CONTROLLABILITY OF THE KIRCHHOFF SYSTEM FOR BEAMS AS LIMIT OF THE MINDLIN-TIMOSHENKO ONE

F. D. ARARUNA * AND E. ZUAZUA †

Abstract. We consider the dynamical one-dimensional Mindlin-Timoshenko system for beams. We analyze how its controllability properties depend on the modulus of elasticity in shear k. In particular we prove that the exact boundary controllability property of the Kirchhoff system may be obtained as singular limit, as $k \to \infty$, of the partial controllability of projections over a sharp subspace of solutions generated by the eigenfunctions that converge, as $k \to \infty$, towards the spectrum of the Kirchhoff system.

Key words. Vibrating beams, controllability, observability, Mindlin-Timoshenko, Kirchhoff, Ingham inequality, Fourier decomposition, singular limit.

AMS subject classifications. 73K05, 93B05, 93B07

1. Introduction.

The Mindlin-Timoshenko system of equations is a widely used and physically fairly complete mathematical model for describing the transverse vibrations of beams. For a beam of length L this one-dimensional system reads as follows:

$$\left| \begin{array}{ll} \dfrac{\rho h^3}{12} u'' - u_{xx} + k\left(u + v_x\right) = 0 & \text{in} \quad Q, \\ \rho h v'' - k\left(u + v_x\right)_x = 0 & \text{in} \quad Q, \end{array} \right. \tag{1.1}$$

where $Q = (0, L) \times (0, T)$, $(0, L)$ being the segment occupied by the beam with $L > 0$ and T a given positive time. In this coupled system of two second order hyperbolic equations, the prime $'$ stands for the partial derivative in time t and the subscript x for the space derivative. The unknown $u = u(x, t)$ represents the angle of rotation and $v = v(x, t)$ the vertical displacement at time t of the cross section located x units from the end-point $x = 0$. The constant $h > 0$ represents the thickness of the beam that, for this model, is considered to be small and uniform, independent of x. The constant ρ is the mass density per unit volume of the beam and the parameter k is the so called modulus of elasticity in shear. It is given by the formula $k = \widehat{k} E h / 2(1 + \mu)$, where $\widehat{k}$ is a shear correction coefficient, E is the Young's modulus and μ is the Poisson's ratio, $0 < \mu < 1/2$.

We impose the following boundary conditions at the left end,

$$u(0, \cdot) = 0, \quad v_x(0, \cdot) = \Theta_k \quad \text{on} \quad (0, T), \tag{1.2}$$

and on the right one:

$$u(L, \cdot) = v_x(L, \cdot) = 0 \quad \text{on} \quad (0, T). \tag{1.3}$$

---

*Departamento de Matemática, Universidade Federal da Paraíba, 58051-900 João Pessoa - PB, Brasil (fagner@mat.ufpb.br). The research of this author was partially supported by PDEE-CAPES (MCT-Brasil) grant BEX0412/02-3.

†IMDEA-Matemáticas & Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, España (enrique.zuazua@uam.es). The research of this author was supported by grant MTM2005-00714, the DOMINO Project CIT-370200-2005-10 in the PROFIT Program, the i-MATH project of the CONSOLIDER program of the Spanish MEC and the SIMUMAT Project of the CAM (Spain).

According to these conditions the angle of rotation is kept fixed both at $x=0$ and $x=L$ and the boundary control Θ_k is a lateral force applied on the vertical displacement at the extreme $x=0$. In particular, no control is applied at $x=L$. To make the system complete, let us include the initial conditions

$$u(\cdot,0)=u_0,\ u'(\cdot,0)=u_1,\ v(\cdot,0)=v_0,\ v'(\cdot,0)=v_1 \quad \text{in} \quad (0,L). \tag{1.4}$$

When assuming that the linear filament of the beam remains perpendicular to the deformed middle surface, the transverse shear effects are neglected and one obtains the so called Kirchhoff system (see Lagnese-Lions [7]):

$$\left| \begin{array}{lll} \rho h v'' - \dfrac{\rho h^3}{12} v''_{xx} + v_{xxxx} = 0 & \text{in} & Q, \\ v_x(0,\cdot) = v_x(L,\cdot) = 0 & \text{on} & (0,T), \\ v_{xxx}(0,\cdot) = \Xi, \quad v_{xxx}(L,\cdot) = 0 & \text{on} & (0,T), \\ v(\cdot,0) = v_0, \quad v'(\cdot,0) = v_1 & \text{in} & (0,L). \end{array} \right. \tag{1.5}$$

The control Ξ enters in this system through the third derivative of the state at $x=0$.

Note that neglecting the shear effects of the beam is formally equivalent to making the modulus k to tend to infinity in (1.1), since k is inversely proportional to the shear angle.

The connections of these two systems and the singular perturbation problem of passing to the limit as k tends to infinity have been intensively investigated recently. We refer for instance to [11], [12] and [13], where these issues are addressed for a number of nonlinear models and under various boundary conditions.

This paper is devoted to the analysis of the controllability properties of these systems and the corresponding singular perturbation problem (as $k\to\infty$). The main goal is analyzing whether the exact controllability property of the Kirchhoff system (1.5) for beams may be obtained as limit of those of the Mindlin-Timoshenko one $(1.1)-(1.4)$ when the singular parameter k tends to infinity.

The problem of exact controllability for the Mindlin-Timoshenko system can be formulated as follows: given $T>0$, large enough, and initial data $\{u_0,u_1,v_0,v_1\}$, to find a control Θ_k such that the solution of system $(1.1)-(1.4)$ satisfies the conditions

$$u(\cdot,T)=u'(\cdot,T)=v(\cdot,T)=v'(\cdot,T)=0 \quad \text{in} \quad (0,L).$$

According to the Hilbert Uniqueness Method (HUM) introduced by J. L. Lions (see [8]), this property is equivalent to a suitable observability inequality for the adjoint system that, after reversing the sense of time, can be written as follows:

$$\left| \begin{array}{lll} \dfrac{\rho h^3}{12}\phi'' - \phi_{xx} + k(\phi+\psi_x) = f & \text{in} & Q, \\ \rho h \psi'' - k(\phi+\psi_x)_x = g & \text{in} & Q, \\ \phi(0,t) = \phi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0 & \text{on} & (0,T), \\ \phi(x,0) = \phi_0(x), \quad \phi'(x,0) = \phi_1(x) & \text{in} & (0,L), \\ \psi(x,0) = \psi_0(x), \quad \psi'(x,0) = \psi_1(x) & \text{in} & (0,L). \end{array} \right. \tag{1.6}$$

To be more precise, in the observability problem, f and g vanish ($f\equiv g\equiv 0$) so that the problem consists in estimating the energy of the initial data in terms of boundary measurements. The general system (1.6) with non-vanishing right hand side terms is useful when analyzing the well-posedness of the non-homogeneous boundary value problem $(1.1)-(1.4)$ by transposition.

We are mainly interested on the behavior of the controls Θ_k, as $k \to \infty$, and whether in the limit as $k \to \infty$ one obtains a control Ξ such that the solution of system (1.5) verifies

$$v(\cdot, T) = v'(\cdot, T) = 0 \quad \text{in} \quad (0, L).$$

This problem was treated initially in Lagnese-Lions [7] with different boundary conditions. The goal in [7] was:

(i) To show that the control time T is independent of k, for any given initial state, and to find, for each k, a control Θ_k driving the system $(1.1)-(1.4)$ to rest at time T.

(ii) To study the behavior of Θ_k as $k \to \infty$.

Combining HUM and multiplier inequalities the authors in [7] obtained a control time independent of k. But, to prove (ii), they imposed the physically unrealistic extra assumption that $L < h$. Moreover, in [7] (Remarks 3.4 and 3.5, pg. 109), it was conjectured that: *as* $k \to \infty$, Θ_k *converges, in some appropriate sense, towards a control driving the system* (1.5) *to equilibrium in time* T.

In this paper we address these questions and obtain the following main results:

- The controls Θ_k of the Mindlin-Timoshenko system may diverge exponentially as $k \to \infty$.
- Analyzing the underlying spectrum it is possible to decompose the adjoint system (1.6) into two subsystems. It is sufficient to obtain a uniform (with relation to k) observability inequality for one of these subsystems.
- Accordingly, the exact controllability requirement on system $(1.1) - (1.4)$ is relaxed to a partial controllability property over a suitable projection of solutions and the controls Θ_k remain bounded as $k \to \infty$.
- The partial controls Θ_k obtained this way converge to an exact control for the limit system (1.5).

With these results, we conclude that the exact controllability property of the Kirchhoff system may be obtained as limit of the partial controllability one of the Mindlin-Timoshenko one. This solves the problem proposed by Lagnese-Lions in [7] for the present boundary conditions. The uniform (with respect to the parameter k) partial controllability result is taken over the subspace of the solutions generated by the eigenfunctions that, in the limit, cover the whole spectrum of the limit Kirchhoff model.

The rest of the paper is divided as follows: in Section 2 we briefly mention some elementary properties (existence, uniqueness and regularity) of solutions for system (1.6) and we rigorously study its limit behavior as $k \to \infty$ towards the Kirchhoff system. In Section 3 we analyze the properties of the spectrum of system (1.6) finding two families of eigenvalues. As $k \to \infty$, one of these families of eigenvalues tends to those of the limit Kirchhoff system, while the other one diverges, disappearing in the limit in the sense that, since they diverge, do not lead to eigenvalues of the limit system. This fact occurs due to (and it is in agreement with) the asymptotic simplification that is produced when passing from a system of two equations and two dependent variables to a scalar equation with only one unknown variable. In Section 4 we discuss some elementary properties of system $(1.1)-(1.4)$ with nonhomogeneous boundary conditions, i. e. in the absence of controls. We also analyze the convergence, as $k \to \infty$, towards the solution of the nonhomogeneous Kirchhoff system. Section 5 is devoted to present and discuss the problem of observability for system (1.6). We show that the observability constant may blow up exponentially as $k \to \infty$. In Section

6, applying the Ingham inequality in the Fourier decomposition of solutions, we get an uniform observability result filtering the eigenfunction components corresponding to eigenvalues that diverge as $k \to \infty$. Filtering corresponds, in other words, to projecting solutions over the subspace of eigencomponents that are well behaved. In Section 7, combining the results of the previous section with HUM, we derive the uniform partial controllability result. More precisely, we prove that the projection over the subspace of solutions of $(1.1)-(1.4)$ generated by the eigenvalues convergent (as $k \to \infty$) and their corresponding eigenfunctions is uniformly controllable with respect to k. In the limit we obtain the exact boundary controllability property of the Kirchhoff system (1.5). Therefore, we see that it suffices to consider only the solutions in a suitable subspace to ensure that the conjecture in [7] is true.

The analysis in this paper depends on the boundary conditions we have chosen that make possible the explicit computation of the spectrum. Similar results are expected for other boundary conditions but a further analysis of this issue is needed.

The results in this paper are related with previous ones on the behavior of controls for systems of vibrations under singular perturbations. We refer to [2] and [3] for the problem of control and homogenization of the wave equation and to [5] and [17] for the behavior of controls under numerical approximations. We also refer to [16] for a discussion and comparison of these two topics. Similar methods have also been used in [10] to analyze the partial controllability of a model for spherical shells.

2. Asymptotic Limit of the Homogeneous System. For the sake of completeness, in this section we study the asymptotic limit of the solutions of the homogeneous system (1.6) as k tends to infinity. Before, we mention some elementary properties of these solutions.

System (1.6) is well-posed in the energy space $\mathcal{X} = H_0^1(0,L) \times L^2(0,L) \times H^1(0,L) \times L^2(0,L)$. More precisely, for any $\{\phi_0, \phi_1, \psi_0, \psi_1\} \in \mathcal{X}$ and $\{f,g\} \in L^1\left(0,T;\left[L^2(0,L)\right]^2\right)$ there exists a unique solution in the class

$$\{\phi,\psi\} \in C^0\left([0,T];H_0^1(0,L) \times H^1(0,L)\right) \cap C^1\left([0,T];\left[L^2(0,L)\right]^2\right) \tag{2.1}$$

satisfying the inequality

$$\begin{aligned} \left\|\{\phi(t),\phi'(t),\psi(t),\psi'(t)\}\right\|_k \leq \quad & C_1 e^{C_2 T}\left(\left\|\{\phi_0,\phi_1,\psi_0,\psi_1\}\right\|_k \right. \\ & \left. + \left\|\{f,g\}\right\|_{L^1\left(0,T;[L^2(0,L)]^2\right)}\right) \end{aligned} \tag{2.2}$$

for all $t \in [0,T]$, where the norm $\|\cdot\|_k$ is defined by

$$\begin{aligned} \left\|\{u_1,u_2,v_1,v_2\}\right\|_k^2 = \quad & \int_0^L |u_{1x}|^2\,dx + \frac{\rho h^3}{12}\int_0^L |u_2|^2\,dx + k\int_0^L |u_1 + v_{1x}|^2\,dx \\ & + \int_0^L |v_1|^2\,dx + \rho h\int_0^L |v_2|^2\,dx. \end{aligned}$$

On the other hand, the energy $E_k(t)$ of the system

$$\begin{aligned} E_k(t) = \quad & \frac{1}{2}\int_0^L \{\frac{\rho h^3}{12}|\phi'(x,t)|^2 + \rho h|\psi'(x,t)|^2 + |\phi_x(x,t)|^2 \\ & + k|\phi(x,t) + \psi_x(x,t)|^2\}dx \end{aligned} \tag{2.3}$$

satisfies

$$\frac{dE_k}{dt}(t) = \int_0^L [f(x,t)\,\phi'(x,t) + g(x,t)\,\psi'(x,t)]\,dx. \tag{2.4}$$

In particular, when $f \equiv g \equiv 0$, the energy E_k is conserved along time.

Estimate (2.2) holds as a consequence of this energy identity and Gronwall's inequality because of the obvious relation between the energy E_k and the norm $\|\cdot\|_k$. The norm $\|\cdot\|_k$ is equivalent to the square root of the sum $E_k + \|v_1\|^2_{L^2(0,L)}$. Note that the canonical norm in $\mathcal{X}$ can be bounded above uniformly in terms of the norm $\|\cdot\|_k$ for all $k \geq 1$, i. e., there exists $C > 0$ independent of k, such that

$$\|\cdot\|_{\mathcal{X}} \leq C\,\|\cdot\|_k, \quad \forall k \geq 1, \tag{2.5}$$

where $\|\cdot\|_{\mathcal{X}}$ stands for the canonical norm in $\mathcal{X}$.

