

HARDY INEQUALITIES, OBSERVABILITY, AND CONTROL FOR THE WAVE AND SCHRÖDINGER EQUATIONS WITH SINGULAR POTENTIALS*

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Abstract. We address the question of exact controllability of the wave and Schrödinger equations perturbed by a singular inverse-square potential. Exact boundary controllability is proved in the range of subcritical coefficients of the singular potential and under suitable geometric conditions. The proof relies on the method of multipliers. The key point in the proof of the observability inequality is a suitable Hardy-type inequality with sharp constants. On the contrary, in the supercritical case, we prove that exact controllability is false.

Key words. Hardy inequalities, observability, control, wave, singular potential

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1. Introduction. In this work, we mainly address the problem of exact controllability for linear wave equations with singular potentials. More precisely, we focus on the so-called *inverse-square* potential arising, for example, in the context of combustion theory [3, 7, 16, 20] and quantum mechanics [1, 10, 27].

Let $N \geq 1$ be given, and consider $\Omega \subset \mathbb{R}^N$ a bounded open set such that $0 \in \Omega$ and whose boundary Γ is of class \mathcal{C}^2 . We denote by Γ_0 some nonempty part of Γ . Then consider the following hyperbolic problem:

$$(1.1) \quad \begin{cases} u_{tt} - \Delta u - \frac{\lambda}{|x|^2}u = 0, & (t, x) \in (0, T) \times \Omega, \\ u(t, x) = h(t, x), & (t, x) \in (0, T) \times \Gamma_0, \\ u(t, x) = 0, & (t, x) \in (0, T) \times \Gamma \setminus \Gamma_0, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \Omega, \end{cases}$$

with (u_0, u_1) given in $L^2(\Omega) \times H'_\lambda$ (the space H'_λ is defined later in section 2. However, to fix ideas, we can assume that $u_1 \in H^{-1}(\Omega)$). Here, $h \in L^2((0, T) \times \Gamma_0)$ is the control that acts on the part Γ_0 of Γ . The solution u of (1.1) is the state of the system.

We are concerned with the property of exact controllability, i.e., whether, for every initial condition $(u_0, u_1) \in L^2(\Omega) \times H'_\lambda$ and every target $(\bar{u}_0, \bar{u}_1) \in L^2(\Omega) \times H'_\lambda$, there exists $h \in L^2((0, T) \times \Gamma_0)$ such that the solution u of (1.1) satisfies

$$(1.2) \quad (u(T, x), u_t(T, x)) = (\bar{u}_0, \bar{u}_1) \quad \text{for a.e. } x \in \Omega.$$

Here and in what follows, λ_* stands for the critical constant

$$(1.3) \quad \lambda_* := \frac{(N-2)^2}{4}$$

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in the Hardy inequality (see for instance [22, 25]) guaranteeing that, when $N \neq 2$, for every $z \in H_0^1(\Omega)$, we have $z/|x| \in L^2(\Omega)$ and

$$(1.4) \quad \forall z \in H_0^1(\Omega), \quad \lambda_* \int_{\Omega} \frac{z^2}{|x|^2} dx \leq \int_{\Omega} |\nabla z|^2 dx.$$

In particular, (1.4) implies that, under the condition $\lambda \leq \lambda_*$, the operator $-\Delta - \lambda|x|^{-2}$ is nonnegative. The critical value λ_* of the parameter plays a key role in the statement of well-posedness results for problems like (1.1), and one expects this value to be important when addressing controllability as well.

Recently, the question of null controllability of heat equations with inverse-square potentials has been studied in [30, 17]. It has been proved that, within the range of subcritical values $\lambda \leq \lambda_*$, null controllability holds.

In this work, we address a similar question in the context of the wave equation. More precisely, we consider the case where the subset Γ_0 of the boundary where the control is active is of the form

$$(1.5) \quad \Gamma_0 = \{x \in \Gamma \mid x \cdot \nu \geq 0\}.$$

Our main result is the exact controllability of (1.1) for $T > 0$ sufficiently large, independent of $\lambda \leq \lambda_*$, in this geometric setting.

This result is actually equivalent to an observability inequality for the solutions of the adjoint system:

$$(1.6) \quad \begin{cases} v_{tt} - \Delta v - \frac{\lambda}{|x|^2} v = 0, & (t, x) \in (0, T) \times \Omega, \\ v(t, x) = 0, & (t, x) \in (0, T) \times \Gamma, \\ v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x), & x \in \Omega. \end{cases}$$

The main contribution of this paper consists precisely in proving this observability inequality. To be more precise, we show that, for T large enough and all $\lambda \leq \lambda_*$, there exists $C > 0$ such that

$$(1.7) \quad \int_{\Omega} \left(|\nabla v(0, x)|^2 - \lambda \frac{|v(0, x)|^2}{|x|^2} + |v_t(0, x)|^2 \right) dx \leq C \int_0^T \int_{\Gamma_0} \frac{\partial v^2}{\partial \nu} d\sigma dt.$$

We refer to Theorem 3.1 for a precise statement of this result. As a consequence of this inequality, it follows that system (1.1) is exactly controllable in time T by a control acting on Γ_0 ; see Theorem 4.1.

The proof of (1.7) relies on the method of multipliers, and a key point in the proof is the following new sharp Hardy-type inequality.

THEOREM 1.1. *Let Ω be a bounded domain of \mathbb{R}^N , $N \geq 1$, and λ_* be as in (1.3).*

Then

$$(1.8) \quad \forall z \in H_0^1(\Omega), \quad \int_{\Omega} |x|^2 |\nabla z|^2 dx \leq R_{\Omega}^2 \int_{\Omega} \left(|\nabla z|^2 - \lambda_* \frac{z^2}{|x|^2} \right) dx + \frac{N^2 - 4}{4} \int_{\Omega} z^2 dx,$$

where $R_{\Omega} := \max_{x \in \Omega} |x|$.

The proof of this theorem is given in an appendix at the end of the paper.

Remark 1.1. The above result is a kind of Hardy–Poincaré inequality in the spirit of the inequalities by Brézis–Vázquez [7], later improved by Vázquez–Zuazua

[31] and Filippas-Tertikas [18] (and extended to the case of multipolar singularities by Bosi-Dolbeault-Esteban [5]). In those works, the norm of z (in L^2 or in $W^{1,q}$ for $1 \leq q < 2$) was estimated by $\|z\|_{H_{\lambda_*}}$. For example, Brézis-Vázquez [7] proved

$$(1.9) \quad \forall z \in H_0^1(\Omega), \quad \int_{\Omega} z^2 dx \leq C \int_{\Omega} \left(|\nabla z|^2 - \lambda_* \frac{z^2}{|x|^2} \right) dx.$$

For our purpose, we need here to provide an estimate of the L^2 -norm of $x \cdot \nabla z$ in terms of $\|z\|_{H_{\lambda_*}}$. Observe that (1.8) combined with (1.9) leads to

$$(1.10) \quad \forall z \in H_0^1(\Omega), \quad \int_{\Omega} |x|^2 |\nabla z|^2 dx \leq C \int_{\Omega} \left(|\nabla z|^2 - \lambda_* \frac{z^2}{|x|^2} \right) dx.$$

However, (1.10) is not sharp enough for our purpose. Indeed, it would be sufficient to prove the required observability inequality, but this would not yield the expected minimal time of controllability.

The question of exact controllability for the wave equation when $\lambda \leq \lambda_*$ is addressed in sections 2–4. On the contrary, in section 5, we prove that exact controllability is false in the supercritical case, i.e., when $\lambda > \lambda_*$. In section 6, we also briefly discuss the case of the Schrödinger equation with an inverse-square potential. Following the proof of Machtyngier [24] (that holds for the standard Schrödinger equation) and using the Hardy inequality stated in Theorem 1.1, we prove similar exact controllability results for this problem.

Finally, section 7 is devoted to discuss further results and issues related to the limitations of the multiplier method. Our main results stated in sections 2–4 are obtained using the simplest radial multiplier $(x - x_0) \cdot \nabla v$ (with $x_0 = 0$); see, e.g., the books by Lions [23] and Komornik [21]. Several variants have been developed in the context of the wave equation. For instance, Osses in [26] introduced an added "rotated multiplier" of the form $A(x - x_0) \cdot \nabla v$, where A is a skew-symmetric matrix. This yields a wider class of geometric situations where explicit observability can be proved for the wave equation. The application of this technique to our problem generates further results, stated in section 7, where the geometric assumption (1.5) above on Γ_0 is relaxed.

The main limitation of the multiplier method, as applied here, is that it has to be centered at the point where the singularity is located ($x_0 = 0$). Thus, our results are limited to the case where Γ_0 is of the form (1.5) or satisfies the assumptions given in section 7. The case of a more general geometry is still to be considered. For the same reason, the case of multipolar inverse-square singular potentials (considered in [17] in the case of the heat equation) cannot be treated with this method. The treatment of these two problems above requires other techniques such as, for example, the derivation of new Carleman estimates for singular wave equations (as done in [30, 17] for the heat equation). The microlocal analysis of these problems is also to be done.

There is an extensive literature on Hardy-type inequalities with various generalizations as, for instance, the case where the singularity is located all along the boundary of the domain (see, for example, [6, 13, 8]). It would be natural to address the extension of the controllability property to those frameworks. Another case of interest that we do not treat here is that in which the singularity is placed on a point of the boundary. Of course, our results apply in that case too. But there one expects

to be able to improve the range of λ 's for which the results hold. This first needs of improvements of the existing Hardy inequalities; see section 7.4.

2. Basic properties for the wave equation with an inverse-square potential. We recall that $N \geq 1$. We fix an arbitrary $T > 0$ and assume that $\Omega \subset \mathbb{R}^N$ is a bounded open set such that $0 \in \Omega$ whose boundary Γ is of class \mathcal{C}^2 . We also use the notations $Q_T := (0, T) \times \Omega$ and $\Sigma_T := (0, T) \times \Gamma$. Finally, we assume that $\lambda \leq \lambda_*$.

