{14}$$

with

$$a_\lambda(x) = a\left(\frac{x}{\lambda}\right). \tag{15}$$

The convergence result (9) is equivalent to the fact that

$$u_\lambda(\cdot,1) \to MG_h(\cdot,1) \text{ in } L^p\left(\mathrm{I\!R}^N\right), \text{ as } \lambda \to \infty. \tag{16}$$

On the other hand, the initial data in (14) converges to $M\delta_0$ in the weak sense of measures as $\lambda \to \infty$, while the coefficients $a_\lambda(x)$ oscillate more and more rapidly. Therefore, according to the classical theory of homogenization, (16) is natural to be expected.

In order to understand the type of the asymptotic expansion one may expect, it is convenient to analyze first the constant coefficient heat equation:

$$\begin{cases} u_t - \triangle u = 0, & \text{in } \mathrm{I\!R}^N \times (0,+\infty) \\ u(\cdot,0) = \varphi, & \text{in } \mathrm{I\!R}^N. \end{cases} \tag{17}$$

In [9] it was proved that, when $\varphi \in L^1\left(\mathrm{I\!R}^N; 1+|\mathrm{x}|^{\mathrm{k+1}}\right)$ for some $k \in \mathrm{I\!N}$, the following holds:

$$\left\| u(\cdot,t) - \sum_{|\alpha|\le k} \frac{(-1)^{|\alpha|}}{\alpha!}\left(\int_{\mathrm{I\!R}^N} \varphi(x)\, x^\alpha dx\right) \partial^\alpha G(\cdot,t) \right\|_q \le t^{-\left(\frac{N}{2}\right)\left(\frac{(k+1)}{N}+\frac{1}{p}-\frac{1}{q}\right)} \left\| |x|^{k+1}\varphi \right\|_p, \tag{18}$$

G being the fundamental solution of the heat equation

$$G(x,t) = (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right). \tag{19}$$

The estimate (18) is a direct consequence of the application of Young's inequality and the following decomposition formula (we refer to [9] for further developments in this direction):

$$\varphi = \sum_{|\alpha|\le k} \frac{(-1)^{|\alpha|}}{\alpha!}\left(\int_{\mathrm{I\!R}^N} \varphi(x)\, x^\alpha dx\right) \partial^\alpha \delta_0 + \sum_{|\alpha|=k+1} \partial^\alpha F_\alpha, \tag{20}$$

with $F_\alpha \in L^1\left(\mathrm{I\!R}^N\right)$ such that

$$\|F_\alpha\|_1 \le C(k,N)\, \|\varphi\|_{1,|x|^{k+1}}. \tag{21}$$

Here and in the sequel $\|\cdot\|_{1,|x|^{k+1}}$ denotes the norm in $L^1\left(\mathrm{I\!R}^N; |\mathrm{x}|^{\mathrm{k+1}}\right)$, that is

$$\|\varphi\|_{1,|x|^{k+1}} = \int_{\mathrm{I\!R}^N} |\varphi(x)|\, |x|^{k+1} dx.$$

The convergence result (18) indicates that, roughly, the solution u of the heat equation (17) may be approximated at any order by a linear combination of the derivatives of the fundamental solution of the heat equation, the coefficients being explicitly given by the moments of the initial datum.

When $k = 0$, (18) is the analogue of (9) since the first term in (18), corresponding to $\alpha = 0$, is the fundamental solution times the total mass of the solution.

The goal of this work is to obtain results of the form (18) for solutions of (1) with periodic coefficients. To do that we use Bloch wave expansions, following closely the work of Conca and Vanninathan [6] which shows how classical homogenization results may be recovered using Bloch wave expansions for elliptic equations. As we shall see, when deriving higher order asymptotic results for (1) two types of terms appear. First, as in the case of constant coefficients, we get those terms that are provided by the moments of the initial datum and then, due to the homogenization process, those that are generated by the microstructure. This second contribution may be obtained by a careful analysis of the first Bloch mode. Here, as in the elliptic results of [6], the contribution of higher Bloch modes may be neglected since they provide terms that decay exponentially as $t \to \infty$.

The Bloch wave expansion provides an orthogonal decomposition of $L^2\left(\mathbb{R}^N\right)$. Thus, our first results will be established in the L^2-setting. Then, we will derive convergence results in $L^\infty\left(\mathbb{R}^N\right)$.

The rest of this work is organized as follows. In section 2 we recall some preliminary results that will be needed to prove rigorously our results. In particular we recall some basic facts about Bloch wave expansions. In section 3 we state without proof the main results of this paper distinguishing those that are valid in $L^2\left(\mathbb{R}^N\right)$ from those that apply in $L^\infty\left(\mathbb{R}^N\right)$. Sections 4 and 5 are devoted to the proof of the L^2 and L^∞ results respectively. Finally, in section 6 we describe how the various terms appearing in the asymptotic expansion may be computed explicitly.

2 Preliminaries

In this section we recall some basic results of Bloch wave expansions and asymptotic analysis that will be needed when stating and proving rigorously the main results of this paper.

2.1 Bloch wave expansions

In this subsection we recall some basic results on Bloch wave expansions. We refer to [6] for the proofs.

We denote by $L^2_\#(Y)$ the (complex) Hilbert space of L^2 periodic functions:

$$L^2_\#(Y) \doteq \left\{ v \in L^2_{loc}\left(\mathbb{R}^N\right) \,:\, v \text{ is } Y\text{-periodic} \right\}. \tag{22}$$

We also introduce the $H^1_\#(Y)$ space:

$$H^1_\#(Y) \doteq \left\{ v \in L^2_\#(Y) \,:\, \frac{\partial v}{\partial y_k} \in L^2_\#(Y),\, \forall k = 1, \ldots, N \right\}. \tag{23}$$

These spaces are endowed with the canonical inner products:

$$(v,w)_{L^2_\#(Y)} = \int_Y v(y)\,\overline{w}(y)\,dy,$$

$$(v,w)_{H^1_\#(Y)} = \int_Y \left[\nabla v(y)\cdot\nabla\overline{w}(y) + v(y)\,\overline{w}(y)\right] dy.$$

Let Y' be the dual cell: $Y' = \left[-\frac{1}{2},\frac{1}{2}\right)^N$. For any $\xi \in Y'$ we consider the second order operator:

$$A(\xi) = -\left(\frac{\partial}{\partial y_k} + i\xi_k\right)\left[a_{kl}\left(\frac{\partial}{\partial y_l} + i\xi_l\right)\right]. \tag{24}$$

When $\xi = 0$ we set $A(\xi) = A$.
We also consider the eigenvalue problem

$$\begin{cases} A(\xi)\,\phi = \lambda\phi, & \text{in } \mathbb{R}^N \\ \phi \text{ is } Y-\text{periodic.} \end{cases} \tag{25}$$

From the ellipticity and symmetry assumptions on the matrix a it is easy to see that for each $\xi \in Y'$ the eigenvalue problem (25) admits a sequence of eigenvalues

$$0 \le \lambda_1(\xi) \le \ldots \le \lambda_m(\xi) \le \ldots \to +\infty,$$

such that the corresponding sequence of eigenfunctions $\{\phi_m(\xi)\}_m$ may be normalized to constitute an orthonormal basis of $L^2_\#(Y)$.

Moreover, as a simple consequence of the min-max principle we have that

$$\ldots \ge \lambda_m(\xi) \ge \ldots \ge \lambda_2(\xi) \ge \lambda_2^{(N)} > 0, \forall \xi \in Y', \tag{26}$$

where $\lambda_2^{(N)}$ is the second eigenvalue of A in the cell Y with Neumann boundary conditions on ∂Y.

In the literature, $\lambda_m(\xi)$ are referred as *Bloch eigenvalues* and $\phi_m(\xi)$ as *Bloch eigenfunctions* or *Bloch waves.*

The following result provides the classical Bloch wave decomposition of $L^2(\mathbb{R}^N)$:

Theorem 2.1 [6, Theorem 2.1] *Let $g \in L^2(\mathbb{R}^N)$ and let us define the m^{th} Bloch coefficient of g as follows:*

$$\widehat{g}_m(\xi) = \int_{\mathbb{R}^N} g(x)\, e^{-i\xi\cdot x}\overline{\phi}_m(x;\xi)\,dx, \qquad m \in N, \xi \in Y', \tag{27}$$

for all $m \in \mathbb{N}$ and $\xi \in Y'$.

Then, we have the following inverse formula:

$$g(x) = \sum_{m=1}^{\infty}\int_{Y'} \widehat{g}_m(\xi)\, e^{i\xi\cdot x}\phi_m(x;\xi)\,d\xi. \tag{28}$$

Furthermore, the following Parseval's identity holds:

$$\int_{\mathbb{R}^N} |g(x)|^2\,dx = \sum_{m=1}^{\infty}\int_{Y'} |\widehat{g}_m(\xi)|^2\,d\xi. \tag{29}$$

Remark 2.1 *Theorem 2.1 guarantees that the set* $\left\{e^{i\xi\cdot x}\phi_m(\xi) : m \geq 1,\ \xi \in Y'\right\}$ *forms a basis of* $L^2\left(\mathbb{R}^N\right)$ *in a generalized sense. Moreover* $L^2\left(\mathbb{R}^N\right)$ *may be identified with* $L^2\left(Y', l^2(\mathbb{N})\right)$ *via Parseval's identity (28), (29).*

We have the following results on the dependence of the eigenvalues with respect to the parameter ξ:

Theorem 2.2 [6, Theorems 2.2 and 2.7] *For all* $m \geq 1$, $\lambda_m(\xi)$ *is a Lipschitz function of* ξ.

Furthermore, in a suitable neighborhood U *of the origin, the first eigenvalue* $\lambda_1(\xi)$ *remains simple and depends on* ξ *analytically.*

Remark 2.2 *As a consequence of Theorem 2.2, we deduce that* $\xi \longrightarrow \phi_1(x,\xi) \in H^1_\#(Y)$ *is also analytic in a neighborhood of* $\xi = 0$ *(see [6, 20]). Moreover* $\phi_1(x;0) = |Y|^{-\frac{1}{2}}$.

In order to obtain a complete asymptotic expansion of the solutions of the parabolic equation, we need further results on the behavior of $\lambda_1(\xi)$ and $\phi_1(x;\xi)$ near $\xi = 0$.

It is easy to see that the homogenized matrix $q = (q_{kl})_{k,l}$ introduced in (11) is symmetric, i. e.

$$q_{kl} = q_{lk}, \tag{30}$$

and elliptic with the ellipticity constant of the matrix a in (3), i. e.,

$$q_{kl}\xi_k\xi_l \geq \alpha |\xi|^2 \quad \forall \xi \in \mathbb{R}^N. \tag{31}$$

We shall denote by A_h the homogenized elliptic constant coefficient operator

$$A_h u = -\frac{\partial}{\partial x_k}\left(q_{kl}\frac{\partial u}{\partial x_l}\right). \tag{32}$$

Remark 2.3 *The homogenized operator* A_h *arises naturally in most homogenization problems. For instance, when* $f \in H^{-1}\left(\mathbb{R}^N\right)$, *the solutions* u_ε *of*

$$\begin{cases} -\dfrac{\partial}{\partial x_k}\left(a_{kl}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial u_\varepsilon}{\partial x_l}\right) + u_\varepsilon = f, & \text{in } \mathbb{R}^N \\ u_\varepsilon \in H^1\left(\mathbb{R}^N\right), & \end{cases} \tag{33}$$

are such that

$$u_\varepsilon \rightharpoonup u^* \text{ as } \varepsilon \to 0, \text{ weakly in } H^1\left(\mathbb{R}^N\right), \tag{34}$$

where u^* *solves the homogenized problem*

$$\begin{cases} A_h u^* + u^* = f, & \text{in } \mathbb{R}^N \\ u^* \in H^1\left(\mathbb{R}^N\right). & \end{cases} \tag{35}$$

We refer to [2, 6, 18] and references therein.

The following holds:

Proposition 2.1 [6, Proposition 3.7] *$\xi=0$ is a critical point of the first eigenvalue, i. e.*

$$\frac{\partial\lambda_1}{\partial\xi_k}(0)=0,\quad \forall k=1,\dots,N. \tag{36}$$

Furthermore, the Hessian of λ_1 at $\xi=0$ satisfies

$$\frac{1}{2}\frac{\partial^2\lambda_1}{\partial\xi_k\xi_l}(0)=q_{kl},\quad \forall k,l=1,\dots,N. \tag{37}$$

The derivatives of the first Bloch eigenfunction $\phi_1(x,\xi)$ satisfy:

$$\frac{\partial\phi_1}{\partial\xi_k}(y;0)=i\,|Y|^{-\frac{1}{2}}\chi_k(y),\quad \forall k=1,\dots,N. \tag{38}$$

Corollary 2.1 *We have*

$$\lambda_1(\xi)=q_{kl}\xi_k\xi_l+\mathcal{O}\left(|\xi|^3\right),\quad \forall\xi\in U, \tag{39}$$

in a neighborhood U of $\xi=0$.

Proof. (39) is an immediate consequence of Proposition 2.1, the fact that $\lambda_1(0)=0$, $\partial_\xi\lambda_1(0)=0$ and the analyticity of λ_1 with respect to ξ at $\xi=0$.

The following result will also be needed.

Lemma 2.1 *The map $\xi\in Y'\to\lambda_1(\xi)\in\mathrm{I\!R}$ has a strict global minimum at $\xi=0$ where $\lambda_1(0)=0$.*

Furthermore, there exists $c>0$ such that

$$\lambda_1(\xi)\geq c\,|\xi|^2,\quad \forall\xi\in\overline{Y'}. \tag{40}$$

Proof. Firstly, we can see that for every $\xi\neq 0$, $\lambda_1(\xi)>0$.

