SOME PROBLEMS AND RESULTS ON THE CONTROLLABILITY OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

We present some recent results on the controllability of Partial Differential Equations. We discuss the different notions of controllability and comment how feasible they are depending on the nature of the system under consideration. We discuss both the wave and heat equations as model examples of conservative and irreversible systems respectively and we describe the different tools that have been developed to address these problems. We also present some recent results on the controllability of the linear system of thermoelasticity which is the simplest one coupling both the hyperbolic and the parabolic nature of the wave and heat equation respectively.

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1 Introduction.

In this paper we present some results obtained recently on the controllability of Partial Differential Equations (PDE). Roughly speaking the problem of controllability can be formulated as follows: Consider an evolution equation or system in which some of the data (the control) is at our disposal (for instance, some boundary condition, the right hand side of the equation, etc.). Given a time interval $(0, T)$ and two values u^0 and v^0 in the space of states where the equation evolves, can one find a control such that the solution u with initial data u^0 and this control takes the value v^0 at $t = T$? The way we have raised the question may be uderstood as an *exact controllability* problem. The control requirement $u(T) = v^0$ can be relaxed in various ways leading to various notions and degrees of controllability.

Many different problems may fit in this general formulation. On the other hand, the solvability of the problem depends very much on the nature of the precise question under consideration and, in particular, the following features may play a crucial role: linearity or nonlinearity of the system, time reversibility, the structure of the set of admissible controls, etc.

The controllability of PDE has been the object of intensive researches during the last decade. However the subject is older than that. In 1978, D. L. Russell [R1] made a rather complete survey of the most relevant results that were available in the literature at that time. In that paper the author described a number of different tools that were developed to address controllability problems, often inspired and related to other subjects of PDE: multipliers, moment problems, nonharmonic Fourier series, etc.

More recently J.-L. Lions introduced the so called HUM (Hilbert Uniqueness Method) (see [L1] and [L2] for instance). That was the starting point of a fruitful decade for this subject.

It would be impossible to present in one single paper all the relevant results that have been proved in this area. This is one of the reasons for having decided to focus on the wave and heat equations. The second one is that these two systems are model examples of conservative and strongly irreversible systems respectively. When studying these two simple models one encounters already some of the most relevant intrinsic difficulties of controllability. We will explain how important is the role that reversibility plays when addressing controllability problems. We will also describe the different methods that have been developed to address controllability issues in these two frameworks: wave and heat equations. In both cases we briefly discuss semilinear equations too.

We also study the linear system of three-dimensional thermoelasticity. It is the simplest system combining the hyperbolic and parabolic nature of the wave and heat equations respectively. In a first approach to the problem one might be tempted to think that controllability results for the system of thermoelasticity can be obtained just by adding what is known about the wave and heat equations separately. But this is not the case. Indeed, the techniques available in the literature for the wave and heat equations are of a rather different nature and it is difficult to combine them to address the system of thermoelasticity. We shall see that some decoupling techniques may be a very useful tool to solve this problem.

As we said above, this survey paper is far from being complete. Even in what concerns the three examples we have chosen to develop (wave and heat equations and the system of thermoelasticity) our list of references is rather limitated. We hope that other authors having proved relevant results in the area will forgive us for the omission. A more complete list of references can be obtained by looking to the bibliographies of the survey papers and the books of our references.

We have left out some other interesting topics like, for instance, controllability of plates, Schrödinger and KdV equations, the system of three-dimensional elasticity, the Stokes and Navier-Stokes equations, Maxwell's system, etc. On the other hand, we have not considered other important related topics like: numerical computation and simulation in controllability problems, stabilizability, networks of flexible structures, connections with finite dimensional controllability theory, the theory of optimal control, etc. We have included in our bibliography a limited number of references for these issues.

2 The linear wave equation.

2.1 Problem formulation.

Let Ω be a bounded domain of $I\!R^n, n \geq 1$, with boundary Γ of class C^2.

Let ω be an open and non-empty subset of Ω. Let $T > 0$ and consider the linear controlled wave equation in the cylinder $Q = \Omega \times (0,T)$:

$$\begin{cases} u_{tt} - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{2.1}$$

In (2.1) Σ represents the lateral boundary of the cylinder Q, i.e. $\Sigma = \Gamma \times (0,T), 1_\omega$ is the characteristic function of the set $\omega, u = u(x,t)$ is the state and $f = f(x,t)$ is the control variable. Since f is multiplied by 1_ω the action of the control is concentrated in ω.

In space dimension $n = 2$, $u = u(x,t)$ may represent, for instance, the vertical displacement of a vibrating membrane occupying Ω and fixed on its boundary Γ under the action of the force f concentrated in ω.

When $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f \in L^2(Q)$ system (2.1) has an unique solution $u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$.

The problem of controllability consists, roughly, on describing the set of reachable final states $(u(T), u_t(T))$ when the control f varies in a given class, for instance, $L^2(Q)$.

One may distinguish the following degrees of controllability:

(a) *Approximate controllability*: System (2.1) is said to be approximately controllable in time T if the set of reachable states

$$R\left(T; (u^0, u^1)\right) = \left\{(u(T), u_t(T)) : f \in L^2(Q)\right\}$$

is dense in $H_0^1(\Omega) \times L^2(\Omega)$ for every $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

(b) *Exact controllability*: System (2.1) is said to be exactly controllable at time T if $R\left(T; (u^0, u^1)\right) = H_0^1(\Omega) \times L^2(\Omega)$ for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

(c) *Null controllability*: System (2.1) is said to be null controllable at time T if $(0, 0) \in R\left(T; (u^0, u^1)\right)$ for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Remark 2.1

(a) Since we are dealing with solutions of the wave equation, for any of these properties to hold the time T has to be sufficiently large due to the finite speed of propagation.

(b) Since system (2.1) is linear and reversible in time null and exact controllability are equivalent notions.

(c) Clearly every exactly controllable system is approximately controllable too. However, system (2.1) may be approximately but not exactly controllable. In those cases for every (u^0, u^1), $(v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\varepsilon > 0$ there exists a control $f_\varepsilon \in L^2(Q)$ such that the solution of (2.1) satisfies

$$\| \, (u(T), u_t(T)) - (v^0, v^1) \, \|_{H_0^1(\Omega) \times L^2(\Omega)} \leq \varepsilon. \tag{2.2}$$

When the system is approximately but not exactly controllable $\| \, f_\varepsilon \, \|_{L^2(Q)} \rightarrow \infty$ as $\varepsilon \rightarrow 0$ for some initial and final data.

(d) Describing the nature of the set of controllable data when exact controllability does not hold is a challenging open problem. We refer to [Le1] for partial but interesting results in that direction.

(e) We have stated controllability problems in $H_0^1(\Omega) \times L^2(\Omega)$ with controls in $L^2(Q)$. Of course this is an arbitrary choice of the functional setting. Many other frameworks make sense.

(f) Null controllability is a physically interesting notion since the state $(0, 0)$ is an equilibrium for system (2.1). More precisely, if the solution of (2.1) reaches this null state at time $t = T$ and we do not introduce further control for $t \geq T$, the solution remains in this equilibrium configuration for all $t \geq T$.

■

Most of the literature on the controllability of the wave equation has been written on the framework of the *boundary control* problem, i.e. when (2.1) is replaced by

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad Q \\ u = v & \text{on} \quad \Sigma \\ u(x, 0) = u^0(x), u_t(x, 0) = u^1(x) & \text{in} \quad \Omega. \end{cases}$$

In this case the control $v = v(x,t)$ acts on the system through the Dirichlet boundary condition.

The control problems above for system (2.1) are refered to as *internal controllability* problems since in that case the control acts on the subset ω of Ω.

In this paper we will only discuss the internal control problem to avoid additional technical difficulties related to the well-posedness of the boundary value problem.

2.2 Approximate controllability.

It is easy to see that approximate controllability is equivalent to an unique continuation property of the adjoint system:

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x), \varphi_t(x,T) = \varphi^1(x) & \text{in} \quad \Omega. \end{cases} \tag{2.3}$$

Indeed, system (2.1) is approximately controllable if and only if the following holds:

$$\varphi \equiv 0 \text{ in } \omega \times (0,T) \Rightarrow \left(\varphi^0, \varphi^1\right) \equiv (0,0). \tag{2.4}$$

By using Holmgren's Uniqueness Theorem it can be easily seen that (2.4) holds if T is large enough. A careful but simple argument leads to the following estimate of the minimal time for approximate controllability:

$$T > 2\delta(\Omega;\omega) \tag{2.5}$$

where

$$\delta(\Omega;\omega) = \sup_{x\in\Omega} \delta(x;\omega), \tag{2.6}$$

$\delta(x,\omega)$ being the infimum of the lengths of the courves in Ω joining x and ω.

The fact that (2.4) holds is a consequence of the following unique continuation result which is of local nature and a consequence of Holgrem's Uniqueness Theorem: *If φ solves the wave equation and*

$$\varphi = 0 \quad in \quad B_\delta \times (0,T)$$

then

$$\varphi = 0 \quad in \quad \bigcup_{0\le r<T/2} \left\{ B_{\delta+r} \times (r, T-r) \right\}.$$

By B_δ we denote the ball of radious δ in $\mathbb{R}^n$ with center in $x = 0$.

Wen the geometry of Ω is complicated, this minimal time of controllability may be very large. But, as T. Cazenave [C] proved, this estimate is sharp.

There are at least two ways of checking that (2.4) implies the approximate controllability:

(a) The application of Hahn-Banach Theorem.

(b) The variational approach developed in [L3].

We prefer to present here the second proof since it is easy to adapt it to other situations.

First of all we observe that, since system (2.1) is linear, it is sufficient to consider the case $u^0 \equiv u^1 \equiv 0$. Then, given any $\left(v^0, v^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\varepsilon > 0$ we look for $f \in L^2(Q)$ such that the solution of (2.1) satisfies (2.2).

To do that we introduce the following functional $J_\varepsilon : L^2(\Omega) \times H^{-1}(\Omega) \to I\!R$:

$$J_\varepsilon\left(\varphi^0, \varphi^1\right) = \frac{1}{2} \int_0^T \int_\omega \mid \varphi \mid^2 dxdt + \varepsilon \left\| \left(\varphi^0, \varphi^1\right) \right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \tag{2.7}$$
$$- < \left(\varphi^0, \varphi^1\right), \left(v^1, -v^0\right) >$$

where $< \cdot, \cdot >$ denotes the duality pairing between $L^2(\Omega) \times H^{-1}(\Omega)$ and its dual $L^2(\Omega) \times H_0^1(\Omega)$.

The functional J_ε is continuous and convex in the Hilbert space $L^2(\Omega) \times H^{-1}(\Omega)$. On the other hand when the unique continuation property (2.4) holds it is easy to see that it is coercive too. More precisely,

$$\lim_{\|(\varphi^0,\varphi^1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} \to \infty} \frac{J_\varepsilon\left(\varphi^0, \varphi^1\right)}{\| \left(\varphi^0, \varphi^1\right) \|_{L^2(\Omega)\times H^{-1}(\Omega)}} \geq \varepsilon. \tag{2.8}$$

We refer to [FPZ1] for the details of the proof in the frame of the heat equation.