Before considering controllability questions, we address the question of well-posedness of problems of the form

$$(2.1) \quad \begin{cases} u_{tt} - \Delta u - \frac{\lambda}{|x|^2} u = 0, & (t, x) \in Q_T, \\ u(t, x) = g(t, x), & (t, x) \in \Sigma_T, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \Omega. \end{cases}$$

In this purpose, we first need to state some basic properties of wave equations of the following type:

$$(2.2) \quad \begin{cases} v_{tt} - \Delta v - \frac{\lambda}{|x|^2} v = f(t, x), & (t, x) \in Q_T, \\ v(t, x) = 0, & (t, x) \in \Sigma_T, \\ v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x), & x \in \Omega. \end{cases}$$

The well-posedness of the heat equation with an inverse-square singular potential has been studied in [2, 8, 31] depending on the value of the parameter λ with respect to λ_* : the problem is well-posed when $\lambda \leq \lambda_*$, whereas instantaneous and complete blow-up of positive solutions occurs when $\lambda > \lambda_*$. We prove here a similar well-posedness result for the wave equation: (2.2) is well-posed in the range of subcritical parameters $\lambda \leq \lambda_*$. As in [31], the case $\lambda = \lambda_*$ is slightly peculiar, since the functional setting differs from the case $\lambda < \lambda_*$ (see Remark 2.1).

2.1. Finite energy solutions for homogeneous Dirichlet boundary conditions. The first step is to prove the existence of finite energy solutions for (2.2). For that we introduce the Hilbert space H_λ obtained as the completion of $H_0^1(\Omega)$ with respect to the norm

$$\|z\|_{H_\lambda} = \left(\int_\Omega \left(|\nabla z|^2 - \lambda \frac{z^2}{|x|^2} \right) dx \right)^{1/2}.$$

In the subcritical case $\lambda < \lambda_*$ when $N \neq 2$, by (1.4), one can easily prove that $\|\cdot\|_{H_\lambda}$ is equivalent to the usual norm in $H_0^1(\Omega)$, and therefore $H_\lambda = H_0^1(\Omega)$. But, as pointed out in [31], when $\lambda = \lambda_*$, H_{λ_*} is strictly larger than $H_0^1(\Omega)$. To be more precise, in the subcritical case, the following holds.

LEMMA 2.1. *Assume that $N \neq 2$ and $\lambda < \lambda_*$. Then there exist two constants $C_{1,\lambda} > 0$ and $C_{2,\lambda} > 0$ such that, for every $z_0 \in H_0^1(\Omega)$,*

$$C_{1,\lambda} \|z_0\|_{H_0^1(\Omega)} \leq \|z_0\|_{H_\lambda} \leq C_{2,\lambda} \|z_0\|_{H_0^1(\Omega)}.$$

More precisely,

$$C_{1,\lambda} = 1 - \frac{\max(0, \lambda)}{\lambda_*}, \quad C_{2,\lambda} = 1 - \frac{\min(0, \lambda)}{\lambda_*}.$$

Remark 2.1. In the critical case $\lambda = \lambda_*$, the space H_{λ_*} has been described by Vázquez-Zuazua [31]. By improving some Hardy–Poincaré inequalities of Brézis–Vázquez [7], they proved that H_{λ_*} is slightly larger than $H_0^1(\Omega)$ and smaller than $\cap_{q < 2} W^{1,q}(\Omega)$. Those inequalities have also been refined by Filippas-Tertakis; see [18].

For all $\lambda \leq \lambda_*$, we also define the domain of the operator $-\Delta - \lambda|x|^{-2}$:

$$D_\lambda := D(-\Delta - \lambda|x|^{-2}) = \{z \in H_\lambda \mid -\Delta z - \lambda|x|^{-2}z \in L^2(\Omega)\}.$$

Remark 2.2. Observe that when $z \in D_\lambda$, then $\Delta z \in L^2(\Omega \setminus B_\varepsilon)$ for all nonempty ball $B_\varepsilon = B(0, \varepsilon) \subset \subset \Omega$. Hence $z \in H^2(\Omega \setminus B_\varepsilon)$, and $\partial z / \partial \nu = \nabla z \cdot \nu|_\Gamma$ exists and belongs, in particular, to $L^2(\Gamma)$.

With the above notations, one can prove that problem (2.2) is well-posed in the energy space $H_\lambda \times L^2(\Omega)$ which, in the subcritical case $\lambda < \lambda_*$ when $N \neq 2$, coincides with the energy space $H_0^1(\Omega) \times L^2(\Omega)$ of the standard wave equation.

PROPOSITION 2.1. *Let $T > 0$ be given and assume $\lambda \leq \lambda_*$.*

(i) *For every $(v_0, v_1) \in H_\lambda \times L^2(\Omega)$ and $f \in L^1(0, T; L^2(\Omega))$, there exists a unique solution v to (2.2) with $v \in \mathcal{C}([0, T]; H_\lambda) \cap \mathcal{C}^1([0, T]; L^2(\Omega))$.*

Moreover, there exists a constant $C > 0$ such that, for every $(v_0, v_1) \in H_\lambda \times L^2(\Omega)$ and $f \in L^1(0, T; L^2(\Omega))$, the solution v of (2.2) satisfies

$$\|(v, v_t)\|_{L^\infty(0, T; H_\lambda \times L^2(\Omega))} \leq C\|(v_0, v_1)\|_{H_\lambda \times L^2(\Omega)} + C\|f\|_{L^1(0, T; L^2(\Omega))}.$$

(ii) *Moreover, if $(v_0, v_1) \in D_\lambda \times H_\lambda$ and $f \in \mathcal{C}([0, T]; L^2(\Omega)) \cap L^1(0, T; H_\lambda)$, then the solution v of (2.2) satisfies*

$$v \in \mathcal{C}([0, T]; D_\lambda) \cap \mathcal{C}^1([0, T]; H_\lambda) \cap \mathcal{C}^2([0, T]; L^2(\Omega)).$$

Furthermore, there exists a constant $C > 0$ such that, for every $(v_0, v_1) \in D_\lambda \times H_\lambda$ and $f \in \mathcal{C}([0, T]; L^2(\Omega)) \cap L^1(0, T; H_\lambda)$, the solution v of (2.2) satisfies

$$\|(v, v_t)\|_{L^\infty(0, T; D_\lambda \times H_\lambda)} \leq C\|(v_0, v_1)\|_{D_\lambda \times H_\lambda} + C\|f\|_{L^1(0, T; H_\lambda)}.$$

Sketch of the proof. As in the case of the heat equation (see [31]), the proof of Proposition 2.1 simply follows from standard semigroup theory. Here we simply indicate how to build the generator of the semigroup. Let us consider the operator A defined by $A(v, w) = (w, \Delta v + \lambda|x|^{-2}v) \forall (v, w) \in D(A) = D_\lambda \times H_\lambda$. Then one can prove that $(A, D(A))$ and $(-A, D(A))$ are m-dissipative in $H_\lambda \times L^2(\Omega)$. (Here H_λ is endowed with the scalar product naturally associated to the norm $\|\cdot\|_{H_\lambda}$.) Hence A generates a unitary C_0 -group in $H_\lambda \times L^2(\Omega)$, and the result follows. \square

As it is typical for the groups of isometries generated by skew-adjoint operators, we also have an energy identity that, in the absence of external forces, ensures the energy conservation property. For all $\lambda \leq \lambda_*$, we define the generalized energy of the solutions of (2.2):

$$E_v^\lambda(t) := \frac{1}{2} \int_\Omega \left(|\nabla v|^2 - \lambda \frac{v^2}{|x|^2} + v_t^2 \right) dx = \frac{1}{2} \left(\|v\|_{H_\lambda}^2 + \|v_t\|_{L^2(\Omega)}^2 \right).$$

In the case $\lambda < \lambda_*$ when $N \neq 2$, since $H_\lambda = H_0^1(\Omega)$, we also introduce the classical energy of v :

$$E_v^0(t) := \frac{1}{2} \int_\Omega (|\nabla v|^2 + v_t^2) dx = \frac{1}{2} \left(\|v\|_{H_0^1(\Omega)}^2 + \|v_t\|_{L^2(\Omega)}^2 \right),$$

which is, in that case, equivalent to $E_v^\lambda(t)$:

$$(2.3) \quad \min \left(1, (C_{1,\lambda})^2 \right) E_v^0(t) \leq E_v^\lambda(t) \leq \max \left(1, (C_{2,\lambda})^2 \right) E_v^0(t).$$

Classical computations show that the generalized energy E_v^λ of the solution is constant when $f = 0$.

LEMMA 2.2. *Assume $\lambda \leq \lambda_*$, and consider $(v_0, v_1) \in H_\lambda \times L^2(\Omega)$ and $f = 0$. Then the energy $t \mapsto E_v^\lambda(t)$ of the solution v of (2.2) is constant in time.*

2.2. Regularity of the normal derivative. Next we need to prove some regularity results of the normal derivative of the solutions of (2.2). Those results are well known in the case of the standard wave equation (i.e., when $\lambda = 0$) and referred to as a property of “hidden regularity”; see [23, Theorem 4.1]. More precisely, we prove the following.

PROPOSITION 2.2. *Let $T > 0$ be given and assume $\lambda \leq \lambda_*$.*

(i) *There exists some constant $C_{T,\lambda} > 0$ such that, for every $(v_0, v_1, f) \in H_\lambda \times L^2(\Omega) \times L^1(0, T; L^2(\Omega))$, the solution v of (2.2) satisfies*

$$(2.4) \quad \int_{\Sigma_T} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt \leq C_{T,\lambda} \left(E_v^\lambda(0) + \|f\|_{L^1(0,T;L^2(\Omega))}^2 \right).$$

(ii) *There exists some constant $C_{T,\lambda} > 0$ such that, for every $f_1 \in C_c^1([0, T]; H_\lambda)$, the solution v of (2.2) with $(v_0, v_1) = (0, 0)$ and $f = df_1/dt$ satisfies*

$$(2.5) \quad \int_{\Sigma_T} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt \leq C_{T,\lambda} \|f_1\|_{L^1(0,T;H_\lambda)}^2.$$

Proof of point (i) of Proposition 2.2. We proceed in two steps. We first prove that (2.4) holds for all $(v_0, v_1, f) \in D_\lambda \times H_\lambda \times C^0([0, T]; L^2(\Omega)) \cap L^1(0, T; H_\lambda)$. Observe that, in that case, the solution v of (2.2) belongs to $\mathcal{C}([0, T]; D_\lambda)$. Hence, by Remark 2.2, $\partial v / \partial \nu$ exists and belongs to $L^2(\Sigma_T)$. Next we conclude that $\partial v / \partial \nu$ can also be defined and that (2.4) also holds for all $(v_0, v_1, f) \in H_\lambda \times L^2(\Omega) \times L^1(0, T; L^2(\Omega))$.

Step 1. Let us first prove that (2.4) holds for all $(v_0, v_1, f) \in D_\lambda \times H_\lambda \times C^0([0, T]; L^2(\Omega)) \cap L^1(0, T; H_\lambda)$. The proof relies on the multiplier method. Let us recall the following classical identity.