Let us note that the energy E_k does not define a norm in $\mathcal{X}$. Accordingly, it is natural to introduce the norm $\|\cdot\|_k$ and the following decomposition of the energy space: $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1$, with

$$\mathcal{X}_0 = H_0^1(0,L) \times L^2(0,L) \times V \times H \quad \text{and} \quad \mathcal{X}_1 = \{\{0,0,c_1,c_2\} \in \mathcal{X};\ c_i \in \mathbb{R},\ i = 1,2\},$$

where $V = H^1(0,L) \cap H$ and $H = \left\{ v \in L^2(0,L);\ \int_0^L v(x)\,dx = 0 \right\}$. In $\mathcal{X}_0$ the energy defines a norm which is equivalent to $\|\cdot\|_k$. On the other hand, the spaces $\mathcal{X}_0$ and $\mathcal{X}_1$ are invariant under the flow generated by system (1.6) in the sense given in the following result:

Proposition 2.1. *Given data $\{\phi_0, \phi_1, \psi_0, \psi_1\}$ and $\{0, f, 0, g\}$ belonging to $\mathcal{X}_i$ and $L^1(0,T;\mathcal{X}_i)$, respectively, with $i = 0,1$, then the associated solution belongs to $\mathcal{X}_i$ for all $t \in [0,T]$.*

Proof. Firstly we show that if $\{\phi_0, \phi_1, \psi_0, \psi_1\} \in \mathcal{X}_0$ and $\{0, f, 0, g\} \in L^1(0,T;\mathcal{X}_0)$ then the corresponding solution belongs to $\mathcal{X}_0$, for all $t \in [0,T]$. In fact, integrating the equation $(1.6)_2$ ($(1.6)_2$ means the second equation in (1.6)) on $]0,L[$ and using the conditions $(1.6)_3$, we have

$$\frac{d^2}{dt^2}\int_0^L \psi(x,t)\,dx = 0,$$

that is,

$$\int_0^L \psi(x,t)\,dx = \int_0^L \psi_0(x)\,dx + t\int_0^L \psi_1(x)\,dx = 0.$$

Now, if $\{\phi_0, \phi_1, \psi_0, \psi_1\} \in \mathcal{X}_1$ and $\{0, f, 0, g\} \in L^1(0,T;\mathcal{X}_1)$, we have $\phi_0 = \phi_1 = f = 0$, $\psi_0 = c_1$, $\psi_1 = c_2$ and $g = g(t) \in L^1(0,T)$. In this way, we get, directly of the system (1.6), that $\phi \equiv \psi_x \equiv 0$. So $\psi = \psi(t)$ and, by $(1.6)_2$, we obtain

$$\rho h \psi''(t) = g(t),\ \forall t \in [0,T]$$

and, therefore,

$$\psi(t) = \int_0^t w(s)\,ds + c_2 t + c_1,\ \forall t \in [0,T],$$

where $w(\sigma) = (1/\rho h)\int_0^\sigma g(\xi)\,d\xi$. ■

We also have the following "hidden regularity" result:

Proposition 2.2. *For any $T > 0$, there exists a constant $C = C(T) > 0$, independent of k, such that the solution $\{\phi, \psi\}$ of* (1.6) *satisfies the inequality*

$$\begin{aligned} \|\{\phi_x(0,\cdot), \psi(0,\cdot)\}\|^2_{L^2(0,T)\times H^1(0,T)} \leq \quad & C\Big\{\|\{\phi_0, \phi_1, \psi_0, \psi_1\}\|^2_k \\ & + \|\{f, g\}\|^2_{L^1\left(0,T;[L^2(0,L)]^2\right)}\Big\} \end{aligned} \tag{2.6}$$

for any $\{\phi_0, \phi_1, \psi_0, \psi_1\} \in \mathcal{X}$ and $\{f, g\} \in L^1\left(0, T; \left[L^2(0, L)\right]^2\right)$.

Proof. It is enough to consider smooth solutions since a classical density argument allows us extending the inequality (2.6) to finite-energy solutions. We use a multiplier method (see [8]). We multiply $(1.6)_1$ by $(L - x)\phi_x$, $(1.6)_2$ by $(L - x)\psi_x$ and after integrating by parts over Q we get

$$\begin{aligned} &\frac{L}{2}\int_0^T \left\{\rho h\,|\psi'(0,t)|^2 + |\phi_x(0,t)|^2\right\} dt = \frac{1}{2}\int_Q \left\{\frac{\rho h^3}{12}\,|\phi'(x,t)|^2 + \rho h\,|\psi'(x,t)|^2\right. \\ &\left. + |\phi_x(x,t)|^2 + k\,|\phi(x,t) + \psi_x(x,t)|^2\right\} dxdt - \int_Q \left\{\frac{\rho h^3}{12}\,|\phi'(x,t)|^2 - |\phi_x(x,t)|^2\right\} dxdt \\ &- \left[\frac{\rho h^3}{12}\int_0^L \phi'(x,t)(L - x)\,\phi_x(x,t)\,dx + \rho h\int_0^L \psi'(x,t)(L - x)\,\psi_x(x,t)\,dx\right]\Bigg|_0^T \\ &+ \int_Q f(x,t)(L - x)\,\phi_x(x,t)\,dxdt + \int_Q g(x,t)(L - x)\,\psi_x(x,t)\,dxdt. \end{aligned} \tag{2.7}$$

Using (2.2) from (2.7) we obtain the estimate

$$\|\{\phi_x(0,\cdot), \psi'(0,\cdot)\}\|^2_{[L^2(0,T)]^2} \leq C\left\{\|\{\phi_0, \phi_1, \psi_0, \psi_1\}\|^2_k + \|\{f, g\}\|^2_{L^1\left(0,T;[L^2(0,L)]^2\right)}\right\}, \tag{2.8}$$

with $C = C(T) > 0$ a constant independent of k.

On the other hand, by the trace theorem, there exists a constant $C_\gamma > 0$, independent of k, such that

$$\|\psi(0,\cdot)\|_{L^2(0,T)} \leq C_\gamma\,\|\psi(\cdot,t)\|_{H^1(Q)}.$$

Thus by (2.2) it follows

$$\|\psi(0,\cdot)\|_{L^2(0,T)} \leq C\left(\|\{\phi_0, \phi_1, \psi_0, \psi_1\}\|_k + \|\{f, g\}\|_{L^1\left(0,T;[L^2(0,L)]^2\right)}\right), \tag{2.9}$$

where $C = C(T) > 0$ is a constant independent of k.

Combining (2.8) and (2.9) we deduce the inequality (2.6), uniformly on $k \geq 1$. ■

Concerning the asymptotic behavior of the solutions of the homogeneous Mindlin-Timoshenko system (1.6), as k tends to infinity, the following result holds:

Theorem 2.1. *Let $\{\phi_k, \psi_k\}$ be the unique solution of* (1.6) *with data $\{\phi_{0k}, \phi_{1k}, \psi_{0k}, \psi_{1k}\} \in \mathcal{X}$ and $\{f, g\} \in L^1\left(0, T; H^1_0(0, L) \times L^2(0, L)\right)$.*

(a) (Weak convergence) Assume that the initial data satisfy

$$\|\{\phi_{0k}, \phi_{1k}, \psi_{0k}, \psi_{1k}\}\|^2_k \leq C, \quad \forall k \geq 1, \tag{2.10}$$

C being a positive constant independent of k and

$$\{\phi_{0k}, \phi_{1k}, \psi_{0k}, \psi_{1k}\} \to \{\phi_0, \phi_1, \psi_0, \psi_1\} \text{ weakly in } \mathcal{X}. \tag{2.11}$$

Then, as $k \to \infty$, the following convergence property holds:

$$\{\phi_k, \phi_k', \psi_k, \psi_k'\} \to \{-\psi_x, -\psi_x', \psi, \psi'\} \text{ weakly} * \text{ in } L^\infty(0,T;\mathcal{X}), \tag{2.12}$$

where ψ solves the homogeneous Kirchhoff system

$$\left|\begin{array}{ll} \rho h\psi'' - \dfrac{\rho h^3}{12}\psi_{xx}'' + \psi_{xxxx} = f_x + g & in \quad Q, \\ \psi_x(0,\cdot) = \psi_x(L,\cdot) = \psi_{xxx}(0,\cdot) = \psi_{xxx}(L,\cdot) = 0 & on \quad (0,T), \\ \psi(\cdot,0) = \psi_0, \quad \left[\psi(\cdot,0) - \dfrac{h^2}{12}\psi_{xx}(\cdot,0)\right]' = \psi_1 + \dfrac{h^2}{12}\phi_{1x} & in \quad (0,L). \end{array}\right. \tag{2.13}$$

(b) *(Strong convergence) If the initial data satisfy the additional conditions*

$$\phi_0 + \psi_{0x} = 0, \quad \lim_{k\to\infty} E_k(0) = \mathcal{E}(0), \tag{2.14}$$

where $\mathcal{E}(t)$ is the energy of (2.13) given by

$$\mathcal{E}(t) = \frac{1}{2}\int_0^L \left\{\rho h|\psi'(x,t)|^2 + \frac{\rho h^3}{12}|\psi_x'(x,t)|^2 + |\psi_{xx}(x,t)|^2\right\} dx, \tag{2.15}$$

then, for $1 < p < \infty$, the following strong convergence holds as $k \to \infty$:

$$\{\phi_k, \phi_k', \psi_k, \psi_k'\} \to \{-\psi_x, -\psi_x', \psi, \psi'\} \text{ strongly in } L^p(0,T;\mathcal{X}). \tag{2.16}$$

Remark 2.1.

- *The existence and uniqueness of weak solutions of the limit system (2.13) can be obtained by classical methods. More precisely, when $\{\psi_0, \psi_1, f_x + g\} \in W \times H^1(0,L) \times L^1(0,T;L^2(0,L))$, where $W = \{v \in H^2(0,L);\ v_x(0) = v_x(L) = 0\}$, there exists a unique finite energy solution ψ in the class*

$$\psi \in C^0([0,T];W) \cap C^1([0,T];H^1(0,L))$$

satisfying the variational formulation of (2.13)

$$\rho h\frac{d}{dt}(\psi'(t), w) + \frac{\rho h^3}{12}\frac{d}{dt}(\psi_x'(t), w_x) + (\psi_{xx}(t), w_{xx}) = (f_x(t) + g(t), w)$$

for all $w \in W$, the boundary conditions $(2.13)_2$ and the initial conditions $(2.13)_3$. Here $(\cdot,\cdot)$ represents the inner product in $L^2(0,L)$. Furthermore the energy $\mathcal{E}(t)$ in (2.15) satisfies

$$\mathcal{E}'(t) = \int_0^L [f_x(x,t) + g(x,t)]\psi'(x,t)\,dx.$$

If $f_x + g \equiv 0$, the energy is conserved.

- *Note however that, in order to identify fully the initial data of the solutions of the limit system* (2.13) *and, more precisely, to determine the initial data of* ψ', *an elliptic equation has to be solved. Namely, the initial datum for the velocity* ψ' *in* $(2.13)_3$ *is determined by solving the elliptic equation*

$$\psi'(\cdot,0) \in H^1(0,L): \quad \rho h\psi'(0) - \frac{\rho h^3}{12}\psi'_{xx}(0) = \rho h\psi_1 + \frac{\rho h^3}{12}\phi_{1x}, \tag{2.17}$$

as the proof of the theorem will show.
To be more precise, this elliptic equation can be written in the variational form

$$-\frac{\rho h^3}{12}(\psi'_x(0), w_x) - \rho h(\psi'(0), w) = \frac{\rho h^3}{12}(\phi_1, w_x) - \rho h(\psi_1, w), \quad \forall w \in H^1(0,L) \tag{2.18}$$

in which the term ϕ_{1x}, *which is an element of* $(H^1(0,L))'$, *is not the derivative of* ϕ_1 *in the sense of transposition but rather the linear mapping so that, when acting on any element* w *of* $H^1(0,L)$, *yields the value* $-(\phi_1, w_x)$. *The same can be said about* $\psi'_{xx}(0)$ *which represents the element of* $(H^1(0,L))'$ *yielding* $-(\psi'_x(0), w_x)$.
- *Similar results hold when the right hand side terms* $\{f_k, g_k\}$ *depend on* k *and converge in a suitable sense. But we shall not discuss this issue since it is not needed for the purpose of this paper.*

Proof of Theorem 2.1. We will prove the theorem in two steps.
Step 1.(*Weak convergence*) Considering $\{f,g\} \in L^2\left(0,T; H_0^1(0,L) \times L^2(0,L)\right)$ and the sequence of initial data $\{\phi_{0k}, \phi_{1k}, \psi_{0k}, \psi_{1k}\} \in \mathcal{X}$ satisfying (2.10), the right side of (2.2) and the energies E_k are uniformly bounded on k. Consequently

$$\left| \begin{array}{l} (\{\phi_k, \phi'_k, \psi_k, \psi'_k\}) \text{ is bounded in } L^\infty(0,T;\mathcal{X}), \\ \left(\sqrt{k}\,[\phi_k + \psi_{kx}]\right) \text{ is bounded in } L^\infty\left(0,T;L^2(0,L)\right). \end{array} \right.$$

This immediately yields an uniform bound for ϕ, ϕ' and ψ' in the corresponding spaces. We also get an uniform bound on ψ in $L^2(0,L)$. The uniform bound on ψ in $H^1(0,L)$ can be easily obtained from the bound in $L^2(0,L)$, that in $\|\cdot\|_k$ and the fact that

$$\begin{aligned} \|\psi_x\|_{L^2(0,L)} &\leq \|\phi + \psi_x\|_{L^2(0,L)} + \|\phi\|_{L^2(0,L)} \leq k\,\|\phi + \psi_x\|_{L^2(0,L)} + \|\phi\|_{L^2(0,L)} \\ &\leq C\,\|\{\phi, \phi', \psi, \psi'\}\|_k . \end{aligned} \tag{2.19}$$

Extracting subsequences, that we still denote by $(\{\phi_k, \psi_k\})$, we get

$$\{\phi_k, \phi'_k, \psi_k, \psi'_k\} \rightarrow \{\phi, \phi', \psi, \psi'\} \text{ weakly} * \text{ in } L^\infty(0,T;\mathcal{X}) \tag{2.20}$$

with

$$\phi + \psi_x = 0. \tag{2.21}$$

For test functions $\{z, w\} \in H_0^1(0,L) \times H^1(0,L)$ satisfying

$$z + w_x = 0, \tag{2.22}$$

the variational formulation of (1.6) reduces to

$$\frac{\rho h^3}{12}\frac{d}{dt}(\phi'_k(t), z) + \rho h\frac{d}{dt}(\psi'_k(t), w) + (\phi_{kx}(t), z_x) = (f(t), z) + (g(t), w). \tag{2.23}$$

Using the convergences (2.20) in (2.23) and applying identities (2.21) and (2.22), the limit weak formulation can be written in terms of ψ as follows

$$\rho h \frac{d}{dt}\left(\psi'(t), w\right) + \frac{\rho h^3}{12}\frac{d}{dt}\left(\psi_x'(t), w_x\right) + \left(\psi_{xx}(t), w_{xx}\right) = \left(f_x(t) + g(t), w\right), \ \forall w \in W. \tag{2.24}$$

This identity is a weak form of the equation $(2.13)_1$. The two boundary conditions

$$\psi_x(0,t) = \psi_x(L,t) = 0 \quad \text{on} \quad (0,T)$$

are deduced form the fact that $\psi_x = -\phi$ and that ϕ satisfies the Dirichlet boundary conditions. The other two in (2.13), namely,

$$\psi_{xxx}(0,t) = \psi_{xxx}(L,t) = 0 \quad \text{on} \quad (0,T),$$

are implicit in the weak form of the equation since the test function w does not vanish of the boundary.