Under these conditions J_ε has a unique minimizer in $L^2(\Omega) \times H^{-1}(\Omega)$. Let us denote it by $(\widehat{\varphi}^0, \widehat{\varphi}^1)$, i.e.

$$\left(\widehat{\varphi}^0, \widehat{\varphi}^1\right) \in L^2(\Omega) \times H^{-1}(\Omega),\ J_\varepsilon\left(\widehat{\varphi}^0, \widehat{\varphi}^1\right) = \min_{(\varphi^0,\varphi^1)\in L^2(\Omega)\times H^{-1}(\Omega)} J_\varepsilon\left(\varphi^0, \varphi^1\right). \tag{2.9}$$

Then if $\widehat{\varphi}$ is the solution of (2.3) with data $(\widehat{\varphi}^0, \widehat{\varphi}^1)$, we have

$$\left| \int_0^T \int_\omega \widehat{\varphi}\varphi dxdt - < \left(\varphi^0, \varphi^1\right), \left(v^1, -v^0\right) > \right| \leq \varepsilon \parallel \left(\varphi^0, \varphi^1\right) \parallel_{L^2(\Omega)\times H^{-1}(\Omega)} \tag{2.10}$$

for all $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

On the other hand, if we set $f = \widehat{\varphi}$ in (2.1) multiplying in (2.3) by u, integrating by parts and taking into account that $u^0 \equiv u^1 \equiv 0$ we get

$$\int_0^T \int_\omega \widehat{\varphi}\varphi dxdt - < \left(\varphi^0, \varphi^1\right), (u_t(T), -u(T)) >= 0. \tag{2.11}$$

Combining (2.10) and (2.11) it follows that

$$\left| < \left(\varphi^0, \varphi^1\right), \left(u_t(T) - v^1,\ v^0 - u(T)\right) > \right| \leq \varepsilon \parallel \left(\varphi^0, \varphi^1\right) \parallel_{L^2(\Omega)\times H^{-1}(\Omega)} \tag{2.12}$$

for all $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and this is equivalent to (2.2).

Remark 2.2 The control one obtains when minimizing J_ε is the one of minimal L^2-norm among all the admissible ones. The set of admissible controls is given by.

$$\mathcal{U}_{ad} = \left\{ f \in L^2(Q) : \| \left(u(T) - v^0,\, u_t(T) - v^1 \right) \|_{H_0^1(\Omega) \times L^2(\Omega)} \leq \varepsilon \right\}.$$

The minimization of J_ε was introduced in [L3] as the dual problem of finding the control of minimal norm in $\mathcal{U}_{ad}$ by Fenchel and Rockafellar's duality principle.

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The unique continuation property (2.4) allows to prove a stronger result. Indeed, in addition to (2.2), one can guarantee that the solution of (2.1) satisfies exactly a finite number of constraints. More precisely, the following holds:

Theorem 2.1 *Let ω be any open and non-empty subset of Ω. Assume that $T > 2\delta(\Omega; \omega)$. Let E be any finite-dimensional subspace of $H_0^1(\Omega) \times L^2(\Omega)$ and let us denote by π_E the orthogonal projection over E.*

Then, for any (u^0, u^1), $(v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\varepsilon > 0$ there exists $f \in L^2(Q)$ such that the solution of (2.1) *satisfies*

$$\left\| \left(u(T) - v^0, u_t(T) - v^1 \right) \right\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq \varepsilon;\ \pi_E \left(u(T), u_t(T) \right) = \pi_E \left(v^0, v^1 \right). \qquad (2.13)$$

This result may be proved at least in two different ways:

(a) By a suitable modification of the functional J_ε to be minimized (in section 3 we give the details in the case of the heat equation).

(b) As a direct consequence of the approximate controllability and the following Theorem of functional Analysis (we refer to [Z6] for a proof):

Theorem 2.2 *Let V and H be two Hilbert spaces and L a bounded linear operator from V to H with dense range. Let E be a finite-dimensional subspace of H and π_E the corresponding orthogonal projection.*

Then, given any $e_0 \in E$, when v runs over the set of elements of v such that $\pi_E L v = e_0$, Lv describes a dense set in $e_0 + E^\perp$.

The results above hold for wave equations with analytic coefficients too. Indeed, the control problem can be reduced to the unique continuation one and the latter may be solved by means of Holmgren's Uniqueness Theorem.

However, the problem is not completely solved in the frame of the wave equation with lower order potentials $a \in L^\infty(Q)$ of the form

$$u_{tt} - \Delta u + a(x,t) u = f 1_\omega \text{ in } Q.$$

Once again the problem may be reduced to show the unique continuation property (2.4) for the adjoint system:

$$\begin{cases} \varphi_{tt} - \Delta\varphi + a(x,t)\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x), \varphi_t(x,T) = \varphi^1(x) & \text{in} \quad \Omega. \end{cases}$$

The following is known:

(a) When $a = a(x)$, D. Tataru [T2] has proved a deep result showing that (2.4) holds with the same estimate in the uniqueness time ($T > 2\delta(\Omega, \omega)$) for bounded, time-independent potentials. The proof of Tataru uses Carleman's estimates in an essential way and is of local nature.

(b) When $a = a(t)$, (2.4) can be proved for $T > 2\delta(\Omega; \omega)$. In [H] a simple proof is given by developing solutions on the spherical harmonics. This reduces the problem to a family of one-dimensional wave equations in the time and radial variables. In one space dimension the unique continuation property is easy to prove. It suffices to view the wave equation as an evolution equation in the radial variable while t plays the role of the space variable. We refer to [Z5] for other applications of this argument.

(c) When $a = a(x,t)$, it is known that the unique continuation property (2.4) does not hold in the class of C^∞ potentials. Indeed, the local uniqueness fails (see S. Alinhac [A]).

(d) When ω is a neighborhood of the boundary of Ω, unique continuation holds for any bounded potential $a = a(x,t)$. We refer to A. Ruiz [Ru] for a proof of this fact that uses Carleman's inequalities. Notice that this result is not of local nature.

2.3 Exact controllability.

As we said above, since the wave equation is reversible in time, exact controllability and null controllability are equivalent notions. Thus, along this section, without loss of generality, we will assume that $v^0 \equiv v^1 \equiv 0$.

If instead of the functional J_ε of section 2.2 we consider

$$\widetilde{J}_\varepsilon\left(\varphi^0, \varphi^1\right) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \varepsilon \parallel \left(\varphi^0, \varphi^1\right) \parallel_{L^2(\Omega)\times H^{-1}(\Omega)} \tag{2.14}$$
$$+ \left\langle \left(\varphi(0), \varphi_t(0)\right), \left(u^1, -u^0\right) \right\rangle$$

it is easy to see that the unique continuation property (2.4) allows to prove that $\widetilde{J}_\varepsilon$ is coercive too. More precisely $\widetilde{J}_\varepsilon$ satisfies (2.8). The minimizer of $\widetilde{J}_\varepsilon$ provides a control f_ε such that the solution of (2.1) satisfies

$$\|(u(T),\, u_t(T))\|_{H_0^1(\Omega)\times L^2(\Omega)} \le \varepsilon. \tag{2.15}$$

The problem of exact controllability consists on analyzing whether one can take $\varepsilon = 0$ in (2.15), or more precisely, if the control f_ε remains bounded in $L^2(Q)$ as $\varepsilon \to 0$.

The unique continuation property (2.4) by itself does not allow to answer to this question. To guarantee that the controls f_ε are bounded we need to know that the minimizers of $\widetilde{J}_\varepsilon$ in $L^2(\Omega) \times H^{-1}(\Omega)$ are bounded. To prove the boundedness of the minimizers for all (u^0, u^1) in $H_0^1(\Omega) \times L^2(\Omega)$ we need the followig inequality:

$$\|(\varphi(0), \varphi_t(0))\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt \tag{2.16}$$

for all solutions of (2.3).

This inequality allows to estimate the total energy of the solution of (2.3) by means of a measurement in the control region $\omega \times (0, T)$. Thus, it establishes the *continuous observability* of system (2.3). The energy $\|(\varphi(t), \varphi_t(t))\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}$ of solutions of (2.3) is conserved along trajectories. Thus, (2.16) is equivalent to

$$\left\|\left(\varphi^0, \varphi^1\right)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt. \tag{2.17}$$

Remark 2.3 When the system to be controlled is irreversible in time, the corresponding adjoint system is irreversible too. In those cases the analogue of (2.16) is of different nature than that corresponding to (2.17). For instance, as we will see in section 3, the analogue of (2.16) is true for the adjoint of the heat equation but the analogue of (2.17) is false.

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When (2.16) holds one can minimize directly the functional $\widetilde{J}_\varepsilon$ with $\varepsilon = 0$, i.e.

$$\widetilde{J}\left(\varphi^0, \varphi^1\right) = \frac{1}{2} \int_0^T \int_\omega \varphi^2 dxdt + \left\langle (\varphi(0), \varphi_t(0)), \left(u^1, -u^0\right)\right\rangle. \tag{2.18}$$

The following is easy to prove: *When the observability inequality (2.16) holds, the functional $\widetilde{J}$ has an unique minimizer $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ in $L^2(\Omega) \times H^{-1}(\Omega)$ for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. The control $f = \widehat{\varphi}$ with $\widehat{\varphi}$ solution of (2.3) corresponding to $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ is such that the solution of (2.1) satisfies*

$$u(T) \equiv u_t(T) \equiv 0. \tag{2.19}$$

Remark 2.4 The control obtained when minimizing $\widetilde{J}$ is the one of minimal L^2-norm among the admissible ones (see [L1,2]).

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Let us now discuss what is known about the observability inequality (2.16):

(a) To our knowledge an inequality of the form (2.16) was proved for the first time by L. F. Ho [Ho] by means of multiplier techniques in the spirit of C. Morawetz [M]. In [Ho] it was shown that if one considers subsets of Γ of the form

$$\Gamma(x^0) = \left\{ x \in \Gamma : (x - x^0) \cdot n(x) > 0 \right\}$$

for some $x^0 \in I\!\!R^n$ (by $n(x)$ we denote the outward unit normal to Ω in $x \in \Gamma$ and by $\cdot$ the scalar product in $I\!\!R^n$) and if $T > 0$ is large enough, the following boundary observability inequality holds:

$$\| (\varphi(0), \varphi_t(0)) \|^2_{H^1_0(\Omega) \times L^2(\Omega)} \leq C \int_0^T \int_{\Gamma(x^0)} \left| \frac{\partial \varphi}{\partial n} \right|^2 d\Gamma dt \tag{2.20}$$

for all $(\varphi^0, \varphi^1) \in H^1_0(\Omega) \times L^2(\Omega)$.

This is the observability inequality that is required to solve the boundary controllability problem mentioned in section 2.1.

Later on inequality (2.20) was proved in [L1,2], for any $T > T(x^0) = 2 \parallel x - x^0 \parallel_{L^\infty(\Omega)}$. This is the optimal observability time that one may derive by means of multipliers.

Proceeding as in [L2], vol. 1, one can easily prove that (2.20) implies (2.16) when ω is a neighborhood of $\Gamma(x^0)$ in Ω, i.e. $\omega = \Omega \cap \Theta$ where Θ is a neighborhood of $\Gamma(x^0)$ in $I\!\!R^n$, with $T > T(x^0)$.

(b) C. Bardos, G. Lebeau and J. Rauch [BLR] proved that, in the class of C^∞ domains, the observability inequality (2.16) holds if and only if (ω, T) satisfy the following *geometric control condition* in Ω*: Every ray of geometric optics that propagates in* Ω *and is reflected on its boundary* Γ *enters* ω *in time less than* T.