LEMMA 2.3 (see [23, Lemma 3.7, page 40]). *Let $q = (q_k)_k \in C^1(\overline{\Omega})^N$ be given, and consider the problem*

$$(2.6) \quad \begin{cases} z_{tt} - \Delta z = F, & (t, x) \in Q_T, \\ z(t, x) = 0, & (t, x) \in \Sigma_T, \\ z(0, x) = z_0(x), \quad z_t(0, x) = z_1(x), & x \in \Omega. \end{cases}$$

Then for every $(z_0, z_1, F) \in H_0^1(\Omega) \times L^2(\Omega) \times L^1(0, T; L^2(\Omega))$, the solution z of (2.6) satisfies the identity

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_T} q \cdot \nu \left(\frac{\partial z}{\partial \nu} \right)^2 d\sigma dt &= \left[\int_{\Omega} z_t q \cdot \nabla z dx \right]_0^T + \frac{1}{2} \int_{Q_T} (z_t^2 - |\nabla z|^2) \operatorname{div} q dx dt \\ &\quad + \sum_{j,k} \int_{Q_T} \frac{\partial q_k}{\partial x_j} \frac{\partial z}{\partial x_j} \frac{\partial z}{\partial x_k} dx dt - \int_{Q_T} F q \cdot \nabla z dx dt. \end{aligned}$$

Since $\Omega \subset \mathbb{R}^N$ is a bounded open set whose boundary Γ is of class \mathcal{C}^2 , it is well known that there exists $q_0 \in \mathcal{C}^1(\overline{\Omega})^N$ such that, for all $x \in \Gamma$, $q_0(x) = \nu(x)$; see [23, Lemma 3.1, page 29]. Let us consider V_0 and V'_0 two open subsets such that

$$0 \in V_0 \subset\subset V'_0 \subset\subset \Omega.$$

Next we introduce $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ a cut-off function of class \mathcal{C}^∞ such that

$$0 \leq \phi \leq 1, \quad \phi \equiv 0 \text{ in } V_0, \quad \phi \equiv 1 \text{ in } \Omega \setminus V'_0.$$

Finally we define $q \in \mathcal{C}^1(\overline{\Omega})^N$ by $q := q_0 \phi$. Notice that q satisfies the following properties:

$$q(x) = \nu(x) \quad \forall x \in \Gamma \text{ and } q(x) = 0 \quad \forall x \in V_0.$$

In order to derive inequality (2.4), we apply Lemma 2.3 with the above choice of q and with $z_0 = v_0$, $z_1 = v_1$, and $F = \lambda v/|x|^2 + f$, where v is the solution of (2.2). We obtain that

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_T} q \cdot \nu \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt &= \left[\int_{\Omega} v_t q \cdot \nabla v dx \right]_0^T + \frac{1}{2} \int_{Q_T} (v_t^2 - |\nabla v|^2) \operatorname{div} q dx dt \\ &\quad + \sum_{j,k} \int_{Q_T} \frac{\partial q_k}{\partial x_j} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_k} dx dt - \lambda \int_{Q_T} \frac{v}{|x|^2} q \cdot \nabla v dx dt \\ &\quad - \int_{Q_T} f q \cdot \nabla v dx dt. \end{aligned}$$

Using the fact that $q \equiv 0$ in V_0 , we deduce that

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_T} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt &\leq C \int_{\Omega} v_t^2(T) dx + C \int_{\Omega \setminus V_0} |\nabla v|^2(T) dx \\ &\quad + C \int_{\Omega} v_t^2(0) dx + C \int_{\Omega \setminus V_0} |\nabla v|^2(0) dx + C \int_{Q_T} v_t^2 dx dt \\ &\quad + C \int_{(0,T) \times (\Omega \setminus V_0)} |\nabla v|^2 dx dt + |\lambda| \int_{Q_T} \frac{|v|}{|x|^2} |q \cdot \nabla v| dx dt \\ &\quad + C \int_{(0,T) \times (\Omega \setminus V_0)} |f| |\nabla v| dx dt. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_T} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt &\leq C(1+T) \|(v, v_t)\|_{L^\infty(0,T;H_0^1(\Omega \setminus V_0)) \times L^2(\Omega)}^2 \\ &\quad + |\lambda| \int_{Q_T} \frac{|v|}{|x|^2} |q \cdot \nabla v| dx dt \\ &\quad + C \|\nabla v\|_{L^\infty(0,T;L^2(\Omega \setminus V_0))} \|f\|_{L^1(0,T;L^2(\Omega))}. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_{Q_T} \frac{|v|}{|x|^2} |q \cdot \nabla v| dx dt &= \int_0^T \int_{\Omega \setminus V_0} \frac{|v|}{|x|^2} |q \cdot \nabla v| dx dt \leq C \int_0^T \int_{\Omega \setminus V_0} |v| |\nabla v| dx dt \\ &\leq C \int_0^T \int_{\Omega \setminus V_0} v^2 dx dt + C \int_0^T \int_{\Omega \setminus V_0} |\nabla v|^2 dx dt \\ &\leq C \int_0^T \int_{\Omega \setminus V_0} |\nabla v|^2 dx dt, \end{aligned}$$

by Poincaré's inequality. It follows that

$$\int_{\Sigma_T} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt \leq C_{T,\lambda} \|(v, v_t)\|_{L^\infty(0,T;H_0^1(\Omega \setminus V_0) \times L^2(\Omega))}^2 + C \|f\|_{L^1(0,T;L^2(\Omega))}^2.$$

Applying [31, page 112], there exists a constant $C = C(V_0) > 0$ such that, for every $z \in H_0^1(\Omega)$,

$$\int_{\Omega} \left(|\nabla z|^2 - \lambda_* \frac{|z|^2}{|x|^2} \right) dx \geq C \|z\|_{H_0^1(\Omega \setminus V_0)}^2.$$

Hence, for $\lambda \leq \lambda_*$,

$$(2.7) \quad \|z\|_{H_0^1(\Omega \setminus V_0)} \leq C \|z\|_{H_{\lambda_*}} \leq C \|z\|_{H_\lambda}.$$

Finally,

$$\int_{\Sigma_T} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt \leq C_{T,\lambda} \|(v, v_t)\|_{L^\infty(0,T;H_\lambda \times L^2(\Omega))}^2 + C \|f\|_{L^1(0,T;L^2(\Omega))}^2.$$

Step 2. By Step 1, the linear mapping $(v_0, v_1, f) \mapsto \partial v / \partial \nu$, that is well defined on $D_\lambda \times H_\lambda \times \mathcal{C}^0([0, T]; L^2(\Omega)) \cap L^1(0, T; H_\lambda)$, can be extended into a linear continuous mapping from $H_\lambda \times L^2(\Omega) \times L^1(0, T; L^2(\Omega))$ to $L^2(\Sigma_T)$. This defines the trace of the normal derivative of the solution v of (2.2) associated to data $(v_0, v_1, f) \in H_\lambda \times L^2(\Omega) \times L^1(0, T; L^2(\Omega))$. And finally, (2.4) also holds for all $(v_0, v_1, f) \in H_\lambda \times L^2(\Omega) \times L^1(0, T; L^2(\Omega))$. \square

Proof of point (ii) of Proposition 2.2. Let us assume that $(v_0, v_1) = (0, 0)$ and $f = df_1/dt$, with $f_1 \in \mathcal{C}_c^1([0, T]; H_\lambda)$, and consider v the solution of (2.2). Let V_0 and q be defined as in the proof of point (i) of Proposition 2.2 and apply once again Lemma 2.3 with this choice of q . Then we obtain

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_T} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt &= \int_{\Omega} v_t(T) q \cdot \nabla v(T) dx + \frac{1}{2} \int_{Q_T} (v_t^2 - |\nabla v|^2) \operatorname{div} q dx dt \\ &\quad + \sum_{j,k} \int_{Q_T} \frac{\partial q_k}{\partial x_j} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_k} dx dt - \lambda \int_{Q_T} \frac{v}{|x|^2} q \cdot \nabla v dx dt \\ &\quad - \int_{Q_T} \frac{df_1}{dt} q \cdot \nabla v dx dt. \end{aligned}$$

Observe that $v = w_t$, where w solves

$$(2.8) \quad \begin{cases} w_{tt} - \Delta w - \frac{\lambda}{|x|^2} w = f_1, & (t, x) \in (0, T) \times \Omega, \\ w(t, x) = 0, & (t, x) \in (0, T) \times \Gamma, \\ w(0, x) = 0, \quad w_t(0, x) = 0, & x \in \Omega. \end{cases}$$

Then let us compute

$$I := - \int_{Q_T} \frac{df_1}{dt} q \cdot \nabla v dx dt = \int_{Q_T} f_1 q \cdot \nabla v_t dx dt = - \int_{Q_T} \operatorname{div}(f_1 q) v_t dx dt.$$

Since $v_t = w_{tt} = \Delta w + \lambda w/|x|^2 + f_1$, we get

$$\begin{aligned} I &= - \int_{Q_T} \operatorname{div}(f_1 q) \left(\Delta w + \lambda \frac{w}{|x|^2} \right) dxdt - \int_{Q_T} \operatorname{div}(q) f_1^2 dxdt - \int_{Q_T} q \cdot \nabla \left(\frac{f_1^2}{2} \right) dxdt \\ &= - \int_{Q_T} \operatorname{div}(f_1 q) \left(\Delta w + \lambda \frac{w}{|x|^2} \right) dxdt - \frac{1}{2} \int_{Q_T} \operatorname{div}(q) f_1^2 dxdt. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_T} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt &= \int_{\Omega} v_t(T) q \cdot \nabla v(T) dx \\ &\quad + \frac{1}{2} \int_{Q_T} \left\{ \left(\Delta w + \lambda \frac{w}{|x|^2} \right)^2 \right. \\ &\quad \left. + f_1^2 + 2f_1 \left(\Delta w + \lambda \frac{w}{|x|^2} \right) - |\nabla v|^2 \right\} \operatorname{div} q dxdt \\ &\quad + \sum_{j,k} \int_{Q_T} \frac{\partial q_k}{\partial x_j} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_k} dxdt - \lambda \int_{Q_T} \frac{v}{|x|^2} q \cdot \nabla v dxdt \\ &\quad - \int_{Q_T} \operatorname{div}(q) f_1 \left(\Delta w + \lambda \frac{w}{|x|^2} \right) dxdt \\ &\quad - \int_{Q_T} q \cdot \nabla f_1 \left(\Delta w + \lambda \frac{w}{|x|^2} \right) dxdt \\ &\quad - \frac{1}{2} \int_{Q_T} \operatorname{div}(q) f_1^2 dxdt. \end{aligned}$$