In fact, for each $\xi\in Y'$, the problem

$$\left\{\begin{array}{lll} A(\xi)\,\phi & =\lambda_1(\xi)\phi, & \text{in } Y\\ \phi & \text{is } Y-\text{periodic} & \end{array}\right. \tag{41}$$

is equivalent to

$$\left\{\begin{array}{lll} A\psi & =\lambda_1(\xi)\psi, & \text{in } Y\\ \psi & \text{is } (Y,\xi)-\text{periodic,} & \end{array}\right. \tag{42}$$

where $\phi=e^{-i\xi\cdot y}\psi$.

Here $(Y,\xi)-$periodicity means that $e^{-i\xi\cdot y}\psi$ is $Y-$periodic.

Multiplying (42) by ψ and integrating by parts, we obtain that

$$\int_Y a_{jk}(x)\frac{\partial\psi}{\partial x_j}\frac{\partial\overline{\psi}}{\partial x_j}=\lambda_1(\xi)\int_Y\psi\overline{\psi}. \tag{43}$$

Hence, if ψ is normalized in $L^2(Y)$, from the ellipticity of the matrix a_{ij}, we obtain that

$$\alpha\int_Y|\nabla\psi|^2\leq\lambda_1(\xi). \tag{44}$$

If $\lambda_1(\xi) = 0$, for some $\xi \neq 0$, we conclude from (44) that ψ is constant in Y and then

$$\phi_1(\xi) = ce^{-i\xi\cdot y}. \tag{45}$$

But the function $\phi_1(\xi) = ce^{-i\xi\cdot y}$ is Y-periodic only for $\xi = 0$ in Y'. This is in contradiction with the fact that $\xi \neq 0$, $\xi \in Y'$.

On the other hand

$$\lambda_1(\xi) = q_{kl}\xi_k\xi_l + \mathcal{O}\left(|\xi|^3\right) \geq \alpha|\xi|^2 + \mathcal{O}\left(|\xi|^3\right), \quad \text{in } U.$$

Therefore, there exists a constant $\beta > 0$ such that

$$\lambda_1(\xi) \geq \beta|\xi|^2, \quad \xi \in U. \tag{46}$$

In view of (46) and taking into account that $\lambda_1(\xi) > 0$ for all $\xi \in Y', \xi \neq 0$ we deduce that (40) holds.

This completes the proof of Lemma 2.1.

Lemma 2.2 *Let $u(x,t)$ be the solution of (1). Then*

$$u(x,t) = \sum_{m=1}^{\infty} \int_{Y'} e^{-\lambda_m(\xi)t}\widehat{\varphi}_m(\xi)\, e^{i\xi\cdot x}\phi_m(x;\xi)\, d\xi, \tag{47}$$

where $\widehat{\varphi}_m(\xi)$ are the Bloch coefficients of the initial datum φ.

Proof. Let $u(x,t)$ be the solution of (1). Since $u(x,t) \in L^2\left(\mathbb{R}^N\right)$ for all $t > 0$, we have that

$$u(x,t) = \sum_{m=1}^{\infty} \int_{Y'} \widehat{u}_m(\xi,t)\, e^{i\xi\cdot x}\phi_m(x;\xi)\, d\xi, \tag{48}$$

where

$$\widehat{u}_m(\xi,t) = \int_{\mathbb{R}^N} u(x,t)\, e^{-i\xi\cdot x}\overline{\phi}_m(x;\xi)\, dx. \tag{49}$$

Furthermore

$$A\left(e^{-i\xi\cdot x}\overline{\phi}_m(x;\xi)\right) = e^{-i\xi\cdot x}\overline{A(\xi)\,\phi_m(x;\xi)} = \lambda_m(\xi)\, e^{-i\xi\cdot x}\overline{\phi}_m(x;\xi).$$

Thus

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\widehat{u}_m(\xi,t)\right) &= \int_{\mathbb{R}^N} u_t(x,t)\, e^{-i\xi\cdot x}\overline{\phi}_m(x;\xi)\, dx \\
&= -\int_{\mathbb{R}^N} Au(x,t)\, e^{-i\xi\cdot x}\overline{\phi}_m(x;\xi)\, dx \\
&= -\int_{\mathbb{R}^N} u(x,t)\, A\left(e^{-i\xi\cdot x}\overline{\phi}_m(x;\xi)\right) dx \\
&= -\int_{\mathbb{R}^N} \lambda_m(\xi)\, u(x,t)\, e^{-i\xi\cdot x}\overline{\phi}_m(x;\xi)\, dx \\
&= -\lambda_m(\xi)\, \widehat{u}_m(\xi,t).
\end{aligned}$$

That is,

$$\left\{ \begin{array}{rcll} \dfrac{\partial}{\partial t}\left(\widehat{u}_m(\xi,t)\right) + \lambda_m(\xi)\,\widehat{u}_m(\xi,t) & = & 0, & \text{in } Y' \times (0,\infty) \\ \widehat{u}_m(\xi,0) & = & \widehat{\varphi}_m(\xi). & \end{array} \right. \tag{50}$$

Then, the m-th Bloch coefficient of u is

$$\widehat{u}_m(\xi,t) = \widehat{\varphi}_m(\xi)\, e^{-\lambda_m(\xi)t}. \tag{51}$$

This proves (47).

2.2 Some basic asymptotic results

The following definition will be useful to simplify the notation of the following sections:

Definition 2.1 *Given $f, g \in C(\mathbb{R};\mathbb{R})$ we say that f and g are of the same order as $t \to \infty$ and we denote it by $f \sim g$ when*

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1. \tag{52}$$

On the other hand, we say that f is negligible with respect to g as $t \to \infty$ and we denote it by $f \ll g$ if

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0. \tag{53}$$

When f is negligible with respect to $g(t) = t$ we shall write $f(t) = o(t)$. Moreover, we shall write $f(t) = O(t)$ when there exists $c > 0$ and $t_0 > 0$ such that

$$|f(t)| \le c\,|t|, \quad \forall t \ge t_0. \tag{54}$$

In the sequel we shall deal with integrals of the form

$$I(t) = \int_D e^{-t\Phi(x)} f(x)\, dx, \qquad \text{as } t \to \infty$$

and we shall be interested on their asymptotic behavior as $t \to \infty$. Watson's Lemma (see [1, p. 263]) plays a crucial role in our developments and guarantees that, if x_0 is a strict global minimum of ϕ in D, the integral in the exterior of a neighborhood of x_0 is negligible as $t \to \infty$.

In particular the following holds:

Lemma 2.3 [1, p. 263] *Let $f : [0,b] \to \mathbb{R}$ be a continuous function, such that it has the uniform asymptotic series expansion*

$$f(x) = x^{\alpha} \sum_{n=1}^{\infty} a_n x^{\beta_n}, \forall x \in [0,b]$$

with $\alpha > -1$ and $\beta_n > 0$. Then

$$\int_0^b e^{-tx} f(x)\, dx \sim \sum_{n=1}^{\infty} a_n \frac{\Gamma(\alpha + \beta_n + 1)}{t^{(\alpha+\beta_n+1)}}, \qquad \text{as } t \to \infty. \tag{55}$$

In the case of $f(x) = x^\alpha$ *we have that*

$$\int_0^b e^{-tx} x^\alpha dx \sim \frac{\Gamma(\alpha+1)}{t^{(\alpha+1)}}, \qquad \text{as } t \to \infty, \tag{56}$$

where

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds, \quad z > 0.$$

As a consequence of Lemma 2.3 the following holds:

Lemma 2.4 *Let* $c > 0$*. Then*

$$\int_{Y'} e^{-c|\xi|^2 t} |\xi|^k \, d\xi \sim c_k t^{-\frac{k+N}{2}}, \qquad \text{as } t \to \infty, \tag{57}$$

for all $k \in \mathbb{N}$*, where* c_k *is a positive constant that may be computed explicitly.*
On the other hand, if $q = (q_{ij})_{ij}$ *is a symetric positive matrix, we also have*

$$\int_{Y'} e^{-cq_{ij}\xi_i\xi_j t} \xi^\beta d\xi \sim c_\beta t^{-\frac{|\beta|+N}{2}}, \qquad \text{as } t \to \infty, \tag{58}$$

for all multi-indexes $\beta \in (\mathbb{N} \cup \{0\})^N$ *, for a suitable* c_β *that may computed as well.*

Proof. We first observe that

$$\int_{Y'} e^{-c|\xi|^2 t} |\xi|^k \, d\xi = \int_B e^{-c|\xi|^2 t} |\xi|^k \, d\xi + \int_{Y' \setminus B} e^{-c|\xi|^2 t} |\xi|^k \, d\xi, \tag{59}$$

where B is the ball in $\mathbb{R}^{\mathrm{N}}$ with centre in $\xi = 0$ and radius 1/2.
We can see that

$$\int_{Y' \setminus B} e^{-c|\xi|^2 t} |\xi|^k \, d\xi \le e^{-\frac{c}{4}t} \int_{Y' \setminus B} |\xi|^k \, d\xi = c_0 e^{-\frac{c}{4}t}, \quad \forall t > 0. \tag{60}$$

On the other hand, using spherical coordinates we have

$$\int_B e^{-c|\xi|^2 t} |\xi|^k \, d\xi = NC_{N-1} \int_0^{\frac{1}{2}} e^{-cr^2 t} r^{N-1+k} dr, \tag{61}$$

C_{N-1} being the measure of the unit sphere in $\mathbb{R}^{\mathrm{N}}$.
Thus, by means of the change of variable $z = cr^2$ we obtain

$$\begin{aligned} \int_0^{\frac{1}{2}} e^{-cr^2 t} r^{N-1+k} dr &= \frac{1}{2c^{\frac{N+k}{2}}} \int_0^{\frac{c}{4}} e^{-zt} z^{\frac{N-2+k}{2}} dz \\ &\sim \frac{1}{2c^{\frac{N+k}{2}}} \Gamma\left(\frac{N+k}{2}\right) t^{-\frac{N+k}{2}}, \text{ as } t \to \infty, \end{aligned} \tag{62}$$

according to Lemma 2.3. In view of (60) and (62) we deduce that

$$\int_{Y'} e^{-c|\xi|^2 t} |\xi|^k \, d\xi \sim \frac{NC_{N-1}}{2c^{\frac{N+k}{2}}} \Gamma\left(\frac{N+k}{2}\right) t^{-\frac{N+k}{2}}, \text{ as } t \to \infty, \tag{63}$$

which proves (57).

Now, we prove (58).

Since $q = (q_{ij})_{i,j=1}^N$ is a symmetric and positive matrix, there exists an orthogonal matrix P, such that

$$P^{-1}qP = D, \tag{64}$$

where D is a diagonal matrix, that is

$$d_{ij} = 0, \quad \text{if } i \neq j, \qquad d_{ii} > 0, \quad \forall i = 1, \ldots, N. \tag{65}$$

We consider the change of variables

$$z = P\xi. \tag{66}$$

Then we have that

$$\sum_{i,j=1}^{N} q_{ij}\xi_i\xi_j = \sum_{i=1}^{N} d_{ii}z_i^2. \tag{67}$$

Therefore

$$\int_{Y'} e^{-cq_{ij}\xi_i\xi_j t}\xi^\beta d\xi = \int_R e^{-cd_{ii}z_i^2 t} \sum_{|\alpha|=|\beta|} c_\alpha z^\alpha dz, \tag{68}$$

R being the new region obtained from Y' by means of the change of variables.

On the other hand, for each $\alpha \in (\mathbb{N} \cup \{0\})^N$ such that $|\alpha| = |\beta|$, we have

$$\int_R e^{-cd_{ii}z_i^2 t} z^\alpha dz \sim \int_I e^{-cd_{ii}z_i^2 t} z^\alpha dz, \tag{69}$$

where $I = [a_1, b_1] \times \ldots \times [a_n, b_n]$ for any $a_i < 0 < b_i$, $i = 1, \ldots N$, since the integral over $R \setminus I$ decays exponentially.

On the other hand

$$\int_I e^{-cd_{ii}z_i^2 t} z^\alpha dz = \prod_{i=1}^{N} \int_{a_i}^{b_i} e^{-cd_{ii}z_i^2 t} z_i^{\alpha_i} dz_i. \tag{70}$$

From Lemma 2.3 we have that

$$\int_{a_i}^{b_i} e^{-cd_{ii}z_i^2 t} z_i^{\alpha_i} dz_i \sim \frac{1}{2(cd_{ii})^{\frac{(1+\alpha_i)}{2}}} \Gamma\left(\frac{1+\alpha_i}{2}\right) t^{-\frac{(1+\alpha_i)}{2}}, \text{ as } t \to \infty, \tag{71}$$

Therefore we obtain that

$$\begin{aligned} \sum_{|\alpha|=|\beta|} c_\alpha \int_I \exp\left(-c \sum_{1\le i\le N} d_{ii}z_i^2 t\right) z^\alpha dz &= \sum_{|\alpha|=|\beta|} c_\alpha \prod_{i=1}^{N} \int_{a_i}^{b_i} e^{-cd_{ii}z_i^2 t} z_i^{\alpha_i} dz_i \\ &\sim \sum_{|\alpha|=|\beta|} \prod_{i=1}^{N} c_{\alpha_i} t^{-\frac{\alpha_i+1}{2}}, \quad \text{as } t \to \infty \\ &\sim \sum_{|\alpha|=|\beta|} c'_\alpha t^{-\frac{|\alpha|+N}{2}}, \quad \text{as } t \to \infty \\ &= C_\beta t^{-\frac{|\beta|+N}{2}}, \quad \text{as } t \to \infty, \end{aligned} \tag{72}$$

and the proof is complete.

Remark 2.4 [1, p. 272] *Note that we are interested in the higher order terms of the asymptotic behavior of integrals of the form*

$$I(t) = \int_D e^{-\phi(\xi)t} f(\xi) d\xi \qquad \text{as } t \to \infty, \tag{73}$$

where D is a neighborhood of $\xi = 0$. We apply the Laplace's Method for Integrals, we refer to [1] and its references for details.