To conclude our result, it remains to identify the initial data of the limit system. In view of the convergences (2.20), and classical compactness arguments, $\psi_k \to \psi$ in $C^0\left([0,T]; L^2(0,L)\right)$. Then $\psi_k(\cdot,0) \to \psi(\cdot,0)$ in $L^2(0,L)$, which combined with (2.11), guarantees that $\psi(\cdot,0) = \psi_0$. In order to identify $\psi'(\cdot,0)$, we multiply both sides of the equation (2.23) by the function $\theta_\delta \in H^1(0,T)$ defined by

$$\theta_\delta(t) = \begin{cases} -\dfrac{t}{\delta} + 1, & \text{if} \quad 0 \le t \le \delta \\ 0, & \text{if} \quad \delta < t \le T \end{cases}$$

and we integrate by parts to obtain

$$\begin{aligned} &-\frac{\rho h^3}{12}\left(\phi_k'(0), z\right) + \frac{\rho h^3}{12\delta}\int_0^\delta \left(\phi_k'(t), z\right) dt - \rho h \left(\psi_k'(0), w\right) + \frac{\rho h}{\delta}\int_0^\delta \left(\psi_k'(t), w\right) dt \\ &+\int_0^\delta \left(\phi_{kx}(t), z_x\right)\theta_\delta(t)\, dt = -\frac{\rho h^3}{12}\left(\phi_{1k}, z\right) - \rho h\left(\psi_{1k}, w\right) + \frac{\rho h^3}{12\delta}\int_0^\delta \left(\phi_k'(t), z\right) dt \\ &+\frac{\rho h}{\delta}\int_0^\delta \left(\psi_k'(t), w\right) dt + \int_0^\delta \left(\phi_{kx}(t), z_x\right)\theta_\delta(t)\, dt = \int_0^\delta \left(f(t), z\right)\theta_\delta(t)\, dt \\ &+\int_0^\delta \left(g(t), w\right)\theta_\delta(t)\, dt. \end{aligned}$$

Passing to the limit in the last equality as $k \to \infty$ and using (2.21)-(2.22) we get

$$\begin{aligned} &\frac{\rho h^3}{12}\left(\phi_1, w_x\right) - \rho h\left(\psi_1, w\right) + \frac{\rho h^3}{12\delta}\int_0^\delta \left(\psi_x'(t), w_x\right) dt + \frac{\rho h}{\delta}\int_0^\delta \left(\psi'(t), w\right) dt \\ &+\int_0^\delta \left(\psi_{xx}(t), w_{xx}\right)\theta_\delta(t)\, dt = \int_0^\delta \left(f_x(t), w\right)\theta_\delta(t)\, dt + \int_0^\delta \left(g(t), w\right)\theta_\delta(t)\, dt. \end{aligned}$$

On the other hand, multiplying in (2.24) by θ_δ and integrating in time we obtain an expression that, compared with the previous one, yields the identity (2.18). This completes the proof of the part (a) of the Theorem.

Step 2. *(Strong convergence)* We know by (2.4) that the energy $E_k(t)$ associated to $\{\phi_k, \psi_k\}$ of (1.6) satisfies

$$E_k(t) = E_k(0) + \int_0^t \int_0^L \left[f(x,s)\,\phi_k'(x,s) + g(x,s)\,\psi_k'(x,s)\right] dx dt. \tag{2.25}$$

On the other hand, in view of the Remark 2.1 and (2.21) it follows that the energy of system (2.13) satisfies

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_0^L \left[-f(x,s)\psi'_x(x,s) + g(x,s)\psi'(x,s)\right] dxdt.$$

Therefore, combining (2.14), (2.20), (2.21) and (2.25), we get

$$\lim_{k\to\infty} E_k(t) = \mathcal{E}(t). \tag{2.26}$$

As a consequence of (2.26), we have the norm convergence which, together with the weak convergence result (2.12), yields the strong convergence one, (b), of the Theorem.

Let us develop this last argument in some more detail. In view of the weak convergence of solutions and the structure of the energy E_k it follows that

$$\liminf_{k\to\infty} \int_0^T E_k(t)dt \geq \frac{1}{2}\int_0^T \int_0^L \left[\frac{\rho h^3}{12}\left|\psi'_x\right|^2 + \rho h\left|\psi'\right|^2 + |\psi_{xx}|^2\right] dxdt = \int_0^T \mathcal{E}(t)\,dt.$$

This fact, together with (2.20), (2.21) and (2.26) implies

$$\lim_{k\to\infty} \int_0^T \int_0^L \left|\phi'_k\right|^2 dxdt = \int_0^T \int_0^L \left|\psi'_x\right|^2 dxdt,$$

$$\lim_{k\to\infty} \int_0^T \int_0^L \left|\psi'_k\right|^2 dxdt = \int_0^T \int_0^L \left|\psi'\right|^2 dxdt,$$

$$\lim_{k\to\infty} \int_0^T \int_0^L \left|\phi_{kx}\right|^2 dxdt = \int_0^T \int_0^L \left|\psi_{xx}\right|^2 dxdt$$

and

$$\lim_{k\to\infty} k\int_0^T \int_0^L \left|\phi_k + \psi_{kx}\right|^2 dxdt \to 0. \tag{2.27}$$

This, combined with the weak convergence, implies the strong convergence of $\{\phi'_k, \psi'_k, \phi_{kx}\}$ to $\{-\psi'_x, \psi', -\psi_{xx}\}$ in $[L^2(Q)]^3$. The strong convergence of ψ_k in $L^2(0,T;H^1(0,L))$ is then a consequence of (2.27) and the fact that ϕ_k strongly converges to $-\psi_x$ in $L^2(Q)$.

Strong convergence in $L^2(0,T;\mathcal{X})$, together with the uniform boundedness in $L^\infty(0,T;\mathcal{X})$ implies strong convergence in $L^p(0,T;\mathcal{X})$ for all $1 < p < \infty$. ■

3. Spectral Analysis. This section is devoted to analyze the asymptotic behavior, as k tends to infinity, of the spectrum of the Mindlin-Timoshenko system. With this aim we write system (1.6) (with $f = g = 0$) in the following abstract form:

$$\Phi' = -i\mathcal{A}\Phi,$$

where $\Phi = [\phi, \phi', \psi, \psi']^T$ and the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ is given by

$$\mathcal{A} = i\begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{12}{\rho h^3}\left(\dfrac{\partial^2}{\partial x^2} - k\right) & 0 & -\dfrac{12k}{\rho h^3}\dfrac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k}{\rho h}\dfrac{\partial}{\partial x} & 0 & \dfrac{k}{\rho h}\dfrac{\partial^2}{\partial x^2} & 0 \end{bmatrix}$$

with domain

$$D(\mathcal{A}) = \left[H_0^1(0,L) \cap H^2(0,L)\right] \times H_0^1(0,L) \times W \times H^1(0,L).$$

The eigenvalue problem for the operator $\mathcal{A}$ reads:

$$\mathcal{A}\Phi = \lambda\Phi. \tag{3.1}$$

Let us compute the eigenvalues and the corresponding eigenfunctions. In view of the various equations involved in (3.1) and the boundary conditions satisfied by the components ϕ and ψ, the solutions $\Phi = [\phi, \phi', \psi, \psi']^T$ associated to the eigenfunctions are such that

$$\{\phi(x,t), \psi(x,t)\} = e^{-i\lambda t}\{\sin(m\pi x/L), c\cos(m\pi x/L)\},$$

where the constant c is to be determined in terms of m and λ. In particular, computing the components ϕ and ψ suffice to identify the 4-component vector.

From (3.1) we have

$$\left|\begin{array}{l} \lambda^2 \dfrac{\rho h^3}{12}\phi - \phi_{xx} + k(\phi + \psi_x) = 0, \\ \lambda^2 \rho h \psi - k(\phi + \psi_x)_x = 0. \end{array}\right. \tag{3.2}$$

Taking the derivative of $(3.2)_1$ with respect to x and substituting in $(3.2)_2$, we get

$$\psi = \frac{1}{\lambda^2 \rho h}\left(\phi_{xxx} - \frac{\lambda^2 \rho h^3}{12}\phi_x\right). \tag{3.3}$$

Now, doing the same in (3.3) and substituting in $(3.2)_1$, it follows

$$\phi_{xxxx} - \left(\frac{\rho h \lambda^2}{k} + \frac{\lambda^2 \rho h^3}{12}\right)\phi_{xx} + \left(\frac{\lambda^4 \rho^2 h^4}{12k} + \lambda^2 \rho h\right)\phi = 0.$$

Since $\phi(x,t) = e^{-i\lambda t}\sin(m\pi x/L)$, we obtain, for λ, the fourth degree equation

$$\lambda^4 - \left(\frac{12\pi^2 m^2}{\rho h^3 L^2} + \frac{\pi^2 k m^2}{\rho h L^2} + \frac{12k}{\rho h^3}\right)\lambda^2 + \frac{12\pi^4 k m^4}{\rho^2 h^4 L^4} = 0, \tag{3.4}$$

while c satisfies

$$c = \frac{\pi^3 m^3}{\lambda^2 \rho h L^3} - \frac{h^2 m \pi}{12L}. \tag{3.5}$$

Solving (3.4) we find the eigenvalues

$$\begin{aligned}\widetilde{\lambda}_{k,m}^{\pm} = \quad & \pm\left[\frac{6\pi^2 m^2}{\rho h^3 L^2} + \frac{\pi^2 k m^2}{2\rho h L^2} + \frac{6k}{\rho h^3}\right. \\ & \left. + \frac{1}{2}\sqrt{\frac{144k^2}{\rho^2 h^6} + \frac{288\pi^2 k m^2}{\rho^2 h^6 L^2} + \frac{24\pi^2 k^2 m^2}{\rho^2 h^4 L^2} + \left(\frac{12\pi^2 m^2}{\rho h^3 L^2} - \frac{\pi^2 k m^2}{\rho h L^2}\right)^2}\,\right]^{\frac{1}{2}}\end{aligned}$$

and

$$\lambda^{\pm}_{k,m} = \pm\left[\frac{6\pi^2 m^2}{\rho h^3 L^2} + \frac{\pi^2 k m^2}{2\rho h L^2} + \frac{6k}{\rho h^3} - \frac{1}{2}\sqrt{\frac{144k^2}{\rho^2 h^6} + \frac{288\pi^2 k m^2}{\rho^2 h^6 L^2} + \frac{24\pi^2 k^2 m^2}{\rho^2 h^4 L^2} + \left(\frac{12\pi^2 m^2}{\rho h^3 L^2} - \frac{\pi^2 k m^2}{\rho h L^2}\right)^2}\,\right]^{\frac{1}{2}}.$$

We denote by c_m and $\tilde{c}_m$ the corresponding values of c according to the definition (3.5).

For m fixed, we see easily that, as k tends to infinity,

$$\widetilde{\lambda}^{\pm}_{k,m} \to \pm\infty. \tag{3.6}$$

This corresponds to that half of the spectrum that disappears when letting k tend infinity, in sense that, since they diverge as $k \to \infty$, do not lead to any eigenvalue of the limit system.

The following result describes the asymptotic behavior of the other family of eigenvalues.

Proposition 3.1. *For fixed $m \in \mathbb{N}$, as $k \to \infty$,*

$$\lambda^{\pm}_{k,m} \to \lambda^{\pm}_{m} = \pm\sqrt{\frac{12\pi^4 m^4}{12\rho h L^4 + \pi^2 \rho h^3 L^2 m^2}}. \tag{3.7}$$

These are the eigenvalues of the limit Kirchhoff system (2.13) *(with $f_x + g = 0$), for which the corresponding eigenfunctions are* $\cos(m\pi x/L)$.

Proof. It is sufficient to prove convergence for the + sign. To simplify the notation we denote by $\lambda_{k,m}$ the eigenvalues $\lambda^{+}_{k,m}$. We have to observe that

$$\left|\lambda_{k,m} - \sqrt{\frac{12\pi^4 m^4}{12\rho h L^4 + \pi^2 \rho h^3 L^2 m^2}}\right| = \left|\frac{\lambda^2_{k,m} - \dfrac{12\pi^4 m^4}{12\rho h L^4 + \pi^2 \rho h^3 L^2 m^2}}{\lambda_{k,m} + \sqrt{\dfrac{12\pi^4 m^4}{12\rho h L^4 + \pi^2 \rho h^3 L^2 m^2}}}\right|$$
$$= \left|\frac{\left(12\rho h L^4 + \pi^2 \rho h^3 L^2 m^2\right)\lambda^2_{k,m} - 12\pi^4 m^4}{\left(12\rho h L^4 + \pi^2 \rho h^3 L^2 m^2\right)\lambda_{k,m} + \sqrt{\left(12\rho h L^4 + \pi^2 \rho h^3 L^2 m^2\right)\left(12\pi^4 m^4\right)}}\right|. \tag{3.8}$$

Let us now analyze the numerator and denominator of this expression separately.

