DECAY RATES
IN THERMOELASTICITY

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ABSTRACT

In this lecture we will present some recent results on the rate of decay of the energy for some systems of thermoelasticity. First we will address the von Kármán system of thermoelastic plates. We will show how the exponential decay of the energy may be proved by using a suitable Lyapunov function. Then, we will consider the linear system of 3-d thermoelasticity with Dirichlet boundary conditions. We will show that, due to the weak interaction between transversal and longitudinal waves, the decay is not uniform for convex domains.

1. The von Kármán system for thermoelastic plates

Let Ω be a bounded smooth domain of $\mathbb{R}^2$. Let us denote by $u = u(x,t) : \Omega \times (0,\infty) \to \mathbb{R}$ the vertical displacement of the plate and by $\theta = \theta(x,t)$ the temperature.

Let us consider the system:

$$\left\{\begin{array}{lll} u_{tt} - h\Delta u_{tt} + \Delta^2 u + \Delta\theta = [u,v] & \text{in} & \Omega \times (0,\infty) \\ \Delta^2 v = [u,u] & \text{in} & \Omega \times (0,\infty) \\ \theta_t - \Delta\theta + \Delta u_t = 0 & \text{in} & \Omega \times (0,\infty) \\ u = \Delta u = 0,\ v = \Delta v = 0, & \text{on} & \partial\Omega \times (0,\infty) \\ \theta = 0 & \text{on} & \partial\Omega \times (0,\infty) \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in} & \Omega \\ \theta(x,0) = \theta^0(x) & \text{in} & \Omega. \end{array}\right. \tag{1}$$

In (1), $h \geq 0$ denotes the constant of rotational inertia of the plate and the bracket $[\cdot,\cdot]$ is defined as follows:

$$[\varphi,\psi] = \frac{\partial^2\varphi}{\partial x_1^2}\frac{\partial^2\psi}{\partial x_2^2} - 2\frac{\partial^2\varphi}{\partial x_1\partial x_2}\frac{\partial^2\psi}{\partial x_1\partial x_2} + \frac{\partial^2\varphi}{\partial x_2^2}\frac{\partial\psi}{\partial x_1^2}. \tag{2}$$

The energy of the system is given by

$$E(t) = \frac{1}{2}\int_\Omega \left[|u_t|^2 + h\,|\nabla u_t|^2 + |\Delta u|^2\right] dx + \frac{1}{4}\int_\Omega |\Delta v|^2\,dx + \frac{1}{2}\int_\Omega \theta^2 dx. \tag{3}$$

For initial data of finite energy and satisfying suitable boundary conditions system (1) has a unique solution.

The energy decreases along trajectories. More precisely,

$$\frac{dE}{dt}(t) = -\int_\Omega |\nabla\theta|^2\,dx \leq 0. \tag{4}$$

The following result, established in a joint work with G. Perla Menzala [5], guarantees the exponential decay of the energy:

Theorem 1 *There exists $C > 0$ and $\omega > 0$ such that*

$$E(t) \leq C \exp\left(-\frac{\omega}{1+R^2} t\right) E(0), \forall t \geq 0 \tag{5}$$

for every solution of (1) *such that $E(0) \leq R$.*

Remark 1

(a) The constant C and ω in the statement of Theorem 1 depend on Ω and h but do not depend on the initial data.

(b) The estimate (5) guarantees an exponential decay rate of the order of R^{-2} as $E(0) = R \to \infty$. We do not know whether this estimate is sharp.

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The method of proof of Theorem 1 consists, roughly, on finding a suitable perturbation F of the energy E for which an inequality f the form

$$\frac{dF}{dt} \leq -cF \tag{6}$$

holds, F being equivalent to E, i.e.

$$\frac{1}{2} F \leq E \leq 2F. \tag{7}$$

It is clear that (6) and (7) provide an exponential decay rate for the energy E.

The Lyapunov function we introduce is of the form

$$F = E + \varepsilon \rho$$

with ε small enough an ρ given by

$$\rho = \int_\Omega \left[h u_t \theta - \frac{h}{2}\theta^2 + u_t(-\Delta)^{-1}\theta + \frac{u u_t}{2} + \frac{h}{2}\nabla u \cdot \nabla u_t \right] dx,$$

where $(-\Delta)^{-1}$ is the inverse of the Dirichlet Laplacian.

In the lecture we will discuss the role that ρ plays in the proof and how the dependence of the decay rate on R appears.