This result was proved by means of microlocal Analysis techniques and has the advantage that:

1.- It characterizes all the control sets ω and control times T that guarantee the exact controllability. At this respect observe that the control sets $\Gamma(x^0)$ that one obtains by multipliers are always large enough subsets of Γ. For instance, when Ω is a ball of $I\!\!R^n$, $\Gamma(x^0)$ is a connected set that always contains more than half of its boundary. On the other hand the estimate one obtains by multipliers on the time of control $T(x^0)$ is not sharp in general.

2.- The methods of [BLR] apply in a large class of wave equations with variable coefficients provided they are smooth and time independent. It is easy to see that the multiplier techniques fail for wave equations with smooth variable coefficients, some particular cases being excepted.

Some of the advantages that the multiplier technique presents are the following:

1.- It applies when the domain is of class C^2 or even when the domain has singularities (see P. Grisvard [G]). However, recently the microlocal approach has been greatly simplified by N. Burq [Bu] by using the microlocal defect measures introduced by P. Gerard [Ge] in the context of the homogenization and the kinetic equations. In [Bu] the geometric control condition was shown to be sufficient for exact controllability for domains Ω of class C^3 and equations with C^2 coefficients.

2.- The multiplier technique can be systematically applied to other systems and equations to obtain easily suboptimal results: plate and Schrödinger equations, KdV equations, Maxwell's system and the system of elasticity, etc. We refer to [L1,2] and to the recent monograph by V. Komornik [K] for an introduction to these techniques.

At this point the works by D. Tataru [T1] are also worth mentioning. In [T1] by using Carleman's inequalities the observability was proved for more general equations than those that one may cover by microlocal tools (since less regularity of the coefficients and on the domain is needed) and for a class of control sets that is larger than the very restrictive one covered by multipliers.

(c) An important problem arising, in particular, when dealing with semilinear problems, is the dependence of the constants in the observability inequality on the coefficients.

Let us consider for instance the adjoint equation

$$\begin{cases} \varphi_{tt} - \Delta\varphi + a(x,t)\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,t) = \varphi^0(x),\ \varphi_t(x,T) = \varphi^1(x) & \text{in} \quad \Omega \end{cases} \tag{2.21}$$

with a bounded potential $a \in L^\infty(Q)$.

Assume that the subset ω of Ω and $T > 0$ are such that the observability holds. More precisely, suppose that there exists a constants $C > 0$ such that

$$\|(\varphi(0), \varphi_t(0))\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \le C(a) \int_0^T \int_\omega \varphi^2 dxdt \tag{2.22}$$

for all $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

This holds, for instance, when ω is a neighborhood of the boundary Γ and $T >$ diam$(\Omega\backslash\omega)$ (see [Z4]). However the proof of (2.22) uses compactness arguments to absorb lower order terms. Thus, no explicit bounds are obtained. In one space dimension in [Z5] it is proved that (2.22) holds with

$$C(a) = O\left(\exp\left(C \parallel a \parallel_\infty^{1/2}\right)\right) \text{ as } \parallel a \parallel_\infty \to \infty. \tag{2.23}$$

This estimate turns to be sharp and it is reasonable to expect it to hold in several space dimensions too.

$\blacksquare$

We have described here the HUM and some tools to prove observability inequalities. However, other methods have been developed to address controllability problems: Moment problems, nonharmonic Fourier series, fundamental solutions, etc. We will not present them here. We refer to the survey paper by D. L. Russell [R1] for the interested reader.

2.4 The semilinear wave equation.

Let us consider the semilinear wave equation

$$\begin{cases} u_{tt} - \Delta u + h(u) = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1 & \text{in} \quad \Omega \end{cases} \tag{2.24}$$

where $h : I\!R \to I\!R$ is a locally Lipschitz function such that

$$| h(s_1) - h(s_2) | \leq C \left(1 + | s_1 |^{p-1} + | s_2 |^{p-1} \right) | s_1 - s_2 |,\ \forall s_1, s_2 \in I\!R \tag{2.25}$$

with

$$1 < p \leq \frac{n}{n-2} \text{ if } n \geq 3;\ 1 < p < \infty \text{ if } n = 1, 2. \tag{2.26}$$

Under assumptions (2.25)-(2.26) system (2.24) is locally well-posed: There exists $\delta > 0$ such that for any $\left(u^0, u^1\right) \in H_0^1(\Omega) \times L^2(\Omega), f \in L^2(Q)$ so that

$$\left\| \left(u^0, u^1 \right) \right\|_{H_0^1(\Omega) \times L^2(\Omega)} + \| f \|_{L^2(Q)} \leq \delta \tag{2.27}$$

system (2.24) admits an unique solution $u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$.

If the non-linearity h satisfies also a "good-sign" condition, for instance

$$H(s) \geq -C \mid s \mid^2 \ \text{ as } \ \mid s \mid \to \infty \tag{2.28}$$

for some $C > 0$ with $H(s) = \int_0^s h(z) dz$, then system (2.24) is globally well-posed and admits a unique solution u in the class above for any initial data $\left(u^0, u^1\right)$ and right hand side f.

The approximate and exact controllability problem can be formulated as in the context of the linear wave equation. At this level it is worth mentioning that, in the non-linear context, the equivalence of the exact and null controllability notions is no longer true.

Concerning the exact controllability the following is known: *If ω is a neighborhood of the boundary Γ, $T > \operatorname{diam}(\Omega \backslash \omega)$ and h is globally Lipschitz, then system (2.24) is exactly controllable.*

This result may be proved at least by two different methods: A fixed point argument based on Schauder's Theorem (see [Z4]) and a global version of the Inverse Function Theorem (see [LT1]).

Very little is known about the exact controllability of (2.24) when h grows superlinearly at infinity. The following was proved in [Z6]:

Theorem 2.3 *Assume that* $n = 1$ *and* $\Omega = (0, l)$. *Let* $\omega = (a, b)$ *be a subinterval of* Ω. *Assume that* $T > 2 \max(a, l - b)$ *and*

$$\lim_{|s| \to \infty} \frac{| h(s) |}{| s | \log^2 | s |} = 0. \tag{2.29}$$

Then, system (2.24) *is exactly controllable.*

Remark 2.5

(a) The control time $T > 2 \max(a, l - b)$ is sharp.

(b) The resctriction (2.29) on the growth of the non-linearity h seems to be too restrictive. However, it turns out to be sharp. Indeed, if $h(s) = -s \log^p(1+ | s |)$ with $p > 2$, then blow-up phenomena occur and due to the finite speed of propagation system (2.24) is not controllable except in the trivial case $\Omega = \omega$ (see [Z6]).

Whether exact controllability holds in the critical case in which

$$| h(s) | \sim C | s | \log^2 | s | \quad \text{as} \quad | s | \to \infty$$

with $C > 0$ large is an open problem.

(c) The proof of this result uses the fixed point argument developed in [Z4] and the observability inequality (2.22) with an explicit bound of the form (2.23).

(d) The method of proof of Theorem 2.3 may be applied also in several space dimensions. However we do not know if the observability inequality (2.22) holds with a constant of the order of $\exp(C \| a \|_\infty^{1/2})$ in several space dimensions.

(e) The proof of Theorem 2.3 is not adapted to take into account whether the non-linearity h satisfies the "good-sign" condition (2.29) or not.

■

When the non-linearity does not satisfy the conditions of the results above, combining the exact controllability of the linearized system and the Inverse Function Theorem one can prove a controllability result of local nature:

Theorem 2.4 *Let* Ω, ω *and* T *be such that the exact controllability of the linear system* (2.1) *holds. Assume that the non-linearity* h *is of class* $C^1, h(0) = 0$, *and that the growth conditions* (2.25)-(2.26) *are satisfied. Then, there exists* $\delta > 0$ *such that for any* (u^0, u^1), $(v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *with*

$$\| \left(u^0, u^1\right) \|_{H_0^1(\Omega) \times L^2(\Omega)} + \| \left(v^0, v^1\right) \|_{H_0^1(\Omega) \times L^2(\Omega)} \leq \delta \tag{2.30}$$

there exists a control $f \in L^2(Q)$ *such that the solution of* (2.24) *satisfies*

$$u(x, T) = v^0(x), \; u_t(x, T) = v^1(x) \; \text{in } \Omega. \tag{2.31}$$

Remark 2.6 Observe that the control time of the small data satisfying (2.30) is the same as for the linear wave equation. This is due to the fact that, as we said above, the result is proved by means of the Inverse Function Theorem by linearizing around the equilibrium $u \equiv 0$.

■

When the non-linearity does not satisfy a "good-sign" condition of the form (2.28) the result is sharp. Indeed, if one considers non-linearities that grow superlinearly at infinity blow-up may occur for large data and therefore the system may not be exactly controllable due to the finite speed of propagation. However, when the non-linearity satisfies a condition like (2.28) the local controllability result may be completed by means of stabilization results.

Let us explain this in the following model example of an homogeneous non-linearity:

$$\begin{cases} u_{tt} - \Delta u + \mid u \mid^{p-1} u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x) & \text{in} \quad \Omega, \end{cases} \tag{2.32}$$

with $1 < p \leq n/(n-2)$ when $n \geq 3$ and $1 < p < \infty$ when $n = 1, 2$.

The following holds:

Theorem 2.5 *Let ω be a neighborhood of Γ in Ω. Then, if p is a above, for every (u^0, u^1), $(v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$ if T is large enough of the order of*

$$T = O\left(\log\left(\left\|\left(u^0, u^1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \left\|\left(v^0, v^1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}\right)\right) \tag{2.33}$$

for large data, there exists $f \in L^2(Q)$ such that the solution of (2.32) satisfies

$$u(x,T) = v^0(x),\ u_t(x,T) = v^1(x) \text{ in } \Omega. \tag{2.34}$$

Whether system (2.32) is exactly controllable in an uniform time or not is an open problem.

The proof of Theorem 2.5 combines the local result of Theorem 2.4 and the fact that solutions of the following locally damped wave equation decay uniformly as $t \to \infty$:

$$\begin{cases} u_{tt} - \Delta u + \mid u \mid^{p-1} u + 1_\omega u_t = 0 & \text{in} \quad \Omega \times I\!R^+ \\ u = 0 & \text{on} \quad \Gamma \times I\!R^+ \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{2.35}$$

Let us discuss briefly the decay properties of (2.35).

The energy of (2.35) is given by

$$E(t) = \frac{1}{2}\int_\Omega \left[\mid u_t(x,t) \mid^2 + \mid \nabla u(x,t) \mid^2\right] dx + \frac{1}{p+1}\int_\Omega \mid u(x,t) \mid^{p+1} dx. \tag{2.36}$$

It follows that

$$\frac{dE}{dt}(t) = -\int_{\omega} \mid u_t(x,t) \mid^2 dx \leq 0. \tag{2.37}$$

Thus, the energy of solutions decreases as time increases. More precisely:

Theorem 2.6 *When ω is a neighborhood of the boundary Γ and $1 < p$, $(n-2)p \leq n$, there exist positive constants $C, \alpha > 0$ such that*

$$E(t) \leq Ce^{-\alpha t}E(0),\ \forall t > 0, \tag{2.38}$$

for every solution of (2.35).

It is easy to see that Theorem 2.5 is a consequence of Theorems 2.4 and 2.6.

The stabilization result of Theorem 2.6 was proved in [Z1] by multiplier techniques. As we said above multipliers are a rather common tool for handeling control problems but also in the theory of stabilizability (we refer to [K] and [La] for an introduction to this topic).