To study the asymptotic behavior of this integral we assume that $\xi = 0$ is the minimum of ϕ in D, $\phi(0) = 0, \phi'(0) = 0$, $\phi''(0) > 0$ and that the functions f and ϕ are analytic in a neighborhood of $\xi = 0$, and then we use the Taylor expansion of these functions. For instance, to compute the first correction of (73) we proceed as follows:

$$\begin{aligned} I(t) &\sim \int_{-\varepsilon}^{\varepsilon} e^{-(\frac{1}{2}\phi''(0)\xi^2+\frac{1}{6}\phi'''(0)\xi^3+\frac{1}{24}\phi''''(0)\xi^4+\ldots)t}(f(0)+f'(0)\xi+\frac{1}{2}f''(0)\xi^2+\ldots)d\xi \\ &\sim \int_{-\varepsilon}^{\varepsilon} e^{-(\frac{1}{2}\phi''(0)\xi^2+\frac{1}{6}\phi'''(0)\xi^3+\frac{1}{24}\phi''''(0)\xi^4)t}(f(0)+f'(0)\xi+\frac{1}{2}f''(0)\xi^2)d\xi, \end{aligned} \tag{74}$$

as $t \to \infty$ for any $\varepsilon > 0$.

But

$$\begin{aligned} &e^{-(\frac{1}{2}\phi''(0)\xi^2+\frac{1}{6}\phi'''(0)\xi^3+\frac{1}{24}\phi''''(0)\xi^4)t} = e^{-\frac{1}{2}\phi''(0)\xi^2 t} e^{-(\frac{1}{6}\phi'''(0)\xi^3+\frac{1}{24}\phi''''(0)\xi^4)t} \\ &= e^{-\frac{1}{2}\phi''(0)\xi^2 t}\left(1 - t\left(\frac{1}{6}\phi'''(0)\xi^3+\frac{1}{24}\phi''''(0)\xi^4\right)+\frac{1}{72}t^2\xi^6(\phi'''(0))^2+\ldots\right). \end{aligned} \tag{75}$$

Substituting (75) in (74) and collecting powers of t gives

$$\begin{aligned} I(t) &\sim \int_{-\varepsilon}^{\varepsilon} e^{-\frac{1}{2}\phi''(0)\xi^2 t}\left(1 - t\left(\frac{1}{6}\phi'''(0)\xi^3+\frac{1}{24}\phi''''(0)\xi^4\right)+\frac{1}{72}t^2\xi^6(\phi'''(0))^2+\ldots\right) \\ &\times\left(f(0)+f'(0)\xi+\frac{1}{2}f''(0)\xi^2\right)d\xi \\ &\sim \int_{-\varepsilon}^{\varepsilon} e^{-\frac{1}{2}\phi''(0)\xi^2 t}\left(1 - t\left(\frac{1}{6}\phi'''(0)\xi^3+\frac{1}{24}\phi''''(0)\xi^4\right)+\frac{1}{72}t^2\xi^6(\phi'''(0))^2\right) \\ &\times\left(f(0)+f'(0)\xi+\frac{1}{2}f''(0)\xi^2\right)d\xi, \end{aligned} \tag{76}$$

as $t \to \infty$.

Next we can apply Watson's Lemma and obtain that

$$\begin{aligned} I(t) &\sim \sqrt{\frac{2\pi}{t\phi''(0)}}\left(f(0)+\frac{1}{t}\left(\frac{f''(0)}{2\phi''(0)} - \frac{f(0)\phi''''(0)}{8(\phi''(0))^2} - \frac{f'(0)\phi'''(0)}{2(\phi''(0))^2}\right.\right. \\ &\left.\left.+\frac{5\phi''(0)(\phi'''(0))^2 f(0)}{24(\phi''(0))^3}\right)\right) + O(t^{-\frac{5}{2}}), \qquad \text{as } t \to \infty. \end{aligned} \tag{77}$$

Observe that all the displayed terms in (76) contribute to the coefficient of $\frac{1}{t}$, the additional terms that we have neglected in going from (74) to (76) contribute to the coefficients of $\frac{1}{t^2}, \frac{1}{t^3}$, and so on. We have excluded odd powers of ξ because they vanish upon integration. If we consider expansions which contain $|\xi|^s$ instead of ξ^s, we can not exclude the terms corresponding to odd powers, because these integrals do not vanish.

3 Main Results

In this section we state without proof the main results of this paper. We first state the asymptotic expansion in $L^2\left(\mathbb{R}^N\right)$ where Bloch waves provide more naturally the results. We then state more explicit results in $L^\infty\left(\mathbb{R}^N\right)$.

3.1 Asymptotic expansion in $L^2\left(\mathbb{R}^N\right)$

Let us consider the functions

$$G_\alpha(x,t) = \int_{Y'} e^{-q_{ij}\xi_i\xi_j t}\xi^\alpha e^{i\xi\cdot x}\phi_1(x;\xi)\,d\xi, \quad (x,t)\in\mathbb{R}^N\times[0,\infty), \tag{78}$$

for any multi-index $\alpha \in (\mathbb{N}\cup\{0\})^N$, $q=(q_{ij})$ being the homogenized matrix.

Remark 3.1 *Observe that, for any $x\in\mathbb{R}^N$ fixed, taking into account that (q_{ij}) is definite positive:*

$$G_\alpha(x,t) \sim \int_{\mathbb{R}^N} e^{-q_{ij}\xi_i\xi_j t}e^{i\xi\cdot x}\xi^\alpha\phi_1(x;\xi)\,d\xi, \quad \text{as } t\to\infty. \tag{79}$$

Note that in (79) the function $\phi_1(x;\xi)$ is assumed to be Y'-periodic with respect to ξ.

If in the right hand side of (79) we replace $\phi_1(x,\xi)$ by its value at $\xi=0$, that is, $\phi_1(x,0)=|Y|^{-\frac{1}{2}}$, we obtain

$$\frac{1}{|Y|^{\frac{1}{2}}}\int_{\mathbb{R}^N} e^{-q_{ij}\xi_i\xi_j t}e^{i\xi\cdot x}\xi^\alpha d\xi, \tag{80}$$

which is the solution of the constant coefficient homogenized heat equation

$$u_t - \frac{\partial}{\partial x_k}\left(q_{kj}\frac{\partial u}{\partial x_j}\right) = 0, \qquad \text{in } \mathbb{R}^N\times(0,+\infty) \tag{81}$$

with initial datum u_0 such that, its Fourier transform $\hat{u}$ satisfies,

$$\widehat{u}(\xi) = \frac{1}{|Y|^{\frac{1}{2}}}\int_{\mathbb{R}^N} u_0(x)e^{-ix\cdot\xi}dx = \frac{1}{|Y|^{\frac{1}{2}}}\xi^\alpha. \tag{82}$$

Thus $u_0 = (-i)^{|\alpha|}(\partial^\alpha\delta_0/\partial x^\alpha)$, δ_0 being the Dirac delta at the origin. Therefore, the function G_α is intimately related to the derivative ∂^α of the fundamental solution of the homogenized heat equation (81). Note however that when approximating G_α by the integral (80) we are not being rigorous. A more careful approximation will require a Taylor expansion of ϕ_1 at $\xi=0$. This problem is addressed in subsection 3.2 below.

We have the following result:

Theorem 3.1 *Assume that $\varphi\in L^1\left(\mathbb{R}^N\right)\cap L^2\left(\mathbb{R}^N\right)$ and that $|x|^{k+1}\varphi\in L^1\left(\mathbb{R}^N\right)$. Then, there exist constants c_α, with $|\alpha|\le k$, and $c_{\gamma,n}$, with $1\le n\le k$, $3n\le|\gamma|\le k+2n$, depending on the initial datum φ and $C>0$ such that the solution u of (1) satisfies*

$$t^{\frac{2k+N}{4}}\left\| u(\cdot,t) - \sum_{|\alpha|\le k} c_\alpha G_\alpha(\cdot,t) - \sum_{n=1}^{k}\sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n}t^n G_\gamma(\cdot,t)\right\|_2 \le Ct^{-\frac{1}{2}}, \tag{83}$$

as $t\to\infty$.

Remark 3.2 *The proof of Theorem 3.1 provides a systematic way of computing the constants c_α and $c_{\gamma,n}$ in (83). We shall return to this question in Section 6. Note that (83) is similar to the result (18) on the solutions of the constant coefficient heat equation. Indeed, as indicated above, the functions G_α are related to the derivatives ∂^α of the fundamental solution of the homogenized heat equation. However, as we shall see below, a more careful analysis shows that the contribution of the microstructure is also present in the functions G_α. On the other hand, when taking $p = 2$ in (18) we obtain the same decay rates as in (83).*

Remark 3.3 *When the coefficients a_{ij} are constant, the constants $c_{\gamma,n}$ appearing in (83) vanish and $G_\alpha = (-i)^{|\alpha|}(\partial^\alpha G_h/\partial x^\alpha)$. We then recover the asymptotic expansion of the constant coefficient heat equation presented in (18).*

3.2 Asymptotic expansion in $L^\infty\left(\mathbb{R}^N\right)$

Analogously, we have the following result for the asymptotic behavior of the solution of (1) in $L^\infty\left(\mathbb{R}^N\right)$.

Theorem 3.2 *Let $\varphi \in L^1\left(\mathbb{R}^N\right) \cap L^2\left(\mathbb{R}^N\right)$ be a function such that $|x|^{k+1}\varphi \in L^1\left(\mathbb{R}^N\right)$. Let $u(x,t)$ be the solution of the problem (1) with initial datum φ. For any $\alpha \in (\mathbb{N} \cup \{0\})^N$, we define the functions*

$$G^*_\alpha(x,t) = \int_{Y'} e^{-q_{ij}\xi_i\xi_j t} e^{i\xi\cdot x}\xi^\alpha d\xi, \quad \forall (x,t) \in \mathbb{R}^{n+1}. \tag{84}$$

Then there exist functions $c_\alpha(x) \in C_\#\left(\overline{Y}\right)$, with $|\alpha| \le k$, and $c_{\gamma,n}(x) \in C_\#\left(\overline{Y}\right)$, with $1 \le n \le k$, $3n \le |\gamma| \le k+2n$, which can be computed explicitly, and $C > 0$ such that

$$t^{\frac{k+N}{2}} \left\| u(\cdot,t) - \sum_{|\alpha|\le k} c_\alpha(\cdot) G^*_\alpha(\cdot,t) - \sum_{n=1}^{k}\sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n}(\cdot) t^n G^*_\gamma(\cdot,t) \right\|_{L^\infty(\mathbb{R}^N)} \le Ct^{-\frac{1}{2}}, \tag{85}$$

as $t \to \infty$.

Corollary 3.1 *Let $\varphi \in L^1\left(\mathbb{R}^N\right) \cap L^2\left(\mathbb{R}^N\right)$ be a function such that $|x|^{k+1}\varphi \in L^1\left(\mathbb{R}^N\right)$. Let $u(x,t)$ be the solution of the problem (1) with initial datum φ. Let*

$$K_h(x,t) = \int_{\mathbb{R}^N} e^{-q_{ij}\xi_i\xi_j t} e^{i\xi\cdot x} d\xi, \quad \forall (x,t) \in \mathbb{R}^{n+1}, \tag{86}$$

be the fundamental solution of the homogenized equation. Then there exist functions $c_\alpha(x) \in C_\#\left(\overline{Y}\right)$, with $|\alpha| \le k$, and $c_{\gamma,n}(x) \in C_\#\left(\overline{Y}\right)$, with $1 \le n \le k$, $3n \le |\gamma| \le k+2n$, which can be computed explicitly, and $C > 0$ such that

$$t^{\frac{k+N}{2}} \left\| u(\cdot,t) - \sum_{|\alpha|\le k} \frac{c_\alpha(\cdot)}{(i)^{|\alpha|}} \partial^\alpha K_h(\cdot,t) - \sum_{n=1}^{k}\sum_{|\gamma|=3n}^{2n+k} \frac{c_{\gamma,n}(\cdot)}{(i)^{|\alpha|}} t^n \partial^\gamma K_h(\cdot,t) \right\|_{L^\infty(\mathbb{R}^N)} \le Ct^{-\frac{1}{2}}, \tag{87}$$

as $t \to \infty$.

Remark 3.4 *The functions $c_\alpha(x)$ and $c_{\gamma,n}(x)$ appearing in the statement of Theorem 3.2 are related to the derivatives of the first Bloch eigenfunction $\phi_1(x,\xi)$ with respect to ξ in $\xi = 0$. We shall describe how to compute them explicitly in section 6. Note that, as indicated in Remark 3.1, G^*_α is related to the ∂^α derivative of the fundamental solution of the homogenized heat equation. Observe, finally, that the rates in (85) and (87) coincide with those one gets for the constant coefficient heat equation (see (18)). The most explicit expansion is obtained in (87) where G^*_α is replaced by the $\alpha-$derivative of the homogenized heat kernel.*

4 Proof of the asymptotic expansion in $L^2\left(\mathbb{R}^N\right)$

In this section we give a detailed proof of Theorem 3.1.

Firstly, we are going to see that, in (47), the terms corresponding to the eigenvalues $\lambda_m(\xi), m \geq 2$, are negligible with respect to the first one, because these terms decay exponentially as $t \to \infty$.