2. The 3-d linear system of linear thermoelasticity

Let Ω be a bounded smooth domain of $\mathbb{R}^3$ and consider the following system:

$$\begin{cases} u_{tt} - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \alpha \nabla \theta = 0 & \text{in} \quad \Omega \times (0, \infty) \\ \theta_t - \Delta \theta + \beta \operatorname{div} u_t = 0 & \text{in} \quad \Omega \times (0, \infty) \\ u = 0, \quad \theta = 0 & \text{on} \quad \partial\Omega \times (0, \infty) \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in} \quad \Omega \\ \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega. \end{cases} \tag{8}$$

This time $u = (u_1, u_2, u_3)$ is a vector field, λ and μ are the Lamé coefficients and $\alpha, \beta > 0$ the coupling parameters.

The energy is given by

$$E(t)=\frac{1}{2}\int_\Omega \left[\mid u_t\mid^2+\mu\mid\nabla u\mid^2+(\lambda+\mu)\mid \operatorname{div} u\mid^2\right]+\frac{\alpha}{2\beta}\int_\Omega \theta^2 \tag{9}$$

and is dissipated along trajectories, i.e.

$$\frac{dE}{dt}(t)=-\frac{\alpha}{\beta}\int_\Omega \mid\nabla\theta\mid^2\leq 0. \tag{10}$$

C. Dafermos [1] proved that the energy of every solution tends to zero as $t\to\infty$ if and only if the following eigenvalue problem has no non–trivial solution:

$$\begin{cases} -\Delta\varphi=\gamma^2\varphi & \text{in} \quad \Omega\\ \operatorname{div}\varphi=0 & \text{in} \quad \Omega\\ \varphi=0 & \text{on} \quad \partial\Omega. \end{cases} \tag{11}$$

When Ω is a ball non–trivial solutions of (11) do exist. However, generically with respect to the domain Ω, the eigenvalues of the Laplacian are simple and in this case the existence of non–trivial solutions of (11) can be easily excluded. We refer to J.-L. Lions and E. Zuazua [4] for other applications of this type of arguments.

In this lecture we will discuss the problem of the uniform decay, i.e. of whether there exist positive constants C and ω such that

$$E(t)\leq C\exp(-\omega t)E(0), \quad \forall t>0 \tag{12}$$

holds for every solution of (8).

We will give a sketch of the proof of the following result obtained in collaboration with G. Lebeau in [3]:

Theorem 2 *When Ω is convex* (12) *does not holds, i.e. the decay rate of solutions of* (8) *is not uniform.*

The proof of the result combines two ingredients. The first one consists on applying the decoupling method by D. Henry, O. Lopes and A. Perissinotto [2]. This allows to reduce the problem to the analysis of the system of elasticity:

$$\begin{cases} u_{tt}-\mu\Delta u-(\lambda+\mu)\operatorname{div} u=0 & \text{in} \quad \Omega\times(0,\infty)\\ u=0 & \text{in} \quad \Omega\times(0,\infty)\\ u(x,0)=u^0(x),\ u_t(x,0)=u^1(x) & \text{in} \quad \Omega \end{cases} \tag{13}$$

and more precisely to the existence of a time $T>0$ and a constant $C>0$ such that

$$\parallel u^0\parallel^2_{(L^2(\Omega))^3}+\parallel u^1\parallel^2_{(H^{-1}(\Omega))^3}\leq C\int_0^T \parallel \operatorname{div} u\parallel^2_{H^{-1}(\Omega)}\,dt \tag{14}$$

for every solution of (13).

A geometric optics construction in the spirit of J. Ralston [6] allows to show that (14) does not hold if there exists a ray in Ω that is always reflected perpendicularly on the boundary. This is obviously the case when Ω is a convex smooth domain.

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4. References

1. C. Dafermos, *Proc. AMS.* **48** (2) (1975), 413–418.
2. D. Henry, O. Lopes and A. Perissinotto, *Nonlinear Anal. TMA.* **21** (1993), 65–75.
3. G. Lebeau and E. Zuazua, *C. R. Acad. Sci. Paris*, to appear.
4. J.-L. Lions and E. Zuazua, in *Partial Differential Equations and Applications*, ed. P. Marcellini, G. Talenti and E. Visentini (LNPAS 177, Marcel–Dekker Inc., 1996).
5. G. Perla-Menzala and E. Zuazua, *C. R. Acad. Sci. Paris*, **324** (1997), 49-54.
6. J. Ralston, in *Studies in Partial Differential Equations*, ed. W. Littman, (MAA Studies in Mathematics, vol. **23**, 1982).