The decay property (2.38) is the same as the one that holds for the linear dissipative wave equation. In particular the constants C and α in (2.38) do not depend on the initial data. The obtention of (2.38) requires structural conditions on the nonlinearity and applies in the particular example (2.35) and slight variations of it. Much more general equations and non-linearities can be handeled if one relaxes the decay requirement (2.38) to the following: For every $R > 0$ there exists $C(R), \alpha(R) > 0$ such that (2.38) holds for every initial data with energy $E(0) \leq R$.

This type of result can be proved easily as a consequence of the uniform decay of the linearized system and a compactness argument. However, according to this type of result the rate of decay may vanish as $E(0) \to \infty$ and therefore, in this way, one can not obtain explicit bounds on the controllability tme of the form (2.33).

3 The heat equation.

3.1 Problem formulation.

With the same notations as in section 2 we consider the linear controlled heat equation:

$$\begin{cases} u_t - \Delta u = f 1_{\omega} & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{3.1}$$

We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (3.1) admits an unique solution $u \in C\left([0,T]\,; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right)$.

We set $R(T;u^0) = \left\{u(T) : f \in L^2(Q)\right\}$. The controllability problems can be formulated as follows:

(a) System (3.1) is said to be approximately controllable if $R(T; u^0)$ is dense in $L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.

(b) System (3.1) is exactly controllable if $R(T; u^0) = L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.

(c) System (3.1) is null controllable if $0 \in R(T; u^0)$ for all $u^0 \in L^2(\Omega)$.

Remark 3.1

(a) As we will see in the next section approximate controllability holds for every open non-empty subset ω of Ω and for every $T > 0$.

(b) It is easy to see that exact controllability may not hold except possibly in the case in which $\omega = \Omega$. Indeed, due to the regularizing effect of the heat equation solutions of (3.1) at time $t = T$ are smooth in $\Omega \backslash \overline{\omega}$. Therefore $R(T; u^0)$ is strictly contained in $L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.

(c) Null controllability implies that all the range of the semigroup generated by the heat equation is reachable too. Let us make this statement more precise.

Let us denote by $S(t)$ the semigroup generated by the heat equation (3.1) without control, i.e. with $f = 0$. Then, if null controllability holds, it follows that for any $u^0 \in L^2(\Omega)$ and $v^0 \in S(T)[L^2(\Omega)]$, there exists a control $f \in L^2(Q)$ such that the solution of (3.1) satisfies $u(T) = v^0$. In other words, $S(T)\left[L^2(\Omega)\right] \subset R(T; u^0)$ for all $u^0 \in L^2(\Omega)$.

To see this, given $u^0 \in L^2(\Omega)$ and $v^0 \in S(T)\left[L^2(\Omega)\right]$ there exists $w^0 \in L^2(\Omega)$ such that $v^0 = S(T)w^0$. We define $w = S(t)w^0$, the solution of (3.1) with $f = 0$ and initial data w^0. Then, the solution u of (3.1) can be decomposed as $u = w + v$ where

$$\left\{ \begin{array}{lll} v_t - \Delta v = f 1_\omega & \text{in} & Q \\ v = 0 & \text{on} & \Sigma \\ v(0) = u^0 - w^0. & & \end{array} \right. \tag{3.2}$$

Since system (3.1) is null controllable there exists $f \in L^2(Q)$ such the solution v of (3.2) satisfies $v(T) = 0$ and this is equivalent to $u(T) = w(T) = v^0$.

The set $S(T)[L^2(\Omega)]$ is dense in $L^2(\Omega)$ and therefore nulll controllability implies approximate controllability. Observe however that the reachable states we obtain by this argument are smooth due to the regularizing effect of the heat equation.

■

3.2 Approximate controllability.

System (3.1) is approximately controllable for any open, non-empty subset ω of Ω and $T > 0$. To see this one can apply Hahn-Banach's Theorem or use the variational approach developed in section 2.2 for the wave equation. In both cases the approximate controllability is reduced to the unique continuation property of the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{3.3}$$

More precisely, approximate controllability holds if and only if the following uniqueness property is true: *If* φ *solves* (3.3) *and* $\varphi = 0$ *in* $\omega \times (0,T)$ *then, necessarily,* $\varphi \equiv 0$, i.e. $\varphi^0 \equiv 0$.

This uniqueness property holds for every open non-empty subset ω of Ω and $T > 0$ by Holmgren's Uniqueness Theorem.

Following the variational approach of section 2.2 the control can be constructed as follows. First of all we observe that it is sufficient to consider the particular case $u^0 \equiv 0$. Then, for any v^0 on $L^2(\Omega), \varepsilon > 0$ and E finite-dimensional subspace of $L^2(\Omega)$ we introduce the functional

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \varepsilon \left\| (I - \pi_E)\varphi^0 \right\|_{L^2(\Omega)} - \int_\Omega \varphi^0 v^0 dx \tag{3.4}$$

where π_E denotes the orthogonal projection from $L^2(\Omega)$ over E.

The functional J_ε is continuous and convex in $L^2(\Omega)$. On the other hand, in view of the unique continuation property above one can prove that

$$\lim_{\|\varphi^0\|_{L^2(\Omega)} \to \infty} \frac{J_\varepsilon(\varphi^0)}{\| \varphi^0 \|_{L^2(\Omega)}} \geq \varepsilon \tag{3.5}$$

(we refer to [Z6] for the details of the proof).

Then, J_ε admits an unique minimizer $\widehat{\varphi}^0$ in $L^2(\Omega)$. The control $f = \widehat{\varphi}$ where $\widehat{\varphi}$ solves (3.3) with $\widehat{\varphi}^0$ as data is such that the solution u of (3.1) with $u^0 = 0$ satisfies

$$\| u(T) - u^0 \|_{L^2(\Omega)} \leq \varepsilon,\ \pi_E\,(u(T)) = \pi_E(v^0). \tag{3.6}$$

A slight change on the functional J_ε allows to build *bang-bang* controls. Indeed, we set

$$\widetilde{J}_\varepsilon(\varphi^0) = \frac{1}{2}\left(\int_0^T \int_\omega \mid \varphi \mid dxdt\right)^2 + \varepsilon \parallel (I - \pi_E)\varphi^0 \parallel_{L^2(\Omega)} - \int_\Omega v^0 \varphi^0 dx. \tag{3.7}$$

The functional $\widetilde{J}_\varepsilon$ is continuous and convex in $L^2(\Omega)$ and satisfies the coercivity property (3.5) too.

Let $\widehat{\varphi}^0$ be the minimizer of $\widetilde{J}_\varepsilon$ in $L^2(\Omega)$ and $\widehat{\varphi}$ the corresponding solution of (3.3). We set

$$f = \int_0^T \int_\omega | \widehat{\varphi} | \, dxdt \, \mathrm{sgn}(\widehat{\varphi}) \tag{3.8}$$

where sgn is the multivalued sign function: $\mathrm{sgn}(s) = 1$ if $s > 0$, $\mathrm{sng}(0) = [-1, 1]$ and $\mathrm{sng}(s) = -1$ when $s < 0$. The control f given in (3.8) is such that the solution u of (3.1) with null initial data satisfies (3.6).

Due to the regularizing effect of the heat equation the zero set of non-trivial solutions of (3.3) is of zero $(n+1)-$dimensional Lebesgue measure. Thus, the control f of (3.8) is of *bang-bang* form, i.e. $f = \pm\lambda$ a.e. in Q where $\lambda = \int_0^T \int_\omega | \widehat{\varphi} | \, dxdt$.

We have proved the following:

Theorem 3.1 *Let ω be any open non-empty subset of Ω and $T > 0$ be any positive control time. Then, for any $u^0, v^0 \in L^2(\Omega)$, $\varepsilon > 0$ and E finite-dimensional subspace of $L^2(\Omega)$ there exists a bang-bang control $f \in L^2(Q)$ such that the solution of* (3.1) *satisfies* (3.6).

Remark 3.2 The control (3.8) obtained by minimizing $\widetilde{J}_\varepsilon$ is the one of minimal $L^\infty(Q)$-norm among the admissible ones (we refer to [FPZ2] for the details of the proof in the particular case where $E = \{0\}$).

■

3.3 Null controllability.

The null controllability problem for system (3.1) is equivalent to the following observability inequality for the adjoint system (3.3):

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{3.9}$$

Due to the regularizing effect of the heat equation the norm in the left hand side of (3.9) is rather weak. However, due to the irreversibility of the system, (3.9) is not easy to prove. For instance, multiplier methods do not apply.

In [R1] the boundary null controllability of the heat equation was proved in one space dimension using moment problems and classical results on the linear independence in $L^2(0, T)$ of families of real exponentials.

Later on in [R2] a deep general result was proved. Roughly speaking it was shown that *if the wave equation is controllable for some $T > 0$ with controls supported in ω, then the heat equation* (3.1) *is null controllable for all $T > 0$ with controls supported in ω.* As a consequence of this and in view of the controllability result presented in section 2.3 it follows that the heat equation (3.1) is null controllable for all $T > 0$ provided ω satisfies the geometric control condition.

However, the geometric control condition does not seem to be natural at all in the context of the heat equation and therefore this result is not totally satisfactory.

More recently G. Lebeau and L. Robbiano [LeR] have proved that *the heat equation* (3.1) *is null controllable for every open, non-empty subset* ω *of* Ω *and* $T > 0$. This result shows, as expected, that the geometric control condition is unnecessary in the context of the heat equation.

A slightly simplified proof of this result was given in [LeZ] where the linear system of thermoelasticity was addressed. Let us describe briefly this proof. The main ingredient of it is an observability estimate for the eigenfunctions of the Laplace operator:

$$\begin{cases} -\Delta\psi_j = \lambda_j\psi_j & \text{in} \quad \Omega \\ \psi_j = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.10}$$

Recall that the eigenvalues $\{\lambda_j\}$ form an increasing sequence of positive numbers such that $\lambda_j \to \infty$ as $j \to \infty$ and that the eigenfunctions $\{\psi_j\}$ may be chosen such that they form an orthonormal basis of $L^2(\Omega)$.

The following holds:

Theorem 3.2 *For any open subset* ω *of* Ω *there exists positive constants* $C_1, C_2 > 0$ *such that*

$$\int_\omega \left| \sum_{\lambda_j \le \mu} a_j\psi_j(x) \right|^2 dx \ge C_1 e^{-C_2\sqrt{\mu}} \sum_{\lambda_j \le \mu} \mid a_j \mid^2 \tag{3.11}$$

for all $\{a_j\} \in \ell^2$ *and for all* $\mu > 0$.

This result was implicitly used in [LeR] and is proved in [LeZ] by means of Carleman's inequalities.

As a consequence of (3.11) one can prove that the observability inequality (3.9) holds for solutions of (3.3) with initial data in $E_\mu = \text{span}\{\varphi_j\}_{\lambda_j \le \mu}$, the constant being of the order of $\exp\left(C\sqrt{\mu}\right)$. This shows that the projection of solutions over E_μ can be controlled to zero with a control of size $\exp\left(C\sqrt{\mu}\right)$. Thus, when controlling the frequencies $\lambda_j \le \mu$ one increases the $L^2(\Omega)$-norm of the high frequencies $\lambda_j > \mu$ by a multiplicative factor of the order of $\exp\left(C\sqrt{\mu}\right)$. However, as it was observed in [LeR], solutions of the heat equation (3.1) without control ($f = 0$) and such that the projection of the initial data over E_μ vanishes, decay in $L^2(\Omega)$ at a rate of the order of $\exp(-\mu t)$.