Using the algebraic identity $a - b = \left(a^2 - b^2\right)/\left(a + b\right)$, we get

$$|\lambda_{k,m}|^2 = \frac{12\pi^4 k m^4}{\rho^2 h^4 L^4}\left[\frac{6\pi^2 m^2}{\rho h^3 L^2} + \frac{\pi^2 k m^2}{2\rho h L^2} + \frac{6k}{\rho h^3} + \frac{1}{2}\sqrt{\frac{144k^2}{\rho^2 h^6} + \frac{288\pi^2 k m^2}{\rho^2 h^6 L^2} + \frac{24\pi^2 k^2 m^2}{\rho^2 h^4 L^2} + \frac{144\pi^4 m^4}{\rho^2 h^6 L^4} - \frac{24\pi^4 k m^4}{\rho^2 h^4 L^4} + \frac{\pi^4 k^2 m^4}{\rho^2 h^2 L^4}}\,\right]^{-1}. \tag{3.9}$$

Then the numerator $\mathcal{N}$ on the right side of (3.8) can be rewritten by

$$\mathcal{N} = \left| -12\pi^4 m^4 + \left(12\rho h L^4 + \pi^2 \rho h^3 L^2 m^2\right) \frac{12\pi^4 k m^4}{\rho^2 h^4 L^4} \left[\frac{6\pi^2 m^2}{\rho h^3 L^2} + \frac{\pi^2 k m^2}{2\rho h L^2} + \frac{6k}{\rho h^3} \right.\right.$$
$$\left.\left. + \frac{1}{2}\sqrt{\frac{144k^2}{\rho^2 h^6} + \frac{288\pi^2 k m^2}{\rho^2 h^6 L^2} + \frac{24\pi^2 k^2 m^2}{\rho^2 h^4 L^2} + \frac{144\pi^4 m^4}{\rho^2 h^6 L^4} - \frac{24\pi^4 k m^4}{\rho^2 h^4 L^4} + \frac{\pi^4 k^2 m^4}{\rho^2 h^2 L^4}} \right]^{-1} \right|,$$

that is,

$$\mathcal{N} = \left| -12\pi^4 m^4 + \frac{\left(12L^2 + \pi^2 h^2 m^2\right) 12\pi^4 m^4}{\frac{6\pi^2 m^2}{k} + \frac{\pi^2 h^2 m^2}{2} + 6L^2 + \frac{\rho h^3 L^2}{2}\sqrt{r}} \right|,$$

where

$$r = \frac{144}{\rho^2 h^6} + \frac{288\pi^2 m^2}{\rho^2 h^6 L^2 k} + \frac{24\pi^2 m^2}{\rho^2 h^4 L^2} + \frac{144\pi^4 m^4}{\rho^2 h^6 L^4 k^2} - \frac{24\pi^4 m^4}{\rho^2 h^4 L^4 k} + \frac{\pi^4 m^4}{\rho^2 h^2 L^4}.$$

Thus, we have the following estimate

$$\begin{aligned}
\mathcal{N} &= \left| \frac{72\pi^4 L^2 m^4 + 6\pi^6 h^2 m^6 - \frac{72\pi^6 m^6}{k} - 6\pi^4 \rho h^3 L^2 m^4 \sqrt{r}}{\frac{6\pi^2 m^2}{k} + \frac{\pi^2 h^2 m^2}{2} + 6L^2 + \frac{\rho h^3 L^2}{2}\sqrt{r}} \right| \\
&\leq \left| \left(\frac{144\pi^2 L^2 m^2}{h^2} + 12\pi^4 m^4 - \frac{144\pi^4 m^4}{h^2 k} \right) - 12\pi^2 \rho h L^2 m^2 \sqrt{r} \right| \\
&= \left| \frac{\left(\frac{144\pi^2 L^2 m^2}{h^2} + 12\pi^4 m^4 - \frac{144\pi^4 m^4}{h^2 k} \right)^2 - \left(12\pi^2 \rho h L^2 m^2\right)^2 r}{\frac{144\pi^2 L^2 m^2}{h^2} + 12\pi^4 m^4 - \frac{144\pi^4 m^4}{h^2 k} + 12\pi^2 \rho h L^2 m^2 \sqrt{r}} \right| \qquad (3.10) \\
&= \left| \frac{4\,(12)^3\,\pi^4 L^2 m^4}{12h^2 L^2 k + \pi^2 h^4 m^2 k - 12\pi^2 h^2 m^2 + \rho h^5 L^2 k \sqrt{r}} \right|.
\end{aligned}$$

Dividing the numerator of the last fraction of (3.10) by the second term of its denominator, it follows that

$$\mathcal{N} \leq \frac{4\,(12)^3\,\pi^2 L^2 m^2}{h^4 k}. \qquad (3.11)$$

On the other hand, the denominator $\mathbf{D}$ of the last term of (3.8) is bounded below by

$$\mathbf{D} \geq 12\sqrt{\rho h}\pi^2 L^2 m^2. \qquad (3.12)$$

From (3.8), (3.11) and (3.12) we obtain that

$$\left| \lambda_{k,m} - \sqrt{\frac{12\pi^4 m^4}{12\rho h L^4 + \pi^2 \rho h^3 L^2 m^2}} \right| \leq \frac{c}{k}, \qquad (3.13)$$

where $c = 4\left(12\right)^2/h^4\sqrt{\rho h}$. The estimate (3.13) immediately implies the statement (3.7) of the proposition. ■

REMARK 3.1. *As we mentioned above, the eigenvalues* $\widetilde{\lambda}^{\pm}_{k,m}$ *tend to* $\pm\infty$. *In other words, they disappear as* k *tends to infinity. This fact is intimately related with the asymptotic simplification that the system undergoes when passing from a system of two equations and two dependent variables to a scalar equation with only one dependent variable. Obviously, for a complete description of the space of solutions of (2.13) the eigenpairs* $(\lambda^{\pm}_m, \cos(m\pi x/L))$ *obtained in the limit as* k *tends to infinity suffice.*

4. Asymptotic Limit of the Controlled System. Our interest now is to study the asymptotic behavior of the solutions $\{u_k, v_k\}$ of the system $(1.1)-(1.4)$, when k tends to infinity. They are defined by transposition (see [9]) as follows. Firstly we consider the solution of the adjoint system

$$\left|\begin{array}{ll} \dfrac{\rho h^3}{12}\phi'' - \phi_{xx} + k\left(\phi + \psi_x\right) = f & \text{in} \quad Q, \\ \rho h\psi'' - k\left(\phi + \psi_x\right)_x = g & \text{in} \quad Q, \\ \phi\left(0,\cdot\right) = \phi\left(L,\cdot\right) = \psi_x\left(0,\cdot\right) = \psi_x\left(L,\cdot\right) = 0 & \text{on} \quad \left(0,T\right), \\ \phi\left(\cdot,T\right) = \phi'\left(\cdot,T\right) = \psi\left(\cdot,T\right) = \psi'\left(\cdot,T\right) = 0 & \text{in} \quad \left(0,L\right). \end{array}\right. \tag{4.1}$$

As indicated in the introduction although, normally, the adjoint system is taken to be homogeneous (i. e. $f \equiv g \equiv 0$), we consider the case where f and g are arbitrary since this is useful to define the solution of $(1.1)-(1.4)$ by transposition.

This system may be reduced to (1.6) by the change of variables $t \to T - t$. Then, when $\{f, g\} \in L^1\left(0, T; H^1_0\left(0,L\right) \times L^2\left(0,L\right)\right)$, it admits a unique solution in the class (2.1) satisfying (2.2) and the hidden regularity property (2.6). Moreover, the conditions of Theorem 2.1 on the initial data and right hand side terms are satisfied for (4.1). Therefore, in the limit as $k \to \infty$,

$$\phi + \psi_x = 0. \tag{4.2}$$

Multiplying both sides of $(1.1)_1$ by ϕ and of $(1.1)_2$ by ψ and integrating, formally, by parts in Q, we obtain the identity

$$\begin{aligned} &\int_Q \left[f\left(x,t\right)u\left(x,t\right) + g\left(x,t\right)v\left(x,t\right)\right]dxdt = \frac{\rho h^3}{12}\int_0^L \phi\left(x,0\right)u_1\left(x\right)dx \\ &-\frac{\rho h^3}{12}\int_0^L \phi'\left(x,0\right)u_0\left(x\right)dx + \rho h\int_0^L \left[\psi\left(x,0\right)v_1\left(x\right) - \psi'\left(x,0\right)v_0\left(x\right)\right]dx \\ &-k\int_0^T \Theta_k\psi\left(0,t\right)dt. \end{aligned} \tag{4.3}$$

In view of (2.1) and (2.6), the right hand side of (4.3) makes sense provided

$$\{u_0, u_1, v_0, v_1\} \in \mathcal{X}' = L^2\left(0,L\right) \times H^{-1}\left(0,L\right) \times L^2\left(0,L\right) \times \left[H^1\left(0,L\right)\right]' \tag{4.4}$$

and

$$\Theta_k \in \left[H^1\left(0,T\right)\right]'. \tag{4.5}$$

Assuming that Θ_k is of the form

$$\Theta_k = \Theta'_{1k}/k, \text{ with } \Theta_{1k} \in L^2\left(0,T\right) \text{ of compact support in } \left(0,T\right), \tag{4.6}$$

the identity (4.3) may be rewritten as

$$\begin{aligned}&\int_Q \left[f(x,t)\,u(x,t)+g(x,t)\,v(x,t)\right]dxdt=\frac{\rho h^3}{12}\left[\langle u_1,\phi(\cdot,0)\rangle_0-(u_0,\phi'(\cdot,0))\right]\\&+\rho h\left[\langle v_1,\psi(\cdot,0)\rangle_1-(v_0,\psi'(\cdot,0))\right]+\int_0^T\Theta_{1k}\psi'(0,t)\,dt,\end{aligned}\tag{4.7}$$

where $\langle\cdot,\cdot\rangle_0$ (resp. $\langle\cdot,\cdot\rangle_1$) represents the duality between $H^{-1}(0,L)$ (resp. $(H^1(0,L))'$) and $H_0^1(0,L)$ (resp. $H^1(0,L)$).

Note that in (4.6) the prime $'$ stands for the classical derivative in the sense of distributions.

We adopt (4.7) as definition of solution of (1.1)$-$(1.4) in the sense of transposition. Arguing as in [8] and in view of the hidden regularity properties in Proposition 2.2 we deduce that system (1.1) $-$ (1.4) has a unique solution in the class

$$\{u,v\}\in C^0\left([0,T];\left[L^2(0,L)\right]^2\right).$$

Moreover, there exists a constant $C>0$, independent of k, such that

$$\|\{u,v\}\|_{L^\infty\left(0,T;[L^2(0,L)]^2\right)}\le C\left(\|\{u_0,u_1,v_0,v_1\}\|_{\mathcal{X}'}+\|\Theta_{1k}\|_{L^2(0,T)}\right).\tag{4.8}$$

Similarly one can show that

$$\{u,v\}\in C^1\left([0,T];H^{-1}(0,L)\times\left[H^1(0,L)\right]'\right)\tag{4.9}$$

and an estimate of the form

$$\|\{u,v\}\|_{W^{1,\infty}(0,T;H^{-1}(0,L)\times[H^1(0,L)]')}\le C\left(\|\{u_0,u_1,v_0,v_1\}\|_{\mathcal{X}'}+\|\Theta_{1k}\|_{L^2(0,T)}\right).\tag{4.10}$$

The solution by transposition of the system (1.5) can be defined in a similar way. Indeed, multiplying system (1.5) by the weak solution ψ of the backward problem (that can be transformed into (2.13) by time-reversal)

$$\left|\begin{array}{ll}\rho h\psi''-\dfrac{\rho h^3}{12}\psi''_{xx}+\psi_{xxxx}=f_x+g & \text{in}\quad Q,\\ \psi_x(0,\cdot)=\psi_x(L,\cdot)=\psi_{xxx}(0,\cdot)=\psi_{xxx}(L,\cdot)=0 & \text{on}\quad (0,T),\\ \psi(\cdot,T)=\psi'(\cdot,T)=0 & \text{in}\quad (0,L)\end{array}\right.\tag{4.11}$$

and after integrating by parts in Q, we get

$$\begin{aligned}&\int_Q\left[g(x,t)+f_x(x,t)\right]v(x,t)\,dxdt=\frac{\rho h^3}{12}\left[\langle v_{1x},\psi_x(\cdot,0)\rangle_0-(v_{0x},\psi'_x(\cdot,0))\right]\\&+\rho h\left[(v_1,\psi(\cdot,0))-(v_0,\psi'(\cdot,0))\right]+\int_0^T\Xi\psi(0,t)\,dt.\end{aligned}\tag{4.12}$$

We adopt identity (4.12) as definition of solution of (1.5) in the sense of transposition. In this sense, when $\{v_0,v_1\}\in H^1(0,L)\times L^2(0,L)$, system (1.5) possesses a unique solution in the class $v\in C^0\left([0,T];H^1(0,L)\right)\cap C^1\left([0,T];L^2(0,L)\right)$.

The following result describes the asymptotic behavior as $k\to\infty$:

THEOREM 4.1. *Consider initial data $\{u_0,u_1,v_0,v_1\}\in\mathcal{X}'$ independent of k such that*

$$u_0+v_{0x}=0;\quad u_1+v_{1x}=0,\tag{4.13}$$

and Θ_k satisfying (4.6) *and*

$$\Theta_{1k} \rightarrow \Theta_1 \text{ weakly in } L^2(0,T), \; \Theta_1 \text{ being of compact support in } (0,T). \tag{4.14}$$

Let $\{u_k, v_k\}$ be the solution of $(1.1)-(1.4)$. *Then, as $k \rightarrow \infty$, the convergence*

$$\{u_k, v_k\} \rightarrow \{-v_x, v\} \text{ weakly}* \text{ in } L^\infty\left(0,T; L^2(0,L) \times L^2(0,L)\right) \tag{4.15}$$

holds, where v is the solution of system (1.5) *with* $\Xi = -\Theta_1'$.

Remark 4.1. *As we shall see in the application to controllability, the controls both for the Midlin-Timoshenko and Kirchhoff system can be taken to be of compact support in* $(0,T)$.