Lemma 4.1 *Let $\varphi \in L^2\left(\mathbb{R}^N\right)$. Let $u(x,t)$ be the solution of (1) with initial datum φ. Then*

$$v(x,t) = \sum_{m=2}^{\infty} \int_{Y'} \widehat{u}_m(\xi,t)\, e^{i\xi\cdot x} \phi_m(x;\xi)\, d\xi \tag{88}$$

satisfies

$$\|v(\cdot,t)\|_{L^2(\mathbb{R}^N)} \leq e^{-\lambda_2^{(N)} t} \|\varphi\|_{L^2(\mathbb{R}^N)}, \tag{89}$$

where $\lambda_2^{(N)} > 0$ is the second eigenvalue for the Neumann eigenvalue problem for the operator $A(0) = -div\,(a(x)\nabla)$ in the cell Y.

Proof. Let $u(x,t)$ be the solution of (1) with initial datum $\varphi \in L^2\left(R^N\right)$. From Parseval's identity (29) we deduce that

$$\begin{aligned} \int_{\mathbb{R}^N} |v(x,t)|^2\, dx &= \sum_{m=2}^{\infty} \int_{Y'} |\widehat{u}_m(\xi,t)|^2\, d\xi = \sum_{m=2}^{\infty} \int_{Y'} \left| e^{-\lambda_m(\xi)t} \widehat{\varphi}_m(\xi) \right|^2 d\xi \\ &= \sum_{m=2}^{\infty} \int_{Y'} e^{-2\lambda_m(\xi)t} |\widehat{\varphi}_m(\xi)|^2\, d\xi. \end{aligned} \tag{90}$$

From (26) we have that $\lambda_m(\xi) \geq \lambda_2^{(N)} > 0$, for each $m \geq 2$ and $\xi \in Y'$. Thus,

$$\begin{aligned} \int_{\mathbb{R}^N} |v(x)|^2\, dx &\leq e^{-2\lambda_2^{(N)} t} \sum_{m=1}^{\infty} \int_{Y'} |\widehat{\varphi}_m(\xi)|^2\, d\xi \\ &= e^{-2\lambda_2^{(N)} t} \int_{\mathbb{R}^N} |\varphi(x)|^2\, dx. \end{aligned} \tag{91}$$

This completes the proof.

Remark 4.1 *Let u be the solution of (1) with initial datum φ and $v(x,t)$ the function defined in (88). In view of Lemma 4.1, if $t \geq \delta > 0$, we have that*

$$\|v(\cdot,t)\|_2 \leq Ce^{-\lambda_2^{(N)}(t-\delta)} \|v(\cdot,\delta)\|_2. \tag{92}$$

Moreover

$$\|v(\cdot,\delta)\|_2 \leq \|u(\cdot,\delta)\|_2\,, \tag{93}$$

and from the $L^1 - L^2$ estimate (8) we have that

$$\|u(\cdot,\delta)\|_2 \leq \frac{C}{\delta^{\frac{N}{4}}}\|\varphi\|_1\,, \quad \forall t \geq \delta > 0. \tag{94}$$

Therefore, from (93) and (94), we obtain that

$$\|v(\cdot,t)\|_2 \leq \frac{Ce^{-\lambda_2^{(N)}(t-\delta)}}{\delta^{\frac{N}{4}}}\|\varphi\|_1\,, \quad \forall t \geq \delta > 0. \tag{95}$$

Thus the exponential decay of v holds even when $\varphi \in L^1\left(\mathbb{R}^{\mathrm{N}}\right)$.

Remark 4.2 *Lemma 4.1 and Remark 4.1 show that the decay of the projection of the solution u of (1) over the Bloch waves with indexes $m \geq 2$ is exponential. Since the Fourier component associated to the first Bloch wave decays with a polynomial rate, it is sufficient to analyze the first term in (47):*

$$I_1(x,t) = \int_{Y'} \widehat{u}_1(\xi,t)\, e^{i\xi\cdot x}\phi_1(x;\xi)\, d\xi = \int_{Y'} e^{-\lambda_1(\xi)t}\widehat{\varphi}_1(\xi)\, e^{i\xi\cdot x}\phi_1(x;\xi)\, d\xi. \tag{96}$$

Then

$$u(x,t) \sim \int_{Y'} e^{-\lambda_1(\xi)t}\widehat{\varphi}_1(\xi)\, e^{i\xi\cdot x}\phi_1(x;\xi)\, d\xi, \qquad \text{as } t \to +\infty, \quad \text{in } L^2(\mathbb{R}^{\mathrm{N}}). \tag{97}$$

Since the first Bloch eigenvalue $\lambda_1(\xi)$ has a strict minimum at $\xi = 0$, the main contribution in (96) is obtained in a neighborhood U of $\xi = 0$, where the functions $\xi \to \lambda_1(\xi)$, $\xi \to \phi_1(x,\xi)$ are analytic (see Remark 2.4 and Theorem 2.2).

Remark 4.3 *It is important to note that, if our initial datum $\varphi \in L^1\left(\mathbb{R}^{\mathrm{N}}\right) \cap L^2\left(\mathbb{R}^{\mathrm{N}}\right)$ satisfies $|x|^{k+1}\varphi(x) \in L^1\left(\mathbb{R}^{\mathrm{N}}\right)$, the first Bloch coefficient $\widehat{\varphi}_1(\xi)$ belongs to $C^{k+1}(U)$, where U is a neighborhood of $\xi = 0$, where the first Bloch wave $\phi_1(x,\xi)$ is analytic. Moreover, since*

$$\widehat{\varphi}_1(\xi) = \int_{\mathbb{R}^{\mathrm{N}}} \varphi(x)\, e^{-i\xi\cdot x}\overline{\phi}_1(x,\xi)\, d\xi, \tag{98}$$

for all $\alpha \in (\mathbb{N} \cup \{0\})^N$ with $|\alpha| \leq k+1$, we have that

$$\begin{aligned} \frac{\partial^\alpha \widehat{\varphi}_1}{\partial \xi^\alpha}(\xi) &= \int_{\mathbb{R}^{\mathrm{N}}} \varphi(x)\frac{\partial^\alpha}{\partial\xi^\alpha}\left[e^{-i\xi\cdot x}\overline{\phi}_1(x,\xi)\right] dx \\ &= \int_{\mathbb{R}^{\mathrm{N}}} \varphi(x)\sum_{\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\frac{\partial^\beta}{\partial\xi^\beta}\left(e^{-i\xi\cdot x}\right)\frac{\partial^{\alpha-\beta}\overline{\phi}_1}{\partial\xi^{\alpha-\beta}}(x,\xi)\, dx \\ &= \sum_{\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\int_{\mathbb{R}^{\mathrm{N}}} \varphi(x)\frac{\partial^\beta}{\partial\xi^\beta}\left(e^{-i\xi\cdot x}\right)\frac{\partial^{\alpha-\beta}\overline{\phi}_1}{\partial\xi^{\alpha-\beta}}(x,\xi)\, dx, \end{aligned} \tag{99}$$

where $\beta \leq \alpha$ *if and only if* $\beta_i \leq \alpha_i$ *for all* $i = 1, \ldots, N$ *and*

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \prod_{i=1}^{N} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}.$$

Hence we have that

$$\frac{\partial \widehat{\varphi}_1}{\partial \xi^\alpha}(0) = \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (-i)^{|\beta|} \int_{\mathbb{R}^N} x^\beta \varphi(x) \frac{\partial^{\alpha-\beta} \overline{\phi}_1}{\partial \xi^{\alpha-\beta}}(x, 0) dx, \tag{100}$$

for $1 \leq |\alpha| \leq k+1$.

Moreover, from Taylor's expansion we have that for all $\xi \in U$

$$\left| \widehat{\varphi}_1(\xi) - \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \frac{\partial^\alpha \widehat{\varphi}_1}{\partial \xi^\alpha}(0) \xi^\alpha \right| \leq c_k \sup_{\nu \in U, |\beta| = k+1} \left| \frac{\partial^\beta \widehat{\varphi}_1}{\partial \xi^\beta}(\nu) \right| |\xi|^{k+1}. \tag{101}$$

But, from (99) and since the function $u \to \phi_1(\cdot, \xi)$ *is analytic in* U *with values in* $C_\#(\overline{Y})$, *we obtain that*

$$\begin{aligned} \left| \frac{\partial^\alpha \widehat{\varphi}_1}{\partial \xi^\alpha}(\xi) \right| &\leq \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \int_{\mathbb{R}^N} \left| \varphi(x) \frac{\partial^\beta}{\partial \xi^\beta} \left(e^{-i\xi \cdot x} \right) \frac{\partial^{\alpha-\beta} \overline{\phi}_1}{\partial \xi^{\alpha-\beta}}(x, \xi) \right| dx \\ &\leq \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} c_\beta \int_{\mathbb{R}^N} \left| x^\beta \varphi(x) \right| dx. \end{aligned} \tag{102}$$

That is

$$\left| \widehat{\varphi}_1(\xi) - \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \frac{\partial^\alpha \widehat{\varphi}_1}{\partial \xi^\alpha}(0) \xi^\alpha \right| \leq C \|\varphi\|_{L^1\left(\mathbb{R}^N; 1+|x|^{k+1}\right)} |\xi|^{k+1}, \qquad \forall \xi \in U, \tag{103}$$

with $C > 0$ *independent of* φ.

The following holds (recall that G_α is defined in (78)):

Lemma 4.2 *Let* $\varphi \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ *be a function such that* $|x|^{k+1} \varphi \in L^1(\mathbb{R}^N)$. *Then, there exist constants* c_α, *with* $|\alpha| \leq k$ *and* $c_{\gamma,n}$, *with* $1 \leq n \leq k$, $3n \leq |\gamma| \leq k + 2n$, *which can be computed explicitly and* $C > 0$ *such that*

$$t^{(2k+N)/4} \left\| I_1(\cdot, t) - \sum_{|\alpha| \leq k} c_\alpha G_\alpha(\cdot, t) - \sum_{n=1}^{k} \sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n} t^n G_\gamma(\cdot, t) \right\|_2 \leq C t^{-1/2}, \tag{104}$$

as $t \to \infty$.

Remark 4.4 *Theorem 3.1 is an immediate consequence of Lemmas 4.1 and 4.2. Thus, in order to complete the proof of Theorem 3.1, it is sufficient to prove Lemma 4.2.*

Proof of Lemma 4.2 Let $\varphi \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ be a function such that $|x|^{k+1}\varphi \in L^1(\mathbb{R}^N)$. From Remark 4.3 we have that there exists a positive constant $C > 0$ such that

$$\left| \widehat{\varphi}_1(\xi) - \sum_{|\alpha| \le k} \frac{1}{\alpha!} \partial^\alpha \widehat{\varphi}_1(0)\, \xi^\alpha \right| \le C\, |\xi|^{k+1}, \quad \forall \xi \in U. \tag{105}$$

From now on we set

$$c_\alpha = \frac{1}{\alpha!} \frac{\partial^\alpha \widehat{\varphi}_1}{\partial \xi^\alpha}(0), \quad \forall \alpha : |\alpha| \le k. \tag{106}$$

Let

$$J_1(x,t) = \int_{Y'} e^{-\lambda_1(\xi)t} \left(\sum_{|\alpha| \le k} c_\alpha \xi^\alpha \right) e^{i\xi\cdot x} \phi_1(x;\xi)\, d\xi, \quad (x,t) \in \mathbb{R}^N \times [0,\infty). \tag{107}$$

Then, from Parseval's identity (29):

$$\|I_1(\cdot,t) - J_1(\cdot,t)\|_2^2 = \int_{Y'} e^{-2\lambda_1(\xi)t} \left| \widehat{\varphi}_1(\xi) - \sum_{|\alpha| \le k} c_\alpha \xi^\alpha \right|^2 d\xi. \tag{108}$$

Again, for the study of the asymptotic behavior of (108), it is enough to study the integral in a neighborhood U of $\xi = 0$, since the integral defined on $Y' \setminus U$ has an exponential decay.

Then, from (105) and Lemma 2.4 we have that

$$\begin{aligned} \int_U e^{-2\lambda_1(\xi)t} \left| \widehat{\varphi}_1(\xi) - \sum_{|\alpha| \le k} c_\alpha \xi^\alpha \right|^2 d\xi \quad &\le c \int_U e^{-2\lambda_1(\xi)t} |\xi|^{2(k+1)}\, d\xi \\ &\le c \int_U e^{-2\gamma|\xi|^2 t} |\xi|^{2(k+1)}\, d\xi \\ &\sim \widetilde{c}\, t^{-\frac{2k+2+N}{2}}, \qquad \text{as } t \to \infty, \end{aligned} \tag{109}$$

where c is the constant obtained in Remark 4.3.

From (109) we can see that, to prove (104), one can replace the integral I_1 by J_1.

Now we are going to study the asymptotic behavior of the integral J_1 by means of Laplace's Method (see Remark 2.4).

Note that the map $\xi \to \lambda_1(\xi)$ is analytic in a neighborhood U of $\xi = 0$ and has a strict minimum at $\xi = 0$. In view of Theorem 2.2, if

$$\nu(\xi) = \lambda_1(\xi) - q_{ij}\xi_i\xi_j, \tag{110}$$

the map $\xi \to \nu(\xi)$ is analytic in U and

$$\nu(\xi) = \sum_{|\alpha| \ge 3} C_\alpha \xi^\alpha, \forall \xi \in U, \tag{111}$$

where

$$C_\alpha = \frac{1}{\alpha!} \frac{\partial^\alpha \lambda_1}{\partial \xi^\alpha}(0). \tag{112}$$

Then, for $t>0$ the function $\left(e^{-\nu(\xi)t}-1\right)$ is analytic for all $\xi\in B(0,\varepsilon)$ with ε small enough. Moreover

$$\begin{aligned} e^{-\nu(\xi)t} &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\nu(\xi)\right)^n t^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^n\left(\sum_{|\alpha|\geq 3}C_\alpha\xi^\alpha\right)^n \\ &= 1+\sum_{n=1}^{\infty}t^n\sum_{|\beta|\geq 3n}\widetilde{c}_{\beta,n}\xi^\beta. \end{aligned} \tag{113}$$

It is important to note that the convergence of the last series is guaranteed by the analyticity of the first Bloch eigenvalue $\lambda_1(\xi)$ in U.