Simplifying the above relation and using the fact that $q \equiv 0$ in V_0 , we deduce that

$$\begin{aligned} (2.9) \quad \frac{1}{2} \int_{\Sigma_T} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt &= \int_{\Omega \setminus V_0} v_t(T) q \cdot \nabla v(T) dx \\ &\quad + \frac{1}{2} \int_0^T \int_{\Omega \setminus V_0} \left\{ \left(\Delta w + \lambda \frac{w}{|x|^2} \right)^2 - |\nabla v|^2 \right\} \operatorname{div} q dxdt \\ &\quad + \sum_{j,k} \int_0^T \int_{\Omega \setminus V_0} \frac{\partial q_k}{\partial x_j} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_k} dxdt - \lambda \int_0^T \int_{\Omega \setminus V_0} \frac{v}{|x|^2} q \cdot \nabla v dxdt \\ &\quad - \int_0^T \int_{\Omega \setminus V_0} q \cdot \nabla f_1 \left(\Delta w + \lambda \frac{w}{|x|^2} \right) dxdt. \end{aligned}$$

By point (ii) of Proposition 2.1, the solution w of (2.8) satisfies

$$\|(w, w_t)\|_{L^\infty(0,T;D_\lambda \times H_\lambda)} \leq C \|f_1\|_{L^1(0,T;H_\lambda)}.$$

Hence, using also (2.7), we get

$$(2.10) \quad \|v\|_{L^\infty(0,T;H_0^1(\Omega \setminus V_0))} \leq C \|v\|_{L^\infty(0,T;H_\lambda)} = C \|w_t\|_{L^\infty(0,T;H_\lambda)} \leq C \|f_1\|_{L^1(0,T;H_\lambda)}.$$

Moreover, we have

$$(2.11) \quad \left\| \Delta w + \lambda \frac{w}{|x|^2} \right\|_{L^\infty(0,T;L^2(\Omega))} = \|w\|_{L^\infty(0,T;D_\lambda)} \leq C \|f_1\|_{L^1(0,T;H_\lambda)}.$$

Besides, using $v_t = w_{tt} = \Delta w + \lambda w/|x|^2 + f_1$ and the fact that $f_1(T) = 0$, we get

$$(2.12) \quad \|v_t(T)\|_{L^2(\Omega)} = \left\| \Delta w(T) + \lambda \frac{w(T)}{|x|^2} \right\|_{L^2(\Omega)} \leq C \|f_1\|_{L^1(0,T;H_\lambda)}.$$

Finally, the estimates (2.10)–(2.12) associated to (2.9) produce (2.5). \square

2.3. Solutions defined by transposition. Finally, as in [23], we also need to introduce the weaker notion of “solution defined by transposition”: for $(u_0, u_1) \in L^2(\Omega) \times H'_\lambda$ and $g \in L^2(\Sigma_T)$, u is called a “very weak solution” of (2.1) if, for all $f \in \mathcal{D}(Q_T)$, it satisfies

$$(2.13) \quad \int_{Q_T} u f \, dx dt = -(u_0, v_t(0))_{L^2(\Omega)} + \langle u_1, v(0) \rangle_{H'_\lambda, H_\lambda} - \int_{\Sigma_T} g \frac{\partial v}{\partial \nu} \, d\sigma dt,$$

where v is the solution of

$$(2.14) \quad \begin{cases} v_{tt} - \Delta v - \frac{\lambda}{|x|^2} v = f(t, x), & (t, x) \in Q_T, \\ v(t, x) = 0, & (t, x) \in \Sigma_T, \\ v(T, x) = 0, \quad v_t(T, x) = 0, & x \in \Omega. \end{cases}$$

Then one can prove what follows.

PROPOSITION 2.3. *Let $T > 0$ be given and assume $\lambda \leq \lambda_*$. For every $(u_0, u_1) \in L^2(\Omega) \times H'_\lambda$ and $g \in L^2(\Sigma_T)$, there exists a unique solution u to (2.1) with*

$$u \in \mathcal{C}([0, T]; L^2(\Omega)) \cap C^1([0, T]; H'_\lambda).$$

Moreover, there exists a constant $C > 0$ such that, for every $(u_0, u_1) \in L^2(\Omega) \times H'_\lambda$ and $g \in L^2(\Sigma_T)$, the solution u of (2.1) satisfies

$$\|(u, u_t)\|_{L^\infty(0,T;L^2(\Omega) \times H'_\lambda)} \leq C \|(u_0, u_1)\|_{L^2(\Omega) \times H'_\lambda} + C \|g\|_{L^2(\Sigma_T)}.$$

The proof of Proposition 2.3 is a consequence of Propositions 2.1 and 2.2 by the method of transposition; see [23, Theorem 4.2, page 46–54].

3. Observability. Our main result guarantees the observability of system (1.6) under the condition $\lambda \leq \lambda_*$ and with an observation on the following part of the boundary:

$$(3.1) \quad \Gamma_0 := \{x \in \Gamma \mid x \cdot \nu \geq 0\}.$$

The above choice of Γ_0 allows us to use a multiplier centered at the singularity, which is a crucial point in the proof of the observability inequality. For simplicity, we proceed here with the more classical multiplier method; see, e.g., [23, 21]. However, some variants, such as the rotated multipliers introduced by Osses [26], allow us to relax assumption (3.1). Such further results are stated in section 7, followed by a discussion concerning more general geometries and the case of multipolar singularities.

Denoting $\Sigma_T^0 := (0, T) \times \Gamma_0$ and

$$(3.2) \quad R_\Omega := \max_{x \in \Omega} |x|,$$

we prove the following inverse or observability inequality.

THEOREM 3.1 (observability). *Assume that $\lambda \leq \lambda_*$ and consider $T > T_0 = 2R_\Omega$. Then there exists $C = C(T, \lambda, \Omega) > 0$ such that, for all $(v_0, v_1) \in H_\lambda \times L^2(\Omega)$, the solution of (1.6) satisfies*

$$(3.3) \quad E_v^\lambda(0) \leq C \int_{\Sigma_T^0} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt.$$

Proof of Theorem 3.1. Let us assume that $\lambda \leq \lambda_*$. In order to prove Theorem 3.1, we apply Lemma 2.3 with $z = v$, $F = \lambda v/|x|^2$, and q defined by $q(x) = x \ \forall x \in \bar{\Omega}$. It follows that

$$\begin{aligned} \left[(v_t, x \cdot \nabla v)_{L^2(\Omega)} \right]_0^T + \frac{N}{2} \int_{Q_T} (v_t^2 - |\nabla v|^2) dx dt + \int_{Q_T} |\nabla v|^2 dx dt \\ = \frac{1}{2} \int_{\Sigma_T} x \cdot \nu \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt + \lambda \int_{Q_T} \frac{v}{|x|^2} x \cdot \nabla v dx dt. \end{aligned}$$

We can compute

$$\int_{Q_T} \frac{v}{|x|^2} x \cdot \nabla v dx dt = -\frac{N-2}{2} \int_{Q_T} \frac{v^2}{|x|^2} dx dt.$$

Hence

$$\begin{aligned} \left[(v_t, x \cdot \nabla v)_{L^2(\Omega)} \right]_0^T + \frac{N}{2} \int_{Q_T} v_t^2 dx dt - \frac{N-2}{2} \int_{Q_T} \left(|\nabla v|^2 - \lambda \frac{v^2}{|x|^2} \right) dx dt \\ \leq \frac{R_\Omega}{2} \int_{\Sigma_T^0} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt, \end{aligned}$$

where $R_\Omega = \max_{x \in \bar{\Omega}} |x|$ and where we used the definition of Σ_T^0 . Using the definition of E_v^λ , this can be rewritten as

$$\begin{aligned} \left[(v_t, x \cdot \nabla v)_{L^2(\Omega)} \right]_0^T + \int_0^T E_v^\lambda(t) dt + \frac{N-1}{2} \int_{Q_T} v_t^2 dx dt \\ - \frac{N-1}{2} \int_{Q_T} \left(|\nabla v|^2 - \lambda \frac{v^2}{|x|^2} \right) dx dt \leq \frac{R_\Omega}{2} \int_{\Sigma_T^0} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt. \end{aligned}$$

Next, multiplying (2.2) by v , we obtain

$$(3.4) \quad \int_{Q_T} \left(v_t^2 - |\nabla v|^2 + \lambda \frac{v^2}{|x|^2} \right) dx dt = \left[\int_{\Omega} v_t v dx \right]_0^T.$$

This implies

$$(3.5) \quad \left[(v_t, x \cdot \nabla v)_{L^2(\Omega)} \right]_0^T + \int_0^T E_v^\lambda(t) dt + \frac{N-1}{2} \left[\int_{\Omega} v_t v dx \right]_0^T \leq \frac{R_\Omega}{2} \int_{\Sigma_T^0} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt.$$

Next, using the fact that $E_v^\lambda(t) = E_v^\lambda(0) \ \forall t \geq 0$, we deduce

$$(3.6) \quad \left[\left(v_t, x \cdot \nabla v + \frac{N-1}{2} v \right)_{L^2(\Omega)} \right]_0^T + T E_v^\lambda(0) \leq \frac{R_\Omega}{2} \int_{\Sigma_T^0} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt.$$

It remains to estimate the quantity

$$\left| \left[\left(v_t, x \cdot \nabla v + \frac{N-1}{2} v \right)_{L^2(\Omega)} \right]_0^T \right|.$$

We proceed as in [23]. The following estimates are valid both for $t = 0$ and $t = T$. First we write

$$\left| \left(v_t, x \cdot \nabla v + \frac{N-1}{2} v \right)_{L^2(\Omega)} \right| \leq \frac{R_\Omega}{2} \|v_t\|_{L^2(\Omega)}^2 + \frac{1}{2R_\Omega} \left\| x \cdot \nabla v + \frac{N-1}{2} v \right\|_{L^2(\Omega)}^2.$$

Next we compute

$$\left\| x \cdot \nabla v + \frac{N-1}{2} v \right\|_{L^2(\Omega)}^2 = \|x \cdot \nabla v\|_{L^2(\Omega)}^2 + \frac{(N-1)^2}{4} \|v\|_{L^2(\Omega)}^2 + (N-1) (x \cdot \nabla v, v)_{L^2(\Omega)}$$

and

$$(x \cdot \nabla v, v)_{L^2(\Omega)} = \frac{1}{2} \int_\Omega x \cdot \nabla (v^2) dx = -\frac{1}{2} \int_\Omega \operatorname{div}(x) v^2 dx = -\frac{N}{2} \int_\Omega v^2 dx.$$

It follows that

$$(3.7) \quad \left| \left(v_t, x \cdot \nabla v + \frac{N-1}{2} v \right)_{L^2(\Omega)} \right| \leq \frac{R_\Omega}{2} \|v_t\|_{L^2(\Omega)}^2 + \frac{1}{2R_\Omega} \|x \cdot \nabla v\|_{L^2(\Omega)}^2 - \frac{1}{2R_\Omega} \left(\frac{N^2-1}{4} \right) \|v\|_{L^2(\Omega)}^2.$$

Let us notice that, in the subcritical case $\lambda < \lambda_*$, one could proceed as in the case of the standard wave equation (when $\lambda = 0$). Indeed, when $\lambda < \lambda_*$, ∇v can be bounded in $L^2(\Omega)$ in terms of E_v^0 . Next one can use the facts that E_v^0 and E_v^λ are equivalent and that $t \mapsto E_v^\lambda$ is constant. However, this would produce the required estimate but with a constant T_0^λ depending on λ and such that $T_0^\lambda \rightarrow +\infty$ as $\lambda \rightarrow \lambda_*$. Hence the time of controllability T_0^λ would not be uniform with respect to the parameter λ . Moreover, by this method, no result could be expected in the critical case $\lambda = \lambda_*$.