Proof of Theorem 4.1. For data $\{u_0, u_1, v_0, v_1, \Theta_k\}$ in the conditions of Theorem 4.1, we consider, for each $k>0$, $\{u_k, v_k\}$ the unique solution of $(1.1)-(1.4)$ in the sense of transposition.

Using (2.1), (2.2), (2.6) and (4.14), it follows, by (4.7), that

$$\{u_k, v_k\} \text{ is bounded in } L^\infty\left(0,T; L^2(0,L) \times L^2(0,L)\right).$$

Then we can extract a subsequence, that we still denote in the same form, such that

$$\{u_k, v_k\} \rightarrow \{u, v\} \text{ weakly}* \text{ in } L^\infty\left(0,T; L^2(0,L) \times L^2(0,L)\right). \tag{4.16}$$

Applying (4.2), (4.14) and (4.16) in (4.7), we obtain, in the limit,

$$\begin{aligned}&\int_Q [f(x,t)\,u(x,t) + g(x,t)\,v(x,t)]\,dxdt = \frac{\rho h^3}{12}\left[-\langle u_1, \psi_x(\cdot,0)\rangle_0 + (u_0, \psi_x'(\cdot,0))\right] \\ &+\rho h\left[\langle v_1, \psi(\cdot,0)\rangle_1 - (v_0, \psi'(\cdot,0))\right] + \int_0^T \Theta_1 \psi'(0,t)\,dt,\end{aligned} \tag{4.17}$$

where ψ is the weak solution of system (4.11). Note that here we have used the fact that the weak convergence property in Theorem 2.1 is also true for the solutions of the adjoint system evaluated at time $t=0$. This result has not been explicitly stated in Theorem 2.1 but can be derived in view of the properties stated there and standard arguments.

On the other hand, from $(1.1)_1$ we have that

$$u_k + v_{kx} = -\frac{1}{k}\left(\frac{\rho h^3}{12} u_k'' - u_{kxx}\right).$$

Then, in the limit as $k \rightarrow \infty$ (convergence takes place in a very weak topology)

$$u + v_x = 0.$$

In this way, using the last equation, and the compatibility conditions on the initial data (4.13), identity (4.17) can be written as in (4.12) with $\Xi = -\Theta_1'$. Thus, v is the unique solution, by transposition, of system (1.5). ■

5. Non Uniform Observability. In this section we consider the adjoint system (1.6) in the particular case where $f \equiv g \equiv 0$. More precisely, assume that $\{\phi, \psi\}$ solves

$$\left|\begin{array}{lll} \dfrac{\rho h^3}{12}\phi'' - \phi_{xx} + k(\phi + \psi_x) = 0 & \text{in} & Q, \\ \rho h \psi'' - k(\phi + \psi_x)_x = 0 & \text{in} & Q, \\ \phi(0,t) = \phi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0 & \text{on} & (0,T), \\ \phi(x,0) = \phi_0(x), \quad \phi'(x,0) = \phi_1(x) & \text{in} & (0,L), \\ \psi(x,0) = \psi_0(x), \quad \psi'(x,0) = \psi_1(x) & \text{in} & (0,L). \end{array}\right. \tag{5.1}$$

We have the following observability result:

THEOREM 5.1. *For $T > 2\alpha L$, with $\alpha = \max\left\{\sqrt{\frac{\rho h^3}{12}}, \sqrt{\frac{\rho h}{k}}\right\}$, $k \geq 1$ and $h \leq \min\left\{\sqrt[3]{\frac{3}{\rho}}, \frac{1}{4\rho}\right\}$, there exists a constant $C_k^* > 0$ such that, for any solution of* (5.1) ,

$$\|\{\phi_0,\phi_1,\psi_0,\psi_1\}\|_k^2 \leq C_k^* \int_0^T \left\{|\phi_x(0,t)|^2 + |\psi(0,t)|^2 + \rho h |\psi'(0,t)|^2\right\} dt. \tag{5.2}$$

More precisely,

$$C_k^* = AC_k,$$

where $A = A(T, L, \rho, h)$ is a positive constant and

$$C_k = \frac{L}{2(T-2\alpha L)} \exp\left(\sqrt{k}L + \frac{2\sqrt{3}L}{h} + L^3 + 3L\right). \tag{5.3}$$

REMARK 5.1.

- *The observability time in Theorem 5.1 is optimal and uniform in the sense that, for k large enough, or more precisely, for $k \geq 12/h^2$, we can take $T > 2L\sqrt{\rho h^3/12}$, independent of k. However, the observability constant C_k^* diverges exponentially as $k \to \infty$. Therefore it is not of use for getting uniform controllability results as k tends to infinity.*
- *The hypotheses of Theorem 5.1 on h and k are natural since, as it was said in the introduction, the Mindlin-Timoshenko's model was deduced for thin beams (which makes the smallness assumption on h natural) and also because we are interested in the singular limit $k \to \infty$.*

Proof of Theorem 5.1.

Step 1. Firstly we consider $\{\phi_0,\phi_1,\psi_0,\psi_1\} \in \mathcal{X}_0$. In this case the energy defines an equivalent norm to the usual one $\|\cdot\|_k$. We will prove that

$$E_k(0) \leq C_k \int_0^T \left\{|\phi_x(0,t)|^2 + |\psi(0,t)|^2 + \rho h |\psi'(0,t)|^2\right\} dt. \tag{5.4}$$

For this, we use a genuinely one-dimensional method which consists roughly on viewing equations $(5.1)_1$ and $(5.1)_2$ as being evolution equations with respect to x, while t plays the role of the space variable. This argument was used in [15] when studying the controllability of the semilinear wave equation in one space dimension.

Let us define the functional

$$\begin{aligned} F_k(x) = \frac{1}{2}\int_{\alpha x}^{T-\alpha x} \{&\frac{\rho h^3}{12}|\phi'(x,t)|^2 + \rho h|\psi'(x,t)|^2 + |\phi_x(x,t)|^2 \\ &+k|\phi(x,t)+\psi_x(x,t)|^2 + |\psi(x,t)|^2\}dt. \end{aligned} \tag{5.5}$$

Note that

$$F_k(0) = \frac{1}{2}\int_0^T \left\{|\phi_x(0,t)|^2 + |\psi(0,t)|^2 + \rho h |\psi'(0,t)|^2\right\} dt. \tag{5.6}$$

The derivative of the functional F_k is

$$\begin{aligned} F_k'(x) = & \int_{\alpha x}^{T-\alpha x} \{ \frac{\rho h^3}{12} \phi'(x,t)\, \phi_x'(x,t) + \rho h \psi'(x,t)\, \psi_x'(x,t) + \phi_x(x,t)\, \phi_{xx}(x,t) \\ & \quad + k\left(\phi(x,t) + \psi_x(x,t)\right) \left(\phi(x,t) + \psi_x(x,t)\right)_x + \psi(x,t)\, \psi_x(x,t) \} dt \\ & - \frac{1}{2} \sum_{t=T-\alpha x, \alpha x} \{ \frac{\rho h^3}{12} \left|\phi'(x,t)\right|^2 + \rho h \left|\psi'(x,t)\right|^2 + \left|\phi_x(x,t)\right|^2 \\ & \qquad + k \left|\phi(x,t) + \psi_x(x,t)\right|^2 + \left|\psi(x,t)\right|^2 \}. \end{aligned} \tag{5.7}$$

Integrating by parts and using $(5.1)_1$, we get

$$\begin{aligned} & \int_{\alpha x}^{T-\alpha x} \frac{\rho h^3}{12} \phi'(x,t)\, \phi_x'(x,t)\, dt = -\int_{\alpha x}^{T-\alpha x} \frac{\rho h^3}{12} \phi''(x,t)\, \phi_x(x,t)\, dt \\ & + \left[\frac{\rho h^3}{12} \phi'(x,t)\, \phi_x(x,t) \right] \Big|_{\alpha x}^{T-\alpha x} = -\int_{\alpha x}^{T-\alpha x} \phi_{xx}(x,t)\, \phi_x(x,t)\, dt \\ & + \int_{\alpha x}^{T-\alpha x} k\left(\phi(x,t) + \psi_x(x,t)\right) \phi_x(x,t)\, dt + \left[\frac{\rho h^3}{12} \phi'(x,t)\, \phi_x(x,t) \right] \Big|_{\alpha x}^{T-\alpha x}. \end{aligned} \tag{5.8}$$

Since $h \le (3/\rho)^{\frac{1}{3}}$, we have

$$\begin{aligned} & \left[\frac{\rho h^3}{12} \phi'(x,t)\, \phi_x(x,t) \right] \Big|_{\alpha x}^{T-\alpha x} \le \frac{1}{4} \sum_{t=T-\alpha x, \alpha x} \left\{ \left(\frac{\rho h^3}{6} \right)^2 \left|\phi'(x,t)\right|^2 + \left|\phi_x(x,t)\right|^2 \right\} \\ & \le \frac{1}{4} \sum_{t=T-\alpha x, \alpha x} \{ \frac{\rho h^3}{12} \left|\phi'(x,t)\right|^2 + \rho h \left|\psi'(x,t)\right|^2 + \left|\phi_x(x,t)\right|^2 \\ & \qquad + k \left|\phi(x,t) + \psi_x(x,t)\right|^2 + \left|\psi(x,t)\right|^2 \}. \end{aligned} \tag{5.9}$$

Using $(5.1)_2$ and integrating by parts, it follows that

$$\begin{aligned} & \int_{\alpha x}^{T-\alpha x} k\left(\phi(x,t) + \psi_x(x,t)\right) \left(\phi(x,t) + \psi_x(x,t)\right)_x dt \\ & = \int_{\alpha x}^{T-\alpha x} \rho h \psi''(x,t) \left[\phi(x,t) + \psi_x(x,t)\right] dt = -\int_{\alpha x}^{T-\alpha x} \rho h \phi'(x,t)\, \psi'(x,t)\, dt \\ & - \int_{\alpha x}^{T-\alpha x} \rho h \psi'(x,t)\, \psi_x'(x,t)\, dt + \left\{ \rho h \psi'(x,t) \left[\phi(x,t) + \psi_x(x,t)\right] \right\} \Big|_{\alpha x}^{T-\alpha x}. \end{aligned} \tag{5.10}$$

We also get

$$\begin{aligned} & \left\{ \rho h \psi'(x,t) \left[\phi(x,t) + \psi_x(x,t)\right] \right\} \Big|_{\alpha x}^{T-\alpha x} \le \frac{1}{4} \sum_{t=T-\alpha x, \alpha x} \{ (2\rho h)^2 \left|\psi'(x,t)\right|^2 \\ & + \left|\phi(x,t) + \psi_x(x,t)\right|^2 \} \le \frac{1}{4} \sum_{t=T-\alpha x, \alpha x} \{ \frac{\rho h^3}{12} \left|\phi'(x,t)\right|^2 + \rho h \left|\psi'(x,t)\right|^2 \\ & + \left|\phi_x(x,t)\right|^2 + k \left|\phi(x,t) + \psi_x(x,t)\right|^2 + \left|\psi(x,t)\right|^2 \}, \end{aligned} \tag{5.11}$$

because $h \le 1/4\rho$ and $k \ge 1$.

Thus, substituting $(5.8)-(5.11)$ in (5.7), we deduce

$$\begin{aligned} F_k'(x) &\le \int_{\alpha x}^{T-\alpha x} k\left(\phi(x,t)+\psi_x(x,t)\right)\phi_x(x,t)\,dt + \int_{\alpha x}^{T-\alpha x} \psi(x,t)\,\psi_x(x,t)\,dt \\ &-\int_{\alpha x}^{T-\alpha x} \rho h \phi'(x,t)\,\psi'(x,t)\,dt \le \left(\sqrt{k}+\frac{2\sqrt{3}}{h}+L^2+3\right) L F_k(x) \end{aligned} \tag{5.12}$$

and, therefore,

$$F_k(x) \le \exp\left(\sqrt{k}L+\frac{2\sqrt{3}L}{h}+L^3+3L\right) F_k(0). \tag{5.13}$$

Integrating (5.13) in $(0,L)$, we have

$$\int_0^L F_k(x)\,dx \le L\exp\left(\sqrt{k}L+\frac{2\sqrt{3}L}{h}+L^3+3L\right) F_k(0). \tag{5.14}$$

Since $T>2\alpha L$, we obtain, by conservation of energy and (5.14),

$$\begin{aligned} &(T-2\alpha L)\,E_k(0) = \int_{\alpha L}^{T-\alpha L} E_k(0)\,dt = \int_{\alpha L}^{T-\alpha L} E_k(t)\,dt \le \int_0^L F_k(x)\,dx \\ &\le L\exp\left(\sqrt{k}L+\frac{2\sqrt{3}L}{h}+L^3+3L\right) F_k(0), \end{aligned} \tag{5.15}$$

which implies (5.4).

Step 2. We consider now $\{\phi_0,\phi_1,\psi_0,\psi_1\}\in\mathcal{X}$ and decompose it in the following way:

$$\{\phi_0,\phi_1,\psi_0,\psi_1\} = \{\phi_0,\phi_1,\psi_0-c_1,\psi_1-c_2\} + \{0,0,c_1,c_2\},$$

where $c_1=(1/L)\int_0^L \psi_0(x)\,dx$ and $c_2=(1/L)\int_0^L \psi_1(x)\,dx$. In this way, according to inequality (5.4), for the initial data $\left\{\widetilde{\phi}_0,\widetilde{\phi}_1,\widetilde{\psi}_0,\widetilde{\psi}_1\right\} = \{\phi_0,\phi_1,\psi_0-c_1,\psi_1-c_2\}\in\mathcal{X}_0$, the corresponding solution $\left\{\widetilde{\phi},\widetilde{\psi}\right\}$ of (5.1) satisfies

$$\left\|\left\{\widetilde{\phi}_0,\widetilde{\phi}_1,\widetilde{\psi}_0,\widetilde{\psi}_1\right\}\right\|_k^2 \le C_{1k}\int_0^T \left\{\left|\widetilde{\phi}_x(0,t)\right|^2+\left|\widetilde{\psi}(0,t)\right|^2+\rho h\left|\widetilde{\psi}'(0,t)\right|^2\right\}dt,$$

with $C_{1k}=2\left(1+L+L^2\right)C_k$ and C_k as in (5.3).