Thus, if we divide the time interval $[0, T]$ in two parts $[0, T/2]$ and $[T/2, T]$, we control to zero the frequencies $\lambda_j \le \mu$ in the interval $[0, T/2]$ and then allow the equation to evolve without control in the interval $[T/2, T]$, it follows that at time $t = T$ the projection of the solution u over E_μ vanishes and the norm of the high frequencies does not exceed the norm of the initial data u^0.

This argument allows to control to zero the projection over E_μ for any $\mu > 0$ but not the whole solution. To do that an iterative argument is needed. We decompose the interval $[0, T)$ in disjoint subintervals of the form $[T_j, T_{j+1})$ for $j \in \mathbb{N}$ with a suitable choice of the sequence $\{T_j\}$. In each interval $[T_j, T_{j+1}]$ we control to zero the frequencies $\lambda_k \leq 2j$. By letting $j \to \infty$ we obtain a control $f \in L^2(Q)$ such that the solution of (3.1) satisfies $u(T) \equiv 0$.

Remark 3.3

(a) Observe that the null controllability has not been obtained as a consequence of the observability inequality (3.9). Actually, (3.9) is derived *a fortiori* as a consequence of the null controllability property.

(b) Once (3.9) is known to hold one can obtain the control with minimal $L^2(Q)$-norm among the admissible ones. To do that it is sufficient to minimize the functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0)u^0 dx \tag{3.12}$$

over $L^2(\Omega)$.

Observe that J is continuous and convex in $L^2(\Omega)$. On the other hand (3.9) guarantees the coercivity of J and the existence of its minimizer.

(c) Once we know that (3.9) holds for any open non-empty subset ω and $T > 0$, by the regularizing effect of the heat equation we deduce that for all $k \in \mathbb{N}$ and $\delta > 0$ there exists $C_k > 0$ such that

$$\| \varphi(0) \|^2_{H^k(\Omega)} \leq C_k \int_\delta^{T-\delta} \int_\omega \varphi^2 dxdt. \tag{3.13}$$

This allows to prove that the null controllability may be achieved by means of C^∞-controls.

(d) As a consequence of the internal null controllability of the heat equation one can deduce easily the null boundary controllability with controls in an arbitrarily small open subset of the boundary. Indeed, let Γ_0 be an open and non-empty subset of $\Gamma = \partial\Omega$. We can extend Ω to a larger open connected set $\widetilde{\Omega}$ such that $\widetilde{\Omega} = \Omega \cup \Gamma_0 \cup \omega$ where ω is a non-empty open set of $\mathbb{R}^n \backslash \Omega$. Given $u^0 \in L^2(\Omega)$ we extend it to $\widetilde{\Omega}$ by setting $\widetilde{u}^0 = 0$ in $\widetilde{\Omega} \backslash \Omega$. Then one can control the initial data $\widetilde{u}^0$ for the heat equation in the cylinder $\widetilde{Q} = \widetilde{\Omega} \times (0,T)$, with control f supported in $\omega \times (0,T)$. The restriction of the controlled solution to $Q = \Omega \times (0,T)$ satisfies

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma_1 = (\Gamma \backslash \Gamma_0) \times (0,T) \\ u = v & \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,T) \\ u(x,0) = u^0(x) & \text{in} \quad \Omega \\ u(x,T) = 0 & \text{in} \quad \Omega \end{cases} \tag{3.14}$$

with control v, the restriction of $\widetilde{u}$ to $\Sigma_0 = \Gamma_0 \times (0,T)$.

As a consequence of the boundary null controllability we deduce the following boundary observability property:

$$\| \varphi(0) \|_{L^2(\Omega)}^2 \leq C \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma dt. \tag{3.15}$$

This observability property (3.15) was derived in [LeR] by using Carleman's inequalities and the iterative argument described above to obtain the null controllability.

(e) The method of proof of the null controllability we have described is based on the possibility of developing solutions in Fourier series. Thus it can be applied in a more general class of heat equations with variable time-independent coefficients. The methods of [R2] apply to variable coefficient equations as well but are also limitated to time independent coefficients.

(f) The null controllability of the heat equation with lower order time-dependent terms of the form

$$\begin{cases} u_t - \Delta u + a(x,t)u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega \end{cases} \tag{3.16}$$

has been studied by A. Fursikov and O. Yu. Imanuvilov (see for instance [CIK], [F], [FI1,2] and [I]). Their approach is different to the one we have presented here. As a consequence of their null controllability results it follows that an observability inequality of the form (3.15) holds for the solution of the adjoint system

$$\begin{cases} -\varphi_t - \Delta \varphi + a(x,t)\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega \end{cases} \tag{3.17}$$

when Γ_0 is any open, non-empty subset of Γ.

As a consequence of this and proceeding as in [L1], vol. 1, (where it is shown that, in general, boundary observability implies internal observability when ω is a neighborhood of the boundary) one can deduce that (3.9) holds when ω is any open and non-empty subset of Ω.

Thus, system (3.16) is null-controllable for any $a \in L^\infty(Q)$ when ω is any open and non-empty subset of Ω and for any $T > 0$.

(g) More recently, E. Fernández-Cara [FC], following the methods of Fursikov and Imanuvilov, has shown that an observability inequality of the form (3.15) holds for solutions of system (3.17) with a constant C of the order of $\exp(C||a||_\infty)$.

■

3.4 The semilinear heat equation.

Let us consider the following semilinear heat equation

$$\begin{cases} u_t - \Delta u + h(u) = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega \end{cases} \tag{3.18}$$

where $h \in C^1(I\!R; I\!R)$ and satisfies

$$\left| h(s_1) - h(s_2) - h'(0)\,(s_1 - s_2) \right| \leq C \left(\mid s_1 \mid^{p-1} + \mid s_2 \mid^{p-1} \right) \mid s_1 - s_2 \mid, \; \forall s_1, s_2 \in I\!R \tag{3.19}$$

for some $C > 0$ with $1 < p \leq (n+4)/n$.

Under assumption (3.19) system (3.18) is locally well posed for $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$. More precisely there exists $\delta > 0$ such that system (3.18) has an unique solution $u \in C\left([0,T]; L^2(\Omega)\right)$ for any $(u^0, f) \in L^2(\Omega) \times L^2(Q)$ such that

$$\| u^0 \|_{L^2(\Omega)} + \| f \|_{L^2(Q)} \leq \delta. \tag{3.20}$$

On the other hand, when h is globally Lipschitz, i.e. when (3.19) holds with $p = 1$ system (3.18) is globally well posed and an unique solution exists for any initial data $u^0 \in L^2(\Omega)$ and control $f \in L^2(Q)$.

The following was proved in [Z6]:

Theorem 3.3 *Assume that h is globally Lipschitz. Let ω be any open non-empty subset Ω and $T > 0$. Then, for any $u^0, v^0 \in L^2(\Omega), \varepsilon > 0$ and E finite-dimensional subspace of $L^2(\Omega)$ there exists $f \in L^2(Q)$ such that the solution u of* (3.18) *satisfies* (3.6).

Remark 3.4

(a) This result was proved in [FPZ1] in the context of approximate controllability, i.e. when $E = \{0\}$. The method of proof is rather similar in both cases. As we will see it is based on the variational approach to approximate controllability described in section 3.2 and a fixed point argument that was introduced in [Z4] to prove the exact controllability of the semilinear wave equation.

(b) The method of proof of Theorem 3.3 does not apply when the nonlinearity grows at infinity in a superlinear way, i.e. when (3.19) holds with $p > 1$. This is due to the fact that the method of proof does not provide any estimate on the control in terms of the Lipschitz constant of the nonlinearity h.

■

Sketch of the proof of Theorem 3.3.

Let us fix $u^0, v^0 \in L^2(\Omega), \varepsilon > 0$ and the finite-dimensional subspace E of $L^2(\Omega)$.

For simplicity we assume that $u^0 = 0$ (notice that since the system under consideration is not linear, the problem can not be reduced to the particular case $u^0 = 0$. However the difficulties are similar in all cases). We also assume that $h(0) = 0$.

We now observe that the heat equation in (3.18) can be written as follows:

$$u_t - \Delta u + g(u)u = f 1_\omega \tag{3.21}$$

where

$$g(s) = h(s)/s \ \text{ if } \ s \neq 0;\ g(0) = h'(0). \tag{3.22}$$

Since h is globally Lipschitz the function g is bounded and

$$\| g \|_{L^\infty(I\!R)} \leq \| h' \|_{L^\infty(I\!R)} . \tag{3.23}$$

For any $w \in L^2(Q)$ we consider the linear system

$$\begin{cases} u_t - \Delta u + g(w)u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{3.24}$$

Notice that the potential $g(w) \in L^\infty(Q)$ and that

$$\| g(w) \|_{L^\infty(Q)} \leq \| g \|_{L^\infty(I\!R)}, \ \forall w \in L^2(Q). \tag{3.25}$$

We look for $f \in L^2(Q)$ such that the solution of (3.24) satisfies (3.6). Of course the control f will depend on w.

To construct this control we introduce the functional

$$J_{\varepsilon,w}(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \varepsilon \parallel (I - \pi_E)\varphi^0 \parallel_{L^2(\Omega)} - \int_\Omega \varphi^0 v^0 dx \tag{3.26}$$

where $\varphi = \varphi(x,t)$ solves the adjoint equation

$$\begin{cases} -\varphi_t - \Delta\varphi + g(w)\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{3.27}$$

The functional $J_{\varepsilon,w}$ is convex and continuous in $L^2(\Omega)$. On the other hand the unique continuation property holds for solutions of (3.27), i.e. *if* φ *solves* (3.27) *and* $\varphi = 0$ *in* $\omega \times (0,T)$, *then* $\varphi \equiv 0$ *and* $\varphi^0 \equiv 0$ (see [SS]). Then $J_{\varepsilon,w}$ is coercive in $L^2(\Omega)$ and satisfies (3.5). Thus, $J_{\varepsilon,w}$ admits an unique minimizer $\widehat{\varphi}^0$ in $L^2(\Omega)$. The control $f = \widehat{\varphi}$ where $\widehat{\varphi}$ solves (3.27) with this data $\widehat{\varphi}^0$, is such that the solution of (3.24) satisfies (3.6).

We have constructed a map $N : L^2(Q) \to L^2(Q)$ such that $N(w) = u$, i.e. to each $w \in L^2(Q)$ we associate the solution u of (3.24) satisfying (3.6) with the control f obtained by minimizing the functional $J_{\varepsilon,w}$. It is easy to see that N is continuous and compact.

On the other hand, it can be proved that the functionals $J_{\varepsilon,w}$ are uniformly coercive and therefore the minimizers $\widehat{\varphi}^0$ are uniformly bounded in $L^2(\Omega)$ for all $w \in L^2(Q)$. This is due to the fact that the potentials $g(w)$ are uniformly bounded in $L^\infty(Q)$. Then, the controls f are uniformly bounded in $L^2(Q)$ and the solutions u of (3.24) too. This shows that the range of N is bounded in $L^2(Q)$.

Thus, we can apply Schauder's fixed point Theorem to N and deduce that it admits a fixed point u. This fixed point solves (3.18) with $u^0 \equiv 0$ and, by construction, satisfies (3.6).