Thus we have that

$$\begin{aligned} e^{-\lambda_1(\xi)t}\left(\sum_{|\alpha|\leq k}c_\alpha\xi^\alpha\right) &= e^{-q_{ij}\xi_i\xi_j t}e^{-\nu(\xi)t}\left(\sum_{|\alpha|\leq k}c_\alpha\xi^\alpha\right) \\ &= e^{-q_{ij}\xi_i\xi_j t}\left(\sum_{|\alpha|\leq k}c_\alpha\xi^\alpha\right)\left(1+\sum_{n\geq 1}\sum_{|\beta|\geq 3n}\widetilde{c}_{\beta,n}\xi^\beta t^n\right) \\ &= e^{-q_{ij}\xi_i\xi_j t}\left(\sum_{|\alpha|\leq k}c_\alpha\xi^\alpha\right) \\ &\quad + e^{-q_{ij}\xi_i\xi_j t}\left(\sum_{|\alpha|\leq k}\sum_{n\geq 1}\sum_{|\beta|\geq 3n}\widetilde{c}_{\beta,n}c_\alpha\xi^{\beta+\alpha}t^n\right) \\ &= e^{-q_{ij}\xi_i\xi_j t}\left(\sum_{|\alpha|\leq k}c_\alpha\xi^\alpha\right) \\ &\quad + e^{-q_{ij}\xi_i\xi_j t}\left(\sum_{n\geq 1}\sum_{|\gamma|\geq 3n}c_{\gamma,n}\xi^\gamma t^n\right), \end{aligned} \tag{114}$$

for suitable constants $c_{\gamma,n}$.

Therefore

$$\begin{aligned} &J_1(x,t) - \sum_{|\alpha|\leq k}c_\alpha G_\alpha(x,t) - \sum_{n=1}^{k}\sum_{|\gamma|=3n}^{2n+k}c_{\gamma,n}t^n G_\gamma(x,t) \\ &= \int_{Y'}\left[e^{-\lambda_1(\xi)t}\left(\sum_{|\alpha|\leq k}c_\alpha\xi^\alpha\right) - e^{-q_{ij}\xi_i\xi_j t}\left(\sum_{|\alpha|\leq k}c_\alpha\xi^\alpha\right)\right]e^{i\xi\cdot x}\phi_1(x,\xi)\,d\xi \\ &- \int_{Y'}e^{-q_{ij}\xi_i\xi_j t}\left(\sum_{n=1}^{k}\sum_{|\gamma|=3n}^{2n+k}c_{\gamma,n}t^n\xi^\gamma\right)e^{i\xi\cdot x}\phi_1(x,\xi)\,d\xi. \end{aligned} \tag{115}$$

From Parseval's identity we have

$$\begin{aligned}
&\left\| J_1(\cdot,t) - \sum_{|\alpha|\le k} c_\alpha G_\alpha(\cdot,t) + \sum_{n=1}^{k}\sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n} t^n G_\gamma(\cdot,t)\right\|_2^2 \\
&= \int_{Y'} \left| \sum_{|\alpha|\le k} c_\alpha \left(e^{-\lambda_1(\xi)t} - e^{-q_{ij}\xi_i\xi_j t}\right)\xi^\alpha - \sum_{n=1}^{k}\sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n} e^{-q_{ij}\xi_i\xi_j t} t^n \xi^\gamma \right|^2 d\xi.
\end{aligned} \tag{116}$$

From (114) we also have

$$\begin{aligned}
& e^{-\nu(\xi)t}\left(\sum_{|\alpha|\le k} c_\alpha \xi^\alpha\right) - \left(\sum_{|\alpha|\le k} c_\alpha\xi^\alpha + \sum_{n=1}^{k}\sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n} t^n \xi^\gamma\right) \\
&= \sum_{n\ge k+1}\sum_{|\gamma|\ge 3n} c_{\gamma,n} t^n\xi^\gamma + \sum_{n=1}^{k}\sum_{|\gamma|\ge 2n+k+1} c_{\gamma,n} t^n \xi^\gamma \\
&= \sum_{|\gamma|=3(k+1)} c_{\gamma,k+1} t^{k+1}\xi^\gamma + \sum_{|\gamma|\ge 3(k+1)+1} c_{\gamma,k+1} t^{k+1}\xi^\gamma \\
&+ \sum_{n\ge k+2}\sum_{|\gamma|\ge 3n} c_{\gamma,n} t^n\xi^\gamma + \sum_{n=1}^{k}\sum_{|\gamma|=2n+k+1} c_{\gamma,n} t^n\xi^\gamma \\
&+ \sum_{n=1}^{k}\sum_{|\gamma|\ge 2n+k+2} c_{\gamma,n} t^n \xi^\gamma \\
&= \sum_{|\gamma|=3(k+1)} c_{\gamma,k+1} t^{k+1}\xi^\gamma + \sum_{n=1}^{k}\sum_{|\gamma|=2n+k+1} c_{\gamma,n} t^n\xi^\gamma \\
&+ \sum_{|\gamma|\ge 3(k+1)+1} c_{\gamma,k+1} t^{k+1}\xi^\gamma + \sum_{n\ge k+2}\sum_{|\gamma|\ge 3n} c_{\gamma,n} t^n\xi^\gamma \\
&+ \sum_{n=1}^{k}\sum_{|\gamma|\ge 2n+k+2} c_{\gamma,n} t^n\xi^\gamma.
\end{aligned} \tag{117}$$

Hence

$$\begin{aligned}
&\left\| J_1(\cdot,t) - \sum_{|\alpha|\le k} c_\alpha G_\alpha(\cdot,t) + \sum_{n=1}^{k}\sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n} t^n G_\gamma(\cdot,t)\right\|_2^2 \\
&\le \int_{Y'} e^{-2q_{ij}\xi_i\xi_j t}\left| \sum_{|\gamma|=3(k+1)} c_{\gamma,k+1} t^{k+1}\xi^\gamma + \sum_{n=1}^{k}\sum_{|\gamma|=2n+k+1} c_{\gamma,n} t^n \xi^\gamma\right|^2 d\xi \\
&+ \int_{Y'} e^{-2q_{ij}\xi_i\xi_j t}\left| \sum_{|\gamma|\ge 3(k+1)+1} c_{\gamma,k+1} t^{k+1}\xi^\gamma + \sum_{n\ge k+2}\sum_{|\gamma|\ge 3n} c_{\gamma,n} t^n\xi^\gamma \right.\\
&\qquad \left. + \sum_{n=1}^{k}\sum_{|\gamma|\ge 2n+k+2} c_{\gamma,n} t^n\xi^\gamma\right|^2 d\xi.
\end{aligned} \tag{118}$$

Note that the convergence of the series entering in (118) is guaranteed by (114) and (117).

Now, we are going to study the asymptotic behavior of the integrals of the right hand side of (118). We can see that

$$\begin{aligned}&\left(\sum_{|\gamma|=3(k+1)} c_{\gamma,k+1}t^{k+1}\xi^\gamma + \sum_{n=1}^{k}\sum_{|\gamma|=2n+k+1} c_{\gamma,n}t^n\xi^\gamma\right)^2\\ &= \left(\sum_{|\gamma|=3(k+1)} c_{\gamma,k+1}t^{k+1}\xi^\gamma\right)^2 + \left(\sum_{n=1}^{k}\sum_{|\gamma|=2n+k+1} c_{\gamma,n}t^n\xi^\gamma\right)^2\\ &+2\left(\sum_{|\gamma|=3(k+1)} c_{\gamma,k+1}t^{k+1}\xi^\gamma\right)\left(\sum_{n=1}^{k}\sum_{|\gamma|=2n+k+1} c_{\gamma,n}t^n\xi^\gamma\right)\\ &= t^{2(k+1)}\sum_{|\gamma|=6(k+1)} \widehat{c}_\gamma\xi^\gamma + \sum_{n=2}^{2k}\sum_{|\gamma|=2n+2k+2} \widehat{c}_{\gamma,n}t^n\xi^\gamma\\ &+\sum_{n=1}^{k}\sum_{|\gamma|=2n+4(k+1)} c'_{\gamma,n}t^{n+k+1}\xi^\gamma.\end{aligned} \tag{119}$$

Remember that, from Lemma 2.4, for each $\beta \in (\mathbb{N}\cup\{0\})^N$ we have that

$$\int_U e^{-q_{ij}\xi_i\xi_j t}t^s\,|\xi|^{|\beta|} \sim Ct^{-\frac{|\beta|+N-2s}{2}}, \qquad \text{as } t\to\infty. \tag{120}$$

Therefore, from (119)

$$\begin{aligned}&\int_{Y'} e^{-2q_{ij}\xi_i\xi_j t}\left|\sum_{|\gamma|=3(k+1)} c_{\gamma,k+1}t^{k+1}\xi^\gamma + \sum_{n=1}^{k}\sum_{|\gamma|=2n+k+1} c_{\gamma,n}t^n\xi^\gamma\right|^2 d\xi\\ &= \int_{Y'} e^{-2q_{ij}\xi_i\xi_j t}\left(\sum_{|\gamma|=6(k+1)} \widehat{c}_\gamma\xi^\gamma t^{2(k+1)} + \sum_{n=2}^{2k}\sum_{|\gamma|=2n+2k+2} \widehat{c}_{\gamma,n}t^n\xi^\gamma\right)d\xi\\ &\int_{Y'} e^{-2q_{ij}\xi_i\xi_j t}\left(\sum_{n=1}^{k}\sum_{|\gamma|=2n+4(k+1)} c'_{\gamma,n}t^{n+k+1}\xi^\gamma\right)d\xi\\ &\sim \sum_{|\gamma|=6(k+1)} \widehat{c}'_\gamma\xi^\gamma t^{-\frac{1}{2}(N+|\gamma|)}t^{2(k+1)} + \sum_{n=2}^{2k}\sum_{|\gamma|=2n+2k+2} \widehat{c}'_{\gamma,n}t^n t^{-\frac{1}{2}(|\gamma|+N)}\\ &+\sum_{n=1}^{k}\sum_{|\gamma|=2n+4(k+1)} c'_{\gamma,n}t^{n+k+1}t^{-\frac{1}{2}(|\gamma|+N)}\xi^\gamma\Bigg)\,d\xi\\ &\sim c''t^{-\frac{1}{2}(N+2k+2)}, \qquad \text{as } t\to\infty.\end{aligned} \tag{121}$$

On the other hand, the Laplace's Method shows us that the terms in the right hand side of (118) have a decay rate $t^{-\beta}$, with $|\beta| \geq \frac{1}{2}(k+2+N)$. In fact

$$
\begin{aligned}
&\left(\sum_{|\gamma|\geq 3(k+1)+1} c_{\gamma,k+1}t^{k+1}\xi^{\gamma} + \sum_{n\geq k+2}\sum_{|\gamma|\geq 3n} c_{\gamma,n}t^n\xi^{\gamma} + \sum_{n=1}^{k}\sum_{|\gamma|\geq 2n+k+2} c_{\gamma,n}t^n\xi^{\gamma}\right)^2 \\
&= \sum_{|\gamma|\geq 6(k+1)+2} \widehat{c}_{\gamma,k+1}t^{2(k+1)}\xi^{\gamma} + \sum_{n\geq 2(k+2)}\sum_{|\gamma|\geq 3n} \widehat{c}_{\gamma,n}t^n\xi^{\gamma} \\
&+ \sum_{n=2}^{2k}\sum_{|\gamma|\geq 2n+2(k+2)} c'_{\gamma,n}t^n\xi^{\gamma} + \sum_{n\geq k+2}\sum_{|\gamma|\geq 3n+3(k+1)+1} c''_{\gamma,n}t^{n+k+1}\xi^{\gamma} \\
&+ \sum_{n=1}^{k}\sum_{|\gamma|\geq 2n+4(k+1)+2} c'''_{\gamma,n}t^{n+k+1}\xi^{\gamma} + \sum_{n=1}^{k}\sum_{m\geq k+2}\sum_{|\beta|\geq 2n+k+2}\sum_{|\gamma|\geq 3m} c_{\gamma,\beta,n,m}t^{n+m}\xi^{\gamma+\beta}.
\end{aligned}
\tag{122}
$$

The terms in the first series in the right hand side of (122) verify

$$|\gamma| + N - 4(k+1) \geq 6(k+1) + 2 + N - 4(k+1) = 2k + 4 + N, \tag{123}$$

provided that $|\gamma| \geq 6(k+1) + 2$.

The terms in the second series, given that $n \geq 2(k+2)$ and $|\gamma| \geq 3n$ satisfy

$$|\gamma| + N - 2n \geq 3n + N - 2n \geq 2k + 4 + N. \tag{124}$$

The terms in the third series verify

$$|\gamma| + N - 2n \geq 2n + 2(k+2) + N - 2n = 2k + 4 + N, \tag{125}$$

provided that $|\gamma| \geq 2n + 2(k+2)$.

The terms in the fourth series verify

$$|\gamma| + N - 2(n+k+1) \geq 3n + 3(k+1) + 1 + N - 2(n+k+1) = n + k + 2 + N \geq 2k + 4 + N, \tag{126}$$

provided that $n \geq k+2$ and $|\gamma| \geq 3n + 3(k+1) + 1$.

Furthermore, the terms in the fifth series, given that $1 \geq n \geq k$, $|\gamma| \geq 2n+4(k+1)+2$ verify

$$|\gamma| + N - 2(n+k+1) \geq 2n + 4(k+1) + 2 + N - 2(n+k+1) = 2k + 4 + N. \tag{127}$$

Finally, the terms in the sixth series satisfy

$$|\beta + \gamma| + N - 2(n+m) \geq 3m + 2n + k + 2 + N - 2n - 2m = m + k + 2 + N \geq 2k + 4 + N, \tag{128}$$

taking into account that $m \geq k+2$, $1 \leq n \leq k$, $|\beta| \geq 2n + k + 2$ and $|\gamma| \geq 3m$.