Taking into account that $\widetilde{\psi}=\psi-c_2t-c_1$, it follows that $\{\phi,\psi\}$ verifies

$$\begin{aligned} &\|\{\phi_0,\phi_1,\psi_0,\psi_1\}\|_k^2 \le 2\,\|\{\phi_0,\phi_1,\psi_0,\psi_1\}\|_k^2 + 2\,\|\{0,0,c_1,c_2\}\|_k^2 \\ &\le 2C_{1k}\int_0^T \left\{|\phi_x(0,t)|^2+|(\psi-c_2t-c_1)(0,t)|^2+\rho h\,|(\psi'-c_2)(0,t)|^2\right\}dt \\ &+2L\left[(c_1)^2+(c_2)^2\right] \le 4C_{1k}\int_0^T \left\{|\phi_x(0,t)|^2+|\psi(0,t)|^2+\rho h\,|\psi'(0,t)|^2\right\}dt \\ &+2L\left[(1+4TC_{1k})(c_1)^2+(1+2\rho hTC_{1k}+\frac{4T^3}{3}C_{1k})(c_2)^2\right]. \end{aligned} \tag{5.16}$$

We now need to estimate the last term of (5.16). Integrating $(5.1)_2$ from 0 to L and using the initial and boundary conditions of system (5.1) we have

$$\int_0^L \psi'(x,t)\,dx = c_2 L \qquad \text{and} \qquad \int_0^L \psi(x,t)\,dx = (c_1 + tc_2)\,L.$$

Hence

$$(c_2)^2 \le \frac{1}{L}\int_0^L |\psi'(x,t)|^2\,dx \le \frac{2}{\rho h L} E_k(t)$$

and

$$\begin{aligned}(c_1)^2 &\le \frac{2}{L}\int_0^L |\psi(x,t)|^2\,dx + 2T^2 (c_2)^2 \\ &\le \frac{4}{L}\max\left\{\frac{T^2}{\rho h},1\right\}\left[E_k(t) + \frac{1}{2}\int_0^L |\psi(x,t)|^2\,dx\right].\end{aligned} \tag{5.17}$$

Combining these estimates the observability inequality in the Theorem follows easily. ■

In the following section we show how the uniform (with respect to k) observability inequality can be proved in a subspace of solutions that, as $k \to \infty$, covers the whole energy space for the limit system.

6. Uniform Observability. To prove the uniform observability of filtered solutions of the Mindlin-Timoshenko system (5.1) we need the following refined version of the classical Ingham inequality on the theory of nonharmonic Fourier series (see Haraux [4] and Micu-Zuazua [14]):

THEOREM 6.1 ([4], [14]). *Let $f = f(t)$ be of the form $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t}$, where $(\lambda_n)_n$ is a sequence of real numbers such that there exist $N \in \mathbb{N}$, $\gamma > 0$ and $\gamma_\infty > 0$ such that*

$$\lambda_{n+1} - \lambda_n \ge \gamma_\infty > 0, \quad \textit{if} \quad |n| > N, \tag{6.1}$$

$$\lambda_{n+1} - \lambda_n \ge \gamma > 0, \quad \forall n \in \mathbb{Z}. \tag{6.2}$$

Let $T > 0$ be such that $T > \dfrac{2\pi}{\gamma_\infty}$. Then, there exist two positive constants $C^{(1)}$ and $C^{(2)}$ such that

$$C^{(1)}\sum_{n\in\mathbb{Z}} |a_n|^2 \le \int_0^T |f(t)|^2\,dt \le C^{(2)}\sum_{n\in\mathbb{Z}} |a_n|^2, \tag{6.3}$$

for all $(a_n)_n \in l^2$. More precisely, $C^{(1)} = C^{(1)}(2N+1)$ and $C^{(2)} = C^{(2)}(2N+1)$, where $C^{(i)}(j)$, $i = 1,2$, are given by the following recurrent formulas:

$$\left|\begin{array}{l} C^{(1)}(j+1) = \left[\left(\dfrac{2C^{(2)}(j)}{|J|} + 1\right)\dfrac{4}{C^{(1)}(j)\left(|J|\gamma_\infty - 2\pi\right)^2\gamma^2} + \dfrac{2}{|J|}\right]^{-1}, \\ C^{(2)}(j+1) = 2\left[|J|(j+1) + C^{(2)}(0)\right], \quad j = 0,1,..., \end{array}\right.$$

and $C^{(1)}(0)$, $C^{(2)}(0)$ *are such that* (6.3) *holds in the particular case in which* $\gamma_\infty = \gamma > 0$.

REMARK 6.1. *The particular case when* $\gamma_\infty = \gamma$ *corresponds to the classical result by Ingham [6] which shows the existence of positive constants* $C^{(1)}$ *and* $C^{(2)}$ *such that* (6.3) *holds when* $T > 2\pi/\gamma$. *Theorem 6.1 allows us to deduce that, for general sequences* $(\lambda_n)_n$, *inequality* (6.3) *holds when* T *is smaller, because the asymptotic gap* γ_∞ *is in general larger than* γ.

To apply Theorem 6.1 and deduce the uniform observability of system (5.1), we need precise estimates on the gap of the spectrum of (5.1). For this, we will look for solutions of this system in separated variables.

According to the asymptotic properties of the two families of eigenvalues $(\lambda_{k,m})_{m\in\mathbb{N}}$ and $\left(\widetilde{\lambda}_{k,m}\right)_{m\in\mathbb{N}}$ of (5.1) given by Proposition 3.1 and by (3.6), respectively, we consider only the family $(\lambda_{k,m})_{m\in\mathbb{N}}$, because it is precisely this one and its corresponding eigenfunctions that generate the solutions that converge to the solutions of the limit Kirchhoff system, while the other one disappears, as k tends to infinity, in the sense that, it does not lead to the eigenvalues of the limit system.

Let us now consider the class of solutions of (5.1) generated by the eigenfunctions associated with the eigenvalues $\lambda_{k,m}$:

$$W_\lambda = \left\{ \{\phi,\psi\} \text{ solution of (5.1) such that } \{\phi,\psi\} = \sum_{m\in\mathbb{N}} \left(a^+_{k,m} e^{-i\lambda^+_{k,m}t} + a^-_{k,m} e^{-i\lambda^-_{k,m}t}\right) \left\{\sin\left(\frac{m\pi x}{L}\right), c_m \cos\left(\frac{m\pi x}{L}\right)\right\} \right\}$$

with c_m being as in (3.5) and $a^\pm_{k,m} = (a^0_{k,m} - i a^1_{k,m}/\lambda^\pm_{k,m})/2$, where $a^0_{k,m}$ and $a^1_{k,m}$ are the Fourier coefficients of the initial data $\{\phi_0,\phi_1\}$ on the basis of sinusoidal eigenfunctions,

$$\{\phi_0,\phi_1\} = \left\{ \sum_{m\in\mathbb{N}} a^0_{k,m} \sin\left(\frac{m\pi x}{L}\right), \sum_{m\in\mathbb{N}} a^1_{k,m} \sin\left(\frac{m\pi x}{L}\right) \right\}.$$

Obviously, this is a strict subspace of the whole space of solutions. Indeed, in this subspace we have excluded all the eigencomponents associated with the eigenvalues $\widetilde{\lambda}^\pm_{k,m}$. In this subspace there is a one-to-one correspondence between the initial data $\{\phi_0,\phi_1\}$ of ϕ and the initial data $\{\psi_0,\psi_1\}$ of ψ. More precisely, the Fourier coefficients of the latter are related to the previous ones by the relations:

$$b^0_{k,m} = c_m a^0_{k,m}; \quad b^1_{k,m} = c_m a^1_{k,m}. \tag{6.4}$$

Let us analyze the gap between consecutive eigenvalues $\lambda^\pm_{km}$. For this, we address the following result:

PROPOSITION 6.1. *Given* $0 < \epsilon < \pi^2\sqrt{12/(12\rho h L^2 + \pi^2\rho h^3)}/L$ *and* $k \geq 8(12)^3 L/\epsilon\pi\sqrt{6\rho h^3}$ *we have*

$$\left|\lambda^\pm_{k,m+1} - \lambda^\pm_{k,m}\right| \geq \gamma_\infty > 0, \quad \text{with} \quad \gamma_\infty = \frac{\pi}{L}\sqrt{\frac{12}{\rho h^3}} - \epsilon, \quad \forall m \geq m_0, \tag{6.5}$$

where

$$m_0 = \frac{2}{h}\sqrt[4]{\frac{9L^2}{\epsilon^2(12\rho h L^2 + \pi^2\rho h^3)}}. \tag{6.6}$$

On the other hand,

$$\left|\lambda^{\pm}_{k,m+1}-\lambda^{\pm}_{k,m}\right|\geq\gamma>0,\quad \textit{with}\quad \gamma=\frac{\pi^2}{L}\sqrt{\frac{12}{12\rho hL^2+\pi^2\rho h^3}}-\epsilon,\quad \forall m\geq 1. \tag{6.7}$$

Proof. To simplify the notation, we denote by λ_{km} both $\lambda^{+}_{k,m}$ and $\lambda^{-}_{k,m}$. In view of (3.13) we get

$$\begin{aligned}|\lambda_{k,m+1}-\lambda_{k,m}| &\geq \frac{(m+1)\,\pi^2}{L}\sqrt{\frac{12}{\frac{12\rho hL^2}{(m+1)^2}+\pi^2\rho h^3}}-\frac{m\pi^2}{L}\sqrt{\frac{12}{\frac{12\rho hL^2}{m^2}+\pi^2\rho h^3}}-\frac{2c}{k}\\ &\geq \frac{\pi^2}{L}\sqrt{\frac{12}{\frac{12\rho hL^2}{m^2}+\pi^2\rho h^3}}-\frac{2c}{k}\geq\frac{\pi}{L}\sqrt{\frac{12}{\rho h^3}}-\left(\frac{24\sqrt{3}L}{h^2m\sqrt{12\rho hL^2+\pi^2\rho h^3m^2}}+\frac{2c}{k}\right),\end{aligned}$$

with c being as (3.13), that is, $c=4\,(12)^2\,/h^4\sqrt{\rho h}$. It is easy to see that when $m\geq m_0$, with m_0 as in (6.6), and $k\geq 4c/\epsilon$, then

$$\frac{24\sqrt{3}L}{h^2m\sqrt{12\rho hL^2+\pi^2\rho h^3m^2}}+\frac{2c}{k}\leq\epsilon.$$

This implies the asymptotic gap condition (6.5) .

Let us analyze now the behavior of the gap for all $m\geq 1$. Proceeding as before we have

$$|\lambda_{k,m+1}-\lambda_{k,m}|\geq\frac{\pi^2}{L}\sqrt{\frac{12}{\frac{12\rho hL^2}{m^2}+\pi^2\rho h^3}}-\frac{2c}{k}\geq\frac{\pi^2}{L}\sqrt{\frac{12}{12\rho hL^2+\pi^2\rho h^3}}-\frac{2c}{k}$$

for all $m\geq 1$. In this way we obtain the gap (6.7) and this concludes the proof of proposition. ■

In view of the gap conditions (6.5) and (6.7) we have all the ingredients we need to prove the following result:

Theorem 6.2. *Let $T>2L\sqrt{\rho h^3/12}$. Then there exist positive constants $c=c\,(T)$ and $C=C\,(T)$ such that*

$$c\,\|\{\phi_0,\phi_1,\psi_0,\psi_1\}\|^2_k\leq\int_0^T|\psi'(0,t)|^2\,dt\leq C\,\|\{\phi_0,\phi_1,\psi_0,\psi_1\}\|^2_k \tag{6.8}$$

for all solution $\{\phi,\psi\}$ of (5.1) *in the class W_λ with initial data $\{\phi_0,\phi_1,\psi_0,\psi_1\}$ satisfying the condition*

$$\phi_0+\psi_{0x}=0. \tag{6.9}$$

Proof. We consider $\{\phi,\psi\}\in W_\lambda$ the solution of (5.1) with initial data $\{\phi_0,\phi_1,\psi_0,\psi_1\}$ satisfying (6.9). Thus

$$\psi'(0,t)=-i\sum_{m\in\mathbb{N}}c_m\left(a^{+}_{k,m}\lambda^{+}_{k,m}e^{-i\lambda^{+}_{k,m}t}+a^{-}_{k,m}\lambda^{-}_{k,m}e^{-i\lambda^{-}_{k,m}t}\right). \tag{6.10}$$

Let $T > 2L\sqrt{\rho h^3/12}$. Applying Theorem 6.1 to the series (6.10) and using the gap conditions, we deduce the existence of positive constants $C^{(1)} = C^{(1)}(T, \gamma)$ and $C^{(2)} = C^{(2)}(T, \gamma)$ such that

$$\begin{aligned} & C^{(1)} \sum_{m\in\mathbb{N}} c_m^2 \left[\left(a_{k,m}^+ \lambda_{k,m}^+ \right)^2 + \left(a_{k,m}^- \lambda_{k,m}^- \right)^2 \right] \\ & \leq \int_0^T |\psi'(0,t)|^2 \, dt \leq C^{(2)} \sum_{m\in\mathbb{N}} c_m^2 \left[\left(a_{k,m}^+ \lambda_{k,m}^+ \right)^2 + \left(a_{k,m}^- \lambda_{k,m}^- \right)^2 \right]. \end{aligned} \tag{6.11}$$

Since the initial data $\{\phi_0, \phi_1, \psi_0, \psi_1\}$ satisfy (6.9), it is easy to see that, for the family of solutions under consideration, the following equivalence holds true:

$$\sum_{m\in\mathbb{N}} c_m^2 \left[\left(a_{k,m}^+ \lambda_{k,m}^+ \right)^2 + \left(a_{k,m}^- \lambda_{k,m}^- \right)^2 \right] \sim \|\{\phi_0, \phi_1, \psi_0, \psi_1\}\|_k^2 \tag{6.12}$$

uniformly on k for all data $\{\phi_0, \phi_1, \psi_0, \psi_1\}$ whose solution $\{\phi, \psi\} \in W_\lambda$. Combining (6.11), (6.12) we complete the proof of the theorem. ∎

Remark 6.2. *Let us compare the observability inequalities in* (6.8) *with Theorem 5.1:*