In order to obtain a uniform time of controllability T_0 and to also treat the critical case $\lambda = \lambda_*$, we need to derive some suitable improved Hardy-type inequalities to produce a uniform bound of the term $\|x \cdot \nabla v\|_{L^2(\Omega)}^2$. Sharp versions of those Hardy-type inequalities are needed to retrieve the expected minimal time of controllability $T_0 = 2R_\Omega$, which coincides with the one that the multiplier method gives for the classical wave equation with $\lambda = 0$. More precisely, we proved the Hardy-type inequality stated in Theorem 1.1

Now we are ready to proceed with the end of the proof of (3.3). By Theorem 1.1, we have

$$\|x \cdot \nabla v\|_{L^2(\Omega)}^2 \leq R_\Omega^2 \|v\|_{H^{\lambda_*}}^2 + \frac{N^2-4}{4} \|v\|_{L^2(\Omega)}^2.$$

Hence (3.7) becomes

$$\begin{aligned} & \left| \left(v_t, x \cdot \nabla v + \frac{N-1}{2}v \right)_{L^2(\Omega)} \right| \\ & \leq \frac{R_\Omega}{2} \|v_t\|_{L^2(\Omega)}^2 + \frac{R_\Omega}{2} \|v\|_{H_{\lambda_\star}}^2 + \frac{1}{2R_\Omega} \left(\frac{N^2-4}{4} - \frac{N^2-1}{4} \right) \|v\|_{L^2(\Omega)}^2 \\ & \leq \frac{R_\Omega}{2} \|v_t\|_{L^2(\Omega)}^2 + \frac{R_\Omega}{2} \|v\|_{H_{\lambda_\star}}^2. \end{aligned}$$

Since $\|\cdot\|_{H_{\lambda_\star}} \leq \|\cdot\|_{H_\lambda} \forall \lambda \leq \lambda_\star$, we get

$$\left| \left(v_t, x \cdot \nabla v + \frac{N-1}{2}v \right)_{L^2(\Omega)} \right| \leq \frac{R_\Omega}{2} (\|v_t\|_{L^2(\Omega)}^2 + \|v\|_{H_\lambda}^2) = R_\Omega E_v^\lambda = R_\Omega E_v^\lambda(0).$$

It follows that

$$\left| \left[\left(v_t, x \cdot \nabla v + \frac{N-1}{2}v \right)_{L^2(\Omega)} \right]_0^T \right| \leq 2R_\Omega E_v^\lambda(0).$$

By (3.6), we finally get

$$(T - T_0)E_v^\lambda(0) \leq \frac{R_\Omega}{2} \int_{\Sigma_T^0} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt,$$

where $T_0 = 2R_\Omega$. This concludes the proof of (3.3). \square

4. Controllability. Our main result guarantees the exact controllability of system (1.1) under the condition $\lambda \leq \lambda_\star$ and when the control acts on the part Γ_0 of the boundary. More precisely, we prove the following.

THEOREM 4.1 (controllability). *Assume $\lambda \leq \lambda_\star$. For every $T > T_0 := 2R_\Omega$, $(u_0, u_1) \in L^2(\Omega) \times H_{\lambda'}'$, and $(\bar{u}_0, \bar{u}_1) \in L^2(\Omega) \times H_{\lambda'}'$, there exists $h \in L^2(\Sigma_T^0)$ such that the solution of (1.1) satisfies (1.2).*

Sketch of proof. As it is classical in controllability problems, the controllability result given in Theorem 4.1 relies on the so-called direct and inverse inequalities for the adjoint system (1.6). The direct inequality has been given in Proposition 2.2 (applied to v with $f = 0$). In this setting, it becomes the following: there exists $C = C(T, \lambda, \Omega) > 0$ such that, for all $(v_0, v_1) \in H_\lambda \times L^2(\Omega)$, the solution of (1.6) satisfies

$$(4.1) \quad \int_{\Sigma_T^0} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt \leq C E_v^\lambda(0).$$

Besides, the inverse inequality has been given in Theorem 3.1. Finally, we refer to [23] for the arguments (Hilbert uniqueness method), proving that (4.1)–(3.3) imply Theorem 4.1. \square

5. Lack of observability and controllability in the supercritical case.

Theorem 4.1 complements the null controllability results obtained in [30, 17] for the heat equation with a subcritical inverse-square potential (i.e., when $\lambda \leq \lambda_\star$). In the supercritical case $\lambda > \lambda_\star$, we follow the arguments by Ervedoza [17] who proved that

null controllability does not hold any more for the heat equation in that range. More precisely, considering a sequence of regularized potentials $-\lambda/(|x|^2 + \varepsilon^2)$ (with $\varepsilon > 0$), he proved that the system cannot be controlled uniformly with respect to $\varepsilon > 0$. The proof relies on the spectral analysis of the associated operator and in particular on the use of the first eigenfunction (the most explosive mode) whose energy is more and more localized around the singularity. It allows in particular to construct a sequence of solutions that contradicts the required observability inequality.

The same arguments hold in our context allowing us to show that problem (1.1) is no more controllable when $\lambda > \lambda_*$. Indeed, let us approximate problem (1.6) by the systems

$$(5.1) \quad \begin{cases} v_{tt} - \Delta v - \frac{\lambda}{|x|^2 + \varepsilon^2} v = 0, & (t, x) \in (0, T) \times \Omega, \\ v(t, x) = 0, & (t, x) \in (0, T) \times \Gamma, \\ v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x), & x \in \Omega, \end{cases}$$

where ε is a positive parameter. To simplify, we consider here the case $N \geq 3$ and $\Omega = B_2$, where B_2 denotes the ball $B(0, 2)$. Therefore, Γ_0 , defined in (3.1), is simply ∂B_2 , and $T_0 = 2R_\Omega = 4$.

For all $\lambda \in \mathbb{R}$, these regularized problems are well-posed in $H_0^1(\Omega) \times L^2(\Omega)$: for all $\lambda \in \mathbb{R}$, $\varepsilon > 0$, and $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a unique solution v to (5.1) with $v \in \mathcal{C}([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))$. Moreover, the following observability inequality holds: for all $\lambda \in \mathbb{R}$, $\varepsilon > 0$, and all $T > 4$, there exists some constant $C_\lambda(\varepsilon) > 0$ such that, for all $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the solution v of (5.1) satisfies

$$(5.2) \quad \int_{B_2} (|\nabla v(0, x)|^2 + |v_t(0, x)|^2) dx \leq C_\lambda(\varepsilon) \int_0^T \int_{\partial B_2} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt.$$

In the supercritical case $\lambda > \lambda_*$, we prove that uniform observability with respect to ε does not hold.

PROPOSITION 5.1. *Assume that $\lambda > \lambda_*$. Then for any time $T > 0$, the constant $C_\lambda(\varepsilon)$ in (5.2) necessarily blows up as $\varepsilon \rightarrow 0^+$.*

Proof of Proposition 5.1. We argue by contradiction: we assume that, for some $T > 0$, there exists $C_\lambda > 0$ (independent of ε) such that the solutions v of (5.1) satisfy

$$(5.3) \quad \int_{B_2} (|\nabla v(0, x)|^2 + |v_t(0, x)|^2) dx \leq C_\lambda \int_0^T \int_{\partial B_2} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt.$$

Step 1. First we prove that the solutions of (5.1) satisfy

$$(5.4) \quad \int_0^T \int_{\partial B_2} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt \leq C_\lambda \int_0^T \int_{B_2 \setminus B_1} (|\nabla v|^2 + v_t^2) dx dt,$$

where $C_\lambda > 0$ is a constant independent of $\varepsilon > 0$. For this, we follow [23, Chapter VII, section 2.3]. We first rewrite Lemma 2.3 in the case of a time-dependent multiplier: assume that $q = (q_k)_k \in \mathcal{C}^1([0, T] \times \overline{\Omega})^N$; then the solutions z of (2.6) satisfy the

identity

$$(5.5) \quad \frac{1}{2} \int_{\Sigma_T} q \cdot \nu \left(\frac{\partial z}{\partial \nu} \right)^2 d\sigma dt = \left[\int_{\Omega} z_t q \cdot \nabla z dx \right]_0^T - \int_{Q_T} v_t q_t \cdot \nabla v dx dt \\ + \frac{1}{2} \int_{Q_T} (z_t^2 - |\nabla z|^2) \operatorname{div} q dx dt \\ + \sum_{j,k} \int_{Q_T} \frac{\partial q_k}{\partial x_j} \frac{\partial z}{\partial x_j} \frac{\partial z}{\partial x_k} dx dt - \int_{Q_T} F q \cdot \nabla z dx dt.$$

Next, we consider $q_0 \in C^1(\overline{\Omega})^N$ such that, for all $x \in \Gamma$, $q_0(x) = \nu(x)$. We also introduce $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ as a cut-off function of class C^∞ such that

$$0 \leq \phi \leq 1, \quad \phi \equiv 0 \text{ in } B_1, \quad \phi \equiv 1 \text{ in } B_2 \setminus B_{3/2}.$$

Finally we define $q \in C^1([0, T] \times \overline{\Omega})^N$ by $q(t, x) := t(T-t)q_0(x)\phi(x)$.