Thus, from (123)-(128), Laplace's Method shows us that

$$
\begin{aligned}
&\int_{Y'} e^{-2q_{ij}\xi_i\xi_j t} \left| \sum_{|\gamma|\geq 3(k+1)+1} c_{\gamma,k+1}t^{k+1}\xi^{\gamma} + \sum_{n\geq k+2}\sum_{|\gamma|\geq 3n} c_{\gamma,n}t^n\xi^{\gamma} \right. \\
&\left. + \sum_{n=1}^{k}\sum_{|\gamma|\geq 2n+k+2} c_{\gamma,n}t^n\xi^{\gamma} \right|^2 d\xi \sim \widehat{C}t^{-\frac{1}{2}(2k+4+N)}, \quad \text{as } t \to \infty.
\end{aligned}
\tag{129}
$$

Then, from (121) and (129), we deduce

$$\left\| J_1(\cdot,t) - \sum_{|\alpha|\leq k} c_\alpha G_\alpha(\cdot,t) - \sum_{n=1}^{k}\sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n} t^n G_\gamma(\cdot,t) \right\|_2 \sim C' t^{-\frac{2k+2+N}{4}}, \tag{130}$$

as $t \to \infty$.

In particular, it follows that

$$t^{\frac{2k+N}{4}} \left\| I_1(\cdot,t) - \sum_{|\alpha|\leq k} c_\alpha G_\alpha(\cdot,t) - \sum_{n=1}^{k}\sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n} t^n G_\gamma(\cdot,t) \right\|_2 \leq C t^{-\frac{1}{2}}, \tag{131}$$

as $t \to \infty$.

Note that the convergence of the series in the proof is guaranteed by the analyticity of the first Bloch eigenvalue $\lambda_1(\xi)$ and the first Bloch wave $\phi_1(\xi)$ in a neighborhood U of the origin $\xi = 0$.

This completes the proof.

Remark 4.5 *The statement of Theorem 3.1 is not completely explicit since in the definition (78) of the functions G_α the first Bloch wave appears. However, this function is only known explicitly for $\xi = 0$. When replacing $\phi_1(x,\xi)$ by its Taylor expansion in $\xi = 0$ we obtain the result stated in Theorem 3.2 in which the function G_α has been replaced by G^*_α.*

Remark 4.6 *From Remark 4.3 we can see that the constant C in (131) depends on the norm of the initial datum φ in $L^1\left(\mathbb{R}^N; 1+|x|^{k+1}\right)$. Indeed, in (109) we use the Lipschitz character of the k-th derivative of the first Bloch coefficient $\widehat{\varphi}_1$ of the initial datum φ in a neighborhood of $\xi = 0$.*

5 Proof of the asymptotic expansion in $L^\infty\left(\mathbb{R}^N\right)$

In this section we present the proof of Theorem 3.2 and Corollary 3.1.

The following result guarantees that, as in the L^2-setting, the terms corresponding to $m \geq 2$ decay exponentially.

Lemma 5.1 *Let $\varphi \in L^1\left(R^N\right) \cap L^2\left(R^N\right)$. Let $u(x,t)$ be the solution of (1) with initial datum φ. Then there exists a constant $C > 0$ such that the function*

$$v(x,t) = \sum_{m=2}^{\infty} \int_{Y'} \widehat{u}_m(\xi,t)\, e^{i\xi\cdot x} \phi_m(x;\xi)\, d\xi \tag{132}$$

satisfies

$$\|v(\cdot,t)\|_\infty \leq C e^{-\lambda_2^{(N)} t} \|\varphi\|_1, \quad \text{as } t \to \infty. \tag{133}$$

Proof. Let $u(x,t)$ be the solution of (1) with initial datum $\varphi \in L^1\left(R^N\right)$.

Firstly, from the $L^2 - L^\infty$ estimate (8), we have that

$$\|v(\cdot,t)\|_\infty \leq C(\infty,2)\, \|v(\cdot,t-1)\|_{L^2(\mathbb{R}^N)}, \quad \forall t > 1. \tag{134}$$

Since in view of Remark 4.1

$$\|v(\cdot,t-1)\|_2 \leq C e^{-\lambda_2^{(N)}(t-1-\delta)} \delta^{-\frac{N}{4}} \|\varphi\|_1, \quad \forall t > 1+\delta, \tag{135}$$

we obtain that

$$\|v(\cdot,t)\|_\infty \leq \widehat{C} e^{-\lambda_2^{(N)} t} \|\varphi\|_1, \quad \forall t > 1+\delta. \tag{136}$$

This completes the proof.

Remark 5.1 *As in Remark 4.3, we can see that if the initial datum* $\varphi \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ *verifies* $|x|^{k+1}\varphi(x) \in L^1(\mathbb{R}^N)$, *the map* $\xi \to \widehat{\varphi}_1(\xi)\phi_1(\cdot,\xi)$ *belongs to* $C^{k+1}(U; C_\#(\overline{Y}))$, *where* U *is a neighborhood of* $\xi = 0$, *because* $\xi \to \widehat{\varphi}_1(\xi)$ *is a map of class* C^{k+1} *in* U *and the first Bloch wave* ϕ_1 *is analytic in* U *with values in* $C_\#(\overline{Y})$. *Thus we have that*

$$\begin{aligned}\left|\widehat{\varphi}_1(\xi)\phi_1(x,\xi) - \sum_{|\alpha|\leq k} \frac{1}{\alpha!}\frac{\partial^\alpha}{\partial\xi^\alpha}\left(\widehat{\varphi}_1(\xi)\phi_1(x,\xi)\right)|_{\xi=0}\xi^\alpha\right| \\ \leq c_k \sup_{\nu\in U, |\beta|=k+1}\left|\frac{\partial^\beta}{\partial\xi^\beta}\left(\widehat{\varphi}_1(\xi)\phi_1(x,\xi)\right)(\nu)\right| |\xi|^{k+1}\end{aligned} \tag{137}$$

where

$$\begin{aligned}\left|\frac{\partial^\alpha}{\partial\xi^\alpha}\left(\widehat{\varphi}_1(\xi)\phi_1(x,\xi)\right)(\xi)\right| &\leq \sum_{\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\left|\frac{\partial^\beta\widehat{\varphi}_1}{\partial\xi^\beta}(\xi)\right|\left|\frac{\partial^{\alpha-\beta}\phi_1(x,\cdot)}{\partial\xi^{\alpha-\beta}}(\xi)\right| \\ &\leq \sum_{\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} c_\beta r_{\alpha-\beta}(x)\int_{\mathbb{R}^N}\left|x^\beta\varphi(x)\right| dx, \quad \forall\xi\in U,\end{aligned} \tag{138}$$

with

$$r_\alpha(x) = \sup_{\xi\in U}\left|\frac{\partial^\alpha\phi_1(x,\xi)}{\partial\xi^\alpha}\right|.$$

Since $\xi \to \phi_1(\cdot,\xi)$ *is analytic in* U *with values in* $C_\#(\overline{Y})$, *there exist constants* $R_\gamma > 0$ *such that*

$$|r_\gamma(x)| \leq R_\gamma, \quad \forall x \in \mathbb{R}^N. \tag{139}$$

Thus we obtain that for all $x \in \mathbb{R}^N$, $\xi \in \mathrm{U}$:

$$\begin{aligned}\left|\widehat{\varphi}_1(\xi)\phi_1(x,\xi) - \sum_{|\alpha|\leq k}\frac{1}{\alpha!}\frac{\partial^\alpha}{\partial\xi^\alpha}\left(\widehat{\varphi}_1(\xi)\phi_1(x,\xi)\right)(0)\xi^\alpha\right| \\ \leq \sum_{\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} R_{\alpha-\beta}c_\beta\int_{\mathbb{R}^N}\left|x^\beta\varphi(x)\right| dx.\end{aligned} \tag{140}$$

In what follows, for all $\alpha \in (\mathbb{N}\cup\{0\})^N$ *we define the functions* $c_\alpha \in C_\#(\overline{Y})$ *by*

$$c_\alpha(x) = \frac{1}{\alpha!}\frac{\partial^\alpha}{\partial\xi^\alpha}\left(\widehat{\varphi}_1(\xi)\phi_1(x,\xi)\right)|_{\xi=0}. \tag{141}$$

Concerning the first term I_1 in the Bloch wave expansion introduced in (96), we have the following result:

Lemma 5.2 *Let $\varphi \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ be a function such that $|x|^{k+1}\varphi \in L^1(\mathbb{R}^N)$. Then there exist functions $c_\alpha(x) \in C_\#(\overline{Y})$, with $|\alpha| \leq k$, $c_{\gamma,n}(x) \in C_\#(\overline{Y})$, with $1 \leq n \leq k$, $3n \leq |\gamma| \leq k+2n$, which can be computed explicitly, and $C>0$ such that*

$$t^{\frac{k+N}{2}} \left\| I_1(\cdot,t) - \sum_{|\alpha|\leq k} c_\alpha(\cdot) G^*_\alpha(\cdot,t) - \sum_{n=1}^{k} \sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n}(\cdot) t^n G^*_\gamma(\cdot,t) \right\|_\infty \leq C t^{-\frac{1}{2}}, \tag{142}$$

as $t \to \infty$, where G^*_α are as in (84).

Proof of Lemma 5.2 To prove this result, we are going to proceed as in the proof of Lemma 4.2. Recall that I_1 is defined as in (96).

Let $\varphi \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ be a function such that $|x|^{k+1}\varphi \in L^1(\mathbb{R}^N)$. Thus the map $\xi \to \phi(x,\xi)\widehat{\varphi}_1(\xi)$ is of class C^{k+1} in a neighborhood U of $\xi = 0$ with values in $C_\#(\overline{Y})$ (see [5, 20]).

Therefore, there exists a constant $c > 0$ such that

$$\left| \phi_1(x,\xi)\widehat{\varphi}_1(\xi) - \sum_{|\alpha|\leq k} c_\alpha \xi^\alpha \right| \leq c\,|\xi|^{k+1}, \ \forall \xi \in U, \forall x \in \overline{Y}. \tag{143}$$

As in the proof of Lemma 4.2, we consider the function

$$J_1(x,t) = \int_{Y'} e^{-\lambda_1(\xi)t} \left(\sum_{|\alpha|\leq k} c_\alpha(x)\xi^\alpha \right) e^{i\xi\cdot x} d\xi, \quad (x,t) \in \mathbb{R}^N \times [0,\infty). \tag{144}$$

Thus

$$|I_1(\cdot,t) - J_1(\cdot,t)| \leq \int_{Y'} e^{-\lambda_1(\xi)t} \left| \widehat{\varphi}_1(\xi)\,\phi_1(x,\xi) - \sum_{|\alpha|\leq k} c_\alpha(x)\xi^\alpha \right| d\xi. \tag{145}$$

In view of (143) the integral defined on $Y' \setminus U$ has an exponential decay, uniform with respect to $x \in Y$.

In fact, if $B(0,\varepsilon) \subset U$, we have that

$$\left| \phi_1(x,\xi)\widehat{\varphi}_1(\xi) - \sum_{|\alpha|\leq k} c_\alpha(x)\xi^\alpha \right| \leq |\phi_1(x,\xi)\widehat{\varphi}_1(\xi)| + \left| \sum_{|\alpha|\leq k} c_\alpha(x)\xi^\alpha \right|. \tag{146}$$

Since $c_\alpha(\cdot), \phi_1(\cdot,\xi) \in C_\#(\overline{Y})$ we have that there exists a constant c' such that

$$|\phi_1(x,\xi)\widehat{\varphi}_1(\xi)| + \left| \sum_{|\alpha|\leq k} c_\alpha(x)\xi^\alpha \right| \leq c', \forall \xi \in Y', x \in Y. \tag{147}$$

Thus

$$\begin{aligned} \int_{Y'\setminus U} e^{-\lambda_1(\xi)t} \left| \widehat{\varphi}_1(\xi)\,\phi_1(x,\xi) - \sum_{|\alpha|\leq k} c_\alpha(x)\xi^\alpha \right| d\xi &\leq c' \int_{Y'\setminus U} e^{-\lambda_1(\xi)t} d\xi \\ &\leq c' \int_{Y'\setminus U} e^{-\gamma|\xi|^2 t} d\xi \leq c' \int_{Y'\setminus U} e^{-\gamma\varepsilon^2 t} d\xi \\ &= \widehat{c}e^{-\gamma\varepsilon^2 t}. \end{aligned} \tag{148}$$

On the other hand, from (143) and Lemma 2.4 we obtain that for all $x \in \mathbb{R}^N$

$$\begin{aligned}\int_U e^{-\lambda_1(\xi)t}\left|\widehat{\varphi}_1(\xi)\,\phi_1(x,\xi)-\sum_{|\alpha|\le k}c_\alpha(x)\xi^\alpha\right|d\xi &\le C\int_U e^{-\lambda_1(\xi)t}\,|\xi|^{(k+1)}\,d\xi\\ &\le C\int_U e^{-\gamma|\xi|^2 t}\,|\xi|^{(k+1)}\,d\xi\\ &\sim \widetilde{C}t^{-\frac{k+1+N}{2}},\qquad \text{as } t\to\infty.\end{aligned}\tag{149}$$

Thus, to prove (142), one can replace the integral I_1 by J_1.

Notice that to analyze the asymptotic behavior of the integral J_1 it is enough to study the asymptotic behavior of the integrals

$$S_\alpha(x,t)=\int_{Y'}e^{-\lambda_1(\xi)t}\xi^\alpha e^{i\xi\cdot x}d\xi,\quad (x,t)\in\mathbb{R}^N\times[0,\infty),\tag{150}$$

for $|\alpha|\le k$.