■

Remark 3.5 In the proof of Theorem 3.3 we can consider the functional $\widetilde{J}_\varepsilon$ as in (3.7) and deduce that the control f can be constructed to be of the form (3.8) with $\widehat{\varphi}$ solution of (3.27) for some $w \in L^2(Q)$. However to our knowledge it is not known whether the zero set of solutions of (3.27) is of zero $(n+1)$-dimensional Lebesgue measure.

■

As we said above the method of proof of Theorem 3.3 does not apply when $p > 1$. However one can apply the Inverse Function Theorem if the control requirement (3.6) is relaxed to

$$\pi_E(u(T)) = \pi_E(v^0), \tag{3.28}$$

i.e. if we only control finite-dimensional projections. Indeed, in this setting one can prove that for any ω non-empty open subset of $\Omega, T > 0$ and finite-dimensional subspace E of $L^2(\Omega)$ there exists $\delta > 0$ such that for any $u^0, v^0 \in L^2(\Omega)$ with

$$\| u^0 \|_{L^2(\Omega)} + \| v^0 \|_{L^2(\Omega)} \leq \delta \tag{3.29}$$

there exists a control $f \in L^2(Q)$ such that the solution of (3.18) satisfies (3.28).

When the non-linearity has the "good-sign", for instance, when dealing with

$$\begin{cases} u_t - \Delta u + | u |^{p-1} u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega \end{cases} \tag{3.30}$$

and taking into account that solutions of (3.30) without control ($f = 0$) decay exponentially in $L^2(\Omega)$ as t increases one deduces that *for any* $u^0 \in L^2(\Omega)$ *if* $T > 0$ *is large enough of the order of* $\log\left(\| u^0 \|_{L^2(\Omega)}\right)$ *there exists* $f \in L^2(Q)$ *such that the solution of* (3.30) *satisfies*

$$\pi_E(u(T)) = 0. \tag{3.31}$$

Remark 3.6

(a) This method does not allow to say anything about the controllability of the projection over $E^{\perp}$, the orthogonal complement of E.

(b) To our knowledge nothing is known about the controllability of blowing-up solutions. For instance let us consider the pais $\{u, f\}$ such that

$$u_t - \Delta u - \mid u \mid^{p-1} u = f 1_{\omega}, \quad \text{in} \quad \Omega \times (0, T)$$

with $p > 1$, $f \in L^2(Q)$, satisfying an initial condition $u(0) = u^0 \in L^2(\Omega)$ and homogeneous Dirichlet boundary conditions.

Observe that for some choices of u^0, the solution of this problem for $f \equiv 0$ may blow-up in time $t \leq T$. Thus, the existence of admissible pairs $\{u, f\}$ is not always obvious.

One of the problems we may consider in this setting is whether the set of $\{u(T)\}$ where $\{u, f\}$ runs over the set of admissible pairs is dense in $L^2(\Omega)$ or not.

As we mentioned in section 2.4, in the context of the wave equation, due to the finite speed of propagation blowing up solutions may not be avoided, the trivial case in which $\omega = \Omega$ being excepted and therefore, for some choices of the initial data, the set of admissible pairs $\{u, f\}$ is empty. However, the possibility of avoiding the blow up for a suitable choice of f may not be excluded in the frame of the heat equation.

■

Recently some local null controllability results have been proved. Lin Guo and Littman in [LiL] by means of a nonlinear version of Cauchy Kovalevski's Theorem have proved that, in one space dimension and when the control acts on one extreme of the boundary, null controllability holds for bounded continuous and sufficiently small initial data provided the nonlinearity is analytic and belongs to Gevrey class 2. On the other hand, Fursikov and Imanuvilov in [FI1] have proved the local null controllability for the semilinear heat equation (3.18) by using Carleman's inequalities. In both cases, when the nonlinearity has the good sign as in (3.30) it can be shown that every initial data u^0 can be driven to zero in a time of the order of $\log \left(\| u^0 \|_{L^2(\Omega)} \right)$. More recently E. Fernández-Cara [FC] has shown that the semilinear heat equation (3.18) is null controllable in an uniform time T for any $T > 0$, provided

$$\mid h(s) \mid / \mid s \mid \log(\mid s \mid) \to 0, \qquad \text{as} \quad |s| \to \infty. \tag{3.32}$$

This is the analogue of he exact controllability result for the semilinear wave equation of Theroem 2.3. Observe that (3.32) is an almost sharp condition to avoid blow-up phenomena. Indeed, when $h(s) \sim -s log^p(|s|)$ as $|s| \to \infty$ with $p > 1$, blow-up may occur. The method of proof of [FC] combines a fixed point argument and the explicit observability estimates mentioned in Remark 3.3, (g).

To our knowledge the only controllability result for a non-linear system with superlinear nonlinearity that holds for all initial data in an uniform time is the one due to J.-M. Coron [Co2] in the context of Navier-Stokes equations in two space dimensions. The results in [Co2] state the approximate controllability of all data in an arbitrarily small control time and are obtained through an approximation argument as a consequence of the exact controllability of Euler equations in the class of smooth data proved in [Co1]. The approach by J.-M. Coron is completely different to the classical one that views the nonlinear control problem as a perturbation of the linear one. In [Co2] the Navier-Stokes equations are viewed as a perturbation of Euler equations and the methods employed are genuinely nonlinear.

4 The linear system of thermoelasticity.

4.1 Problem formulation.

Let Ω be a bounded domain of $I\!R^3$ with boundary Γ and let ω be an open non-empty subset of Ω.

Let us consider the system

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\nabla\theta = f1_\omega & \text{in} \quad Q \\ \theta_t - \Delta\theta + \beta \operatorname{div} u_t = 0 & \text{in} \quad Q \\ u = 0\,,\ \theta = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x) & \text{in} \quad \Omega \\ \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega. \end{cases} \tag{4.1}$$

In (4.1), $u = (u_1, u_2, u_3)$ describes the displacement and θ the temperature. λ and μ are Lamé's constants and, as well as the coupling constants α and β, are assumed to be positive. The control $f = (f_1, f_2, f_3)$ acts on the system as a volume force.

Notice that we have not introduced any control in the heat equation. Thus, we try to control both displacement and temperature by means of the sole control f. Of course, this requires the system to be strongly coupled.

System (4.1) combines the hyperbolic nature of Lamé's system in linear elasticity and the parabolic character of the heat equation.

System (4.1) is well-posed for $(u^0, u^1, \theta^0) \in H = \left(H_0^1(\Omega)\right)^3 \times \left(L^2(\Omega)\right)^3 \times L^2(\Omega)$ and $f \in \left(L^2(Q)\right)^3$ and admits an unique solution in the class $(u, u_t, \theta) \in C\left([0,T]; H\right)$.

The different notions of controllability make sense for this system too. However, the exact controllability has to be excluded by the irreversible character of the system. Approximate controllability can easily be deduced when T is large enough as a consequence of Holmgren's Uniqueness Theorem.

Other notions of controllability can also be introduced. For instance, Lagnese and Lions in [LaL] proved *partial controllability* results for various systems of thermoelasticity.

System (4.1) is said to be partially controllable if for every $(u^0, u^1, \theta^0) \in H$ there exists $f \in (L^2(Q))^3$ such that the solution of (4.1) satisfies

$$u(x,T) = u_t(x,T) = 0 \text{ in } \Omega.$$

In other words, partial controllability guarantees that the displacement u can be driven to zero but nothing is said about the temperature.

In [LaL] it was shown that when ω and T are such that Lamé's system is exactly controllable then system (4.1) is partially controllable when α and β are sufficiently small.

More recently in [Z2] the following notion of *exact-approximate controllability* was introduced: System (4.1) is said to be exact-approximately controllable if for any (u^0, u^1, θ^0), $(v^0, v^1, \eta^0) \in H$ and $\varepsilon > 0$ there exists $f \in (L^2(Q))^3$ such that the solution of (4.1) satisfies

$$\left\{ \begin{array}{l} u(x,T) = v^0(x),\ u_t(x,T) = v^1(x) \quad \text{in} \quad \Omega \\ \| \theta(T) - \eta^0 \|_{L^2(\Omega)} \leq \varepsilon. \end{array} \right. \tag{4.2}$$

Observe that in (4.2) the displacement is required to reach exactly (v^0, v^1) while the temperature only reaches the target in an approximate way.

Clearly exact-approximate controllability is a stronger notion than partial controllability. On the other hand one expects null controllability to be stronger than exact-approximate controllability. However to prove this fact one needs the backward uniqueness for the system of thermoelasticity (4.1) with $f = 0$ and, to our knowledge, this is an open problem.

In section 4.2 we will briefly describe the results from [Z2] on exact-approximate controllability. In section 4.3 we mention the results from [LeZ] on null controllability that apply when an extra control is added in the heat equation or when the Dirichlet boundary conditions are replaced by periodic ones.

One expects the system (4.1) to be null controllable when ω and T are such that the Lamé system is exactly controllable. However this is by now an open problem.

We will not describe here some other relevant results on the controllability of systems in thermoelasticity. For instance the work by S. Hansen [H] shows that the system of thermoelasticity in one space dimension is null boundary controllable. The methods of [H] are different from ours and are based on the reduction of the control problem to a moment problem and the theory of nonharmonic Fourier series.

4.2 Exact-approximate controllability.

The following result was proved in [Z2]:

Theorem 4.1 *Let Ω be a bounded domain of $\mathbb{R}^3$ with boundary Γ of class C^2. Assume that ω is a neighborhood of Γ in Ω and $T > \mathrm{diam}(\Omega \backslash \omega) / \sqrt{\mu}$.*

Then, system (4.1) *is exact-approximately controllable,* i.e. *for every (u^0, u^1, θ^0), $(v^0, v^1, \eta^0) \in H$ and $\varepsilon > 0$ there exists $f \in (L^2(Q))^3$ such that the solution of* (4.1) *satisfies* (4.2)

In [Z2] this result was reduced to a suitable observability inequality for the adjoint system:

$$\left\{\begin{array}{lll}\varphi_{tt}-\mu\Delta\varphi-(\lambda+\mu)\nabla\operatorname{div}\varphi+\beta\nabla\psi_t=0 & \text{in} & Q\\ -\psi_t-\Delta\psi-\alpha\operatorname{div}\varphi=0 & \text{in} & Q\\ \varphi=0,\psi=0 & \text{on} & \Sigma\\ \varphi(x,T)=\varphi^0(x),\varphi_t(x,T)=\varphi^1(x) & \text{in} & \Omega\\ \psi(x,T)=\psi^0(x) & \text{in} & \Omega.\end{array}\right. \tag{4.3}$$

More precisely it was shown that Theorem 4.1 is a consequence of the following two results:

Theorem 4.2 *Under the assumptions of Theorem 4.1, for every bounded set B of $L^2(\Omega)$ there exists $\delta=\delta(B)>0$ such that*

$$\delta\leq\int_0^T\int_\omega|\varphi|^2\,dxdt \tag{4.4}$$

holds for every solution of (4.3) *with initial data such that*

$$\|(\varphi^0,\varphi^1+\beta\nabla\psi^0)\|_{(L^2(\Omega))^3\times(H^{-1}(\Omega))^3}\geq 1,\psi^0\in B. \tag{4.5}$$

Theorem 4.3 *Under the assumptions of Theorem* 4.1*, if $\varphi=0$ in $\omega\times(0,T)$ then, necessarily, $\varphi^0\equiv\varphi^1\equiv 0$ and $\psi^0\equiv 0$ in Ω.*

Remark 4.1.