Applying (5.5) with the above choice of q and with $z = v$ and $F = \lambda v/(|x|^2 + \varepsilon^2)$ and using the fact that $q \equiv 0$ in B_1 , we obtain

$$\int_0^T \int_{\partial B_2} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt \leq C \int_0^T \int_{B_2 \setminus B_1} (v_t^2 + |\nabla v|^2) dx dt \\ + \lambda \int_0^T \int_{B_2 \setminus B_1} \frac{1}{|x|^2 + \varepsilon^2} |v q \cdot \nabla v| dx dt.$$

Since the potentials $1/(|x|^2 + \varepsilon^2)$ are uniformly bounded in $B_2 \setminus B_1$, we finally obtain

$$\int_0^T \int_{\partial B_2} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt \leq C_\lambda \int_0^T \int_{B_2 \setminus B_1} (v_t^2 + |\nabla v|^2) dx dt$$

for some constant C_λ dependent of λ but independent of ε .

Step 2. By Step 1, it is now sufficient to contradict the following inequality:

$$(5.6) \quad \int_{B_2} (|\nabla v(0, x)|^2 + |v_t(0, x)|^2) dx \leq C_\lambda \int_0^T \int_{B_2 \setminus B_1} (|\nabla v|^2 + v_t^2) dx dt.$$

In this purpose, we consider the radial solutions of (5.1), that is, the solutions of

$$(5.7) \quad \begin{cases} v_{tt} - v_{rr} - \frac{N-1}{r} v_r - \frac{\lambda}{r^2 + \varepsilon^2} v = 0, & (t, r) \in (0, T) \times (0, 2), \\ v_r(t, 0) = 0 = v(t, 2), & t \in (0, T), \\ v(0, r) = v_0(r), \quad v_t(0, r) = v_1(r), & r \in (0, 2). \end{cases}$$

Hence it remains to contradict

$$(5.8) \quad \int_0^2 (|v_r(0, r)|^2 + |v_t(0, r)|^2) r^{N-1} dr \leq C_\lambda \int_0^T \int_1^2 (v_r^2 + v_t^2) r^{N-1} dr dt.$$

Next, by the change of variable

$$\bar{v}(t, r) = r^{(N-1)/2} v(t, r),$$

problems (5.7) become

$$(5.9) \quad \begin{cases} \bar{v}_{tt} - \bar{v}_{rr} - \frac{K}{r^2 + \varepsilon^2} \bar{v} = 0, & (t, r) \in (0, T) \times (0, 2), \\ \bar{v}(t, 0) = 0 = v(t, 2), & t \in (0, T), \\ \bar{v}(0, r) = \bar{v}_0(r), \quad \bar{v}_t(0, r) = \bar{v}_1(r), & r \in (0, 2), \end{cases}$$

where

$$K = \lambda - \frac{(N-1)(N-3)}{4} > \frac{1}{4},$$

since $\lambda > \lambda_*$. Moreover, (5.8) becomes

$$(5.10) \quad \begin{aligned} & \int_0^2 \left(|\bar{v}_r(0, r)|^2 + \frac{(N-1)(N-3)}{4} \frac{|\bar{v}(0, r)|^2}{r^2 + \varepsilon^2} + |\bar{v}_t(0, r)|^2 \right) dr \\ & \leq C_\lambda \int_0^T \int_1^2 \left(\bar{v}_r^2 + \frac{(N-1)(N-3)}{4} \frac{\bar{v}^2}{r^2 + \varepsilon^2} + \bar{v}_t^2 \right) dr dt. \end{aligned}$$

Step 3. Finally, we contradict (5.10). Under the assumption that $K > 1/4$, Ervedoza [17] proved that the operator

$$L^\varepsilon \Phi := -\Phi_{rr} - \frac{K}{r^2 + \varepsilon^2} \Phi$$

with Dirichlet conditions admits a first eigenfunction Φ_0^ε such that

$$\begin{cases} L^\varepsilon \Phi_0^\varepsilon = \lambda_0^\varepsilon \Phi_0^\varepsilon, & \lambda_0^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} -\infty, \\ \|\Phi_0^\varepsilon\|_{L^2(0,2)} = 1, & \|\Phi_0^\varepsilon\|_{H^1(1,2)} \xrightarrow{\varepsilon \rightarrow 0} 0. \end{cases}$$

For $\varepsilon < 0$ small enough, $\lambda_0^\varepsilon > 0$, and we denote $\omega_0^\varepsilon = \sqrt{-\lambda_0^\varepsilon}$. Next we introduce

$$\bar{v}(t, r) = e^{-\omega_0^\varepsilon t} \Phi_0^\varepsilon.$$

It is easy to see that \bar{v} solves (5.9). Moreover, since $N \geq 3$, we have

$$\begin{aligned} & \int_0^2 \left(|v_r(0, r)|^2 + \frac{(N-1)(N-3)}{4} \frac{|v(0, r)|^2}{r^2 + \varepsilon^2} + |v_t(0, r)|^2 \right) dr \\ & \geq \int_0^2 |v_t(0, r)|^2 dr = (\omega_0^\varepsilon)^2 \|\Phi_0^\varepsilon\|_{L^2(0,2)}^2 = (\omega_0^\varepsilon)^2. \end{aligned}$$

On the other hand, we compute

$$\begin{aligned} & \int_0^T \int_1^2 \left(v_r^2 + \frac{(N-1)(N-3)}{4} \frac{v^2}{r^2 + \varepsilon^2} + v_t^2 \right) dr dt \\ & \leq C \int_0^T \int_1^2 (v_r^2 + v^2 + v_t^2) dr dt \leq C \int_0^T \int_1^2 (v_r^2 + v_t) dr dt, \end{aligned}$$

by Poincaré inequality. Since $\|\Phi_0^\varepsilon\|_{L^2(0,2)} = 1$ and $\|\Phi_0^\varepsilon\|_{H^1(1,2)}$ is bounded, we deduce that

$$\begin{aligned} & \int_0^T \int_1^2 \left(v_r^2 + \frac{(N-1)(N-3)}{4} \frac{v^2}{r^2 + \varepsilon^2} + v_t^2 \right) dr dt \\ & \leq C \frac{1 - e^{-2\omega_0^\varepsilon T}}{2\omega_0^\varepsilon} \left(\|\Phi_0^\varepsilon\|_{H_0^1(1,2)}^2 + (\omega_0^\varepsilon)^2 \|\Phi_0^\varepsilon\|_{L^2(1,2)}^2 \right) \leq \frac{C}{2} \left(\frac{1}{\omega_0^\varepsilon} + \omega_0^\varepsilon \right). \end{aligned}$$

From (5.10), we finally get

$$(\omega_0^\varepsilon)^2 \leq C_\lambda \left(\frac{1}{\omega_0^\varepsilon} + \omega_0^\varepsilon \right),$$

which provides a contradiction since $\omega_0^\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. \square

6. The Schrödinger equation with an inverse-square potential. In this section, we briefly discuss the case of the Schrödinger equation with an inverse-square singular potential. Here $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded open set such that $0 \in \Omega$ and whose boundary Γ is of class C^3 . We still denote by Γ_0 the subset of Γ defined in (3.1).

In the Hilbert spaces $L^2(\Omega; \mathbb{C})$ and $H_0^1(\Omega; \mathbb{C})$, we consider the following inner products:

$$\langle u, v \rangle_{L^2(\Omega; \mathbb{C})} = \operatorname{Re} \int_{\Omega} u(x) \overline{v(x)} dx \quad \forall u, v \in L^2(\Omega; \mathbb{C})$$

and

$$\langle u, v \rangle_{H_0^1(\Omega; \mathbb{C})} = \operatorname{Re} \int_{\Omega} \nabla u(x) \cdot \nabla \overline{v(x)} dx \quad \forall u, v \in H_0^1(\Omega; \mathbb{C}).$$

For all $\lambda \leq \lambda_*$, we also define the Hilbert space $H_\lambda(\Omega; \mathbb{C})$ as the completion of $H_0^1(\Omega; \mathbb{C})$ with respect to the norm associated to the inner product:

$$\langle u, v \rangle_{H_\lambda(\Omega; \mathbb{C})} = \operatorname{Re} \int_{\Omega} \left(\nabla u(x) \cdot \nabla \overline{v(x)} - \lambda \frac{u(x) \overline{v(x)}}{|x|^2} \right) dx.$$

In order to simplify the notations, in the following, we denote by $L^2(\Omega)$, $H_0^1(\Omega)$, and H_λ the spaces $L^2(\Omega; \mathbb{C})$, $H_0^1(\Omega; \mathbb{C})$, and $H_\lambda(\Omega; \mathbb{C})$, respectively.

Then we consider the following problem:

$$(6.1) \quad \begin{cases} iu_t + \Delta u + \frac{\lambda}{|x|^2} u = 0, & (t, x) \in (0, T) \times \Omega, \\ u(t, x) = h(t, x), & (t, x) \in (0, T) \times \Gamma_0, \\ u(t, x) = 0, & (t, x) \in (0, T) \times \Gamma \setminus \Gamma_0, \\ u(0, x) = u_0(x), & x \in \Omega. \end{cases}$$

Following the proof of Machtyngier [24] (developed when $\lambda = 0$) and using the Hardy inequality stated in Theorem 1.1, we can also prove exact controllability results for the above problem.

As in [24], the proof relies on some direct and inverse inequalities for the adjoint system:

$$(6.2) \quad \begin{cases} iv_t + \Delta v + \frac{\lambda}{|x|^2}v = 0, & (t, x) \in (0, T) \times \Omega, \\ v(t, x) = 0, & (t, x) \in (0, T) \times \Gamma, \\ v(0, x) = v_0(x), & x \in \Omega. \end{cases}$$

More precisely, we prove what follows.

THEOREM 6.1. *Let $T > 0$ be given and assume $\lambda \leq \lambda_*$. Then there exist some constants $C_1, C_2 > 0$ such that, for every $v_0 \in H_\lambda$, the solution v of (6.2) satisfies*

$$(6.3) \quad \int_{\Sigma_T} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt \leq C_1 \|v_0\|_{H_\lambda}^2$$

and

$$(6.4) \quad \|v_0\|_{H_\lambda}^2 \leq C_2 \int_{\Sigma_T^0} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt.$$

Proof of Theorem 6.1. Before proving (6.3) and (6.4), let us first state a preliminary identity. Multiplying (6.2) by \overline{v}_t , one can prove

$$\int_{\Omega} \left(\nabla v \cdot \nabla \overline{v}_t - \lambda \frac{v \overline{v}_t}{|x|^2} \right) dx = i \int_{\Omega} |v_t|^2 dx \in i\mathbb{R}.$$

Therefore,

$$\frac{d}{dt} \|v(t)\|_{H_\lambda}^2 = 2\operatorname{Re} \left(\int_{\Omega} \nabla v \cdot \nabla \overline{v}_t - \lambda \frac{v \overline{v}_t}{|x|^2} \right) dx = 0,$$

and we deduce that

$$(6.5) \quad \forall t \geq 0, \quad \|v(t)\|_{H_\lambda} = \|v_0\|_{H_\lambda}.$$

Next, we recall the following identity given by the method of multipliers.