From (113), if $\xi\in U$, we have that

$$\begin{aligned}e^{-\lambda_1(\xi)t}\xi^\alpha &= e^{-q_{ij}\xi_i\xi_j t}\xi^\alpha+e^{-q_{ij}\xi_i\xi_j t}\left(\sum_{n\ge 1}\sum_{|\gamma|\ge 3n}\widetilde{c}_{\gamma,n}\xi^{\gamma+\alpha}t^n\right)\\ &= e^{-q_{ij}\xi_i\xi_j t}\xi^\alpha+e^{-q_{ij}\xi_i\xi_j t}\left(\sum_{n\ge 1}\sum_{|\gamma|\ge 3n+|\alpha|}\widehat{c}_{\gamma,n}\xi^{\gamma}t^n\right),\end{aligned}\tag{151}$$

for suitable constants $\widehat{c}_{\gamma,n}$.

Concerning S_α we have the following:

$$\begin{aligned}&\left|S_\alpha(x,t)-G^*_\alpha(x,t)-\sum_{n=1}^{k-|\alpha|}\sum_{|\gamma|=3n+|\alpha|}^{2n+k}\widehat{c}_{\gamma,n}t^nG^*_\gamma(x,t)\right|\\ &=\left|\int_{Y'}\left(e^{-\lambda_1(\xi)t}\xi^\alpha-e^{-q_{ij}\xi_i\xi_j t}\left(\xi^\alpha+\sum_{n=1}^{k-|\alpha|}\sum_{|\gamma|=3n+|\alpha|}^{2n+k}\widehat{c}_{\gamma,n}t^n\xi^\gamma\right)\right)e^{i\xi\cdot x}d\xi\right|\\ &\le\int_{Y'}\left|e^{-\lambda_1(\xi)t}\xi^\alpha-e^{-q_{ij}\xi_i\xi_j t}\left(\xi^\alpha+\sum_{n=1}^{k-|\alpha|}\sum_{|\gamma|=3n+|\alpha|}^{2n+k}\widehat{c}_{\gamma,n}t^n\xi^\gamma\right)\right|d\xi\\ &\le\int_{Y'}e^{-q_{ij}\xi_i\xi_j t}\left|e^{-\nu(\xi)t}\xi^\alpha-\xi^\alpha-\sum_{n=1}^{k-|\alpha|}\sum_{|\gamma|=3n+|\alpha|}^{2n+k}\widehat{c}_{\gamma,n}t^n\xi^\gamma\right|d\xi.\end{aligned}\tag{152}$$

Notice that the last integral is independent on x. In view of (152), we can see that the integral defined on $Y'\setminus U$ has an exponential decay, uniform with respect to $x\in\mathbb{R}^N$.

Analogously to (117), from (151) we have that

$$
\begin{aligned}
& e^{-\nu(\xi)t}\xi^\alpha - \xi^\alpha - \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|=3n+|\alpha|}^{2n+k} \widehat{c}_{\gamma,n} t^n \xi^\gamma \\
&= \sum_{n\geq k+1-|\alpha|} \sum_{|\gamma|\geq 3n+|\alpha|} \widehat{c}_{\gamma,n} t^n \xi^\gamma + \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|\geq 2n+k+1} \widehat{c}_{\gamma,n} t^n \xi^\gamma \\
&= \sum_{|\gamma|=3(k+1-|\alpha|)+|\alpha|} \widehat{c}_{\gamma,k+1-|\alpha|} t^{k+1-|\alpha|} \xi^\gamma + \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|=2n+k+1} \widehat{c}_{\gamma,n} t^n \xi^\gamma \\
&+ \sum_{|\gamma|\geq 3(k+1-|\alpha|)+1+|\alpha|} \widehat{c}_{\gamma,k+1-|\alpha|} t^{k+1-|\alpha|} \xi^\gamma + \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|\geq 2n+k+2} \widehat{c}_{\gamma,n} t^n \xi^\gamma \\
&+ \sum_{n\geq k+1-|\alpha|} \sum_{|\gamma|\geq 3n+1+|\alpha|} \widehat{c}_{\gamma,n} t^n \xi^\gamma .
\end{aligned}
\tag{153}
$$

Hence we can write

$$
\begin{aligned}
& \left| S_\alpha(x,t) - G^*_\alpha(x,t) - \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|=3n+|\alpha|}^{2n+k} \widehat{c}_{\gamma,n} t^n G^*_\gamma(x,t) \right| \\
&\leq \int_{Y'} e^{-q_{ij}\xi_i\xi_j t} \left| \sum_{|\gamma|=3(k+1)-2|\alpha|} \widehat{c}_{\gamma,k+1-|\alpha|} t^{k+1-|\alpha|} \xi^\gamma + \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|=2n+k+1} \widehat{c}_{\gamma,n} t^n \xi^\gamma \right| d\xi \\
&+ \int_{Y'} e^{-q_{ij}\xi_i\xi_j t} \left| \sum_{|\gamma|\geq 3k+4-2|\alpha|} \widehat{c}_{\gamma,k+1-|\alpha|} t^{k+1-|\alpha|} \xi^\gamma + \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|\geq 2n+k+2} \widehat{c}_{\gamma,n} t^n \xi^\gamma \right. \\
&\left. + \sum_{n\geq k+1-|\alpha|} \sum_{|\gamma|\geq 3n+1+|\alpha|} \widehat{c}_{\gamma,n} t^n \xi^\gamma \right| d\xi .
\end{aligned}
\tag{154}
$$

Now, we will study the asymptotic behavior of the integrals of the right hand side of (154). From Lemma 2.4 we have that

$$
\begin{aligned}
& \int_{Y'} e^{-q_{ij}\xi_i\xi_j t} \left| \sum_{|\gamma|=3(k+1)-2|\alpha|} \widehat{c}_{\gamma,k+1-|\alpha|} t^{k+1-|\alpha|} \xi^\gamma + \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|=2n+k+1} \widehat{c}_{\gamma,n} t^n \xi^\gamma \right| d\xi \\
&\leq \int_{Y'} e^{-q_{ij}\xi_i\xi_j t} \left(\sum_{|\gamma|=3(k+1)-2|\alpha|} \left| \widehat{c}_{\gamma,k+1-|\alpha|} t^{k+1-|\alpha|} \xi^\gamma \right| + \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|=2n+k+1} |\widehat{c}_{\gamma,n} t^n \xi^\gamma| \right) d\xi \\
&\leq c_1 \sum_{|\gamma|=3(k+1)-2|\alpha|} \int_{Y'} e^{-q_{ij}\xi_i\xi_j t} t^{k+1-|\alpha|} |\xi|^{|\gamma|} d\xi \\
&+ c_2 \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|=2n+k+1} \int_{Y'} e^{-q_{ij}\xi_i\xi_j t} t^n |\xi|^{|\gamma|} d\xi \\
&\sim \sum_{|\gamma|=3(k+1)-2|\alpha|} t^{k+1-|\alpha|} s_\gamma t^{-\frac{1}{2}(|\gamma|+N)} d\xi + \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|=2n+k+1} s_{\gamma,n} t^n t^{-\frac{1}{2}(|\gamma|+N)} d\xi \\
&\sim c'_\alpha t^{-\frac{1}{2}(k+1+N)}, \qquad \text{as } t \to \infty,
\end{aligned}
\tag{155}
$$

for suitable constants $s_\gamma, s_{\gamma,n}$.

Furthermore, analogously to (122)-(128), by using the Laplace's Method (see Remark 2.4), we can neglect the terms of the second integral of the right hand side of (154)

$$\int_{Y'} e^{-q_{ij}\xi_i\xi_j t} \left| \sum_{|\gamma|\geq 3k+4-2|\alpha|} \widehat{c}_{\gamma,k+1-|\alpha|} t^{k+1-|\alpha|}\xi^\gamma + \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|\geq 2n+k+2} \widehat{c}_{\gamma,n} t^n \xi^\gamma \right. \\ \left. + \sum_{n\geq k+1-|\alpha|} \sum_{|\gamma|\geq 3n+1+|\alpha|} \widehat{c}_{\gamma,n} t^n \xi^\gamma \right| d\xi, \tag{156}$$

because they contribute to the coefficients of t^{-s} with $s > \frac{1}{2}(k+1+N)$.

In fact, we have the following asymptotic behavior for the integral (156):

The terms of the first series in (156) satisfy

$$|\gamma| + N - 2(k+1-|\alpha|) \geq 3k+4-2|\alpha| + N - 2(k+1-|\alpha|) = k+2+N \tag{157}$$

provided that $|\gamma| \geq 3k+4-2|\alpha|$.

For the terms of the second series in (156) we have that

$$|\gamma| + N - 2n \geq 2n+k+2+N-2n = k+2+N, \tag{158}$$

provided that $1 \leq n \leq k-|\alpha|$ and $|\gamma| \geq 2n+k+2$.

Finally, the terms of the third series in (156) verify

$$|\gamma| + N - 2n \geq 3n + |\alpha| + 1 + N - 2n = n + |\alpha| + 1 + N \geq k+2+N \tag{159}$$

provided that $n \geq k+1-|\alpha|$ and $|\gamma| \geq 3n+1+|\alpha|$.

Therefore, from (154)-(159) we obtain that there exists a positive constant M such that

$$\left| S_\alpha(x,t) - G^*_\alpha(x,t) - \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|=3n+|\alpha|}^{2n+k} \widehat{c}_{\gamma,n} t^n G^*_\gamma(x,t) \right| \sim M t^{-\frac{1}{2}(k+1+N)}, \tag{160}$$

as $t \to \infty$, uniformly with respect to $x \in \mathrm{I\!R}^{\mathrm{N}}$.

Moreover, from (160) and since $c_\alpha \in C_\#(\overline{Y})$ we have that

$$\begin{aligned} &\left| J_1(\cdot,t) - \sum_{|\alpha|\leq k} c_\alpha(x) G^*_\alpha(\cdot,t) - \sum_{n=1}^{k} \sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n}(x) t^n G^*_\gamma(\cdot,t) \right| \\ &\leq \sum_{|\alpha|\leq k} |c_\alpha(x)| \left| S_\alpha(x,t) - G^*_\alpha(x,t) - \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|=3n+|\alpha|}^{2n+k} \widehat{c}_{\gamma,n} t^n G^*_\gamma(x,t) \right| \\ &\leq \sum_{|\alpha|\leq k} \|c_\alpha(\cdot)\|_\infty \left| S_\alpha(x,t) - G^*_\alpha(x,t) - \sum_{n=1}^{k-|\alpha|} \sum_{|\gamma|=3n+|\alpha|}^{2n+k} \widehat{c}_{\gamma,n} t^n G^*_\gamma(x,t) \right| \\ &\sim \widetilde{C} t^{-\frac{1}{2}(k+1+N)}, \end{aligned} \tag{161}$$

as $t \to \infty$, uniformly with respect to $x \in \mathbb{R}^N$. Therefore, there exists a positive constant $C > 0$ such that

$$\left\| J_1(\cdot,t) - \sum_{|\alpha|\le k} c_\alpha(x) G_\alpha^*(\cdot,t) - \sum_{n=1}^{k} \sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n}(x) t^n G_\gamma^*(\cdot,t) \right\|_\infty \le C t^{-\frac{k+1+N}{2}}, \tag{162}$$

as $t \to \infty$.

Finally we conclude that

$$t^{\frac{k+N}{2}} \left\| I_1(\cdot,t) - \sum_{|\alpha|\le k} c_\alpha(x) G_\alpha^*(\cdot,t) - \sum_{n=1}^{k} \sum_{|\gamma|=3n}^{2n+k} c_{\gamma,n}(x) t^n G_\gamma^*(\cdot,t) \right\|_\infty \le C t^{-\frac{1}{2}}, \tag{163}$$

as $t \to \infty$.

Analogously to the L^2 case, the uniform convergence on x of the series in the proof is guaranteed by the analyticity of the first Bloch eigenvalue $\lambda_1(\xi)$ and the first Bloch wave $\phi_1(\xi)$ in a neighborhood U of the origin $\xi = 0$. This completes the proof.

Proof of Corollary 3.1 To prove (87), it is enough to show that

$$\left\| G_\alpha(\cdot,t) - (i)^{-|\alpha|} \frac{\partial^\alpha G_h}{\partial x^\alpha}(\cdot,t) \right\|_\infty \tag{164}$$

has an exponential rate decay as $t \to \infty$.

Notice that

$$\begin{aligned} \frac{\partial^\alpha G_h}{\partial x^\alpha}(x,t) &= \frac{\partial^\alpha}{\partial x^\alpha}\left[\int_{\mathbb{R}^N} e^{-q_{kl}\xi_k\xi_l t} e^{i\xi\cdot x} \right] \\ &= \int_{\mathbb{R}^N} e^{-q_{kl}\xi_k\xi_l t} \frac{\partial^\alpha}{\partial x^\alpha}\left[e^{i\xi\cdot x}\right] d\xi \\ &= \int_{\mathbb{R}^N} e^{-q_{kl}\xi_k\xi_l t} (i)^{|\alpha|} \xi^\alpha e^{i\xi\cdot x} d\xi \\ &= (i)^{|\alpha|} \int_{\mathbb{R}^N} e^{-q_{kl}\xi_k\xi_l t} \xi^\alpha e^{i\xi\cdot x} d\xi. \end{aligned} \tag{165}$$

Thus for all $\gamma \in (\mathbb{N} \cup \{0\})^N$ and $x \in \mathbb{R}^N$

$$\begin{aligned} \left| G_\gamma(x,t) - (i)^{-|\gamma|} \frac{\partial^\gamma G_h}{\partial x^\gamma}(x,t) \right| &\le \int_{\mathbb{R}^N \setminus Y'} e^{-q_{kl}\xi_k\xi_l t} |\xi|^{|\gamma|} d\xi \\ &\le \int_{\mathbb{R}^N \setminus Y'} e^{-\alpha|\xi|^2 t} |\xi|^{|\gamma|} d\xi \\ &\le c \int_{\frac{\pi}{2}}^{\infty} e^{-\alpha r^2 t} r^{|\gamma|} r^{N-1} dr \\ &= c \int_{\frac{\pi}{2}}^{\infty} e^{-\left(\frac{\alpha}{2}+\frac{\alpha}{2}\right) r^2 t} r^{|\gamma|} r^{N-1} dr \\ &\le c \exp\left(-\frac{\alpha}{2}\frac{\pi^2}{4} t \right) \int_{\frac{\pi}{2}}^{\infty} e^{-\frac{\alpha}{2} r^2 t} r^{|\gamma|} r^{N-1} dr \\ &\le c \exp\left(-\frac{\alpha}{2}\frac{\pi^2}{4} t \right) \int_{0}^{\infty} e^{-\frac{\alpha}{2} r^2 t} r^{|\gamma|} r^{N-1} dr \\ &\sim c'_\gamma \exp\left(-\frac{\alpha}{2}\frac{\pi^2}{4} t \right) t^{-\frac{|\gamma|+N}{2}} \end{aligned} \tag{166}$$

as $t \to \infty$.