(a) The partial controllability result of [LaL] mentioned in section 4.1 is a consequence of the observability estimate (4.4) in the particular case $B=\{0\}$. In this case, assuming that $\alpha\beta$ is sufficiently small, (4.4) can be derived as a consequence of the observability of the linear system of elasticity.

Indeed, under the assumptions of Theorem 4.1 it follows that

$$\|(\varphi^0,\varphi^1)\|^2_{(L^2(\Omega))^3\times(H^{-1}(\Omega))^3}\leq C\int_0^T\int_\omega|\varphi|^2\,dxdt \tag{4.6}$$

for every solution of

$$\left\{\begin{array}{lll}\varphi_{tt}-\mu\Delta\varphi-(\lambda+\mu)\nabla\mathrm{div}\varphi=0 & \text{in} & Q\\ \varphi=0 & \text{on} & \Sigma\\ \varphi(x,T)=\varphi^0(x),\varphi_t(x,T)=\varphi^1(x) & \text{in} & \Omega.\end{array}\right. \tag{4.7}$$

From (4.6) it follows that (4.4) holds when $B=\{0\}$ and $\alpha\beta$ is small enough.

(b) In Theorem 4.2 B is any bounded set and no restriction on the product of the coupling parameters $\alpha\beta$ is imposed. Therefore, Theorem 4.2 can not be derived by a perturbation argument from the observability inequality (4.6) in Lamé's system (4.7).

(c) Theorem 4.2 provides an observability estimate that is uniform provided ψ^0 remains a priori bounded. But it does not say anything about the behavior of the constant δ in (4.4) as $\| \psi^0 \|_{L^2(\Omega)} \to \infty$.

Observe however that, due to the irreversibility of system (4.3), an estimate of the form

$$\| \left(\varphi^0, \varphi^1 + \beta \nabla \psi^0, \psi^0\right) \|^2_{(L^2(\Omega))^3 \times (H^{-1}(\Omega))^3 \times L^2(\Omega)} \leq C \int_0^T \int_\omega | \varphi |^2 \, dxdt$$

may not hold for all $(\varphi^0, \varphi^1, \psi^0)$ with a uniform constant C.

However the following estimate is very probably true

$$\| (\varphi(0), \varphi_t + \beta \nabla \psi(0), \psi(0)) \|^2_{(L^2(\Omega))^3 \times (H^{-1}(\Omega))^3 \times L^2(\Omega)} \leq C \int_0^T \int_\omega | \varphi |^2 \, dxdt \quad (4.8)$$

and is equivalent to the null controllability of the system.

(d) Theorem 4.3 is a consequence of Homgren's Uniqueness Theorem and guarantees the approximate controllability of the system.

■

Now, assuming that Theorem 4.2 and 4.3 hold, let us briefly sketch the proof of Theorem 4.1.

First of all we observe that the problem can be reduced to consider the case where $u^0 \equiv u^1 \equiv 0$ and $\theta^0 \equiv 0$. Then, given $(v^0, v^1, \eta^0) \in H$ and $\varepsilon > 0$ we introduce the functional

$$\begin{aligned} J_\varepsilon\left(\varphi^0, \varphi^1, \psi^0\right) &= \frac{1}{2} \int_0^T \int_\omega | \varphi |^2 \, dxdt + \varepsilon \| \psi^0 \|_{L^2(\Omega)} - \\ &- \int_\Omega v^1 \cdot \varphi^0 dx + < v^0, \varphi^1 > - \int_\Omega \left(\eta^0 + \beta \operatorname{div} v^0\right) \psi^0 dx \end{aligned} \quad (4.9)$$

defined for $(\varphi^0, \varphi^1, \psi^0) \in H' = \left(L^2(\Omega)\right)^3 \times \left(H^{-1}(\Omega)\right)^3 \times L^2(\Omega)$ where $\cdot$ denotes the scalar product in $I\!R^3$ and $< \cdot, \cdot >$ the duality pairing between $\left(H_0^1(\Omega)\right)^3$ and $\left(H^{-1}(\Omega)\right)^3$.

In view of Theorems 4.2 and 4.3 one can prove the following coercivity property:

$$\lim_{\|(\varphi^0, \varphi^1 + \beta \nabla \psi^0, \psi^0)\|_{H'} \to \infty} \frac{J_\varepsilon\left(\varphi^0, \varphi^1, \psi^0\right)}{\| \left(\varphi^0, \varphi^1 + \beta \nabla \psi^0, \psi^0\right) \|_{H'}} \geq \varepsilon. \quad (4.10)$$

Since J_ε is continuous and convex in H', from (4.10) it follows that J_ε has a unique minimizer in H' that we denote by $\left(\widehat{\varphi}^0, \widehat{\varphi}^1, \widehat{\psi}^0\right)$. The control $f = \widehat{\varphi}$ where $\left(\widehat{\varphi}, \widehat{\psi}\right)$ solves (4.3) with the minimizer as initial data is such that the solution of (4.1) with null initial data satisfies (4.2).

Remark 4.2. It is easy to see that Theorem 4.1 can be improved to guarantee that the temperature reaches exactly a finite number of constraints as well. In other words, if E is a finite-dimensional subspace of $L^2(\Omega)$ the control f can be chosen such that the solution u of (4.1) satisfies

$$\left\{\begin{array}{lll} u(x,T)=u^0(x),\ u_t(x,T)=u^1(x) & \text{in} & \Omega \\ \parallel \theta(T)-\eta^0 \parallel_{L^2(\Omega)}\leq \varepsilon,\ \pi_E\left(\theta(T)\right)=\pi_E(\eta^0). & & \end{array}\right. \tag{4.11}$$

To see this it is sufficient to replace the functional J_ε above by

$$\widetilde{J}_\varepsilon\left(\varphi^0,\varphi^1,\psi^0\right)=\frac{1}{2}\int_0^T\int_\omega \mid\varphi\mid^2 dxdt+\varepsilon \parallel (I-\pi_E)\psi^0 \parallel_{L^2(\Omega)}$$
$$-\int_\Omega v^1\cdot\varphi^0 dx+<v^0,\varphi^1>-\int_\Omega\left(\eta^0+\beta\operatorname{div}v^0\right)\psi^0 dx.$$

■

As we said above Theorem 4.3 can be proved by means of Holmgren's Uniqueness Theorem. Thus, let us briefly indicate the main steps of the proof of Theorem 4.2.

Step 1. First of all we introduce the new variable

$$\phi(x,t)=-\int_t^T\varphi(x,s)ds+\phi^0(x)$$

with

$$-\mu\Delta\phi^0-(\lambda+\mu)\nabla\operatorname{div}\phi^0=-\varphi^1-\beta\nabla\psi^0 \ \text{ in } \ \Omega;\phi^0\in\left(H_0^1(\Omega)\right)^3$$

and observe that (ϕ,ψ) satisfy

$$\left\{\begin{array}{lll} \phi_{tt}-\mu\Delta\phi-(\lambda+\mu)\nabla\operatorname{div}\phi+\beta\nabla\psi=0 & \text{in} & Q \\ -\psi_t-\Delta\psi-\alpha\operatorname{div}\phi_t=0 & \text{in} & Q \\ \phi=0,\psi=0 & \text{on} & \Sigma \\ \phi(x,T)=\phi^0(x),\phi_t(x,T)=\phi^1(x),\psi(x,T)=\psi^0(x) & \text{in} & \Omega. \end{array}\right. \tag{4.12}$$

The problem is reduced to prove that for every bounded set B of $L^2(\Omega)$ there exists $\delta=\delta(B)>0$ such that

$$\delta\leq\int_0^T\int_\omega \mid\phi_t\mid^2 dxdt \tag{4.13}$$

for every solution of (4.12) with data

$$\parallel\left(\phi^0,\phi^1\right)\parallel_{\left(H_0^1(\Omega)\right)^3\times(L^2(\Omega))^3}\geq 1,\ \psi^0\in B. \tag{4.14}$$

Step 2. Following [HLP] we introduce the decoupled system

$$\left\{\begin{array}{lll} \widetilde{\phi}_{tt}-\mu\Delta\widetilde{\phi}-(\lambda+\mu)\nabla\operatorname{div}\widetilde{\phi}-\alpha\beta P\widetilde{\phi}_t=0 & \text{in} & Q \\ -\widetilde{\psi}_t-\Delta\widetilde{\psi}-\alpha\operatorname{div}\widetilde{\phi}_t=0 & \text{in} & Q \\ \widetilde{\phi}=0,\widetilde{\psi}=0 & \text{on} & \Sigma \\ \widetilde{\phi}(x,T)=\phi^0(x),\widetilde{\phi}_t(x,T)=\phi^1(x),\widetilde{\psi}(x,T)=\psi^0(x) & \text{in} & \Omega \end{array}\right. \tag{4.15}$$

where P is the orthogonal projection from $\left(L^2(\Omega)\right)^3$ into the closed subspace

$$\left\{\nabla\varphi :\ \varphi \in H_0^1(\Omega)\right\}.$$

Both system (4.12) and (4.15) generate a continuous semigroup in H that we denote by $S(t)$ and $\widetilde{S}(t)$ respectively.

From [HLP] we know that $S-\widetilde{S} : H \to C([0,T];H)$ is a compact map.

Step 3. In view of the compactness of the difference between the two semigroups S and $\widetilde{S}$ it is sufficient to prove the observability for the hyperbolic system that $\widetilde{\phi}$ satisfies in (4.15).

More precisely, let us consider the system

$$\begin{cases} \widetilde{\phi}_{tt} - \mu\Delta\widetilde{\phi} - (\lambda+\mu)\nabla \operatorname{div}\widetilde{\phi} - \alpha\beta P\widetilde{\phi}_t = 0 & \text{in} \quad Q \\ \widetilde{\phi} = 0 & \text{on} \quad \Sigma \\ \widetilde{\phi}(T) = \phi^0, \widetilde{\phi}_t(T) = \phi^1 & \text{in} \quad \Omega, \end{cases} \tag{4.16}$$

and let us assume that the following observability inequality holds:

$$\parallel \left(\phi^0,\phi^1\right) \parallel^2_{\left(H_0^1(\Omega)\right)^3\times\left(L^2(\Omega)\right)^3} \leq C\int_0^T\int_\omega \mid \widetilde{\phi}_t \mid^2 dxdt. \tag{4.17}$$

Then, by a classical compactness-uniqueness argument (see for instance [L2], vol. 1, Appendix I) in view of the compactness of $S-\widetilde{S}$ and the uniqueness result of Theorem 4.3 one can deduce that (4.13) holds for solutions of (4.12) with initial data in (4.14).

Step 4. In [Z2] using multiplier techniques the observability inequality (4.17) has been proved when ω is a neighborhood of the boundary and $T > \operatorname{diam}(\Omega\backslash\omega)/\sqrt{\mu}$.

■

Remark 4.3. The proof of Theorem 4.1 shows that the exact-approximate controllability of system (4.1) holds whenever one has the observability inequality (4.17) for system (4.16). In [Z2] we proved (4.17) in the simple case in which ω is a neighborhood of the boundary and $T > \operatorname{diam}(\Omega\backslash\omega)/\sqrt{\mu}$. However, the problem of characterizing the pairs (ω, T) such that (4.17) holds in the spirit of the geometric control condition of [BLR] for the controllability of the wave equation is open.