LEMMA 6.1 (see [24, Lemma 2.2, page 26]). *Let $q = (q_k)_k \in \mathcal{C}^2(\overline{\Omega})^N$ be given, and consider the problem*

$$(6.6) \quad \begin{cases} iz_t + \Delta z = F, & (t, x) \in Q_T, \\ z(t, x) = 0, & (t, x) \in \Sigma_T, \\ z(0, x) = z_0(x), & x \in \Omega. \end{cases}$$

Then for every $(z_0, z_1, F) \in H_0^1(\Omega) \times L^2(\Omega) \times L^1(0, T; L^2(\Omega))$, the solution z of (2.6) satisfies the identity

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_T} q \cdot \nu \left| \frac{\partial z}{\partial \nu} \right|^2 d\sigma dt &= \frac{1}{2} \operatorname{Im} \left[\int_{\Omega} z q \cdot \nabla \overline{z} dx \right]_0^T + \frac{1}{2} \operatorname{Re} \int_{Q_T} z \nabla(\operatorname{div} q) \cdot \nabla \overline{z} dx dt \\ &+ \operatorname{Re} \sum_{j,k} \int_{Q_T} \frac{\partial q_k}{\partial x_j} \frac{\partial \overline{z}}{\partial x_k} \frac{\partial z}{\partial x_j} dx dt + \operatorname{Re} \int_{Q_T} F q \cdot \nabla \overline{z} dx dt \\ &+ \frac{1}{2} \operatorname{Re} \int_{Q_T} F \overline{z} \operatorname{div} q dx dt. \end{aligned}$$

Proof of (6.3). We apply Lemma 6.1 with $z = v$, $F = -\lambda v/|x|^2$, and q defined as in the proof of Proposition 2.2. (Observe that, since Γ is here of class \mathcal{C}^3 , then q can be chosen such that $q \in \mathcal{C}^2(\overline{\Omega})^N$.) Using the fact that $q \equiv 0$ in V_0 , we get

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_T} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt &\leq C \|v(T)\|_{L^2(\Omega \setminus V_0)}^2 + C \|\nabla v(T)\|_{L^2(\Omega \setminus V_0)}^2 \\ &\quad + C \|v(0)\|_{L^2(\Omega \setminus V_0)}^2 + C \|\nabla v(0)\|_{L^2(\Omega \setminus V_0)}^2 \\ &\quad + \int_0^T \int_{\Omega \setminus V_0} |v|^2 dx dt + \int_0^T \int_{\Omega \setminus V_0} |\nabla v|^2 dx dt. \end{aligned}$$

Using Poincaré inequality and next (2.7), we deduce that

$$\begin{aligned} \int_{\Sigma_T} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt &\leq C \|\nabla v(T)\|_{L^2(\Omega \setminus V_0)}^2 + C \|\nabla v(0)\|_{L^2(\Omega \setminus V_0)}^2 + \int_0^T \int_{\Omega \setminus V_0} |\nabla v|^2 dx dt \\ &\leq C \|v(T)\|_{H_\lambda}^2 + C \|v(0)\|_{H_\lambda}^2 + \int_0^T \|v(t)\|_{H_\lambda}^2 dx dt. \end{aligned}$$

From (6.5), we finally get the result

$$\int_{\Sigma_T} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt \leq C(2+T) \|v_0\|_{H_\lambda}^2.$$

Proof of (6.4). We proceed in two steps. First, we prove that the following inequality holds for any given $\varepsilon > 0$ such that $T - \varepsilon > 0$:

$$(6.7) \quad (T - \varepsilon) \|v_0\|_{H_\lambda}^2 \leq \frac{1}{2} \int_{\Sigma_T^0} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt + c_\varepsilon \|v_0\|_{L^2(\Omega)}^2.$$

In a second step, to conclude the proof, it is enough to prove the following estimate:

$$(6.8) \quad \|v_0\|_{L^2(\Omega)}^2 \leq K \int_{\Sigma_T^0} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt.$$

Step 1. In order to prove (6.7), we apply Lemma 6.1 with $z = v$, $F = -\lambda v/|x|^2$, and q defined by $q(x) = x \forall x \in \overline{\Omega}$. It follows that

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_T} x \cdot \nu \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt &= \frac{1}{2} \text{Im} \left[\int_{\Omega} vx \cdot \nabla \bar{v} dx \right]_0^T + \int_{Q_T} |\nabla v|^2 dx dt \\ &\quad - \lambda \text{Re} \int_{Q_T} \frac{v}{|x|^2} x \cdot \nabla \bar{v} dx dt - \frac{\lambda}{2} N \int_{Q_T} \frac{|v|^2}{|x|^2} dx dt. \end{aligned}$$

Next, we compute

$$-\lambda \text{Re} \int_{Q_T} \frac{v}{|x|^2} x \cdot \nabla \bar{v} dx dt = \frac{\lambda}{2} (N-2) \int_{Q_T} \frac{|v|^2}{|x|^2} dx dt.$$

Thus,

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_T} x \cdot \nu \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt &= \frac{1}{2} \text{Im} \left[\int_{\Omega} vx \cdot \nabla \bar{v} dx \right]_0^T + \int_{Q_T} \left(|\nabla v|^2 - \lambda \frac{|v|^2}{|x|^2} \right) dx dt \\ &= \frac{1}{2} \text{Im} \left[\int_{\Omega} vx \cdot \nabla \bar{v} dx \right]_0^T + \int_0^T \|v(t)\|_{H_\lambda}^2 dt. \end{aligned}$$

Using (6.5), the following identity holds:

$$\frac{1}{2} \int_{\Sigma_T} x \cdot \nu \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt = \frac{1}{2} \operatorname{Im} \left[\int_{\Omega} vx \cdot \nabla \bar{v} dx \right]_0^T + T \|v_0\|_{H_\lambda}^2.$$

Furthermore, for all $\varepsilon > 0$ such that $T - \varepsilon > 0$, we can write

$$\left| \operatorname{Im} \int_{\Omega} vx \cdot \nabla \bar{v} dx \right| \leq c_\varepsilon \|v\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{C} \|x \cdot \nabla \bar{v}\|_{L^2(\Omega)}^2,$$

where C denotes here the constant in (1.10). Hence, using (1.10), we get

$$\left| \operatorname{Im} \int_{\Omega} vx \cdot \nabla \bar{v} dx \right| \leq c_\varepsilon \|v\|_{L^2(\Omega)}^2 + \varepsilon \|v\|_{H_\lambda}^2.$$

Finally, we obtain (6.7).

Step 2. As in [24], we argue by contradiction by the so-called compactness-uniqueness argument. If (6.8) is not satisfied, one can construct a solution v of (6.2) (obtained as limit of a sequence v_n) that both satisfies

$$(6.9) \quad \|v(0)\|_{L^2(\Omega)} = 1$$

and

$$(6.10) \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \Sigma_T^0.$$

Now it remains to prove that $v \equiv 0$, which is in contradiction with (6.9). In this purpose, we use standard unique continuation properties showing that (6.2) combined with (6.10) implies $v \equiv 0$. (Due to the local nature of those properties, one can use standard unique continuation results that hold, for example, for bounded potentials since we need only to apply them away from the singularity. For such results, we refer, for example, to [29, section 5.2] and the references therein.) \square

Still following [24] and applying the Hilbert uniqueness method, we deduce the exact controllability of system (6.1) under the condition $\lambda \leq \lambda_*$ and when the control acts on the subset Γ_0 of the boundary. More precisely, we prove the following.

THEOREM 6.2 (controllability). *Assume $\lambda \leq \lambda_*$, and let $T > 0$ be given. Then for any $u_0 \in H_\lambda'$, there exists $h \in L^2(\Sigma_T^0)$ such that the solution of (6.1) satisfies $u(T) \equiv 0$.*

Remark 6.1. As done for the wave equation in section 5, using the same arguments, one can also prove that exact controllability is false when $\lambda > \lambda_*$.

7. Further results and open problems.

7.1. Rotated multipliers. The results stated in sections 2–4 rely on the use of the simplest radial multiplier $x \cdot \nabla v$. We may also use the rotated multiplier introduced by Osses in [26] in order to relax the geometric assumption (1.5) on Γ_0 . To do this, consider a skew-symmetric matrix $A \in \mathbb{R}^N \times \mathbb{R}^N$ ($A = -A^t$), some positive real number $d > 0$, and denote by I the identity matrix in $\mathbb{R}^N \times \mathbb{R}^N$. Without loss of generality, we may assume that

$$(7.1) \quad d^2 + \|A\|_2^2 = 1,$$

where $\|A\|_2 = \sup\{|Ax|, |x| = 1\}$ is the Euclidean norm in \mathbb{R}^N . We also define

$$r(d, A) = \max\{x \cdot (dI + A)v, x \in \Gamma_0\}.$$

Then, we replace assumption (1.5) by the following one:

$$(7.2) \quad \Gamma_0 = \{x \in \Gamma \mid x \cdot (dI + A)\nu \geq 0\}.$$

Using the rotated multiplier $(dI - A)x \cdot \nabla v$ instead of $x \cdot \nabla v$ and arguing as in the proof of Theorem 3.1, we can prove that, for all $\lambda \leq \lambda_*$, the solutions of (1.6) satisfy

$$(7.3) \quad (dT - T_0)E_v^\lambda(0) \leq \frac{R_\Omega}{2} \int_{\Sigma_T^0} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt,$$

where $T_0 = 2R_\Omega$. Hence Theorem 3.1 still holds for any $T > T_0/d$. Arguing as in the proof of Theorem 4.1, this yields the controllability of (1.1) in time $T > T_0/d$ with controls in a subset of the boundary of the form (7.2).

We refer to [26, page 781, Figures 2.1 and 2.2] for several examples of geometric configurations that enter in that framework.