- *In Theorem 5.1, the observed quantity in the right hand side term of* (5.2) *depends both on ϕ and ψ. This would imply a controllability result for system* $(1.1)-(1.4)$, *but with an extra control entering on u at $x = 0$. In* (6.8) *the observed quantity depends only on ψ (more precisely on $\psi'(0,t)$) and this corresponds to using one simple control in* $(1.1)-(1.4)$.
- *The time of controllability in* (6.8) *is smaller and the observability constant remains bounded as $k \to \infty$.*
- *We obtained inequalities* (6.8) *only for the solutions in the subspace W_λ, since the other family of eigenvalues $\left(\widetilde{\lambda}_{k,m}\right)_{m\in\mathbb{N}}$ diverges (as $k \to \infty$) and, consequently, the subspace they generate does not contribute to our main goal which is to show the controllability of the Kirchhoff system as limit of Mindlin-Timoshenko one, as we shall see in the following section.*

Remark 6.3. *Let us finally mention a variant of the observability result in* (6.8) *that will be used in the sequel. Consider a function $\beta : (0,T) \to [0,1]$ in the class C^∞, such that*

$$\beta(t) = \begin{cases} 1, & \text{if } t \in (2\epsilon, T - 2\epsilon) \\ 0, & \text{if } t \in (0, \epsilon) \cup (T - \epsilon, T) \end{cases} \tag{6.13}$$

with $\epsilon > 0$ sufficiently small such that $T - 2\epsilon > 2L\sqrt{\rho h^3/12}$. In view of the time invariance of system (5.1), *we deduce*

$$c \left\| \{\phi(\cdot,\epsilon), \phi'(\cdot,\epsilon), \psi(\cdot,\epsilon), \psi'(\cdot,\epsilon)\} \right\|_k^2 \leq \int_0^T \beta(t) |\psi'(0,t)|^2 \, dt \tag{6.14}$$

for all solution $\{\phi, \psi\}$ of (5.1) *in the class W_λ.*

7. Uniform Controllability in Optimal Time. Due to the results of uniform observability obtained in the previous section, we can apply HUM to obtain an uniform (with respect to k) controllability result for suitable projections of solutions of the Mindlin-Timoshenko system. To be more precise, since only the eigenvalues of the family $(\lambda_{k,m})_{m\in\mathbb{N}}$ tend to eigenvalues of the limit Kirchhoff system, it is sufficient to obtain the control result on the projections Π_λ over the eigencomponents entering in the subspace W_λ^0 of W_λ:

$$W_\lambda^0 = \{\{\phi, \psi\} \in W_\lambda \text{ such that the initial data } \{\phi_0, \phi_1, \psi_0, \psi_1\} \text{ satisfy } (6.9)\}.$$

The partial controllability condition we shall achieve at the final time $t = T$ reads

$$\Pi_\lambda \{u_k(\cdot, T), u_k'(\cdot, T), v_k(\cdot, T), v_k'(\cdot, T)\} = 0. \tag{7.1}$$

This means that

$$\begin{aligned} &\frac{\rho h^3}{12}\langle u_k'(\cdot, T), \sin(m\pi x/L)\rangle_0 + \rho h \langle v_k'(\cdot, T), c_m \cos(m\pi x/L)\rangle_1 = 0, \\ &\frac{\rho h^3}{12}(u_k(\cdot, T), \sin(m\pi x/L)) + \rho h (v_k(\cdot, T), c_m \cos(m\pi x/L)) = 0, \ \forall m \in \mathbb{N}. \end{aligned} \tag{7.2}$$

Furthermore, we also describe the asymptotic behavior of the controls, as $k \to \infty$. As we shall see, they converge to exact controls for the limit system. The following holds:

THEOREM 7.1. *Let $T > 2L\sqrt{\rho h^3/12}$. Then, for all initial data $\{u_0, u_1, v_0, v_1\} \in \mathcal{X}'$, satisfying the compatibility condition (4.13), there exists a control $\Theta_k \in H^{-1}(0, T)$, with*

$$k\Theta_k = \Theta_{1k}' : \Theta_{1k} \in L^2(0, T) \textit{ of compact support in } (0, T),$$

such that the solution $\{u_k, v_k\}$ of (1.1) – (1.4) satisfies (7.1).

Moreover, the function Θ_{1k} may be written in the form $\Theta_{1k} = -\rho h\beta(\cdot)\widehat{\psi}_k'(0, \cdot)$, where $\left\{\widehat{\phi}_k, \widehat{\psi}_k\right\} \in W_\lambda^0$ is the solution of system (5.1) with initial data $\left\{\widehat{\phi}_{0k}, \widehat{\phi}_{1k}, \widehat{\psi}_{0k}, \widehat{\psi}_{1k}\right\}$ minimizing the functional

$$\begin{aligned} J_k\{\phi_{0k}, \phi_{1k}, \psi_{0k}, \psi_{1k}\} = {} & \frac{\rho h}{2}\int_0^T \beta(t)\left|\psi_k'(0, t)\right|^2 dt - \frac{\rho h^3}{12}\langle u_1, \phi_{0k}\rangle_0 \\ & + \frac{\rho h^3}{12}(u_0, \phi_{1k}) - \rho h\left[\langle v_1, \psi_{0k}\rangle_1 - (v_0, \psi_{1k})\right], \end{aligned} \tag{7.3}$$

over W_λ^0, where $\{\phi_k, \psi_k\} \in W_\lambda^0$ is the solution of (5.1) with initial data $\{\phi_{0k}, \phi_{1k}, \psi_{0k}, \psi_{1k}\}$. Furthermore, as $k \to \infty$,

$$\Theta_{1k} \to \Theta_1 \textit{ strongly in } L^2(0, T)$$

with Θ_1 of compact support in $(0, T)$. The limit control $\Xi = -\Theta_1' \in H^{-1}(0, T)$ is an exact control driving system (1.5) to equilibrium in time T. Moreover, the function Θ_1 may be written in the form $\Theta_1 = -\rho h\beta(\cdot)\widehat{\psi}'(0, \cdot)$, where $\widehat{\psi}$ is the solution of the adjoint system

$$\left| \begin{array}{lll} \rho h\psi'' - \dfrac{\rho h^3}{12}\psi_{xx}'' + \psi_{xxxx} = 0 & \textit{in} & Q, \\ \psi_x(0, \cdot) = \psi_x(L, \cdot) = \psi_{xxx}(0, \cdot) = \psi_{xxx}(L, \cdot) = 0 & \textit{on} & (0, T), \\ \psi(\cdot, 0) = \psi_0, \quad \psi'(\cdot, 0) = \psi_1 & \textit{in} & (0, L), \end{array} \right. \tag{7.4}$$

with initial data $\left\{\widehat{\psi}_0, \widehat{\psi}_1\right\} \in W \times H^1(0,L)$ *minimizing the functional*

$$\begin{aligned} J\{\psi_0, \psi_1\} = & \frac{\rho h}{2}\int_0^T \beta(t)\left|\psi'(0,t)\right|^2 dt - \frac{\rho h^3}{12}\left[\langle v_{1x}, \psi_{0x}\rangle_0 - (v_{0x}, \psi_{1x})\right] \\ & -\rho h\left[(v_1, \psi_0) - (v_0, \psi_1)\right], \end{aligned} \tag{7.5}$$

where ψ *is the solution of* (7.4) *with initial data* $\{\psi_0, \psi_1\}$.

REMARK 7.1. *In the hypotheses of Theorem 7.1, there are many possible controls* $\Theta_k \in H^{-1}(0,T)$ *and* $\Xi \in H^{-1}(0,T)$ *fulfilling the controllability requirements. The construction we develop below, presented in the statement of the Theorem, provides controls of the form* $\Theta_k = \mu_k'$ *and* $\Xi = \mu'$*, with* $\mu_k, \mu \in L^2(0,T)$ *having compact support in time and minimal* L^2_β*-norm. The weight function* β *is chosen as in* (6.13).

Proof of Theorem 7.1. We proceed in several steps.
Step 1.(Existence of the control) Consider $\{\phi_k, \psi_k\} \in W_\lambda^0$ the unique solution of the adjoint system (5.1) with initial data $\{\phi_{0k}, \phi_{1k}, \psi_{0k}, \psi_{1k}\}$. Multiplying $(1.1)_1$ and $(1.1)_2$ by ϕ_k and ψ_k, respectively, and integrating by parts in Q, we get

$$\begin{aligned} & \frac{\rho h^3}{12}\left\{\left[\langle u_k'(\cdot,T), \phi_k(\cdot,T)\rangle_0 - (u_k(\cdot,T), \phi_k'(\cdot,T))\right] - \left[\langle u_1, \phi_{0k}\rangle_0 - (u_0, \phi_{1k})\right]\right\} \\ & +\rho h\left\{\left[\langle v_k'(\cdot,T), \psi_k(\cdot,T)\rangle_1 - (v_k(\cdot,T), \psi_k'(\cdot,T))\right] - \left[\langle v_1, \psi_{0k}\rangle_1 - (v_0, \psi_{1k})\right]\right\} \\ & -\int_0^T \Theta_{1k}\psi_k'(0,t)\,dt = 0. \end{aligned}$$

Thus to prove (7.1) in the sense of (7.2) it is sufficient to prove the existence of $\Theta_{1k} \in L^2(0,T)$ such that

$$\begin{aligned} & -\frac{\rho h^3}{12}\left[\langle u_1, \phi_{0k}\rangle_0 - (u_0, \phi_{1k})\right] - \rho h\left[\langle v_1, \psi_{0k}\rangle_1 - (v_0, \psi_{1k})\right] \\ & -\int_0^T \Theta_{1k}\psi_k'(0,t)\,dt = 0 \end{aligned} \tag{7.6}$$

for all data $\{\phi_{0k}, \phi_{1k}, \psi_{0k}, \psi_{1k}\}$ whose solution $\{\phi_k, \psi_k\} \in W_\lambda^0$.

In view of the structure of β and due to (6.8), the quadratic functional J_k defined in (7.3) is continuous, strictly convex and coercive. So, there exists an unique minimizer $\left\{\widehat{\phi}_{0k}, \widehat{\phi}_{1k}, \widehat{\psi}_{0k}, \widehat{\psi}_{1k}\right\}$, whose solution $\left\{\widehat{\phi}_k, \widehat{\psi}_k\right\} \in W_\lambda^0$, can be characterized by the formula

$$\begin{aligned} & -\frac{\rho h^3}{12}\left[\langle u_1, \phi_{0k}\rangle_0 - (u_0, \phi_{1k})\right] - \rho h\left[\langle v_1, \psi_{0k}\rangle_1 - (v_0, \psi_{1k})\right] \\ & +\rho h\int_0^T \beta(t)\,\widehat{\psi}_k'(0,t)\,\psi_k'(0,t)\,dt = 0 \end{aligned} \tag{7.7}$$

for all data $\{\phi_{0k}, \phi_{1k}, \psi_{0k}, \psi_{1k}\}$ whose solution $\{\phi_k, \psi_k\} \in W_\lambda^0$.

According to (7.7), the function $\Theta_{1k} = -\rho h\beta(\cdot)\,\widehat{\psi}_k'(0,\cdot) \in L^2(0,T)$, where $\left\{\widehat{\phi}_k, \widehat{\psi}_k\right\} \in W_\lambda^0$ solves (5.1) with the minimizer $\left\{\widehat{\phi}_{0k}, \widehat{\phi}_{1k}, \widehat{\psi}_{0k}, \widehat{\psi}_{1k}\right\}$ as data, verifies (7.6). Therefore

$$\Theta_k = -\frac{\rho h}{k}\left[\beta(\cdot)\,\widehat{\psi}_k'(0,\cdot)\right]' \in H^{-1}(0,T) \tag{7.8}$$

is the control we were looking for.

Step 2.(Uniform bound of the control) Let us observe that, since $\left\{\widehat{\phi}_{0k},\widehat{\phi}_{1k},\widehat{\psi}_{0k},\widehat{\psi}_{1k}\right\}$ is the minimizer of J_k, we have

$$J_k\left\{\widehat{\phi}_{0k},\widehat{\phi}_{1k},\widehat{\psi}_{0k},\widehat{\psi}_{1k}\right\}\leq J_k\left\{0,0,0,0\right\}=0.$$

Consequently

$$\int_0^T\left|\Theta_{1k}\left(t\right)\right|^2dt\leq C\left\|\left\{u_0,u_1,v_0,v_1\right\}\right\|_{\mathcal{X}'}\left\|\left\{\widehat{\phi}_{0k},\widehat{\phi}_{1k},\widehat{\psi}_{0k},\widehat{\psi}_{1k}\right\}\right\|_k. \tag{7.9}$$

In view of the first inequality of (6.8), we can estimate the last term in (7.9) by

$$C\left\|\left\{u_0,u_1,v_0,v_1\right\}\right\|_{\mathcal{X}'}\left(\int_0^T\rho h\beta\left(t\right)\left|\widehat{\psi}_k'\left(0,t\right)\right|^2dt\right)^{\frac{1}{2}}. \tag{7.10}$$

Combining (7.9) and (7.10), we obtain

$$\left\|\Theta_{1k}\right\|_{L^2(0,T)}\leq C\left\|\left\{u_0,u_1,v_0,v_1\right\}\right\|_{\mathcal{X}'}. \tag{7.11}$$

Step 3.(Convergence of controls) Thanks to (7.11) there exists a subsequence of (Θ_{1k}) (still denoted by the index k to simplify the notation) such that

$$\Theta_{1k}\rightarrow\Theta_1\text{ weakly in }L^2\left(0,T\right). \tag{7.12}$$

We consider now $\{u_k,v_k\}$ the solution of $(1.1)-(1.4)$ with Θ_k given in (7.8). Thus, we are in the conditions of Theorem 4.1 and we can assert that the convergence (4.15) holds.

It remains to prove that $\Xi=-\Theta_1'$ is the control such that the solution v of (1.5) satisfies

$$v\left(\cdot,T\right)=v'\left(\cdot,T\right)=0\quad\text{in}\quad\left(0,L\right), \tag{7.13}$$

with

$$\Theta_1=-\rho h\beta\left(\cdot\right)\widehat{\psi}'\left(0,\cdot\right), \tag{7.14}$$

where $\widehat{\psi}$ is the solution of (7.4) with initial data $\left\{\widehat{\psi}_0,\widehat{\psi}_1\right\}\in W\times H^1\left(0,L\right)$ minimizing the functional (7.5). For this, it is sufficient to prove that

$$\begin{aligned}&-\frac{\rho h^3}{12}\left[\left\langle v_{1x},\psi_{0x}\right\rangle_0-\left(v_{0x},\psi_{1x}\right)\right]-\rho h\left[\left(v_1,\psi_0\right)-\left(v_0,\psi_1\right)\right]\\&+\rho h\int_0^T\beta\left(t\right)\widehat{\psi}'\left(0,t\right)\psi'\left(0,t\right)dt=0,\quad\forall\left\{\psi_0,\psi_1\right\}\in W\times H^1\left(0,L\right),\end{aligned} \tag{7.15}$$

where ψ is the solution of (7.4) with initial data $\left\{\psi_0,\psi_1\right\}$.