■

4.3 Null controllability.

Let us assume that Ω is the unit cube of $\mathbb{R}^3$, i.e. $\Omega = (0,1)^3$ and consider system (4.1) with periodic boundary conditions instead of Dirichlet ones. This case feets in the framework of [LeZ] where M is assumed to be a compact, connected and C^∞ Riemannian manifold without boundary of dimension three.

Let us consider the linear system of thermoelasticity in the manifold M:

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\nabla\theta = f 1_\omega & \text{in} \quad Q \\ \theta_t - \Delta\theta + \beta \operatorname{div} u_t = 0 & \text{in} \quad Q \\ u(0) = u^0, u_t(0) = u^1, \theta(0) = \theta^1 & \text{in} \quad M, \end{cases} \tag{4.18}$$

where now $Q = M \times (0,T)$.

Let ω be an open subset of M. We are interested on controlling (4.18) by means of control functions f with support in $\overline{\omega} \times [0,T]$.

In (4.18) θ is again a scalar function and u is a vector-field. We denote by $\widetilde{H^s}(M)$ the space of vector fields with H^s regularity.

The following result was proved in [LeZ]:

Theorem 4.4 *Suposse that any geodesic curve in M of length $\sqrt{\mu}T$ intersects ω. Then for any $(u^0, u^1, \theta^0) \in H = \widetilde{H^0}(M) \times \widetilde{H^{-1}}(M) \times H^{-1}(M)$ such that $\int_M \theta^0 dx = 0$ there exists $f \in L^2\left(0,T; \widetilde{H^{-1}}(M)\right)$ with support in $\overline{\omega} \times [0,T]$ such that the solution of (4.18) satisfies*

$$u(T) = u_t(T) = 0,\ \theta(T) = 0 \ \text{ in } M. \tag{4.19}$$

Moreover, there exists $C > 0$ such that

$$\| f \|_{L^2(0,T;\widetilde{H^{-1}}(M))} \leq C \left\| \left(u^0, u^1, \theta^0\right) \right\|_H, \tag{4.20}$$

for all $(u^0, u^1, \theta^0) \in H$.

Remark 4.4.

(a) The condition we have imposed on (ω, T) is the geometric control condition for the linear system of elasticity in the manifold without boundary M. In this sense the result is sharp.

(b) The condition $\int_M \theta^0 dx = 0$ is necessary to drive the solution of (4.18) to zero by means of the sole control f acting on the hyperbolic equation. Indeed, integrating the heat equation in (4.18) on M one observes that $\int_M \theta(x,t)dx$ remains constant along trajectories. Therefore $\int_M \theta^0(x)dx = 0$ is a necessary condition for the existence of f such that the solution of (4.18) satisfies $\int_M \theta(x,T)dx = 0$.

(c) For simplicity of the proof we have chosen to state the result in the context of weak solutions rather than finite energy solutions as in Theorem 4.1. Consequently the control f is less regular and belongs to $L^2\left(0,T; \widetilde{H^{-1}}(M)\right)$. For this reason we have not multiplied the control by the characteristic function of the set ω. However, the controls under consideration have the support contained in $\overline{\omega} \times [0,T]$ as explicitely stated in the Theorem.

(d) As a consequence of Theorem 4.4 and when considering the Dirichlet problem in a bounded domain, one can obtain the null controllability with Dirichlet boundary controls acting on the whole boundary on both the displacement u and the temperature θ in time $T > \mathrm{diam}(\Omega)/\sqrt{\mu}$. To see this it sufficient to build a cube $\widetilde{\Omega}$ containing Ω and to construct controls for the system in $\widetilde{\Omega}$ with support in $\widetilde{\Omega}\backslash\Omega$. Then the restriction of the controlled solution to Ω satisfies the required conditions.

The extension of Theorem 4.4 to the case of Dirichlet boundary conditions with controls acting only on the hyperbolic equation and located on a subset ω of Ω satisfying the geometric control condition is an open problem.

■

Sketch of the proof of Theorem 4.4.
First of all we observe that the null controllability of system (4.18) is equivalent to an observability inequality for the adjoint system

$$\left\{\begin{array}{lll} \varphi_{tt} - \mu\Delta\varphi - (\lambda+\mu)\nabla\,\mathrm{div}\,\varphi + \beta\nabla\psi_t = 0 & \text{in} & Q \\ -\psi_t - \Delta\psi - \alpha\,\mathrm{div}\,\varphi = 0 & \text{in} & Q \\ \varphi(T) = \varphi^0, \varphi_t(T) = \varphi^1, \psi(T) = \psi^0 & \text{in} & M. \end{array}\right. \tag{4.21}$$

More precisely the controllability result of Theorem 4.4 is equivalent to the existence of a positive constant $C > 0$ such that:

$$\| \varphi(0) \|^2_{\widetilde{H^1}(M)} + \| \varphi_t(0) + \beta\nabla\psi(0) \|^2_{\widetilde{H^0}(M)} + \| \psi(0) \|^2_{H^1(M)} \leq C \int_0^T \| \varphi \|^2_{\widetilde{H^1}(\omega)} dt, \tag{4.22}$$

for every solution of (4.19) with $\displaystyle\int_M \psi^0 dx = 0$.

To prove (4.20) we proceed in several steps.

Step 1. We consider first the vector field $\sigma = \mathrm{curl}\,\varphi$ that satisfies

$$\left\{\begin{array}{lll} \sigma_{tt} - \mu\Delta\sigma = 0 & \text{in} & Q \\ \sigma(T) = \mathrm{curl}\,\varphi^0 = \sigma^0, \sigma_t(T) = \mathrm{curl}\,\varphi^1 = \sigma^1 & \text{in} & M. \end{array}\right. \tag{4.23}$$

By [BLR] we know that, under the geometric control condition imposed on ω and T , there exists $C > 0$ such that

$$\| \sigma(0) \|^2_{\widetilde{H^0}(M)} + \| \sigma_t(0) \|^2_{\widetilde{H^{-1}}(M)} \leq C \int_0^T \int_\omega | \sigma |^2 \, dxdt \tag{4.24}$$

holds for every solution of (4.23).

Step 2. We consider the pair (ρ, ψ) with $\rho = \mathrm{div}\,\varphi$ that satisfies

$$\left\{\begin{array}{lll} \rho_{tt} - (\lambda+2\mu)\Delta\varphi + \beta\Delta\psi_t = 0 & \text{in} & Q \\ -\psi_t - \Delta\psi - \alpha\rho = 0 & \text{in} & Q \\ \rho(T) = \mathrm{div}\,\varphi^0 = \rho^0, \rho_t(T) = \mathrm{div}\,\varphi^1 = \rho^1, \psi(T) = \psi^0 & \text{in} & M. \end{array}\right. \tag{4.25}$$

By means of the change of variable $\xi = -\Delta\psi$ this system is reduced to

$$\begin{cases} \rho_{tt} - (\lambda + 2\mu + \alpha\beta)\Delta\rho + \beta\Delta\xi = 0 & \text{in} \quad Q \\ -\xi_t - \Delta\xi + \alpha\Delta\rho = 0 & \text{in} \quad Q \\ \rho(T) = \rho^0, \rho_t(T) = \rho^1, \xi(T) = -\Delta\psi^0 & \text{in} \quad M. \end{cases} \tag{4.26}$$

In view of (4.24) to deduce (4.20) it is sufficient to prove that

$$\| \rho(0) \|^2_{L^2(M)} + \| \rho_t(0) - \beta\xi(0) \|^2_{H^{-1}(M)} + \| \xi(0) \|^2_{H^{-1}(M)} \leq C \int_0^T \int_\omega \rho^2 dxdt. \tag{4.27}$$

Step 3. The observability inequality (4.27) is equivalent to the null controllability of the system

$$\begin{cases} w_{tt} - c^2\Delta w + \alpha\Delta\eta = g1_\omega & \text{in} \quad Q \\ \eta_t - \Delta\eta + \beta w_t = 0 & \text{in} \quad Q \\ w(0) = w^0, w_t(0) = w^1, \eta(0) = \eta^0 & \text{in} \quad M, \end{cases} \tag{4.28}$$

with $c^2 = \lambda + 2\mu$. Observe that in (4.28) both w and η are scalar functions.

More precisely (4.27) holds if and only if for every $(w^0, w^1, \eta^0) \in H^1(M) \times L^2(M) \times H^1(M)$ with $\int_M (\eta^0 + \beta u^0)dx = 0$ there exists $g \in L^2(Q)$ such that the solution of (4.28) satisfies

$$w(T) = w_t(T) = \eta(T) = 0 \text{ in } M$$

and there exists $C > 0$ such that

$$\| g \|_{L^2(Q)} \leq C \| (w^0, w^1, \eta^0) \|_{H^1(M)\times L^2(M)\times H^1(M)}$$

for every (w^0, w^1, η^0) as above.

Step 4. The null controllability of (4.28) was proved in [LeZ] both in the case where $\partial M = \emptyset$ and $\partial M \neq \emptyset$ with Dirichlet boundary conditions.

The proof is based in a spectral decomposition that allows to split the semigroup generated by (4.28) with $f = 0$ into its hyperbolic and parabolic projection. This can be easily done since the spectrum can be computed almost explicitely because all the differential operators (with respect to the space variables) involved in (4.28) are powers of the laplacian.

Once the semigroup is decomposed into its hyperbolic and parabolic projection one proceeds as follows:

(a) The hyperbolic component is shown to be exactly controllable with controls supported in ω provided ω satisfies the geometric control condition of Theorem 4.4 in time $\sqrt{\lambda + 2\mu}T$ as a consequence of the results in [BLR].

(b) The parabolic component is shown to be null controllable for any open subset ω of M and $T > 0$ by the method described in section 3.3 to prove null controllability of the heat equation.

(c) Finally the problem of controlling to zero both the hyperbolic and parabolic components simultaneously is formulated and solved by means of Fredholm's alternative. At this level the compactness of the map that associates the $L^2(Q)$-control to every $L^2(M)$-initial data of the parabolic component is essential. In this way the problem is reduced to control a finite-dimensional subspace and this is done by means of a compactness-uniqueness argument.

■

Remark 4.5.

(a) The main point in which the fact that M has no boundary is used is when deriving the equations satisfied by $\operatorname{curl}\varphi = \sigma$ and $(\rho = \operatorname{div}\varphi, \psi)$. Obviously, when $\partial M \neq \emptyset$ systems (4.23) and (4.25) are coupled on the boundary ∂M and therefore they can not be treated separately as above.

(b) Observe that the restriction on the control time is imposed by the observability of (4.23) since system (4.25) is controllable in a shorter interval of time due to the higher speed of propagation.

(c) Similar problems can be formulated when the control acts through the parabolic equation, i.e. when the equations in (4.18) are replaced by

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\nabla\theta = 0 & \text{in} \quad Q \\ \theta_t - \Delta\theta + \beta \operatorname{div} u_t = g & \text{in} \quad Q. \end{cases}$$

In this case and always when $\partial M = \emptyset$ since $v = \operatorname{curl} u$ satisfies

$$v_{tt} - \mu\Delta u = 0 \ \text{ in } \ Q$$

i.e. v is not controlled at all, one can not expect the null controllability to hold. However, in this context and under the assumptions of Theorem 4.4 one can prove that $\operatorname{div} u$ and θ can indeed be driven to zero.

■

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