Sketch of proof of (7.3). The proof of Theorem 3.1 is modified as follows: with $q(x) = (dI - A)x$ instead of $q(x) = x$ and with similar computations, (3.6) becomes (observe that $\operatorname{div}(Ax) = 0$; hence $\operatorname{div}(q(x)) = dN$)

$$(7.4) \quad \left[\left(v_t, q(x) \cdot \nabla v + \frac{d(N-1)}{2}v \right)_{L^2(\Omega)} \right]_0^T + dTE_v^\lambda(0) \\ - \int_{Q_T} A \nabla v \cdot \nabla v \, dx dt + \lambda \int_{Q_T} \frac{Ax}{|x|^2} \cdot \nabla \left(\frac{v^2}{2} \right) \, dx dt \leq \frac{r(d, A)}{2} \int_{\Sigma_T^0} \left(\frac{\partial v}{\partial \nu} \right)^2 d\sigma dt.$$

Next, using $A^t = -A$, we have

$$\int_{\Omega} A \nabla v \cdot \nabla v \, dx = 0 \quad \text{and} \quad \int_{\Omega} \frac{Ax}{|x|^2} \cdot \nabla \left(\frac{v^2}{2} \right) \, dx = 0.$$

Besides, (3.7) is replaced by

$$\left| \left(v_t, q(x) \cdot \nabla v + \frac{d(N-1)}{2}v \right)_{L^2(\Omega)} \right| \\ \leq \frac{R_\Omega}{2} \|v_t\|_{L^2(\Omega)}^2 + \frac{1}{2R_\Omega} \|q(x) \cdot \nabla v\|_{L^2(\Omega)}^2 - \frac{1}{2R_\Omega} \left(\frac{d^2(N^2-1)}{4} \right) \|v\|_{L^2(\Omega)}^2.$$

Next we estimate

$$\|q(x) \cdot \nabla v\|_{L^2(\Omega)}^2 \leq \int_{\Omega} (d^2 + \|A\|_2^2) |x|^2 |\nabla v|^2 = \int_{\Omega} |x|^2 |\nabla v|^2.$$

Therefore, by Theorem 1.1, we have

$$\|q(x) \cdot \nabla v\|_{L^2(\Omega)}^2 \leq R_\Omega^2 \|v\|_{H_{\lambda_*}}^2 + \frac{N^2-4}{4} \|v\|_{L^2(\Omega)}^2.$$

Hence, using the fact that $d \leq 1$ and following the end of the proof of Theorem 3.1, we finally get (7.3). \square

7.2. Geometric conditions. In the proof of the observability inequality, both for the wave and the Schrödinger equations, it has been necessary to choose a multiplier centered at the singularity. This choice of the multiplier limits our result to the case where the control acts on a subset $\Gamma_0 \subset \Gamma$ of the form given by (3.1) (or of the form (7.2) by using rotated multipliers).

Now it would be interesting to consider more general geometries for the subset Γ_0 of the boundary Γ where the control acts. For example, with respect to the literature concerning the standard wave and Schrödinger equations (see, for example, [23]), it would be natural to assume that

$$(7.5) \quad \Gamma_0 = \{x \in \Gamma \mid (x - x_0) \cdot \nu \geq 0\}$$

for some $x_0 \in \mathbb{R}^N$.

But new difficulties arise when doing that. Let us comment on them in the case of the wave equation, for instance. Following the proof of Theorem 3.1 with $q(x) = x - x_0$ instead of $q(x) = x$, we see that, when using this new multiplier, one needs to estimate two extra terms:

$$\left[\int_{\Omega} v_t x_0 \cdot \nabla v \, dx \right]_0^T \quad \text{and} \quad \int_{Q_T} \frac{v}{|x|^2} x_0 \cdot \nabla v \, dx.$$

The first term could be estimated by $E_v^\lambda(0)$, at least in the subcritical case $\lambda < \lambda_*$, using the fact that $v \in \mathcal{C}([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ and the equivalence (2.3) between the classical and the generalized energy that holds in that case. However, this estimate would not be uniform with respect to λ and would not hold for $\lambda = \lambda_*$.

The situation concerning the second term is even worse, since

$$\int_{Q_T} \frac{v}{|x|^2} x_0 \cdot \nabla v \, dx dt = \int_{Q_T} \frac{x \cdot x_0}{|x|^4} v^2 \, dx dt,$$

which cannot be estimated by $E_v^\lambda(0)$. (Moreover, in the situation that we are interested with, that is to say when $0 \in \Omega$, this term may not have a definite sign.)

Hence the general case of a subset Γ_0 of the form (3.1) cannot be treated by the method of multipliers, at least without additional developments. However, there is no obvious reason (for example, in terms of propagation of optic rays) to exclude similar results when $x_0 \neq 0$. This case could possibly be addressed by means of suitable hyperbolic Carleman estimates (see section 7.5 below).

7.3. Multipolar singularities. The fact that, with the method of multipliers, one needs to choose a multiplier centered at the singularity also limits our proof to the case of a single singularity. However, it would also be interesting to study the case of multipolar inverse-square singular potentials. This situation has been considered in [17] for the heat equation, and the problem has been solved by proving parabolic Carleman estimates with singular potentials. Here again, the solution should rely on the derivation of hyperbolic Carleman estimates for wave and Schrödinger equations with singular potentials. In the spirit of [5], one should also extend the Hardy inequality proved in Theorem 1.1 to the case of multipolar singularities. On that subject, let us also mention the related work of Duyckaerts [15], who studied the dispersive properties (Strichartz estimates) for the Schrödinger equations with multipolar potentials.

7.4. Other singularities. In this paper we have considered the case where the singularity is in the interior of the domain. But the same results apply when the singularity is placed on a point of the boundary. However, in that case, one expects an improvement of the Hardy inequalities in the sense that λ_* might be larger. There are partial results in that direction, but the complete picture is still to be clarified (see [11]). It would be natural to first address in a systematic manner the problem of the optimal Hardy constant for boundary singularities and then the corresponding controllability problems. The same can be said about the case where the singularities are localized all over the boundary or in internal submanifolds.

7.5. Perspectives. The purpose of this work was to analyze the controllability properties of the wave and Schrödinger equations with an inverse-square potential by using the multiplier method. As it has been underlined in sections 7.2–7.3, it does not seem possible to go further with this method. A possible way to improve our results would be to derive hyperbolic Carleman estimates (see [32, 14] among others) for equations with singular potentials. The difficulty here relies in the choice of suitable weight functions allowing us to compensate the singularity of the potential term. This strategy has been used successfully for the heat equation with singularities arising in a diffusion coefficient [9] or in a potential term [30, 17]. This will be the object of a forthcoming work.

8. Appendix: A sharp Hardy-type inequality. This section is devoted to the proof of Theorem 1.1. The proof that is given here is elementary, as the proof of the standard Hardy inequality recently given in [5]. Contrary to [31], there is no need to use cut-off arguments in order to work in a ball and to consider separately the radial and nonradial components of the functions. As a consequence, it allows us to sharply determine the constants that appear in the inequality (which is crucial in order to get the expected minimal time of controllability).

The main point in the proof is the following change of variables (inspired by [7, 31]):

$$Z(x) = |x|^{(N-2)/2} z(x), \quad \text{i.e.,} \quad z(x) = \frac{1}{|x|^{(N-2)/2}} Z(x).$$

Let us first observe that, for any $\alpha, \beta \geq 0$, we have

$$\operatorname{div} \left(\frac{x}{|x|^\alpha} \right) = \frac{N - \alpha}{|x|^\alpha} \quad \text{and} \quad \nabla \left(\frac{1}{|x|^\beta} \right) = -\beta \frac{x}{|x|^{\beta+2}}.$$

Next we compute

$$\begin{aligned} \int_{\Omega} |x|^2 |\nabla z|^2 &= \int_{\Omega} |x|^2 \left(\frac{\nabla Z}{|x|^{(N-2)/2}} - \frac{N-2}{2} \frac{xZ}{|x|^{(N+2)/2}} \right)^2 \\ &= \int_{\Omega} \frac{|\nabla Z|^2}{|x|^{N-4}} + \frac{(N-2)^2}{4} \int_{\Omega} \frac{Z^2}{|x|^{N-2}} - \frac{N-2}{2} \int_{\Omega} \frac{x \cdot \nabla (Z^2)}{|x|^{N-2}} \\ &= \int_{\Omega} \frac{|\nabla Z|^2}{|x|^{N-4}} + \frac{(N-2)^2}{4} \int_{\Omega} \frac{Z^2}{|x|^{N-2}} + \frac{N-2}{2} \int_{\Omega} \operatorname{div} \left(\frac{x}{|x|^{N-2}} \right) Z^2 \\ &= \int_{\Omega} \frac{|\nabla Z|^2}{|x|^{N-4}} + \left[\frac{(N-2)^2}{4} + (N-2) \right] \int_{\Omega} \frac{Z^2}{|x|^{N-2}} \\ &= \int_{\Omega} \frac{|\nabla Z|^2}{|x|^{N-4}} + \frac{N^2 - 4}{4} \int_{\Omega} \frac{Z^2}{|x|^{N-2}}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_{\Omega} |\nabla z|^2 &= \int_{\Omega} \left| \frac{\nabla Z}{|x|^{(N-2)/2}} - \frac{N-2}{2} \frac{x}{|x|^{(N+2)/2}} Z \right|^2 \\
&= \int_{\Omega} \frac{|\nabla Z|^2}{|x|^{N-2}} + \frac{(N-2)^2}{4} \int_{\Omega} \frac{Z^2}{|x|^N} - \frac{N-2}{2} \int_{\Omega} \frac{x}{|x|^N} \nabla (Z^2) \\
&= \int_{\Omega} \frac{|\nabla Z|^2}{|x|^{N-2}} + \frac{(N-2)^2}{4} \int_{\Omega} \frac{Z^2}{|x|^N} + \frac{N-2}{2} \int_{\Omega} \operatorname{div} \left(\frac{x}{|x|^N} \right) Z^2 \\
&= \int_{\Omega} \frac{|\nabla Z|^2}{|x|^{N-2}} + \frac{(N-2)^2}{4} \int_{\Omega} \frac{Z^2}{|x|^N}.
\end{aligned}$$

It follows that

$$\int_{\Omega} |\nabla z|^2 - \lambda_{\star} \frac{z^2}{|x|^2} = \int_{\Omega} \frac{|\nabla Z|^2}{|x|^{N-2}}.$$

Hence (1.8) may be rewritten exactly as follows:

$$\int_{\Omega} \frac{|\nabla Z|^2}{|x|^{N-4}} + \frac{N^2-4}{4} \int_{\Omega} \frac{Z^2}{|x|^{N-2}} \leq R_{\Omega}^2 \int_{\Omega} \frac{|\nabla Z|^2}{|x|^{N-2}} + \frac{N^2-4}{4} \int_{\Omega} \frac{Z^2}{|x|^{N-2}}.$$

And this inequality is trivially true by the definition of R_{Ω} . \square

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