We know that, for $\Theta_{1k}=-\rho h\beta\left(\cdot\right)\widehat{\psi}_k'\left(0,\cdot\right)$, where $\left\{\widehat{\phi}_k,\widehat{\psi}_k\right\}\in W_\lambda^0$ is the solution of (5.1) with the minimizer $\left\{\widehat{\phi}_{0k},\widehat{\phi}_{1k},\widehat{\psi}_{0k},\widehat{\psi}_{1k}\right\}$ as data, the solution of system $(1.1)-(1.4)$ satisfies (7.1). Hence, we get

$$-\frac{\rho h^3}{12}\left[\left\langle u_1,\phi_{0k}\right\rangle_0-\left(u_0,\phi_{1k}\right)\right]-\rho h\left[\left\langle v_1,\psi_{0k}\right\rangle_1-\left(v_0,\psi_{1k}\right)\right]-\int_0^T\Theta_{1k}\psi_k'\left(0,t\right)dt=0 \tag{7.16}$$

for all data $\{\phi_{0k}, \phi_{1k}, \psi_{0k}, \psi_{1k}\}$ whose solution $\{\phi_k, \psi_k\} \in W^0_\lambda$.

Combining the first inequality in (6.8) and (7.11) we deduce that the sequence $\left(\left\{\widehat{\phi}_{0k}, \widehat{\phi}_{1k}, \widehat{\psi}_{0k}, \widehat{\psi}_{1k}\right\}\right)$ is uniformly bounded in $\mathcal{X}$. So, extracting a subsequence, that we still denote by $\left(\left\{\widehat{\phi}_{0k}, \widehat{\phi}_{1k}, \widehat{\psi}_{0k}, \widehat{\psi}_{1k}\right\}\right)$, we get

$$\left\{\widehat{\phi}_{0k}, \widehat{\phi}_{1k}, \widehat{\psi}_{0k}, \widehat{\psi}_{1k}\right\} \to \left\{\overline{\phi}_0, \overline{\phi}_1, \overline{\psi}_0, \overline{\psi}_1\right\} \text{ weakly in } \mathcal{X},$$

and, by (2.2) (in this case $f = g = 0$), we can pass to the limit as $k \to \infty$ on the corresponding solutions and see that the limit $\overline{\psi}$ is the weak solution of (7.4) with initial data $\left\{\overline{\psi}_0, \overline{\psi}_1\right\}$.

Multiplying $(5.1)_1$ and $(5.1)_2$ by $(-\eta\sigma_{xx})$ and $\eta\sigma_x$, respectively, where η belongs to $C^1_0(2\epsilon, T-2\epsilon)$ and $\sigma(x) = (L-x)\, e^{(x-L)^3 x^3}$ and integrating in $(0, L) \times (2\epsilon, T-2\epsilon)$, with $\epsilon > 0$ small enough, we get the following identity for the solution $\left\{\widehat{\phi}_k, \widehat{\psi}_k\right\}$:

$$\begin{aligned}&\frac{\rho h^3}{12}\int_{2\epsilon}^{T-2\epsilon}\int_0^L \widehat{\phi}'_k(x,t)\,\eta'(t)\,\sigma_{xx}(x)\,dxdt - \int_{2\epsilon}^{T-2\epsilon}\int_0^L \widehat{\phi}_{kx}(x,t)\,\eta(t)\,\sigma_{xxx}(x)\\ &+\rho h\int_{2\epsilon}^{T-2\epsilon}\int_0^L \widehat{\psi}'_{xk}(x,t)\,\eta'(t)\,\sigma(x)\,dxdt + \rho h L\int_{2\epsilon}^{T-2\epsilon}\beta(t)\,\widehat{\psi}'_k(0,t)\,\eta'(t)\,dt = 0,\end{aligned} \tag{7.17}$$

with β being the function in (6.13).

Passing to the limit in (7.17) we deduce that $\overline{\psi}$ satisfies

$$\begin{aligned}&-\frac{\rho h^3}{12}\int_{2\epsilon}^{T-2\epsilon}\int_0^L \overline{\psi}'_x(x,t)\,(x,t)\,\eta'(t)\,\sigma_{xx}(x)\,dxdt - L\int_{2\epsilon}^{T-2\epsilon}\Theta_1\eta'(t)\,dt\\ &+\int_{2\epsilon}^{T-2\epsilon}\int_0^L \overline{\psi}_{xx}(x,t)\,\eta(t)\,\sigma_{xxx}(x) + \rho h\int_{2\epsilon}^{T-2\epsilon}\int_0^L \overline{\psi}'_x(x,t)\,\eta'(t)\,\sigma(x)\,dxdt = 0.\end{aligned} \tag{7.18}$$

On the other hand, multiplying $(7.4)_1$ by $\eta\sigma_x$, we have the following identity for the limit solution $\overline{\psi}$:

$$\begin{aligned}&\rho h\int_{2\epsilon}^{T-2\epsilon}\int_0^L \overline{\psi}'_x(x,t)\,\eta'(t)\,\sigma(x)\,dxdt + \rho h L\int_{2\epsilon}^{T-2\epsilon}\beta(t)\,\overline{\psi}'(0,t)\,\eta'(t)\,dt\\ &-\frac{\rho h^3}{12}\int_{2\epsilon}^{T-2\epsilon}\int_0^L \overline{\psi}'_x(x,t)\,\eta'(t)\,\sigma_{xx}(x)\,dxdt + \int_{2\epsilon}^{T-2\epsilon}\int_0^L \overline{\psi}_{xx}(x,t)\,\eta(t)\,\sigma_{xxx}(x) = 0.\end{aligned} \tag{7.19}$$

Combining (7.18) and (7.19) we finally deduce

$$\int_{2\epsilon}^{T-2\epsilon}\left[\Theta_1 + \rho h\beta(t)\,\overline{\psi}'(0,t)\right]\eta'(t)\,dt = 0,\ \forall\eta \in C^1_0(2\epsilon, T-2\epsilon) \tag{7.20}$$

and then

$$\Theta_1 = -\rho h\beta(\cdot)\,\overline{\psi}'(0,\cdot), \tag{7.21}$$

where $\overline{\psi}$ is the solution of the adjoint system (7.4).

To show that (7.15) is satisfied it is sufficient to pass to the limit in (7.16) using as test functions the solutions of the corresponding adjoint systems in separated variables. In this way, one reproduces at the variational level, rigorously, the proof that, heuristically, would consist in passing to the limit in (7.2), and using the fact that, in the limit, $u = -v_x$, to deduce

$$-\frac{\rho h^3}{12}\left[\langle v_{1x},\psi_{0x}\rangle_0 - (v_{0x},\psi_{1x})\right] - \rho h\left[(v_1,\psi_0) - (v_0,\psi_1)\right] - \int_0^T \Theta_1 \psi'(0,t)\,dt = 0 \quad (7.22)$$

and, consequently,

$$(v(\cdot,T),\cos(m\pi x/L)) = (v'(\cdot,T),\cos(m\pi x/L)) = 0, \quad \forall m \in \mathbb{N}.$$

To conclude the proof of the theorem, it remains to prove that the function Θ_1 can be identified as in (7.14). In fact, it follows from (7.15), (7.21) and (7.22) that

$$\int_0^T \beta(t)\left[\widehat{\psi}'(0,t) - \overline{\psi}'(0,t)\right]\psi'(0,t)\,dt = 0$$

for all solution ψ of the adjoint problem (7.4).

Taking $\psi = \widehat{\psi} - \overline{\psi}$, it follows that

$$\int_0^T \beta(t)\left[\widehat{\psi}'(0,t) - \overline{\psi}'(0,t)\right]^2 dt = 0$$

and, therefore, we obtain (7.14). Considering the equations (7.15) and (7.16) with data $\left\{\widehat{\psi}_0,\widehat{\psi}_1\right\}$ and $\left\{\widehat{\phi}_{0k},\widehat{\phi}_{1k},\widehat{\psi}_{0k},\widehat{\psi}_{1k}\right\}$, respectively, we get

$$-\frac{\rho h^3}{12}\left[\left\langle u_1,\widehat{\phi}_{0k}\right\rangle_0 - \left(u_0,\widehat{\phi}_{1k}\right)\right] - \rho h\left[\left\langle v_1,\widehat{\psi}_{0k}\right\rangle_1 - \left(v_0,\widehat{\psi}_{1k}\right)\right]$$
$$+\rho h\int_0^T \beta(t)\left|\widehat{\psi}_k'(0,t)\right|^2 dt = 0$$

and

$$-\frac{\rho h^3}{12}\left[\left\langle v_{1x},\widehat{\psi}_{0x}\right\rangle_0 - \left(v_{0x},\widehat{\psi}_{1x}\right)\right] - \rho h\left[\left(v_1,\widehat{\psi}_0\right) - \left(v_0,\widehat{\psi}_1\right)\right]$$
$$+\rho h\int_0^T \beta(t)\left|\widehat{\psi}'(0,t)\right|^2 dt = 0.$$

It follows from the last two equations

$$\int_0^T \beta(t)\left|\widehat{\psi}_k'(0,t)\right|^2 dt \to \int_0^T \beta(t)\left|\widehat{\psi}'(0,t)\right|^2 dt$$

that, together with the weak convergence (7.12), yields

$$\Theta_{1k} \to \Theta_1 \text{ strongly in } L^2(0,T)$$

and

$$\lim_{k\to\infty} J_k\left\{\widehat{\phi}_{0k},\widehat{\phi}_{1k},\widehat{\psi}_{0k},\widehat{\psi}_{1k}\right\} = J\left\{\widehat{\psi}_0,\widehat{\psi}_1\right\},$$

proving the theorem. ∎

REMARK 7.2. *According to Theorems 7.1 we can recover the exact controllability property of the Kirchhoff system as a limit of the partial controllability properties of the Mindlin-Timoshenko one.*

REMARK 7.3. *Let us observe also that these results are obtained for the optimal control time* $T > 2L\sqrt{\rho h^3/12}$ *which is the best possible one both for the Mindlin-Timoshenko system and the Kirchhoff one.*

Acknowledgments. This paper was part of F. D. Araruna's doctoral thesis at Universidade Federal do Rio de Janeiro in 2004. Most of this research was done while this author was visiting Universidad Autónoma de Madrid and he expresses his thanks for their kind hospitality.

REFERENCES

[1] F. D. ARARUNA, *Controlabilidade Exata do Sistema de Kirchhoff como Limite do Sistema de Mindlin-Timoshenko, Ph.D. Thesis, Instituto de Matemática, Universidade Federal do Rio de Janeiro, (2004).*

[2] C. CASTRO & E. ZUAZUA, Concentration and Lack of Observability of Waves in Highly Heterogeneous Media. Archive Rational Mechanics and Analysis, **164** (1) (2002), pp. 39-72.

[3] C. CASTRO & E. ZUAZUA, Low Frequency Asymptotic Analysis of a String with Rapidly Oscillating Density, SIAM J. Appl. Math., **60** (2000), pp. 1205-1233.

[4] A. HARAUX, Séries Lacunaires et Contrôle Semi-Interne des Vibrations d'une Plaque Retangulaire, J. Math. Pures Appl., **68** (1989) , pp. 457-465.

[5] J. A. INFANTE, & E. ZUAZUA, Boundary Observability for the Space-Discretizations of the One-Dimensional Wave Equation, M2AN, **33** (2) (1999), pp. 407-438.

[6] A. E. INGHAM, Some Trigonometrical Inequalities with Applications to the Theory of Series, Math. Zeits., **41** (1936) , pp. 367-379.

[7] J. E. LAGNESE, & J. L. LIONS, *Modelling Analysis and Control of Thin Plates, RMA 6, Masson, Paris, (1988).*

[8] J. L. LIONS, *Contrôlabilité Exacte, Pertubations et Estabilisation de Systèmes Distribuées, Tome I, Contrôlabilité Exacte, RMA 8, Masson, Paris, (1988).*

[9] J. L. LIONS & E. MAGENES, *Problèmes aux Limites Non Homogènes et Applications, Dunod, Gauthier-Villars, Paris, vol. 1, (1968).*

[10] P. LORETI & V. VALENTE, Partial Exact Controllability for Spherical Membranes, SIAM J. Cont. Optim., **35** (2), (1997) , pp. 641-653.

[11] G. P. MENZALA & E. ZUAZUA, The Beam Equation as a Limit of 1-D Nonlinear Von Kármán Model, Appl. Math. Lett., **12** (1999) , pp. 47-52.

[12] G. P. MENZALA & E. ZUAZUA, Timoshenko's Beam Equation as Limit of a Nonlinear One-Dimensional Von Kármán System, Proceedings of the Royal Society of Edinburgh, **130A**, (2000) , pp. 855-875.

[13] G. P. MENZALA & E. ZUAZUA, Timoshenko's Plate Equation as a Singular Limit of the Dynamical Von Kármán System, J. Math. Pures Appl., **79**, (1) (2000) , pp. 73-94.

[14] S. MICU & E. ZUAZUA, Boundary Controllability of a Linear Hybrid System Arising in the Control of Noise, SIAM J. Control Optm., **35**, (5) (1997) , pp. 1614-1637.

[15] E. ZUAZUA, Exact Controllability for the Semilinear Wave Equation in One Space Dimension, Ann. Inst. H. Poincaré Analyse Non Linéaire, **10** (1993) , pp. 109-129.

[16] E. ZUAZUA, Observability of $1-d$ Waves in Heterogeneous and Semi-Discrete Media. In "Advances in Structural Control", J. Rodellar et al. eds, CIMNE, Barcelona, (1999), pp. 1-30.

[17] E. ZUAZUA, Propagation, Observation, Control and Numerical Approximation of Waves Approximated by Finite Difference Methods, SIAM Review, **47** (2) (2005), pp. 197-243.