Notice that the integrals of the right hand side of (166) are independent on x. Thus we have that

$$\begin{aligned}\left\| G_\gamma(\cdot,t) - (i)^{-|\gamma|}\frac{\partial^\gamma G_h}{\partial x^\gamma}(\cdot,t)\right\|_\infty &\leq \int_{\mathbb{R}^N\setminus Y'} e^{-q_{kl}\xi_k\xi_l t}\,|\xi|^{|\gamma|}\,d\xi \\ &\sim c'_\gamma \exp\left(-\frac{\alpha}{2}\frac{\pi^2}{4}t\right) t^{-\frac{|\gamma|+N}{2}}\end{aligned} \tag{167}$$

as $t \to \infty$.

Thus the norm (164) has an exponential rate decay. This completes the proof.

6 Effective computation of the constants and periodic functions appearing in the asymptotic expansions

In this section we describe in detail how the constants and functions appearing in the expansions of Theorems 3.1 and 3.2 can be computed explicitly.

For the asymptotic expansion of $u(x,t)$ in $L^2\left(\mathbb{R}^N\right)$, i. e., in the context of Theorem 3.1, we have that

$$c_\alpha = \frac{1}{\alpha!}\frac{\partial^\alpha \widehat{\varphi}_1}{\partial \xi^\alpha}(0), \tag{168}$$

where

$$\widehat{\varphi}_1(\xi) = \int_{\mathbb{R}^N} \varphi(x)\, e^{-i\xi\cdot x}\overline{\phi}_1(x,\xi)\,dx. \tag{169}$$

According to (99),

$$\partial^\alpha \widehat{\varphi}_1(0) = \sum_{\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(-i)^{|\beta|}\int_{\mathbb{R}^N} x^\beta \varphi(x)\,\partial_\xi^{\alpha-\beta}\overline{\phi}_1(x,0)\,dx. \tag{170}$$

Let us recall how the constants $c_{\gamma,n}$, in Theorem 3.1 are obtained.

Let $\nu(\xi) = \lambda_1(\xi) - q_{ij}\xi_i\xi_j$, which is an analytic function in U. Since

$$\nu(\xi) = \sum_{|\alpha|\geq 3}\frac{1}{\alpha!}\frac{\partial^\alpha\lambda_1}{\partial\xi^\alpha}(0)\xi^\alpha, \tag{171}$$

then

$$e^{-\nu(\xi)t} = 1 + \sum_{n\geq 1}\frac{(-1)^n}{n!}t^n\left(\sum_{|\alpha|\geq 3}\frac{1}{\alpha!}\frac{\partial^\alpha\lambda_1}{\partial\xi^\alpha}(0)\xi^\alpha\right)^n. \tag{172}$$

Moreover

$$\left(\sum_{|\alpha|\geq 3}\frac{1}{\alpha!}\frac{\partial^\alpha\lambda_1}{\partial\xi^\alpha}(0)\xi^\alpha\right)^n = \sum_{|\beta|\geq 3n}\widehat{c}_{\beta,n}\xi^\beta. \tag{173}$$

where the coefficients $\widehat{c}_{\beta,n}$ may be computed explicitly in terms of the derivatives $(\partial^\alpha\lambda_1/\partial\xi^\alpha)(0)$.

Thus

$$\left(\sum_{|\alpha|\le k} c_\alpha \xi^\alpha\right)\left(\sum_{n\ge 1}\frac{(-1)^n}{n!}t^n\left(\nu(\xi)\right)^n\right)$$

$$= \sum_{|\alpha|\le k}\sum_{n\ge 1}\sum_{|\beta|\ge 3n} c_\alpha \frac{(-1)^n}{n!}\widehat{c}_{\beta,n} t^n \xi^{\beta+\alpha} \tag{174}$$

$$= \sum_{n\ge 1}\sum_{|\gamma|\ge 3n} c_{\gamma,n} t^n \xi^\gamma .$$

where the coefficients $c_{\gamma,n}$ depend on c_α and $\widehat{c}_{\beta,n}$.

Thus

$$e^{-\nu(\xi)t}\left(\sum_{|\alpha|\le k} c_\alpha \xi^\alpha\right) = \sum_{|\alpha|\le k} c_\alpha \xi^\alpha + \sum_{n\ge 1}\sum_{|\gamma|\ge 3n} c_{\gamma,n} t^n \xi^\gamma . \tag{175}$$

This shows that the coefficients $c_{\gamma,n}$ in Theorem 3.1 may be obtained as a combination of the coefficients c_α and $\widehat{c}_{\beta,n}$, the latter being combinations of the derivatives of λ_1 at $\xi = 0$.

Let us now analyze the functions entering in the statement of Theorem 3.2. We have

$$c_\alpha(x) = \frac{1}{\alpha!}\frac{\partial^\alpha}{\partial \xi^\alpha}\left(\widehat{\varphi}_1(\cdot)\phi_1(x,\cdot)\right)(0). \tag{176}$$

that may developed as in (99). Moreover, the functions $c_{\gamma,n}(x)$ are obtained from the asymptotic expansion (151) with the functions $c_\alpha(x)$ given in (141). That is: If $\varphi \in L^1\left(\mathrm{I\!R}^{\mathrm{N}}; 1+|\mathrm{x}|^{\mathrm{k}+1}\right)$, the function

$$\xi \to \widehat{\varphi}_1(\xi)\phi(\cdot,\xi) \in C^{k+1}\left(U, C_\#(\overline{Y})\right). \tag{177}$$

Then we can approximate the function (177) by the $k-$th Taylor expansion

$$\sum_{|\alpha|\le k}\frac{1}{\alpha!}\frac{\partial^\alpha}{\partial \xi^\alpha}\left(\phi_1(x,\cdot)\widehat{\varphi}_1\right)(0)\,\xi^\alpha,\ \forall \xi \in U,\ \forall x \in \mathrm{I\!R}^{\mathrm{N}}. \tag{178}$$

Let

$$c_\alpha(x) = \frac{1}{\alpha!}\frac{\partial^\alpha}{\partial \xi^\alpha}\left(\phi_1(x,\cdot)\widehat{\varphi}_1\right)(0) \tag{179}$$

Hence, as in (175), we obtain that

$$\begin{aligned} e^{-\lambda_1(\xi)t}\left(\sum_{|\alpha|\le k} c_\alpha(x)\xi^\alpha\right) &= e^{-\nu(\xi)t}e^{-q_{ij}\xi_i\xi_j t}\left(\sum_{|\alpha|\le k} c_\alpha(x)\xi^\alpha\right) \\ &= e^{-q_{ij}\xi_i\xi_j t}\left(\sum_{|\alpha|\le k} c_\alpha(x)\xi^\alpha\right) \\ &+ e^{-q_{ij}\xi_i\xi_j t}\left(\sum_{n\ge 1}\sum_{|\gamma|\ge 3n} c_{\gamma,n}(x)\xi^\gamma t^n\right). \end{aligned} \tag{180}$$

We can see that in the definitions of the constants and functions involved in these asymptotic expansions, the derivatives of the functions $\phi_1(x,\xi)$ and $\lambda_1(\xi)$ with respect to ξ at $\xi = 0$ appear.

The higher order derivatives of λ_1 and ϕ_1 with respect to ξ may be computed taking derivatives in the equation

$$A(\xi)\,\phi_1(\xi) = \lambda_1(\xi)\,; \quad \phi_1 \text{ is } Y\text{-periodic.} \tag{181}$$

In fact, if $\xi \in U$ we have that

$$\begin{aligned} A(\xi)\,\phi_1(\xi) &= -\left(\frac{\partial}{\partial x_k} + i\xi_k\right)\left[a_{kl}\left(\frac{\partial}{\partial x_l} + i\xi_l\right)\phi_1(\xi)\right] \\ &= -\frac{\partial}{\partial x_k}\left(a_{kl}\frac{\partial \phi_1}{\partial x_l}\right) - \frac{\partial}{\partial x_l}\left(a_{kl} i\xi_l \phi_1(\xi)\right) \\ &\quad -i\xi_k\left(a_{kl}\left(\frac{\partial \phi_1(\xi)}{\partial x_l}\right)\right) - i\xi_k\left(a_{kl}\phi_1(\xi) i\xi_l\right) \\ &= A(0)\phi_1(\xi) - i\xi_l\frac{\partial}{\partial x_l}\left(a_{kl}\phi_1(\xi)\right) \\ &\quad -i\xi_k\left(a_{kl}\left(\frac{\partial \phi_1(\xi)}{\partial x_l}\right)\right) + \xi_k\xi_l a_{kl}\phi_1(\xi). \end{aligned} \tag{182}$$

Moreover, we obtain that, if $\xi \in U$, we have

$$\begin{aligned} \frac{\partial^2}{\partial \xi_r \xi_s}\left(A(\cdot)\,\phi_1(\cdot)\right)(0) &= A(0)\frac{\partial^2 \phi_1}{\partial \xi_r \xi_s}(0) - i\frac{\partial}{\partial x_k}\left(a_{ks}\frac{\partial \phi_1}{\partial \xi_r}(0)\right) \\ &\quad -i\frac{\partial}{\partial x_k}\left(a_{kr}\frac{\partial \phi_1}{\partial \xi_s}(0)\right) - i\left(a_{sl}\frac{\partial}{\partial x_l}\left(\frac{\partial \phi_1}{\partial \xi_r}(0)\right)\right) \\ &\quad -i\left(a_{rl}\frac{\partial}{\partial x_l}\left(\frac{\partial \phi_1}{\partial \xi_s}(0)\right)\right) + a_{sr}\phi_1(0) + a_{rs}\phi_1(0). \end{aligned} \tag{183}$$

On the other hand

$$\begin{aligned} \frac{\partial^2}{\partial \xi_r \xi_s}\left(\lambda_1(\cdot)\,\phi_1(\cdot)\right)(0) = \quad & \frac{\partial^2 \lambda_1}{\partial \xi_r \xi_s}(0)\phi_1(0) + \frac{\partial \lambda_1}{\partial \xi_s}(0)\frac{\partial \phi_1}{\partial \xi_r}(0) \\ & + \frac{\partial \lambda_1}{\partial \xi_r}(0)\frac{\partial \phi_1}{\partial \xi_s}(0) + \lambda_1(0)\frac{\partial^2 \phi_1}{\partial \xi_r \xi_s}(0). \end{aligned} \tag{184}$$

We also know that (Proposition 2.1)

$$\lambda_1(0) = \frac{\partial \lambda_1}{\partial \xi_r}(0) = 0 \quad , \quad \frac{\partial^2 \lambda_1(0)}{\partial \xi_s \partial \xi_j} = 2q_{sj}, \tag{185}$$

and

$$\phi_1(0) = \frac{1}{|Y|^{\frac{1}{2}}} \quad , \quad \frac{\partial \phi_1}{\partial \xi_s}(0) = \frac{i}{|Y|^{\frac{1}{2}}}\chi_s. \tag{186}$$

Then, the second derivative of $\phi_1(\cdot;\xi)$ at $\xi = 0$ may be characterized as the solution of

$$\begin{aligned} & A(0)\frac{\partial^2 \phi}{\partial \xi_s \partial \xi_j}(0) = -\frac{1}{|Y|^{\frac{1}{2}}}\left[\frac{\partial}{\partial x_k}\left(a_{ks}\chi_s\right) + \frac{\partial}{\partial x_k}\left(a_{kr}\chi_s\right) + a_{sl}\frac{\partial \chi_r}{\partial x_l} a_{rl}\frac{\partial \chi_s}{\partial x_l} + 2a_{rs}\right] \\ & \frac{\partial^2 \phi}{\partial \xi_s \partial \xi_j}(0) \quad \text{is } Y\text{-periodic.} \end{aligned} \tag{187}$$

It is important to see that, when the coefficients a_{kl} of the equation are constant, the terms corresponding to the constants $c_{\gamma,n}$ in Lemma 4.2 and Theorem 3.1 do not appear.

In this way we recover results of the form (18). This is due to the fact that, in this case, $\lambda_1(\xi) = q_{ij}\xi_i\xi_j$, q_{ij} being constant coefficients.

Recently, in [4], it has been proved that all odd order derivatives of the first Bloch eigenvalue at $\xi = 0$ vanish, that is,

$$\frac{\partial^\beta \lambda_1}{\partial \xi^\beta}(0) = 0, \quad \forall \beta \in \mathbb{Z}_+^N,\ |\beta| \text{ odd.}$$

Of course, this result may be useful to simplify the expressions we have obtained in the asymptotic expansions.

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